14.12 Game Theory

Professor: Muhamet Yildiz (office hours: M 4:00-5:30)

TA: Kenichi Amaya (office hours: TBD)

Place: 4-153 [**THE CLASS ROOM WILL BE CHANGED IN THE FUTURE; CHECK THE HOME-PAGE**]

Time: MW 2:30-4:00.

Recitation: F 10 or F 3, at E51-085.

Home-page: http://web.mit.edu/14.12/www/index.html

Game Theory is a misnomer for Multiperson Decision Theory, the analysis of situations in which payoffs to agents depend on the behavior of other agents. It involves the analysis of conflict, cooperation, and (tacit) communication. Game theory has applications in several fields, such as economics, politics, law, biology, and computer science. In this course, I will introduce the basic tools of game theoretic analysis. In the process, I will outline some of the many applications of game theory, primarily in economics and political science.

Game Theory has emerged as a branch of mathematics and is still quite mathematical. Our emphasis will be on the conceptual analysis, keeping the level of math to a minimum, especially at a level that should be quite acceptable to the average MIT student. Yet bear in mind that this still implies that you should be at ease with basic probability theory and calculus, and more importantly, you should be used to thinking in mathematical terms. Intermediate Microeconomics is also a prerequisite (simultaneous attendance to one of the intermediate courses is also acceptable). In any case, if you are taking this course, you should be prepared to work hard.

Textbook The main textbook will be

> Robert Gibbons, *Game Theory For Applied Economists,* Princeton University Press, 1992.

This is the only required textbook and covers the majority of this course's topics. I recommend that you buy it. The book

> Prajit Dutta, *Strategies and Games*, 1999

will also be very useful, especially for the exercises. (You need to solve a lot of problems to learn Game Theory.) I will also refer to

> David Kreps, *A Course in Microeconomic Theory,* 1990, Harvester.

The last two books will be on reserve at Dewey; you are not required to buy it. All the lectures will be supplemented with detailed notes as well.

Those who want more advanced treatment should look at Drew Fudenberg and Jean Tirole, *Game Theory*, MIT Press, 1991 or Martin Osborne and Ariel Rubinstein, *A Course in Game Theory,* MIT Press, 1994. These two books are very good but harder than the level at which the course is pitched. Those who need an easier — and longer — exposition of the topics can read Avinash Dixit and Susan Sekeath, *Games of Strategy*, 2000.

Grading There will be two midterms and a comprehensive final exam. ALL EXAMS WILL BE OPEN BOOK. Also approximately 6 problem sets to be handed in. Each midterm is worth 25%, the final is worth 40%, and the problem sets will make up the remaining 10% of the final grade. The first midterm will be on October 10th and the second one on November 14th.

The final exam will be in the week of finals. A portion of the last class before each exam will be devoted to problem solving and the review of the material. (The dates for these review sessions are October 3rd, November 7th, and December 12th.)

In addition, there will be in-class quizzes. In these quizzes you will be asked to play various games. In most of these games you will not know who the other players are. The points (normalized to 5%) you get in these games will be bonuses. They will be added to your final grade after the cut off values for the letter grades are determined. (In this way, you will not be given any incentive to care about the other players' payoffs in the game.)

Course Outline The following is a rough outline. Depending on the interests and the inclinations of the group, the topics and their weight may change a little. The number in square brackets denotes the expected time to be devoted to each topic. G. refers to Gibbons' textbook.

1. Introduction to game theory [1 lecture]
2. Payoffs in games: Rational Choice Under Uncertainty [1 lecture]
   (a) Expected Utility Theory; Risk aversion, Kreps, Chapters 3.1-3.3
   (b) Applications; risk sharing, insurance, option value.
3. A More Formal Introduction to Games [3 lectures]
   (a) Extensive Forms and Normal Forms, G., Ch. 1.1A and 2.1A
   (b) Strategies, Dominant Strategies and Iterative elimination of strictly dominated strategies, G. 1.1B
   (c) Nash Equilibrium, G. 1.1C
   (d) Applications of Nash Equilibrium, G. 1.2
4. Backward Induction, Subgame Perfection, and Forward Induction [3 lectures]

(a) Analysis of Extensive-Form Games, G. 2.1A
(b) Backward induction
(c) Subgame Perfection, G. 2.2A
(d) Applications, G. 2.2B,C,D and 2.1B,C.
(e) Bargaining and negotiations, G. 2.1D
(f) Forward induction.
(g) Applications.

