

# Analysis II – Lecture Notes

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April 17, 2017

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# Chapter 1

## Differentiation in several variables

### Lecture 1

#### Motivation

It's important to study the way functions in several variables change (leading to the notions of continuity and differentiability).

#### Example 1.1.

- Newton's gravitational law  $F = G \frac{Mm}{r^2}$
- Lagrange energy field  $L(q_1, q_2, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n)$ , where  $G_i = g_i(t), \forall i \in \{1, \dots, n\}$  are positive variables and  $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$  are the temporal derivatives of the position. Now we can write equations like  $L = K - V$ , where  $K$  denotes the kinetic energy and  $V$  the potential energy.

**Base idea** Understand differentiation as linearisation. In one dimension this corresponds to finding the tangent line to the graph of a function. In two dimensions we can look either for tangent lines (partial differentiability) or a tangent plane (total differentiability). We will see that total differentiability  $\implies$  partial differentiability.

### Norms, Metrics and Banach and Hilbert spaces

**Definition 1.2.** Let  $\mathcal{X}$  be a set. Now  $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}^{\geq 0}$  is called a norm iff:

1.  $\|x\| = 0 \Leftrightarrow x = 0$ .
2.  $\|\lambda x\| = |\lambda| \|x\|, \forall \lambda \in \mathbb{R}$
3.  $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in \mathcal{X}$

**Definition 1.3.** On  $\mathbb{R}^n$  we define the  $p$ -norm  $\|\cdot\|_p : \mathbb{R}^n \rightarrow \mathbb{R} \geq 0$  by:

$$\forall p \in \mathbb{N}, \forall x \in \mathbb{R}^n. \|x\|_p := \|(x_1, x_2, \dots, x_n)\|_p := \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}.$$

For  $p = 1$  we call the norm the Manhattan norm, for  $p = 2$  we call the norm the Euclidean norm, which is defined by the inner product  $\langle \cdot | \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\langle x | y \rangle := \sum_i x_i y_i$  in the sense that

$$\|x\|_2 := (\langle x | x \rangle)^{\frac{1}{2}}$$

**Exercise 1.** The  $p$ -norm is a norm on  $\mathbb{R}^n$ .

**Definition 1.4.** A vector space equipped with a norm is called normed vector space.

**Definition 1.5.** A complete normed vector space is called Banach space.

**Definition 1.6.** A Banach space with an inner product  $\langle x | y \rangle$  is called a Hilbert space iff its norm is defined by  $\|x\| := \sqrt{\langle x | x \rangle}$ .

**Definition 1.7.** Let  $\mathcal{X}$  be a set,  $\text{dist} : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R} \geq 0$  with

1.  $\text{dist}(x, y) \geq 0$  ;non-negativity
2.  $\text{dist}(x, y) = 0 \Leftrightarrow x = y$  ;identity of indiscernibles
3.  $\text{dist}(x, y) = \text{dist}(y, x)$  ;symmetry
4.  $\text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z)$  ;triangle inequality

, then  $\text{dist}$  is called a metric on  $\mathcal{X}$  and  $(\mathcal{X}, \text{dist})$  is called a metric space.

**Exercise 2.** If  $\mathcal{X}$  is a normed vector space, then  $\text{dist}(x, y) := \|x - y\|$  is a metric. In particular all  $p$ -norms on  $\mathbb{R}^2$  are equivalent i.e.

$$\exists C \in \mathbb{R}, C = C(x, p, q) \geq 1 : \frac{1}{C} \leq \frac{\|x\|_p}{\|x\|_q} \leq C, \quad \forall x \in \mathbb{R}^n.$$

This implies that all these norms define the same topology on  $\mathbb{R}^n$ .

**Remark.** Given a metric space  $(\mathcal{X}, \text{dist})$ , for any  $R \in \mathbb{R} \geq 0, x \in \mathcal{X}$  sets of the form

$$B(x, R) := \{y \in \mathcal{X} : \text{dist}(x, y) < R\},$$

are called “open balls in  $\mathcal{X}$ ” (centered at  $x$  with radius  $R$ ). For example, we define a topology in  $\mathbb{R}^n$  by saying that a set  $\mathcal{O} \subset \mathbb{R}^n$  is open iff  $\forall x \in \mathcal{O}, \exists R = R(x) > 0. B(x, R) \subset \mathcal{O}$ .

**Definition 1.8.** A sequence  $(x_k)_{k \in \mathbb{N}}$  in  $\mathbb{R}^n$  converges to  $x \in \mathbb{R}^n$  iff  $\|x_k - x\| \xrightarrow{k \rightarrow \infty} 0$ , or equivalently if  $\forall \epsilon > 0. \exists N \in \mathbb{N}, \forall n \geq N \in \mathbb{N}. \|x_n - x\| < \epsilon$  or equivalently  $\forall x \in \mathbb{R}^n, \forall \mathcal{O}$  open,  $x \in \mathcal{O}. \exists N \in \mathbb{N}. \forall n \geq N \in \mathbb{N}. x_n \in \mathcal{O}$ . If the limit exists we write  $\lim_{k \rightarrow \infty} x_k$ .

**Lemma 1.9.** Let  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ ,  $(x_k)_{k \in \mathbb{N}} = (x_{k_1}, x_{k_2}, x_{k_n})$ ,  $k \in \mathbb{N}$ . Now we have

$$\lim_{n \rightarrow \infty} x_k = x \Leftrightarrow \forall j \in \{1, 2, \dots, n\}. \lim_{n \rightarrow \infty} x_{n_j} = x_j.$$

*Proof.* “ $\Rightarrow$ ”: We easily observe

$$\forall x \in \mathbb{R}^n, \forall i \in \{1, 2, \dots, n\}. \|x\| = \sqrt{x_1^2 + \dots + x_n^2} \geq \sqrt{x_i^2} = |x_i|$$

Fix  $\epsilon > 0$ , for any  $i \in \{1, 2, \dots, n\}$  we have  $\epsilon'_i := \frac{\epsilon}{\sqrt{n}}$ ,  $\exists |x_{k_i} - x_i| < \epsilon' = \frac{\epsilon}{\sqrt{n}}$ .  $\forall k \geq N_\epsilon^i$ . Now take  $N_\epsilon := \max\{N_\epsilon^1, N_\epsilon^2, \dots, N_\epsilon^n\}$ , then

$$\|x_k - x\| = \left( \sum_i (x_{k_i} - x_i)^2 \right)^{\frac{1}{2}} < \sqrt{\sum_i \epsilon'^2} = \epsilon \quad \forall k \geq N_\epsilon.$$

□

**Example 1.10.**  $x_k := (\frac{1}{k}, \frac{1}{k})_{k \in \mathbb{N}} \Rightarrow \lim_{k \rightarrow \infty} x_k = (0, 0) = 0$  We can use this sequence, by applying sandwich theorem, on the components sequences of a sequence of vectors.

**Definition 1.11.** Let  $\mathcal{X}$  be any set. Now a point  $a \in \mathcal{X}$  is called an accumulation point of  $\mathcal{X}$  iff  $\forall \epsilon' > 0 : (B(a, \epsilon') \setminus \{a\}) \cap \mathcal{U} \neq \emptyset$ .

**Definition 1.12.** Let  $f : \mathcal{U} \rightarrow \mathbb{R}^n$ ,  $\mathcal{U} \subset \mathbb{R}^n$ , let  $a$  be an accumulation point for  $\mathcal{U}$ . If  $\exists L \in \mathbb{R}^n$ ,  $\forall \epsilon > 0. \exists \delta > 0. \|f(x) - L\| < \epsilon \quad \forall x \in \mathcal{U} \cap (B(a, \delta) \setminus \{a\})$ , then  $L$  is the limit of  $f$  at  $a$ , denoted by  $L = \lim_{x \rightarrow a} f(x)$ .

**Remark.** One could also write (equivalent definition):  $\forall \epsilon > 0, \exists \delta > 0. x \neq a \Rightarrow \|x - a\| < \delta \Rightarrow \|f(x) - L\| < \epsilon$ .

**Lemma 1.13** (Another equivalent definition). Given a function  $f : \mathcal{U} \rightarrow \mathbb{R}^n$  with  $\mathcal{U} \subset \mathbb{R}^n$  and given an accumulation point  $a$  for  $\mathcal{U}$ , we have

$$\exists \lim_{x \rightarrow a} f(x) = L \in \mathbb{R}^n \Leftrightarrow \forall (x_k) \in \mathcal{U} \parallel \lim_{k \rightarrow \infty} x_k = a \Rightarrow \lim_{k \rightarrow \infty} f(x_k) = L.$$

**Example 1.14.**

$$\bullet f : \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) := \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$\text{-- Approach } (0, 0) \text{ along the } x\text{-axis: } \longrightarrow f(x_k, y_k) = f(\frac{1}{k}, 0) = \frac{\frac{1}{k} \cdot 0}{\frac{1}{k^2} + 0^2} = 0 \xrightarrow{k \rightarrow \infty} 0.$$

$$\text{-- Approach } (0, 0) \text{ along the } y\text{-axis: } \longrightarrow f(x_k, y_k) = f(0, \frac{1}{k}) = \frac{0 \cdot \frac{1}{k}}{0^2 + \frac{1}{k^2}} = 0 \xrightarrow{k \rightarrow \infty} 0.$$

$$\text{-- Approach } (0, 0) \text{ along the main diagonal: } \longrightarrow f(x_k, y_k) = f(\frac{1}{k}, \frac{1}{k}) = \frac{\frac{1}{k^2}}{\frac{1}{k^2} + \frac{1}{k^2}} = \frac{1}{2} \xrightarrow{k \rightarrow \infty} \frac{1}{2}.$$

Hence,  $f$  is not continuous at 0.

$$\bullet \ g : \mathbb{R}^n \rightarrow \mathbb{R}, g(x, y) := \begin{cases} \frac{xy^3}{x^4+y^4} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

– Take  $y_k := cx_k$  and yield.

$$g(x_k, y_k) = \frac{x_k cx_k^3}{x_k^4 + c^4 x_k^4} = \frac{c^3}{1 + c^4} \Rightarrow \nexists \text{ limit.}$$

$$\bullet \ h : \mathbb{R}^n \rightarrow \mathbb{R}, h(x, y) := \begin{cases} \frac{x^2}{x+y} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$

As it turns out this does not converge either.