5. Evolutionary foundations of equilibrium; evolutionarily stable strategies and replicator dynamics. [1 lecture]

6. Moral Hazard [2 lectures]
   (a) The principal-agent formulation, Kreps chapters 16.1 and 16.2
   (b) Applications; insurance, efficiency wages

7. Repeated Games and Cooperation [2 lectures] G. 2.3

8. Incomplete Information [2 lectures]
   (a) Bayesian Nash Equilibrium, G., 3.1A,C
   (b) Auctions
   (c) Applications, G. 3.2

9. Dynamic Games of Incomplete Information [3 lectures]
   (a) Perfect Bayesian Equilibrium, G. 4.1
   (b) Sequential Bargaining Under Asymmetric Information, G. 4.3B
   (c) Reputation, G. 4.3C

10. Problems of Asymmetric Information in Economics [3 lectures]

(a) Signaling and the Intuitive Criterion, G. 4.2A and 4.4

(b) Applications of Signaling, G. 4.2B,C

(c) The principal-agent problem, Kreps Chapter 17

(d) Applications; lemons, efficiency wages, credit-rationing, price-discrimination.

14.12 Game Theory Lecture Notes
Introduction

Muhamet Yildiz

(Lecture 1)

Game Theory is a misnomer for Multiperson Decision Theory, analyzing the decision-making process when there are more than one decision-makers where each agent's payoff possibly depends on the actions taken by the other agents. Since an agent's preferences on his actions depend on which actions the other parties take, his action depends on his beliefs about what the others do. Of course, what the others do depends on their beliefs about what each agent does. In this way, a player's action, in principle, depends on the actions available to each agent, each agent's preferences on the outcomes, each player's beliefs about which actions are available to each player and how each player ranks the outcomes, and further his beliefs about each player's beliefs, ad infinitum.

Under perfect competition, there are also more than one (in fact, infinitely many) decision makers. Yet, their decisions are assumed to be decentralized. A consumer tries to choose the best consumption bundle that he can afford, given the prices – without paying attention what the other consumers do. In reality, the future prices are not known. Consumers' decisions depend on their expectations about the future prices. And the future prices depend on consumers' decisions today. Once again, even in perfectly competitive environments, a consumer's decisions are affected by their beliefs about what other consumers do – in an aggregate level.

When agents think through what the other players will do, taking what the other players think about them into accout, they may find a clear way to play the game. Consider the following "game":

$1 \setminus 2$	L	m	R
T	$(1,1)$	$(0,2)$	$(2,1)$
M	$(2,2)$	$(1,1)$	$(0,0)$
B	$(1,0)$	$(0,0)$	$(-1,1)$

Here, Player 1 has strategies, T, M, B and Player 2 has strategies L, m, R. (They pick their strategies simultaneously.) The payoffs for players 1 and 2 are indicated by the numbers in parentheses, the first one for player 1 and the second one for player 2. For instance, if Player 1 plays T and Player 2 plays R, then Player 1 gets a payoff of 2 and Player 2 gets 1. Let's assume that each player knows that these are the strategies and the payoffs, each player knows that each player knows this, each player knows that each player knows that each player knows this,... ad infinitum. [In that case, we formally say that the strategies and the payoffs are *common knowledge*.]

Now, player 1 looks at his payoffs, and realizes that, no matter what the other player plays, it is better for him to play M rather than B. That is, if 2 plays L, M gives 2 and B gives 1; if 2 plays m, M gives 1, B gives 0; and if 2 plays R, M gives 0, B gives -1. Therefore, he realizes that he should not play B.[1] Now he compares T and M. He realizes that, if Player 2 plays L or m, M is better than T, but if she plays R, T is definitely better than M. Would Player 2 play R? What would she play? To find an answer to these questions, Player 1 looks at the game from Player 2's point of view. He realizes that, for Player 2, there is no strategy that is outright better than any other strategy. For instance, R is the best strategy if 1 plays B, but otherwise it is strictly worse than m. Would Player 2 think that Player 1 would play B? Well, she knows that Player 1 is trying to maximize his expected payoff, given by the first entries as everyone knows. She must then deduce that Player 1 will not play B. Therefore, Player 1 concludes, she will not play R (as it is worse than m in this case). Ruling out the possibility that Player 2 plays R, Player 1 looks at his payoffs, and sees that M is now better than T, no matter what. On the other side, Player 2 goes through similar reasoning, and concludes that 1 must play M, and therefore plays L.

This kind of reasoning does not always yield such a clear prediction. Imagine that you want to meet with a friend in one of two places, about which you both are indifferent.

[1] After all, he cannot have any belief about what Player 2 plays that would lead him to play B when M is available.

Unfortunately, you cannot communicate with each other until you meet. This situation is formalized in the following game, which is called *pure coordination game*:

1 \ 2	Left	Right
Top	(1,1)	(0,0)
Bottom	(0,0)	(1,1)

Here, Player 1 chooses between Top and Bottom rows, while Player 2 chooses between Left and Right columns. In each box, the first and the second numbers denote the von Neumann-Morgenstern utilities of players 1 and 2, respectively. Note that Player 1 prefers Top to Bottom if he knows that Player 2 plays Left; he prefers Bottom if he knows that Player 2 plays Right. He is indifferent if he knows thinks that the other player is likely to play either strategy with equal probabilities. Similarly, Player 2 prefers Left if she knows that player 1 plays Top. There is no clear prediction about the outcome of this game.

One may look for the *stable* outcomes (strategy profiles) in the sense that no player has incentive to deviate if he knows that the other players play the prescribed strategies. Here, Top-Left and Bottom-Right are such outcomes. But Bottom-Left and Top-Right are not stable in this sense. For instance, if Bottom-Left is known to be played, each player would like to deviate – as it is shown in the following figure:

1 \ 2	Left	Right
Top	(1,1)	$\Leftarrow\Downarrow$(0,0)
Bottom	(0,0)$\Uparrow\Longrightarrow$	(1,1)

(Here, $\Uparrow$ means player 1 deviates to Top, etc.)

Unlike in this game, mostly players have different preferences on the outcomes, inducing *conflict*. In the following game, which is known as the *Battle of the Sexes*, conflict and the need for coordination are present together.

1 \ 2	Left	Right
Top	(2,1)	(0,0)
Bottom	(0,0)	(1,2)

Here, once again players would like to coordinate on Top-Left or Bottom-Right, but now Player 1 prefers to coordinate on Top-Left, while Player 2 prefers to coordinate on Bottom-Right. The stable outcomes are again Top-Left and Bottom- Right.

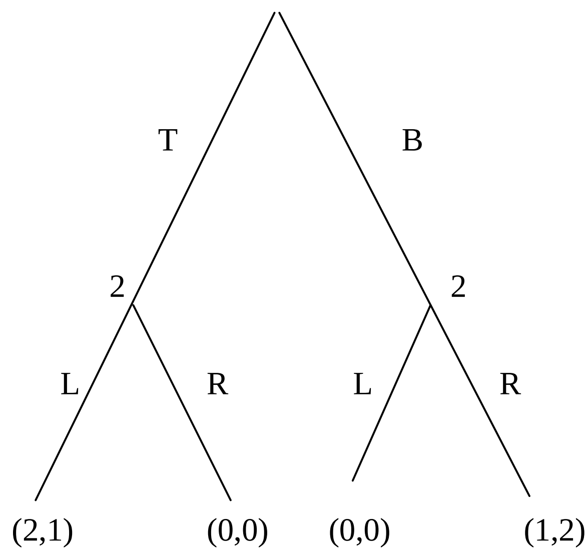


Figure 1:

Now, in the Battle of the Sexes, imagine that Player 2 knows what Player 1 does when she takes her action. This can be formalized via the following tree:

Here, Player 1 chooses between Top and Bottom, then (knowing what Player 1 has chosen) Player 2 chooses between Left and Right. Clearly, now Player 2 would choose Left if Player 1 plays Top, and choose Right if Player 1 plays Bottom. Knowing this, Player 1 would play Top. Therefore, one can argue that the only reasonable outcome of this game is Top-Left. (This kind of reasoning is called *backward induction*.)