## Lecture 2

**Definition 2.15.** Let  $f : \mathcal{U} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$  and assume let  $a \in \mathbb{R}^n$  be an accumulation point. Now if  $a \in \mathcal{U}$  and if  $\exists \lim_{n \rightarrow \infty} f(x) = f(a)$ , then we say that  $f$  is continuous at  $a$ . If all points in  $\mathcal{U}$  are accumulation points for  $\mathcal{U}$  and  $f$  is continuous in every point of  $\mathcal{U}$ , then we call  $f$  continuous on  $\mathcal{U}$ .

**Example 2.16.** 1.  $f : \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) := \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq 0 \\ 0 & (x, y) = 0 \end{cases}$ . We

have seen that  $\nexists \lim_{(x,y) \rightarrow (0,0)} f(x, y)$ , hence  $f$  is not continuous in 0. How about the other points in the domain?

$$\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) \Leftrightarrow \forall y_n \xrightarrow{x_n \rightarrow x_0} y_0. \exists \lim_{n \rightarrow \infty} f(x_n, y_n)$$

By the usual properties of sequences of reals and their limits, we obtain:

$$\begin{matrix} x_n & \xrightarrow{n \rightarrow \infty} & x_0 \neq 0 \\ y_n & \xrightarrow{n \rightarrow \infty} & y_0 \neq 0 \end{matrix} \Rightarrow \frac{x_n y_n}{x_n^2 + y_n^2} \xrightarrow{n \rightarrow \infty} \frac{x_0 y_0}{x_0^2 + y_0^2},$$

hence  $f$  is continuous on  $\mathbb{R}^2 \setminus \{(0, 0)\}$ , since  $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = \lim_{(x,y) \rightarrow (x_0, y_0)} \frac{xy}{x^2+y^2} = \frac{x_0 y_0}{x_0^2 + y_0^2} = f(x_0, y_0)$

$$2. \ f : \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) := \begin{cases} \frac{x^4 + 2x^2 y^2}{x^2 + y^2} & (x, y) \neq 0 \\ 0 & (x, y) = 0 \end{cases}. \text{ As before, we can}$$

conclude that  $g$  is continuous on  $\mathbb{R}^2 \setminus \{0\}$ . Furthermore, we have

$$g(x, y) = \frac{x^2 + 2y^2}{x^2 + y^2} x^2 \leq \frac{x^4 + 2x^2 y^2 + x^2}{x^2 + x^2} = x^2 + y^2 = \|(x, y)\|^2 \xrightarrow{(x,y) \rightarrow (x_0, y_0)} 0.$$

Thus  $g$  is continuous on its domain.

**Theorem 2.17.** Let  $f, g : \mathcal{U} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $a$  be an accumulation point for  $\mathcal{U}$ . Then assuming that the limits on the right-hand-side exist, we obtain the following identities:

(i)

$$\lim_{x \rightarrow a} f(x) + g(x) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

(ii)

$$\lim_{x \rightarrow a} \langle f(x) | g(x) \rangle = \left\langle \lim_{x \rightarrow a} f(x) \middle| \lim_{x \rightarrow a} g(x) \right\rangle$$

(iii)

$$\lim_{x \rightarrow a} f(x)g(x) = \left( \lim_{x \rightarrow a} f(x) \right) \left( \lim_{x \rightarrow a} g(x) \right)$$

If  $g(x) \neq 0$ ,  $\forall x \in \mathcal{U}$  and if  $\exists \lim_{x \rightarrow a} g(x)$ , then we further have:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$$

1. If we write  $f : \mathcal{U} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$  with  $f = (f_1, f_2, f_3, \dots, f_n)$  with  $f_j : \mathcal{U} \rightarrow \mathbb{R} \forall j \in \{1, 2, \dots, n\}$  then

$$\exists \lim_{x \rightarrow a} f(x) = L \in \mathbb{R}^m = (L_1, L_2, \dots, L_n) \Leftrightarrow \forall j \in \{1, 2, \dots, m\}. \exists \lim_{x \rightarrow a} f_j(x) = L_j$$

*Proof.* (i), (iii) follow from the definitions, from lemma 1.9 and the additivity of limits on  $\mathbb{R}$ . (iv) follows from:

$$\|f(x) - L\| \leq |f_j(x) - L_j| \quad \forall j \in \{1, 2, \dots, n\}.$$

(iii) follows from (i), (iii) and (iv).  $\square$

**Remark.** The analogous rules also hold for continuity i.e. sums, products, inner product, components of continuous functions are also continuous. Furthermore, sum, inner product, product with scalar are continuous themselves.

**Definition 2.18** (Cauchy Sequence). Let  $(\mathcal{X}, \text{dist})$  be a metric space and let  $a_n$  in  $\mathcal{X}$ . Now  $a_n$  is called Cauchy (or a Cauchy sequence) iff  $\forall \epsilon > 0. \exists N \in \mathbb{N}. \forall n, m \geq N. \text{dist}(a_n, a_m) < \epsilon$ .

**Definition 2.19** (A complete metric space). A metric space  $(\mathcal{X}, \text{dist})$  is called complete iff every Cauchy sequence in the space is convergent.

**Theorem 2.20.**  $(\mathbb{R}^n, \|\cdot\|)$  is Banach.

*Proof.* Let  $(x_k)$  be a Cauchy sequence. Then  $\exists N_\epsilon \in \mathbb{N}. \|x_k - x_l\| < \epsilon, \forall k, l \geq N_\epsilon \Rightarrow (x_{k_i})$  Cauchy  $\forall i \in \{1, 2, \dots, n\}$ . Then  $x_{k_i}$  converges to some  $x_i \in \mathbb{R} \forall i \in \{1, 2, \dots, n\}$ . Now put  $x := (x_1, x_2, \dots, x_n)$  and apply lemma 1.9. It follows  $x_k \rightarrow x$ . Therefore,  $(\mathbb{R}^n, \|\cdot\|)$  is Banach.  $\square$

### Lecture 3

**Remark.** An inner product, allows us to “measure” angles between vectors.

## Differentiability; linear maps, matrices and the operator norm

**Definition 3.21.** Let  $\mathcal{X}, \mathcal{Y}$  be two vector spaces over  $\mathbb{R}$ . A map  $A : \mathcal{X} \rightarrow \mathcal{Y}$  is called linear iff  $A(\alpha x + \beta y) = \alpha A(x) + \beta A(y) \quad \forall \alpha, \beta \in \mathbb{R}, \forall x, y \in \mathcal{X}$ . The set of linear maps  $\mathcal{X} \rightarrow \mathcal{Y}$  is denoted by  $L(\mathcal{X}, \mathcal{Y})$ .  $L(\mathcal{X}, \mathcal{X})$  is denoted  $L(\mathcal{X})$ . Often we write  $Ax$  instead of  $A(x)$ .

**Facts:**

1.  $A0 = 0 \quad \forall A \in L(\mathcal{X}, \mathcal{Y})$ .
2. If  $\mathcal{X}$  is finite dimensional (e.g.  $\mathbb{R}^n$ ), then  $A \in L(\mathcal{X})$  bijective  $\Leftrightarrow A$  injective  $\Leftrightarrow A$  surjective.

*Proof Sketch.*  $A$  surjective  $\Leftrightarrow \langle A\mathcal{X} \rangle = \langle \mathcal{X} \rangle \Leftrightarrow \langle A^{-1}\mathcal{X} \rangle = \langle \mathcal{X} \rangle \Leftrightarrow A$  injective (by linearity).  $\square$

3.  $\forall A_1, A_2 \in L(\mathcal{X}, \mathcal{Y}), \forall \alpha_1, \alpha_2 \in \mathbb{R} :$

$$(\alpha_1 A_1 + \alpha_2 A_2) \in L(\mathcal{X}, \mathcal{Y}).$$

Thus  $L(\mathcal{X}, \mathcal{Y})$  is a vector space.

**Definition 3.22.** The set  $\mathcal{M}^{m \times n}(\mathbb{R})$  of  $(m \times n)$ -matrices  $\forall m, n \in \mathbb{N}$  fixed consists of all vectors in  $\mathbb{R}^{m \cdot n}$  written in a “separable form” i.e.

$$A := (a_{i,j})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} =: \begin{pmatrix} a_{1,1}, a_{1,2}, \dots, a_{1,n} \\ a_{2,1}, a_{2,2}, \dots, a_{2,n} \\ \vdots \\ a_{m,1}, a_{m,2}, \dots, a_{m,n} \end{pmatrix}.$$

**Definition 3.23.** Matrix multiplication is defined by: Let  $A \in \mathcal{M}^{n \times m}(\mathbb{R}), B \in \mathcal{M}^{m \times p}(\mathbb{R})$ . Then  $C := A \cdot B := (c_{k,j})_{\substack{1 \leq j \leq n \\ 1 \leq k \leq p}}$ , where  $c_{k,j} = \sum_{i=1}^m a_{i,j} b_{k,i} \in \mathcal{M}^{n \times p}(\mathbb{R})$ . Furthermore,  $A \in \mathcal{M}^{m \times n}(\mathbb{R})$  is called invertible iff  $\exists A^{-1} \in \mathcal{M}^{n \times m}(\mathbb{R})$ .

$$AA^{-1} = A^{-1}A = I_n := \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots \\ & & \vdots & & \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad (\Rightarrow m = n), \text{ where } I_n \text{ is the identity}$$

matrix.