When Player 2 is to check what the other player does, he gets only 1, while Player 1 gets 2. (In the previous game, two outcomes were stable, in which Player 2 would get 1 or 2.) That is, Player 2 prefers that Player 1 has information about what Player 2 does, rather than she herself has information about what player 1 does. When it is common knowledge that a player has some information or not, the player may prefer not to have that information – a robust fact that we will see in various contexts.

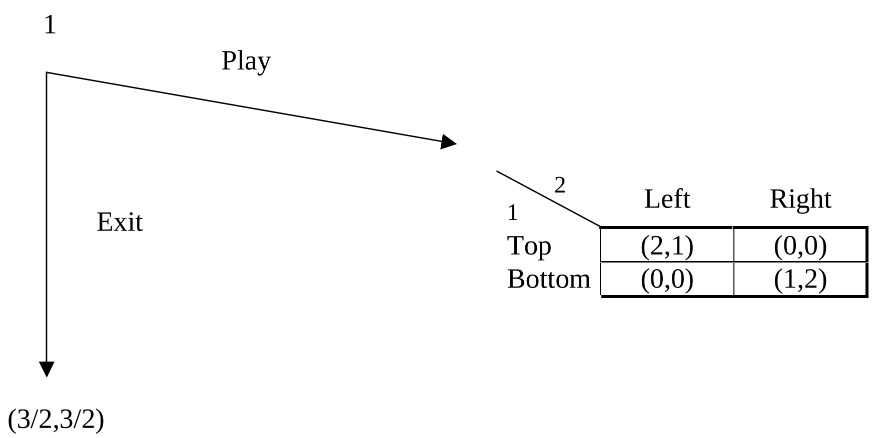
Exercise 1 *Clearly, this is generated by the fact that Player 1 knows that Player 2 will know what Player 1 does when she moves. Consider the situation that Player 1 thinks that Player 2 will know what Player 1 does only with probability* $\pi < 1$*, and this probability does not depend on what Player 1 does. What will happen in a "reasonable" equilibrium? [By the end of this course, hopefully, you will be able to formalize this* *situation, and compute the equilibria.]*

Another interpretation is that Player 1 can communicate to Player 2, who cannot communicate to player 1. This enables player 1 to commit to his actions, providing a strong position in the relation.

Exercise 2 *Consider the following version of the last game: after knowing what Player 2 does, Player 1 gets a chance to change his action; then, the game ends. In other words, Player 1 chooses between Top and Bottom; knowing Player 1's choice, Player 2 chooses between Left and Right; knowing 2's choice, Player 1 decides whether to stay where he is or to change his position. What is the "reasonable" outcome? What would happen if changing his action would cost player 1 c utiles?*

Imagine that, before playing the Battle of the Sexes, Player 1 has the option of exiting, in which case each player will get 3/2, or playing the Battle of the Sexes. When asked to play, Player 2 will know that Player 1 chose to play the Battle of the Sexes.

There are two "reasonable" equilibria (or stable outcomes). One is that Player 1 exits, thinking that, if he plays the Battle of the Sexes, they will play the Bottom-Right equilibrium of the Battle of the Sexes, yielding only 1 for player 1. The second one is that Player 1 chooses to Play the Battle of the Sexes, and in the Battle of the Sexes they play Top-Left equilibrium.



Some would argue that the first outcome is not really reasonable. Because, when asked to play, Player 2 will know that Player 1 has chosen to play the Battle of the Sexes, forgoing the payoff of 3/2. She must therefore realize that Player 1 cannot possibly be planning to play Bottom, which yields the payoff of 1 max. That is, when asked to play, Player 2 should understand that Player 1 is planning to play Top, and thus she should play Left. Anticipating this, Player 1 should choose to play the Battle of the Sexes game, in which they play Top-Left. Therefore, the second outcome is the only reasonable one. (This kind of reasoning is called *Forward Induction*.)

Here are some more exmples of games:

1. Prisoners' Dilemma:

1 \ 2	Confess	Not Confess
Confess	(-1, -1)	(1, -10)
Not Confess	(-10, 1)	(2, 2)

   This is a well known game that most of you know. [It is also discussed in Gibbons.] In this game no matter what the other player does, each player would like to confess, yielding (-1,-1), which is dominated by (2,2).

2. Hawk-Dove game

1 \ 2	Hawk	Dove
Hawk	$\left(\frac{V-C}{2},\frac{V-C}{2}\right)$	$(V, 0)$
Dove	$(0,V)$	$(\frac{V}{2},\frac{V}{2})$

   This is a generic biological game, but is also quite similar to many games in economics and political science. V is the value of a resource that one of the players will enjoy. If they shared the resource, their values are $V/2$. Hawk stands for a "tough" strategy, whereby the player does not give up the resource. However, if the other player is also playing hawk, they end up fighting, and incur the cost $C/2$ each. On the other hand, a Hawk player gets the whole resource for itself when playing a Dove. When $V > C$, we have a Prisoners' Dilemma game, where we would observe fight.

   When we have $V < C$, so that fighting is costly, this game is similar to another well-known game, inspired by the movie Rebel Without a Cause, named "Chicken", where two players driving towards a cliff have to decide whether to stop or continue. The one who stops first loses face, but may save his life. More generally, as class of games called "wars of attrition" are used to model this type of situations. In this case, a player would like to play Hawk if his opponent plays Dove, and play Dove if his opponent playes Hawk.

14.12 Game Theory Lecture Notes
Theory of Choice

Muhamet Yildiz

(Lecture 2)

1 The basic theory of choice

We consider a set X of alternatives. Alternatives are mutually exclusive in the sense that one cannot choose two distinct alternatives at the same time. We also take the set of feasible alternatives exhaustive so that a player's choices will always be defined. Note that this is a matter of modeling. For instance, if we have options Coffee and Tea, we define alternatives as $C =$ Coffee but no Tea, $T =$ Tea but no Coffee, $CT =$ Coffee and Tea, and $NT =$ no Coffee and no Tea.

Take a relation $\succeq$ on X. Note that a relation on X is a subset of $X \times X$. A relation $\succeq$ is said to be *complete* if and only if, given any $x, y \in X$, either $x \succeq y$ or $y \succeq x$. A relation $\succeq$ is said to be *transitive* if and only if, given any $x, y, z \in X$,

$$[x \succeq y \text{ and } y \succeq z] \Rightarrow x \succeq z.$$

A relation is a *preference relation* if and only if it is complete and transitive. Given any preference relation $\succeq$, we can define strict preference $\succ$ by

$$x \succ y \iff [x \succeq y \text{ and } y \not\succeq x],$$

and the indifference $\sim$ by

$$x \sim y \iff [x \succeq y \text{ and } y \succeq x].$$

A preference relation can be *represented* by a utility function $u : X \to \mathbb{R}$ in the following sense:

$$x \succeq y \iff u(x) \geq u(y) \qquad \forall x, y \in X.$$

The following theorem states further that a relation needs to be a preference relation in order to be represented by a utility function.

Theorem 1 *Let X be finite. A relation can be presented by a utility function if and only if it is complete and transitive. Moreover, if $u : X \to \mathbb{R}$ represents $\succeq$, and if $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then $f \circ u$ also represents $\succeq$.*

By the last statement, we call such utility functions ordinal.

In order to use this ordinal theory of choice, we should know the agent's preferences on the alternatives. As we have seen in the previous lecture, in game theory, a player chooses between his strategies, and his preferences on his strategies depend on the strategies played by the other players. Typically, a player does not know which strategies the other players play. Therefore, we need a theory of decision-making under uncertainty.

2 Decision-making under uncertainty

We consider a finite set Z of prizes, and the set P of all probability distributions $p : Z \to [0, 1]$ on Z, where $\sum_{z \in Z} p(z) = 1$. We call these probability distributions lotteries. A lottery can be depicted by a tree. For example, in Figure 1, Lottery 1 depicts a situation in which if head the player gets \$10, and if tail, he gets \$0.