**Remark.** Matrix multiplication is not commutative (only associative, as is function composition in general).

**Remark.**  $\mathcal{M}^{m \times n}$  is a vector space over  $\mathbb{R}$  for all  $m, n \in \mathbb{N}$ .

4. Let  $\mathcal{X}, \mathcal{Y}$  be finite dimensional vector spaces with basis  $\{x_1, x_2, \dots, x_n\}, \{y_1, y_2, \dots, y_m\}$ . We can now use these basis to define the natural isomorphism between  $\mathcal{M}^{m \times n}$  and

$$L(\mathcal{X}, \mathcal{Y}) : A_{i,j} \rightarrow \left( (x_k)_{k \in \{1,2,\dots,n\}} \rightarrow \left( \sum_{k=1}^m a_{i,k} \right)_{i \in \{1,2,\dots,n\}} \right).$$

By linearity and item (2) we see that this mapping is a linear bijection, thus it is an isomorphism. Sometimes we will denote the matrix associated to an linear map (given a fixed basis)  $A$  by  $[A]$ . Usually we will not distinguish between  $A$  and  $[A]$ .



5. Let  $A \in L(\mathcal{X}, \mathcal{Y})$ ,  $B \in L(\mathcal{Y}, \mathcal{Z})$  be linear maps, then  $B \circ A \in L(\mathcal{X}, \mathcal{Z})$  and  $[B \circ A] = [B] \cdot [A]$ .
6.  $A \in L(\mathcal{X})$  invertible  $\Leftrightarrow [A]$  invertible (under multiplication)  $\Leftrightarrow \det(A) \neq 0$ .

**Remark.**  $\mathcal{M}^{m \times n}$  is topologically just  $\mathbb{R}^{m \cdot n}$ .

**Definition 3.24** (operator norm). For  $A \in L(\mathcal{X}, \mathcal{Y})$ , define the operator norm by  $\|A\| := \sup_{\|x\| \leq 1} \|Ax\|$ .

**Remark.** operator norm not isometric to  $\|\cdot\|_2$ .

**Lemma 3.25.** If  $\alpha \in \mathbb{R}$  has the property that  $\|Ax\| \leq \alpha \|x\| \forall x \in \mathbb{R}^n$ , then  $\|A\| \leq \alpha$ .

*Proof.*  $\forall x \in \mathbb{R}^n, \|x\| \leq 1$ , we have that  $\|Ax\| \leq \alpha \|x\| \leq \alpha$ . Thus, by definition of the operator norm  $\|A\| = \sup_{\|x\| \leq 1} \|Ax\| \leq \alpha$ .  $\square$

**Theorem 3.26** (Properties of the operator norm). 1.  $\|Ax\| \leq \|A\| \|x\|, \forall x \in \mathbb{R}^n$ .

2.  $\|A\| < \infty, \forall A \in L(\mathbb{R}^n, \mathbb{R}^m)$ .

3.  $\forall A \in L(\mathbb{R}^n, \mathbb{R}^m)$ .  $A$  Lipschitz i.e.  $\exists$  a real constant (here the operator norm) s.t.  $\|Ax - Ay\| \leq \|A\| \|x - y\|, \forall x, y \in \mathbb{R}^n$ .

**Exercise 3.** This implies uniform continuity.

4. The operator norm is a norm i.e.  $\forall A, B \in L(\mathbb{R}^n, \mathbb{R}^m), \forall \alpha \in \mathbb{R}$ , we have:

(a)  $\|A + B\| \leq \|A\| + \|B\|$ , the triangular inequality

(b)  $\|\alpha A\| = |\alpha| \|A\|$

5.  $\forall A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ .  $\|BA\| \leq \|B\| \|A\|$ .

6. With  $I(\mathbb{R}^n) \subset L(\mathbb{R}^n) := \{A \in L(\mathbb{R}^n) \text{ invertible}\}$  we have that for  $A \in I(\mathbb{R}^n)$ ,  $B \in L(\mathbb{R}^n)$  and  $\|B - A\| \|A^{-1}\| < 1$  it follows  $B \in I(\mathbb{R}^n)$ . In other words,  $\forall A \in I(\mathbb{R})$ , we have that  $B(A, \frac{1}{\|A^{-1}\|}) \subset I(\mathbb{R}^n)$ . Thus,  $I(\mathbb{R}^n)$  is open in  $\mathbb{R}^n$ .

7.  $\forall A \in I(\mathbb{R}^n)$ .  $A^{-1}$  is continuous.

*Proof.*

1. Fix  $x \in \mathbb{R}, x \neq 0$  then for  $\frac{x}{\|x\|}$  we obtain  $\left\| \frac{x}{\|x\|} \right\| = \frac{1}{\|x\|} \|x\| = 1$ . By definition  $\|A\| \geq \left\| A \left( \frac{x}{\|x\|} \right) \right\| = \frac{1}{\|x\|} \|A(x)\| \Rightarrow \|A\| \|x\| \geq \|A(x)\|$ . The case  $x = 0$  is trivial.
2.  $\forall A \in L(\mathbb{R}^m, \mathbb{R}^n)$ .  $\|A\| \leq m \cdot n \cdot \max(A) < \infty$ .
3. Follows from linearity and item 2.
4. (a) Obvious.  
(b) By linearity (preserved by sup)

5. By item 1 and Lemma 3.25:

$$\|BAx\| \leq \|BA\| \|x\| \leq \|B\| \|A\| \|x\| \quad (1.1)$$

6. Using item 1 and linearity:

$$\begin{aligned} \frac{\|x\|}{\|A^{-1}\|} &= \frac{\|A^{-1}Ax\|}{\|A^{-1}\|} \stackrel{\text{item 1}}{\leq} \frac{\|A^{-1}\|}{\|A^{-1}\|} \|Ax\| = \|Ax\| \\ &= \|(A - B)x + Bx\| \stackrel{\text{item 4}}{\leq} \|A - B\| \|x\| + \|Bx\| \end{aligned}$$

Therefore by assumption  $0 < \frac{1}{\|A^{-1}\|} \|x\| \leq \|Bx\|$ . Thus,  $x \neq 0 \Rightarrow 0 < \|Bx\| \Rightarrow B$  invertible.

7. By item 5 and item 6:  $\|BA\| \leq \|B\| \|A\|$ .

□

**Lemma 3.27.**  $\|A\| = 0 \Rightarrow A$  not invertible.

*Proof.* Assume  $\|A\| = 0 \Rightarrow \forall x \in \mathbb{R}^n. 0 = \|A\| \|x\| \geq \|Ax\| \Rightarrow \|Ax\| = 0$ . Thus  $A$  is not invertible at all. □

## Lecture 4

**Definition 4.28.** Let  $f : \mathcal{U} \rightarrow \mathbb{R}$  be some function and  $x_0$  an accumulation point of  $\mathcal{U}$ . We say that  $f$  is differentiable in  $x_0$  iff  $\exists \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} =: f'(x)$ . Another way to write this is  $\exists \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} =: f'(x)$ .

**Remark.** Equivalently,  $f$  is differentiable in  $x_0 \Leftrightarrow \exists$  linear map  $A \in L(\mathbb{R})$  s.t.  $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0) - A(x - x_0)}{|x - x_0|} = 0$ .

So how can we generalize this to several dimensions?

**Remark.** From now on we will mostly consider functions on ope sets, thus all points in the domain will be open.

**Definition 4.29.** Let  $f : \mathcal{U} \rightarrow \mathbb{R}^n, \mathcal{U} \subset \mathbb{R}^m$  open, be a map and let  $x_0 \in \mathcal{U}$ . The map  $f$  is called (totally) differentiable in  $x_0$  iff  $\exists A \in L(\mathbb{R}^n, \mathbb{R}^m)$ , so that

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0) - A(x - x_0)}{\|x - x_0\|} = 0,$$

or

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0) - A(x_0)h}{h} = 0.$$

The linear approximation  $A$  of  $f$  is called the total differential of  $f$  in  $x_0$  and is denoted by  $D_x f$  or  $Df(x_0)$  or  $df(x_0)$  or  $f'(x_0)$ .

**Lemma 4.30** (Uniqueness). If  $A_1, A_2 \in L(\mathbb{R}^n, \mathbb{R}^m)$  both satisfy the above condition, then  $A_1 = A_2$ .

*Proof.* Before applying the definition of differentiability twice, we observe:

$$\begin{aligned} & \|(A_1 - A_2)h\| \\ &= \|A_1h - A_2h\| \\ &= \|f(x_0 + h) - f(x_0) - A_2h - (f(x_0 + h) - f(x_0) - A_1h)\| \\ &\leq \|f(x_0 + h) - f(x_0) - A_2h\| + \|f(x_0 + h) - f(x_0) - A_1h\| \end{aligned}$$

Now divide by  $\|h\|$  (assuming  $h \in \mathbb{R}^n \setminus \{0\}$ ):

$$0 \leq \frac{\|(A_1 - A_2)h\|}{\|h\|} \leq \frac{\|f(x_0 + h) - f(x_0) - A_1h\|}{\|h\|} + \frac{\|f(x_0 + h) - f(x_0) - A_2h\|}{\|h\|}.$$

Thus,  $\lim_{h \rightarrow 0} \frac{\|(A_1 - A_2)h\|}{\|h\|} = 0$ .

Consider some (fixed but arbitrary)  $h \in \mathbb{R}^n$  and look the  $th, \forall t \in \mathbb{R}$ . The limit being 0 means in particular that  $\lim_{t \rightarrow 0} \frac{\|(A_1 - A_2)(th)\|}{\|th\|} = 0$ . By linearity of  $A_1, A_2$ :  $\frac{\|t(A_1 - A_2)h\|}{\|th\|} = \frac{\|(A_1 - A_2)h\|}{\|h\|}$ . But doesn't depend on  $t$  anymore.  $\forall h \in \mathbb{R}^n \setminus \{0\} \Rightarrow \frac{\|(A_1 - A_2)h\|}{\|h\|} = 0$  and  $\|(A_1 - A_2)h\| = 0$ . It follows that  $A_1 = A_2$ .  $\square$

**Remark.** In the definition of differentiability one can also write  $\lim_{x \rightarrow x_0} \frac{\|f(x) - f(x_0) - A(x - x_0)\|}{\|x - x_0\|} = 0$ .

**Remark.**  $\lim_{x \rightarrow x_0} \|g(x)\| = c \neq 0 \not\Rightarrow \exists \lim_{x \rightarrow x_0} g(x)$ .

**Counter-Example 4.31.**  $h(x) := \begin{cases} 1, & x \in (0, \frac{1}{2}) \\ 0, & x \in [\frac{1}{2}, 1] \end{cases}$ . Define  $g : [0, 1] \rightarrow \mathbb{R}^2$  by  $g(x) := \begin{pmatrix} c \cos(x) \\ c \sin(x) \end{pmatrix}$ , and  $\|g\|$  at  $x = \frac{1}{2}$ :  $\lim_{x \rightarrow \frac{1}{2}} \|g(x)\| = 0$ , but  $\exists \lim_{x \rightarrow \frac{1}{2}} g(x)$ .

### Example 4.32.

- The differential of  $A \in L(\mathbb{R}^n, \mathbb{R}^m)$  in any point is  $A$  itself.
- Affine maps: Consider any affine map  $\mathbb{R}^n \rightarrow \mathbb{R}^m, x \mapsto Ax + b, \forall b \in \mathbb{R}^m, A \in L(\mathbb{R}^n, \mathbb{R}^m)$ . Here the differential is  $A$  again.

**Proposition 4.33.**  $f : \mathcal{U} \rightarrow \mathbb{R}^m$  is differentiable in  $x_0 \in \mathcal{U} \Leftrightarrow \exists \varphi : \mathcal{U} \rightarrow \mathbb{R}^m, \exists A \in L(\mathbb{R}^n, \mathbb{R}^m)$  s.t.  $f(x) = f(x_0) + A(x - x_0) + \varphi(x - x_0)$ , where  $\lim_{x \rightarrow x_0} \frac{\varphi(x - x_0)}{\|x - x_0\|} = 0$ .

## Lecture 5

**Proposition 5.34.** If any function  $f : \mathcal{U} \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ , for  $\mathcal{U}$  open is differentiable in  $x_0 \in \mathcal{U}$ , it is also continuous in  $x_0$ .

*Proof.* By differentiability in  $x_0$ ,  $\exists D$  s.t.

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + \varphi(x - x_0)$$

with  $\lim_{x \rightarrow x_0} \frac{\varphi(x - x_0)}{\|x - x_0\|} = 0$ . Thus  $\lim_{x \rightarrow x_0} \varphi(x - x_0) = 0$ , so  $\lim_{x \rightarrow x_0} f(x) = f(x_0) + Df(x_0)(x - x_0) = f(x_0)$ .  $\square$

**Proposition 5.35** (linearity). *Let  $f, g : \mathcal{U} \rightarrow \mathbb{R}^m$  differentiable in  $x_0 \in \mathcal{U}$ ,  $\alpha, \beta \in \mathbb{R}$ ,  $\mathcal{U} \subset \mathbb{R}^n$  open, then  $\alpha f + \beta g : \mathcal{U} \rightarrow \mathbb{R}^m$  is differentiable in  $x_0$  and  $D(\alpha f + \beta g)(x_0) = \alpha Df(x_0) + \beta Dg(x_0)$ .*

*Proof.* Plugging in the definition of differentiability, we yield:

$$\begin{aligned} & \lim_{x \rightarrow x_0} f(\alpha x + \beta x) \\ &= \lim_{x \rightarrow x_0} f(\alpha x_0 + \beta x_0) + Df(\alpha x + \beta x - \alpha x_0 - \beta x_0) + \varphi(\alpha x + \beta x - \alpha x_0 - \beta x_0) \\ &\stackrel{\varphi \rightarrow 0}{=} \lim_{x \rightarrow x_0} f(\alpha x_0 + \beta x_0) + Df(\alpha x + \beta x - \alpha x_0 - \beta x_0) \end{aligned}$$

By definition of differentiability  $Df$  is linear:

$$\Rightarrow Df(\alpha x + \beta x - \alpha x_0 - \beta x_0) = \alpha Df(x_0) + \beta Dg(x_0)$$

□

**Theorem 5.36** (chain rule). *Let  $f : \mathcal{U} \rightarrow \mathbb{R}^m$ ,  $\mathcal{U} \subset \mathbb{R}^n$  open be differentiable in  $x_0 \in \mathcal{U}$  and  $g : \mathcal{V} \rightarrow \mathbb{R}^p$ , for  $\mathcal{V} \subset \mathbb{R}^m$  open with  $y_0 := f(x_0) \in \mathcal{V}$ . Then  $g \circ f : \mathcal{U} \rightarrow \mathbb{R}^p$  is differentiable in  $x_0$  and  $D(g \circ f)(x_0) = Dg(f(x_0)) \circ Df(x_0)$ .*

*Proof.* From differentiable of  $f$  in  $x_0$  and  $g$  in  $y_0$ , we obtain:

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + \varphi(x - x_0), \lim_{x \rightarrow x_0} \frac{\varphi(x - x_0)}{\|x - x_0\|} = 0 \quad (1.2)$$

$$f(y) = f(y_0) + Df(y_0)(y - y_0) + \varphi(y - y_0), \lim_{y \rightarrow y_0} \frac{\varphi(y - y_0)}{\|y - y_0\|} = 0 \quad (1.3)$$

Putting  $y := f(x)$ ,  $x \in \mathcal{U}$ , we get (allowed since  $f$  continuous):

$$y - y_0 = f(x) - f(x_0) = Df(x_0)(x - x_0) + \varphi(x - x_0) \quad (1.4)$$

Using this equations, we want to give a linear approximation of  $g \circ f$  in  $g(f(x_0)) = g(y_0)$ :

$$\begin{aligned} & g(f(x)) \\ &\stackrel{(1.3)}{=} g(f(x_0)) + (Dg(f(x_0))(x - x_0) + \varphi(x - x_0)) + \varphi(f(x) - f(x_0)) \\ &\stackrel{(1.2)}{=} g(f(x_0)) + Dg(f(x_0))(Df(x_0)(x - x_0) + \varphi(x - x_0)) + \varphi(f(x) - f(x_0)) \\ &\stackrel{\text{linearity}}{=} g(f(x_0)) + Dg(f(x_0))(Df(x_0)(x - x_0)) + Dg(f(x_0))\varphi(x - x_0) + \psi(f(x) - f(x_0)). \end{aligned}$$

Defining  $\psi(\theta(fx - x_0)) := \psi(f(x) - f(x_0)) + Dg(f(x_0))\varphi(x - x_0)$ :

$$= (g \circ f)(x_0) + D(g \circ f)(x_0)(x - x_0) + \theta(x - x_0).$$

Firstly, since  $Dg(f(x_0)) \in L(\mathbb{R}^m, \mathbb{R}^n)$ , it follows from

$$\begin{aligned} & \lim_{x \rightarrow x_0} \frac{\varphi(x - x_0)}{\|x - x_0\|} = 0, \text{ that} \\ & \lim_{x \rightarrow x_0} \frac{Dg(f(x_0))\varphi(x - x_0)}{\|x - x_0\|} = 0 \end{aligned}$$

Secondly, define  $\varphi_0$  by  $\varphi(y-y_0) =: \|y - y_0\| \Psi_0(y-y_0)$ , so  $\lim_{x \rightarrow x_0} \Psi_0(y-y_0) = 0$  and use (1.2), the triangular inequality and the operator norm to estimate:

$$\begin{aligned} \|\Psi(f(x) - f(x_0))\| &= \|f(x) - f(x_0)\| \|\Psi_0(f(x) - f(x_0))\| = \|df(x_0)(x - x_0) + \varphi(x - x_0)\| \\ &= \|\Psi_0(f(x)) - f(x_0)\| \leq (\|Df(x_0)(x - x_0)\| + \|\varphi(x - x_0)\|) \|\Psi_0(f(x) - f(x_0))\| \\ &= \|Df(x_0)\| \|x - x_0\| \end{aligned}$$

Division by  $\|x - x_0\|$  yields:

$$\begin{aligned} 0 &\leq \frac{\|\Psi(f(x) - f(x_0))\|}{\|x - x_0\|} \\ &\leq \left( \|Df(x_0)\| + \frac{\|\varphi(x - x_0)\|}{\|x - x_0\|} \right) \|\Psi(f(x) - f(x_0))\| \end{aligned}$$

Since  $\lim_{z \rightarrow 0} \Psi_0(z) = 0$  and  $f$  is continuous in  $x_0$ , so  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ , we obtain  $\lim_{x \rightarrow x_0} \Psi_0(f(x) - f(x_0)) = 0$ . We can conclude

$$\lim_{x \rightarrow x_0} \text{RHS} = 0 \Rightarrow \lim_{x \rightarrow x_0} \frac{\|\Psi(f(x) - f(x_0))\|}{\|x - x_0\|} = 0.$$

□

**Proposition 5.37** (Lipschitz continuity and  $Df = 0 \Rightarrow f$  constant). *Let  $f : \mathcal{U} \rightarrow \mathbb{R}^m$ ,  $\mathcal{U} \subset \mathbb{R}^n$  open and convex, i.e.  $\forall u, v \in \mathcal{U}, \forall t \in [0, 1] : (1-t)u + tv \in \mathcal{U}$ , be differentiable in  $\mathcal{U}$  s.t.  $\exists M \geq 0$  with  $\|Df(x)\| \leq M, \forall x \in \mathcal{U}$ . Then,  $f$  is  $M$ -Lipschitz continuous on  $\mathcal{U}$ , i.e.  $\|f(x) - f(y)\| \leq M \|x - y\|, \forall x, y \in \mathcal{U}$ . In particular, if  $M = 0$  meaning that  $Df$  is constant on  $\mathcal{U}$  then  $f$  is a constant function.*

*Proof.* To be continued... □

**Remark.** The special case is a generalisation of the statement that if  $f : I \rightarrow \mathbb{R}$  is differentiable with  $f' \equiv 0$ , then  $f$  must be constant.