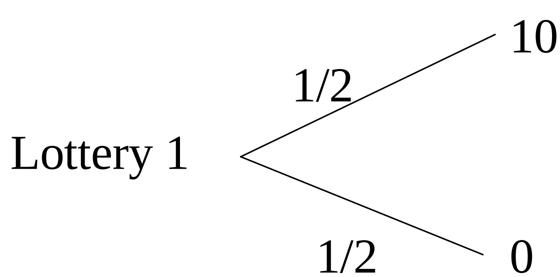


Figure 1:

Unlike the situation we just described, in game theory and more broadly when agents make their decision under uncertainty, we do not have the lotteries as in casinos where the probabilities are generated by some machines or given. Fortunately, it has been shown by Savage (1954) under certain conditions that a player's beliefs can be represented by a (unique) probability distribution. Using these probabilities, we can represent our acts by lotteries.

We would like to have a theory that constructs a player's preferences on the lotteries from his preferences on the prizes. The most well-known such theory is the theory of expected utility maximization by Von Neumann and Morgenstern. A preference relation $\succeq$ on P is said to be represented by a von Neumann-Morgenstern utility function $u : Z \to \mathbb{R}$ if and only if

$$p \succeq q \iff U(p) \equiv \sum_{z \in Z} u(z)p(z) \geq \sum_{z \in Z} u(z)q(z) \equiv U(q) \tag{1}$$

for each $p, q \in P$. Note that $U : P \to \mathbb{R}$ represents $\succeq$ in ordinal sense. That is, the agent acts as if he wants to maximize the expected value of u. For instance, the expected utility of Lottery 1 for our agent is $E(u(\text{Lottery 1})) = \frac{1}{2}u(10) + \frac{1}{2}u(0)$.[1]

The necessary and sufficient conditions for a representation as in (1) are as follows:

Axiom 1 *$\succeq$ is complete and transitive.*

This is necessary by Theorem 1, for U represents $\succeq$ in ordinal sense. The second condition is called *independence* axiom, stating that a player's preference between two lotteries p and q does not change if we toss a coin and give him a fixed lottery r if "tail" comes up.

Axiom 2 *For any $p, q, r \in P$, and any $a \in (0, 1]$, $ap + (1 - a)r \succ aq + (1 - a)r \iff p \succ q$.*

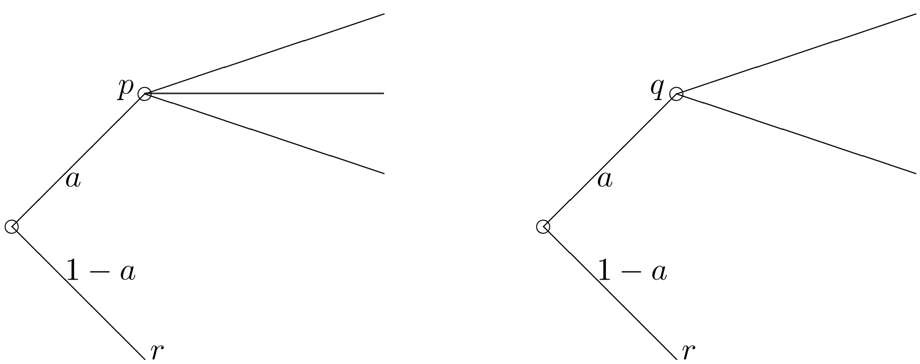
Let p and q be the lotteries depicted in Figure 2. Then, the lotteries $ap + (1 - a)r$ and $aq + (1 - a)r$ can be depicted as in Figure 3, where we toss a coin between a fixed lottery r and our lotteries p and q. Axiom 2 stipulates that the agent would not change his mind after the coin toss. Therefore, our axiom can be taken as an axiom of "dynamic consistency" in this sense.

The third condition is purely technical, and called *continuity* axiom. It states that there are no "infinitely good" or "infinitely bad" prizes.

Axiom 3 *For any $p, q, r \in P$, if $p \succ r$, then there exist $a, b \in (0, 1)$ such that $ap + (1 - a)r \succ q \succ bp + (1 - r)r$.*

[1] If Z were a continuum, like $\mathbb{R}$, we would compute the expected utility of p by $\int u(z)p(z)dz$.