**Definition 5.38.** Let  $f : \mathcal{U} \rightarrow \mathbb{R}, \mathcal{U} \subset \mathbb{R}^n$  open and consider  $x_0 \in \mathcal{U}$  and  $v \in \mathbb{R}^n$  so that  $\|v\| = 1$ . If the limit exists  $D_v f(x_0) := \lim_{t \rightarrow 0} \frac{f(x_0 + tv) - f(x_0)}{t}$ , it is called the directional derivative of  $f$  in  $x_0$  in direction of  $v$ .

**Example 5.39.**  $f : \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) := \begin{cases} \frac{xy}{x^2 + y^2} & (x, y) \neq 0 \\ 0 & \text{otherwise} \end{cases}$ . We can check

that the partial and even the total differential exist in all  $(x, y) \neq 0$ , but  $f$  is not continuous in 0. It is, however, more interesting to look for  $f$  in 0. So we will later come back to this example.

**Definition 5.40.** If  $v = e_j \quad \forall j \in \{1, 2, \dots, n\}$  in previous definition, then the corresponding directional derivatives the  $j$ -th partial derivatives of  $f$  in  $x_0$  denoted by  $f_{x_j}(x_0), D_j(f(x_0)), \frac{\partial f}{\partial x_j}(x_0)$ .

**Example 5.41.**

$$1. f : \mathbb{R} \rightarrow \mathbb{R}, f(x, y) := x^2 + y^2, \frac{\partial f}{\partial x}((x, y)) = 2x, \frac{\partial f}{\partial y}((x, y)) = 2y$$

$$2. \ g : \mathbb{R}^2 \rightarrow \mathbb{R}, g(x, y) := \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq 0 \\ 0 & \text{otherwise} \end{cases} . \quad \frac{\partial g}{\partial x}((0, 0)) = \lim_{t \rightarrow 0} \frac{g((t, 0))}{t} = \lim_{t \rightarrow 0} \frac{0}{t} = 0, \quad \frac{\partial g}{\partial y}((0, 0)) = \lim_{t \rightarrow 0} \frac{g((0, t))}{t} = \lim_{t \rightarrow 0} \frac{0}{t} = 0. \text{ However, } g \text{ is not continuous in } 0 \text{ let alone differentiable: } y := \alpha x_i, \lim_{x \rightarrow 0} \frac{x \alpha x}{x^2 + \alpha^2 x^2} = \frac{\alpha}{1 + \alpha^2}$$

3. Show that  $f$  has partial derivatives everywhere apart from the origin.

**Definition 5.42.**  $f : \mathcal{U} \rightarrow \mathbb{R}$ ,  $\mathcal{U} \subset \mathbb{R}^n$  open is called partially differentiable iff partial derivatives exist and are continuous on  $\mathcal{U}$ .

**Example 5.43.** 1.  $f : \mathbb{R}^2 \rightarrow \mathbb{R}, f((x, y)) := x^2 + y^2$ , is continuously partially differentiable on  $\mathbb{R}^2$ , since both  $\mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \rightarrow 2x, (x, y) \rightarrow 2y$  are continuous on  $\mathbb{R}^2$ .

2. Norms on  $\mathbb{R}^k$  are continuously partially differentiable on  $\mathbb{R}^k \setminus \{0\}$ .

**Definition 5.44** (Jacobian).  $f : \mathcal{U} \rightarrow \mathbb{R}^m$ ,  $\mathcal{U} \subset \mathbb{R}^n$  open, so that all partial derivatives  $\frac{\partial f_i}{\partial x_j}(x_0), \forall i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}$  in  $\times_o$  exist. Then the matrix  $J_f(x_0) := \left( \frac{\partial f_i}{\partial x_j} \right)_{1 \leq i \leq m, 1 \leq j \leq n} \in \mathcal{M}^{m \times n}(\mathbb{R})$  is called the Jacobian matrix of  $f$  in  $x_0$ .

**Remark.** We will see that in the special case  $m = 1$ , the Jacobian is also called gradient of  $f$  in  $x_0$  and has a special geometric meaning.

**Example 5.45.**  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2, f((x, y, z)) := \begin{pmatrix} x^2 + y^2 + z^2 \\ xyz \end{pmatrix}$ .  $f$  is continuously partially differentiable everywhere and the Jacobian is given by  $J_f((x, y, z)) = \begin{pmatrix} 2x & 2y & 2z \\ yz & xz & xy \end{pmatrix} \in \mathcal{M}^{2 \times 3}(\mathbb{R})$ .

## Lecture 6

**Theorem 6.46** (total differentiability implies partial differentiability). Let  $f : \mathcal{U} \rightarrow \mathbb{R}^m, \mathcal{U}$  open be totally differentiable in  $x_0 \in \mathcal{U}$ . Then  $f$  is partially in  $x_0$  and  $[df(x_0)] = J_f(x_0)$ .

*Proof.*  $f$  is totally differentiable in  $x_0 \Rightarrow \forall h \in \mathbb{R}^n. f(x_0 + h) = f(x_0) + Ah + \varphi(h)$ , where  $\lim_{h \rightarrow 0} \frac{\varphi(h)}{h} = 0$ . Define  $[Df(x_0)] := A(x_0) = (a_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$ . Note we

$$\text{can write } Df(x_0)(h) = Ah = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n a_{1,j} h_j \\ \vdots \\ \sum_{j=1}^n a_{m,j} h_j \end{pmatrix}.$$

Written componentwise, this means for  $i \in \{1, 2, \dots, m\} : f_i(x_0 + h) = f_i(x_0) +$

$$\sum_{j=1}^n a_{i,j} h_j + \varphi_i(h), \text{ where } \varphi \left( \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} \right) \text{ and } \lim_{h \rightarrow 0} \frac{\varphi_i(h)}{\|h\|} = 0. \text{ Now putting}$$

$h = te_j, t \in \mathbb{R} \Rightarrow x_0 + te_j \in \mathcal{U}, \forall j \in \{1, 2, \dots, n\}$ . Thus,  $f_j(x_0 + te_j) = f_j(x_0) + \sum_{l=1}^n a_{jl} th_l + \varphi_j(te_j) = f_j(x_0) + ta_{ij} + \varphi_j(te_j)$ . Finally, let us look at the

$j$ -th partial derivatives  $\frac{\partial f_i}{\partial x_j}(x_0)$  of the  $i$ -th component  $f_i$  of  $f$  in  $x_0$ . By definition  $\frac{\partial f_i}{\partial x_j}(x_0) = \lim_{t \rightarrow 0} \frac{f_i(x_0 + te_j) - f_i(x_0)}{t} = \lim_{t \rightarrow 0} \frac{ta_{ij} + \varphi_i(te_j)}{t} = a_{ij} + \lim_{t \rightarrow 0} \frac{\varphi_i(te_j)}{t} = 0$ , since  $\lim_{h \rightarrow 0} \frac{\varphi_i(h)}{\|h\|} = 0$ .

Thus, the matrix  $A$  giving the differential is exactly the Jacobean in  $x_0$ .  $\square$

**Theorem 6.47** ( $\exists$  and continuity of partial derivatives  $\Rightarrow$  total differentiability).

**Remark.** Now given a partial differentiable function we can first determine the Jacobean, check its partial derivatives for continuity and if they are construct the total differential using the Jacobean.

**Example 6.48.**  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2, f((x, y, z)) = \begin{pmatrix} x^2 + y^2 + z^2 \\ xyz \end{pmatrix}, J_f((x_0, y_0, z_0)) = \begin{pmatrix} 2x_0 & 2y_0 & 2z_0 \\ y_0 z_0 & x_0 z_0 & x_0 y_0 \end{pmatrix} \in \mathcal{M}^{2 \times 3}(\mathbb{R})$ . Therefore,  $Df((x_0, y_0, z_0)) \in L(\mathbb{R}^3, \mathbb{R}^2)$  is defined as  $Df((x_0, y_0, z_0))(x, y, z) = \begin{pmatrix} 2(x_0 x + y_0 y + z_0 z) \\ x y_0 z_0 + x_0 y z_0 + x_0 y_0 z \end{pmatrix}$ .

**Remark.** The converse is not true,  $f$  totally differentiable  $\nRightarrow$  continuous partially differentiable. For instance in case of

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, f((x, y)) = \begin{cases} (x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) & (x, y) \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

**Remark.** One can show that continuous partial differentiability is equivalent to continuous total differentiability in the sense that  $x \mapsto Df(x)$  is continuous.

## Lecture 7

**Theorem 7.49.** Let  $f : \mathcal{U} \rightarrow \mathbb{R}^m, \mathcal{U} \subset \mathbb{R}^n$  open be any function and let its partial derivatives  $\frac{\partial f}{\partial x_i} : \mathcal{U} \rightarrow \mathbb{R}^m$   $i \in \{1, 2, \dots, n\}, j \in \{1, 2, \dots, m\}$  exist and are continuous ( $f$  is continuously differentiable). Then  $f$  is totally differentiable and the total differential is given as  $[Df(x_0)] = J_f(x_0) \quad \forall x_0 \in \mathcal{U}$ .

*Proof.* Wlog. let  $m = 1$  (component-wise total differentiability is equivalent to total differentiability as already shown). We need to show (given fixed  $x_0$ )  $f(x_0 + h) = f(x_0) + Ah + \varphi(h)$  with  $\lim_{h \rightarrow 0} \frac{\varphi(h)}{\|h\|} = 0, \forall h \in \mathbb{R}^n$ , put  $h^0 := 0 \in \mathbb{R}^n, \dots, h^j := \sum_{l=1}^j h_l e_l = (h_1 e_1, \dots, h_j e_j, 0, \dots, 0), \dots, h^n = h$ , where  $h = \sum_{l=1}^n h_l e_l$  with  $h_j \in \mathbb{R}$  and  $\{e_1, e_2, \dots, e_n\}$  the standard basis in  $\mathbb{R}^n$ . We can write

$$f(x_0 + h) - f(x_0) = f(x_0 + h) - f(x_0 + h^{n-1}) + f(x_0 + h^{n-1}) - \dots + f(x_0 + h^1) - f(x_0)$$

$$= \sum_{j=1}^n f(x_0 + h^j) - f(x_0 + h^{j-1}) \quad (1.5)$$

Applying the mean value theorem to  $f$  restricted to the segments  $[x_0 + h^{j-1}, x_0 + h^j]$  leads to  $\exists \xi_j \in (x_0 + h^{j-1}, x_0 + h^j)$ , so that  $\frac{\partial f}{\partial x_j}(x_0 + h^j + \xi_j) = \frac{f(x_0 + h^j) - f(x_0 + h^{j-1})}{h_j}$ .

**Local definition 7.49.1.** Here a segment in  $\mathbb{R}^n$  is defined as  $[x, y] := \{(1 - t)x + ty : t \in [0, 1]\}$   $\forall x, y \in \mathbb{R}^n$ .

Substituting these equalities into equation 1.5 leads to:

$$f(x_0 + h) - f(x_0) = \sum_{j=1}^n [f(x_0 + h^j) - f(x_0 + h^{j-1})] = \sum_{j=1}^n \frac{\partial f}{\partial x_j} (x_0 + h^{j-1} + \xi^j h_j e_j) h_j.$$

Coming back to the definition we want to show that  $\lim_{h \rightarrow 0} \frac{\varphi(h)}{\|h\|} = 0$ , where

$$S\varphi(h) = f(x_0) - f(x_0 + h) - Ah = \sum_{j=1}^n \left( \frac{\partial f}{\partial x_j} (x_0 + h^{j-1} + \xi^j h_j e_j) - \frac{\partial f}{\partial x_j} (x_0) \right) h_j.$$

By construction,  $(x_0 + h^{j-1} + \xi^j h_j e_j) \xrightarrow{h \rightarrow 0} x_0$ . Therefore, by continuity of  $\frac{\partial f}{\partial x_j} \Rightarrow (x_0 + h^{j-1} + \xi^j h_j e_j) \rightarrow \frac{\partial f}{\partial x_j} (x_0)$  and hence  $\lim_{h \rightarrow 0} \varphi(h) = 0$ . Now putting  $\varphi_j(h) := \frac{\partial f}{\partial x_j} (x_0 + h^{j-1} + \xi^j h_j e_j) - \frac{\partial f}{\partial x_j} (x_0)$  we obtain a function  $\Psi : h \rightarrow \sum_{j=1}^n \Psi_j(h) e_j$ , with  $\varphi(h) = \langle \Psi(h) | h \rangle$ . Then, we can estimate  $|\varphi(h)| = \langle \Psi(h) | h \rangle \leq \|\Psi(h)\| \cdot \|h\|$ , where the last estimation is by Cauchy-Schwartz-Inequality. Hence,

$$0 < \frac{|\varphi(h)|}{\|h\|} \leq \frac{\|\varphi(h)\| \|h\|}{\|h\|} = \|\Psi(h)\| \xrightarrow{h \rightarrow 0} 0.$$

□

**Corollary 7.50.** Recalling the chain rule for the total differential:  $\mathbb{R}^m \supset \mathcal{U} \xrightarrow{f} \mathcal{V} \subset \mathbb{R}^n \xrightarrow{g} \mathbb{R}^p$ ,  $f$  differentiable in  $x_0$ ,  $g$  differentiable in  $y_0 := f(x_0) \Rightarrow g \circ f$  differentiable in  $x_0$  and  $Dg \circ f(x_0) = Dg(x_0) \circ Df(x_0)$ . Knowing  $[Df(x_0)] = J_f(x_0)$ ,  $[Dg(x_0)] = J_g(y_0)$  and  $[Dg \circ f(x_0)] = J_{g \circ f}(x_0)$  we can conclude that  $J_{g \circ f}(x_0) = J_g(f(x_0)) \cdot J_f(x_0)$ .

**Example 7.51.** 1.  $f : \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x^2 + y^2 \\ 2xy \end{pmatrix}$ ,  $g : \begin{pmatrix} u \\ v \end{pmatrix} \rightarrow u + v$ , then  $g \circ f : x^2 + 2xy + y^2 = (x + y)^2$  and  $J_{g \circ f}(x, y) = (2(x + y), 2(x + y)) = 2(x + y, x + y)$ . On the other hand:  $J_g(u, v) = (1, 1)$  and  $J_f(x, y) = \begin{pmatrix} 2x & 2y \\ 2y & 2x \end{pmatrix} = 2 \begin{pmatrix} x & y \\ y & x \end{pmatrix} \Rightarrow J_g(f(x, y)) \cdot J_f(x, y) = 2(1, 1) \cdot \begin{pmatrix} x & y \\ y & x \end{pmatrix} = 2(x + y, x + y)$ .

2.

$$\begin{aligned} f : (x, y) &\xrightarrow{1}{xy}, g : t \rightarrow \begin{pmatrix} t \\ t^2 \end{pmatrix} \\ J_f((x, y)) &= \left( \frac{1}{x^2 y}, \frac{1}{xy^2} \right), J_g((x, y)) = \begin{pmatrix} 1 \\ 2t \end{pmatrix} \xrightarrow{\text{chain rule}} J_{g \circ f}((x, y)) = \\ J_g(f((x, y))) \cdot J_f((x, y)) &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} -\frac{1}{x^2 y} & -\frac{1}{xy^2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{x^2 y} & -\frac{1}{xy^2} \\ -\frac{2}{x^3 y^2} & -\frac{2}{x^2 y^3} \end{pmatrix}. \text{ On} \\ \text{the other hand: } g \circ f((x, y)) &= g(f((x, y))) = g\left(\frac{1}{xy}\right) = \begin{pmatrix} \frac{1}{xy} \\ \frac{1}{x^2 y^2} \end{pmatrix} \Rightarrow \\ J_{g \circ f}((x, y)) &= \begin{pmatrix} -\frac{1}{x^2 y} & -\frac{1}{xy^2} \\ -\frac{2}{x^3 y^2} & -\frac{2}{x^2 y^3} \end{pmatrix}. \end{aligned}$$



$$\begin{aligned}
 3. \quad f : t &\rightarrow \begin{pmatrix} f \\ \frac{1}{t} \end{pmatrix}, g : (u, v) \rightarrow uv \Rightarrow J_f\left(\begin{pmatrix} 1 \\ -\frac{1}{t^2} \end{pmatrix}\right), J_g((u, v)) = (v, u) \xrightarrow{\text{chain rule}} \\
 J_{g \circ f}(t) &= J_g \cdot J_f = \begin{pmatrix} 1 \\ t \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{t^2} \end{pmatrix} = \frac{1}{t} - \frac{1}{t} = 0.
 \end{aligned}$$

**Definition 7.52.** Let  $f : \mathcal{U} \rightarrow \mathbb{R}, \mathcal{U} \subset \mathbb{R}^n$  open, partially differentiable in  $x_0 \in \mathcal{U}$ . Then, the Jacobian of  $f$  in  $x_0$  is also called the gradient of  $f$  in  $x_0$  and denoted by  $\nabla f(x_0)$ . Thus, we have  $\nabla f(x_0) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(x_0) e_j = \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$ .

**Remark.** We will see that the gradient can be used to compare directional derivatives and we can interpret it geometrically as the direction of maximal increase of  $f$  in  $x_0$ .

**Proposition 7.53.** Let  $f : \mathcal{U} \rightarrow \mathbb{R}, \mathcal{U} \subset \mathbb{R}^n$  open, be continuously differentiable on  $\mathcal{U}$ . Then,  $\forall x \in \mathcal{U}, \forall v \in \mathbb{R}^n$  with  $\|v\| = 1$  we have  $D_v f(x) = \langle \nabla f(x) | v \rangle$ .

*Proof.* Consider the line  $\{x + tv : t \in \mathbb{R}\}$  in  $\mathbb{R}^n$ , which is parallel to the unit vector  $v$ . Since  $\mathcal{U}$  is open,  $\exists \epsilon > 0$  so that  $x + tv \in \mathcal{U}, \forall t \in (-\epsilon, \epsilon)$ . Define  $\varphi : (-\epsilon, \epsilon) \rightarrow \mathcal{U}; \varphi(t) := x + tv$  and consider  $F := f \circ \varphi : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ . Applying the chain rule in this situation yields  $J_f(t) = J_F(\varphi(t)) \cdot J_\varphi(t) = \nabla f(\varphi(t)) \cdot$

$$\begin{pmatrix} \varphi'_1(t) \\ \vdots \\ \varphi'_n(t) \end{pmatrix} \stackrel{\text{definition}}{=} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \langle \nabla f(\varphi(t)) | v \rangle. \text{ In particular for } t = 0, J_F(0) = \langle \nabla f(x) | v \rangle.$$

By definition of the directional derivative, we have that  $D_v f(x) = \lim_{t \rightarrow 0} \frac{f(x+tv) - f(x)}{t} = \lim_{t \rightarrow 0} \frac{F(t) - F(0)}{t} = F'(0)$ .  $\square$

**Remark.** The assumption of continuity of partial derivatives is not necessary.

**Corollary 7.54.** In the situation of the last proposition the gradient gives the direction of maximal increase in each point in  $x \in \mathcal{U}$ . This means that  $D_v f(x)$  is maximal for  $v = \frac{\nabla f(x)}{\|\nabla f(x)\|}$ . Furthermore, if  $\nabla f(x)$  is not zero  $D_v f(x) = 0$  iff  $v \perp \nabla f(x)$ , and always if  $\nabla f(x) = 0$ .

*Proof.* Assume  $\nabla f(x) \neq 0$  (otherwise the claim is obvious). Thus  $\theta := \angle(\nabla f(x), v)$  is well defined and we obtain  $\cos(\theta) = \frac{\langle \nabla f(x) | v \rangle}{\|\nabla f(x)\| \|v\|}$  and thus  $D_v f(x) = \langle \nabla f(x) | v \rangle = \|\nabla f(x)\| \cos(\theta)$ , which is maximal iff  $\cos(\theta) = 1 \Leftrightarrow \theta = \frac{\pi}{2}$ . Also  $v \perp \nabla f(x) \Leftrightarrow \theta = \frac{\pi}{2} \Leftrightarrow \cos(\theta) = 0 = \langle \nabla f(x) | v \rangle$ .  $\square$

**Remark.** We can visualize this statement by the ‘‘Hill Billy’’ example; here the gradient corresponds to the direction of steepest ascend  $\perp$  constant height lines. Also a river bed  $g(t)$  satisfies  $\varphi'(t) = -\nabla f(\varphi(t))$ .

**Example 7.55.** 1. the paraboloid:  $f : \mathbb{R}^2 \rightarrow \mathbb{R}, f((x, y)) = x^2 + y^2$ . What is the direction  $\frac{v}{\|v\|}$  and magnitude  $\|v\|$  of maximal increase  $v$  of  $f$  in  $(1, 1)$ ?

$$\nabla f((1, 1)) = (2, 2) \Rightarrow \frac{v}{\|v\|} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \wedge \|v\| = 2\sqrt{2}.$$

2.  $f : \mathbb{R}^2 \rightarrow \mathbb{R}, f((x, y)) := x^2 + e^{xy} \sin(y)$ . Some question; Find the direction  $\frac{v}{\|v\|}$  and magnitude  $\|v\|$  of maximal increase  $v$  of  $f$  in  $(0, 1)$ .

$$\begin{aligned}
 \nabla f &= (2xy + ye^{xy} \sin(y), x^2 + xe^{xy} \sin(y) + e^{xy} \cos(y)) \\
 &\Rightarrow \nabla f((0, 1)) = (0, 2) \Rightarrow \frac{v}{\|v\|} = (0, 1) \wedge \|v\| = 2.
 \end{aligned}$$

## Lecture 8

### The inverse function theorem

**Motivation** In analysis of functions of one variable we had that

**Theorem 8.56.** *Given*

1.  $f : I \rightarrow \mathbb{R}$ .  $I \subset \mathbb{R}$  ( $I$  being an open interval)
2.  $f$  is continuously differentiable on its domain (interval  $I$ )
3.  $f'(x) \neq 0 \forall x \in I$  (i.e.  $f$  is strictly increasing or decreasing)

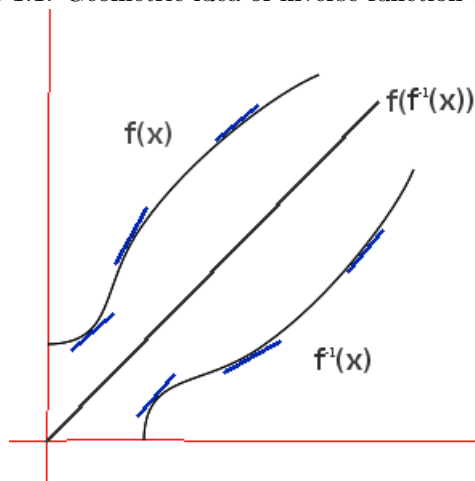
Then  $f$  is invertible (bijective), its inverse  $f^{-1}$  is also continuously differentiable and

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

**Remark.** Can be derived as

$$f^{-1}(f(x)) = x \xrightarrow{\text{differentiate}} (f^{-1})'(f(x)) \cdot f'(x) = 1 \iff (f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

Figure 1.1: Geometric idea of inverse function theorem



We wish to generalize this statement to functions of multiple variables.

**Local definition 8.56.1.** We call  $f|_u$  a restriction of  $f$  at  $u$ ; we only consider  $f$  on the sub-domain  $u$  and its image from  $u$ .

**Theorem 8.57** (Inverse function theorem for functions of several variables).  
Given

1.  $f : u \rightarrow \mathbb{R}^n$ .  $u \subset \mathbb{R}^n$  ( $u$  is an open  $n$ -ball)

2.  $f$  is differentiable on  $u$ , or, equivalently,  $f|_u$  is differentiable
3.  $Df : u \rightarrow L(\mathbb{R}^n)$  is continuous.
4.  $Df(x_0)$  is invertible  $x_0 \in u$ , or, equivalently, that the jacobian  $J_f(x_0)$  is invertible in  $M^{n \times n}$

Then  $\exists$  neighborhood  $u_0 \subset u$  of  $x_0$  so that  $f|_{u_0}$  is bijective onto its image  $v_0 = f(u_0)$  (invertible), the inverse  $g = (f|_{u_0})^{-1}$  is also differentiable on  $v_0$  and  $\forall y = f(x) \in v_0$

$$Dg(y) = (Df(x))^{-1}$$

or, equivalently

$$J_g(y) = (J_f(x))^{-1}$$

Proof sketch:

I  $\exists u_0$  neighbourhood of  $x_0$  such that  $f|_{u_0}$  is bijective with inverse  $g : v_0 \rightarrow u_0$ , where  $v_0 = f(u_0)$  (using *Banach fixed point theorem*)

II  $v_0$  is open

III  $g$  is differentiable on  $v_0$  and  $Dg(y) = (Df(g(y)))^{-1}, \forall y \in v_0$

**Definition 8.58.** A function defined from metric space  $(\mathcal{X}, d)$  to itself  $\kappa : \mathcal{X} \rightarrow \mathcal{X}$  is called a contraction if

$$d(\kappa(x), \kappa(y)) < cd(x, y). \forall x, y \in \mathcal{X}. c \in [0, 1)$$

**Remark.** Intuitive picture of a contraction  $\kappa$  involves a function where the preimage always changes faster than the image; graph would be 'flatter' the smaller  $c$  is.

**Local definition 8.58.1.** A complete metric space is a metric space where all cauchy sequences converge. i.e.

$$((x_p) \in \mathcal{X} \implies \forall \epsilon > 0. \exists N_\epsilon \in \mathbb{N} : d(x_p, x_q) < \epsilon. \forall p, q \geq N_\epsilon) \implies x_p \xrightarrow{p \rightarrow \infty} x \in \mathcal{X}$$

**Theorem 8.59** (Banach fixed point theorem). Let  $(\mathcal{X}, d)$  be a complete metric space. If  $\kappa : \mathcal{X} \rightarrow \mathcal{X}$  is a contraction mapping, then  $\kappa$  has a unique fixed point  $x \in \mathcal{X}$ , i.e. a point  $x \in \mathcal{X}$  where  $\kappa(x) = x$ .

**Remark.** Intuitive picture of Banach fixed point theorem: because  $\kappa$  is a contraction (the preimage  $x$  is always changing faster than the image  $\kappa(x)$ ), you know that somewhere as you're moving towards either its maximal point (e.g:  $+\infty$ ) or minimal point (e.g:  $-\infty$ ) the value of  $x$  will catch up to  $\kappa(x)$

*Proof.* 1. Take some  $x_1 \in \mathcal{X}$

2. Define sequence  $(x_p)$  inductively such that  $x_{p+1} = \kappa^p(x_1)$ .  $p \in \mathbb{N}$

3. Define inequality

$$d(x_{p+1}, x_p) \leq c^{p-1}d(x_2, x_1). p \in \mathbb{N}. c \in [0, 1) \quad (1.6)$$

where  $c$  is the contraction constant for  $\kappa$ .

4. Proving above inequality by induction.

Base case ( $p = k = 1$ ):

$$d(x_{1+1}, x_1) \leq c^{1-1} d(x_2, x_1)$$

$$d(x_2, x_1) \leq (c^0) d(x_2, x_1) \implies d(x_2, x_1) = d(x_2, x_1)$$

Inductive step ( $p = k + 1$ )

$$d(x_{k+2}, x_{k+1}) = d(\kappa(x_{k+1}), \kappa(x_k))$$

Since  $\kappa$  is a contraction

$$= d(\kappa(x_{k+1}), \kappa(x_k)) \leq c d(x_{k+1}, x_k)$$

Activate induction

$$\leq c c^{k-1} d(x_2, x_1) = c^k d(x_2, x_1)$$

5. We will show the sequence  $(x_p)$  is cauchy. Let  $m, n \in \mathbb{N}$ .  $m > n$

6. By triangle inequality

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n)$$

By inequality (1.6)

$$\leq c^{m-2} d(x_2, x_1) + c^{m-3} d(x_2, x_1) + \cdots + c^{n-1} d(x_2, x_1)$$

$$= c^{n-1} d(x_2, x_1) \sum_{k=0}^{m-n-2} c^k$$

$$\leq c^{n-1} d(x_2, x_1) \sum_{k=0}^{\infty} c^k = c^{n-1} d(x_2, x_1) \frac{1}{1-c}$$

So finally

$$d(x_m, x_n) = c^{n-1} \frac{d(x_2, x_1)}{1-c}$$

7. Let  $\epsilon = c^{N-1} \frac{d(x_2, x_1)}{1-c} > 0 \iff N = \lceil \log_c \left( \frac{\epsilon(1-c)}{d(x_2, x_1)} \right) \rceil + 1$

8.  $\epsilon$  is arbitrary, so the sequence  $(x_p)$  is cauchy. And since the metric space it is defined on,  $(\mathcal{X}, d)$ , is complete, it converges to some  $x^* \in \mathcal{X}$ .

9.  $\kappa$  is a contraction  $\implies$  Lipschitz (with constant  $< 1$ )  $\iff$  Lipschitz continuous. Thus we can have

$$\kappa\left(\lim_{p \rightarrow \infty} x_p\right) = \lim_{p \rightarrow \infty} x_{p+1}$$

$$\kappa(x^*) = x^*$$

So  $x^*$  is a fixed point of  $\kappa$

10. Suppose there exists another  $y \in \mathcal{X}$  such that  $\kappa(y) = y$ , then

$$0 \leq d(x^*, y) = d(\kappa(x^*), \kappa(y)) \leq cd(x^*, y)$$

Since  $|c| < 1$ ,  $d(x^*, y) = 0$

$$0 \leq d(\kappa(x^*), \kappa(y)) \leq 0$$

$$x^* = \kappa(x^*) = \kappa(y) = y$$

So  $x^*$  is the unique fixed point. □

*The inverse function theorem proof. :*

I ‘ $\exists u_0$  neighbourhood of  $x_0$  such that  $f|_{u_0}$  is bijective with inverse  $g : v_0 \rightarrow u_0$ , where  $v_0 = f(u_0)$ ’

(a)  $A = Df(x_0) \in L(\mathbb{R}^n)$  is invertible (by assumption),  $\implies \|A\| \neq 0 \iff \|A^{-1}\| \neq 0$ . Define

$$\lambda = \frac{1}{2\|A^{-1}\|} > 0$$

(b) The differential map  $Df : u \rightarrow L(\mathbb{R}^n)$  is continuous in  $x_0$  (by assumption); there exists (for the previously defined  $\lambda > 0$ ) an open neighbourhood  $u_0 \subset u$  of  $x_0$  so that

$$\|Df(x) - Df(x_0)\| < \lambda \forall x \in u_0 \quad (1.7)$$

(c)  $\forall y \in \mathbb{R}^n$  define a map  $\kappa = \kappa_y : u \rightarrow \mathbb{R}^n$  by

$$\kappa(x) = x + A^{-1}(y - f(x)) \quad (1.8)$$

**Remark.** *Contrary to what the notation suggests, we do not have  $f(x) = y$  always; this is the case exactly when  $x$  is a fixed point of  $\kappa$ . Note also here that  $\kappa$  is not a contraction yet either, but the plan to show that  $\kappa$  is a contraction.*

(d) Finding the derivative of  $\kappa$  by the chain rule

$$D\kappa(x) = I + A^{-1} \circ (-Df(x)) = I - A^{-1} \circ Df(x) = A^{-1} \circ (A - Df(x))$$

(e) By properties of the operator norm we get  $\forall x \in u_0$

$$\|D\kappa(x)\| = \|A^{-1} \circ (A - Df(x))\| \leq \|A^{-1}\| \|A - Df(x)\|$$

recall that  $A = Df(x_0)$ , and by inequality (1.7) is

$$< \|A^{-1}\| \lambda = \|A^{-1}\| \frac{1}{2\|A^{-1}\|} = \frac{1}{2} < 1$$

(f) By proposition, since the differential is bounded  $\implies$  map is Lipschitz. We have that  $\|\kappa(u) - \kappa(v)\| \leq \frac{1}{2}\|u - v\|$ ,  $\forall u, v \in u_0 \implies \kappa$  is a contraction.

- (g) By Banach fixed point theorem,  $\forall y \in \mathbb{R}^n$ , there is at most one fixed point  $x \in u_0$  of  $\kappa = \kappa_y \implies f(x) = y$ . In particular,  $\forall y \in v_0 = f(u_0)$  there exists exactly one <sup>1</sup>  $x \in u_0$  with  $f(x) = y \implies f|_{u_0}$  is injective  $\implies f|_{u_0}$  is bijective.

II ‘ $v_0$  is open’

- (a) Let  $y_1 \in v_0$  and let  $x_1 = f^{-1}(y_1)$ .  
 (b) Since  $u_0$  is open,  $\exists \rho > 0$  so that closed ball  $\overline{B}(x_1, \rho) = \{u \in \mathbb{R}^n : \|x_1 - u\| \leq \rho\} \subset u_0$   
 (c) We will show that  $B(y_1, \lambda\rho) \subset v_0$  which would imply that  $v_0$  is open; we need to prove that

$$\|y - y_1\| < \lambda\rho = \frac{\rho}{2\|A^{-1}\|} \implies y \in v_0 = f(u_0)$$

- (d) Let  $y \in B(y_1, \lambda\rho)$ . For  $\kappa = \kappa_y$ , we can have

$$\|\kappa(x_1) - x_1\| = \|A^{-1}(y - f(x_1))\|$$

by property of the operator norm is

$$\leq \|A^{-1}\| \|y - f(x_1)\| < \|A^{-1}\| \lambda\rho = \|A^{-1}\| \frac{\rho}{2\|A^{-1}\|} = \rho/2$$

thus,  $\forall x \in \overline{B}(x_1, \rho) := \overline{B}$ ,

$$\|\kappa(x) - x_1\| = \|\kappa(x) + (-\kappa(x_1) + \kappa(x_1)) - x_1\|$$

by triangle inequality

$$\leq \|\kappa(x) - \kappa(x_1)\| + \|\kappa(x_1) - x_1\| \leq \frac{\rho}{2} + \frac{\rho}{2} = \rho$$

- (e) But this means that  $\kappa(\overline{B}) \subset \overline{B} \implies \kappa$  is a contraction of a complete metric space  $\overline{B}$  with the euclidean metric.  
 (f) By Banach fixed point theorem,  $\exists! x \in \overline{B} : \kappa(x) = x \iff f(x) = y \iff y \in v_0 \implies v_0$  is open.

III ‘ $g$  is differentiable on  $v_0$  and  $D_g(y) = (Df(g(y)))^{-1}, \forall y \in v_0$ ’

- (a) Let  $g = (f|_{u_0})^{-1}$   
 (b) Since  $f|_{u_0}$  is bijective, we have a one to one correspondance

$$v_0 \rightarrow \begin{cases} y & \leftrightarrow x = g(y) \in u_0 \\ y + k & \leftrightarrow x + h = g(y + k) \in u_0 \end{cases}$$

- (c) With  $\kappa = \kappa_y$  defined in (1.8), and knowing  $f(x) = y, f(x+h) = y+k$ , we now have

$$\begin{aligned} \kappa(x+h) - \kappa(x) &= x+h + A^{-1}(y - f(x+h)) - x - A^{-1}(y - f(x)) \\ &= h + A^{-1}(f(x) - f(x+h)) = h - A^{-1}k \end{aligned}$$

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<sup>1</sup>Banach fixed point theorem says that this implies that there exists *at most one*  $x$ , but I think it's exactly one since it works for all  $y$ .

(d) We can have

$$\|h\| = \|x + h - x\|$$

since  $\kappa$  is a contraction with constant  $\frac{1}{2}$ , it is

$$\geq 2\|\kappa(x + h) - \kappa(x)\| = 2\|h - A^{-1}k\|$$

(e) On the other hand

$$\|h\| = \|h - A^{-1}k + A^{-1}k\|$$

by triangle inequality is

$$\leq \|h - A^{-1}k\| + \|A^{-1}k\|$$

by the previous inequality is

$$\leq \frac{\|h\|}{2} + \|A^{-1}k\|$$

(f) Thus,  $\|A^{-1}k\| \geq \frac{\|h\|}{2}$  and finally

$$\begin{aligned} \|h\| &\leq 2\|A^{-1}k\| \leq 2\|A^{-1}\| \|k\| = \frac{2\|k\|}{2\lambda} \\ &\implies \|h\| \leq \frac{\|k\|}{\lambda} \end{aligned} \tag{1.9}$$

(g) We need to show that  $Df(x)$  is invertible. Recall from property 6 of the operator norm that if  $\|B - A\| < \frac{1}{\|A^{-1}\|} \implies B$  is invertible. We have by inequality (1.7) (recall  $A = Df(x_0)$ )

$$\|Df(x) - A\| < \lambda = \frac{1}{2\|A^{-1}\|} < \frac{1}{\|A^{-1}\|}$$

*implies*  $Df(x)$  is invertible.

(h) Now to identify the differential of  $g$  in  $y = f(x)$ , let  $B = (Df(x))^{-1}$  (note  $h = Ih = (BDf(x))h$ )

$$\begin{aligned} g(y + k) - g(y) - Bk &= x + h - x - Bk = h - B(y + k - y) \\ &= BDf(x)h - B(f(x + h) - f(x)) = -B(f(x + h) - f(x) - Df(x)h) \end{aligned}$$

(i) Then  $g$  is differentiable if following limit equals 0 as  $k \rightarrow 0$

$$\begin{aligned} 0 &\leq \frac{\|g(y + k) - g(y) - Bk\|}{\|k\|} = \frac{1}{\|k\|} \|B(f(x + h) - f(x) - Df(x)h)\| \\ &\leq \frac{\|B\|}{\|k\|} \|f(x + h) - f(x) - Df(x)h\| \end{aligned}$$

by inequality (1.9) is

$$\leq \frac{\|B\|}{\lambda} \frac{\|f(x + h) - f(x) - Df(x)h\|}{\|h\|}$$

- (j) As  $k \rightarrow 0, h \rightarrow 0$  by inequality (1.9), and as  $f$  is differentiable  $\frac{\|B\|}{\|k\|} \|f(x+h)-f(x)-Df(x)h\| \xrightarrow{h \rightarrow 0} 0$ .  $\frac{\|B\|}{\lambda}$  is a constant (by definition,  $\lambda > 0$ ). So, finally,

$$\lim_{k \rightarrow 0} \frac{\|g(y+k) - g(y) - Bk\|}{|k|} = 0$$

thus  $g$  is differentiable on  $v_0$ , and the differential is  $B = (Df(x))^{-1}$ .

□

**Example 8.60.** of a map that is locally invertible in every point but not globally injective is a map of polar coordinates in the plane  $f : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}^2$  defined by  $f(r, \theta) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$ . Take  $r > 0, \theta \in \mathbb{R}$ , then  $J_f(r, \theta) = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$ .  $\det J_f = r \cos^2 \theta + r \sin^2 \theta \implies f$  is locally invertible in  $(r, \theta)$ . But  $f(r, \theta) = f(r, \theta + 2\pi)$ , so  $f$  is not injective.

**Remark.** One can strengthen the statement of the theorem by assuming  $f$  is continuously differentiable on all of  $u$ , then one can show (using property 7 of operator norm) that  $g$  is also continuously differentiable. <sup>2</sup>

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<sup>2</sup>Refer to Rudin