Figure 2: Two lotteries



$ap + (1 - a)r$ $\qquad$ $aq + (1 - a)r$

Figure 3: Two compound lotteries

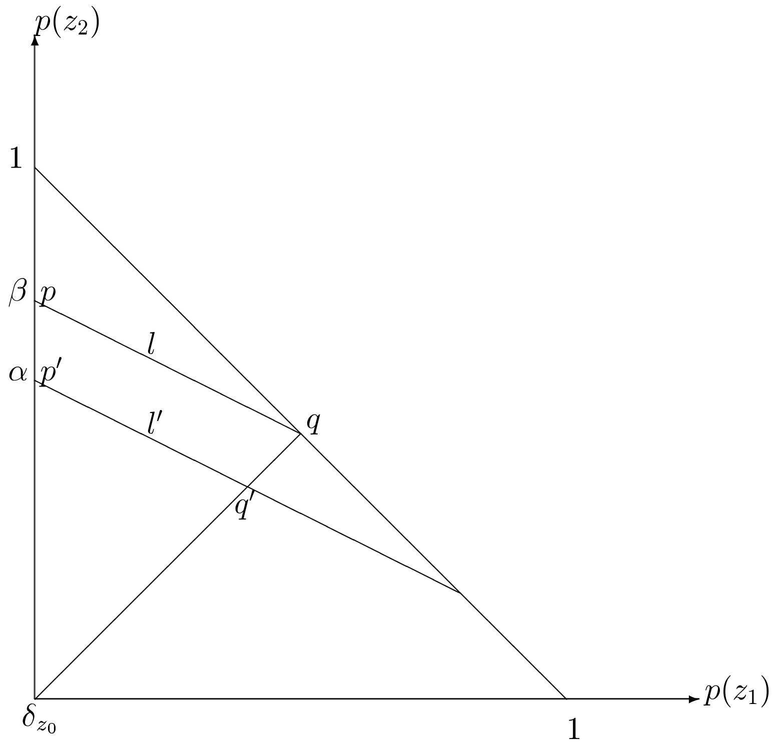


Figure 4: Indifference curves on the space of lotteries

Axioms 2 and 3 imply that, given any $p, q, r \in P$ and any $a \in [0, 1]$,

$$\text{if } p \sim q, \text{ then } ap + (1-a)\, r \sim aq + (1-a)r. \tag{2}$$

This has two implications:

1. The indifference curves on the lotteries are straight lines.

2. The indifference curves, which are straight lines, are parallel to each other.

To illustrate these facts, consider three prizes z_0, z_1, and z_2, where $z_2 \succ z_1 \succ z_0$. A lottery p can be depicted on a plane by taking $p(z_1)$ as the first coordinate (on the horizontal axis), and $p(z_2)$ as the second coordinate (on the vertical axis). $p(z_0)$ is $1 - p(z_1) - p(z_2)$. [See Figure 4 for the illustration.] Given any two lotteries p and q, the convex combinations $ap + (1-a)\, q$ with $a \in [0, 1]$ form the line segment connecting p to q. Now, taking $r = q$, we can deduce from (2) that, if $p \sim q$, then

$ap + (1-a)\,q \sim aq + (1-a)q = q$ for each $a \in [0,1]$. That this, the line segment connecting p to q is an indifference curve. Moreover, if the lines l and l' are parallel, then $\alpha/\beta = |q'|\,/\,|q|$, where $|q|$ and $|q'|$ are the distances of q and q' to the origin, respectively. Hence, taking $a = \alpha/\beta$, we compute that $p' = ap + (1-a)\,\delta_{z_0}$ and $q' = aq + (1-a)\,\delta_{z_0}$, where δ_{z_0} is the lottery at the origin, and gives z_0 with probability 1. Therefore, by (2), if l is an indifference curve, l' is also an indifference curve, showing that the indifference curves are parallel.

Line l can be defined by equation $u_1 p\,(z_1) + u_2 p\,(z_2) = c$ for some $u_1, u_2, c \in \mathbb{R}$. Since l' is parallel to l, then l' can also be defined by equation $u_1 p\,(z_1) + u_2 p\,(z_2) = c'$ for some c'. Since the indifference curves are defined by equality $u_1 p\,(z_1) + u_2 p\,(z_2) = c$ for various values of c, the preferences are represented by

$$\begin{aligned} U\,(p) &= 0 + u_1 p\,(z_1) + u_2 p\,(z_2) \\ &\equiv u(z_0)p(z_0) + u(z_1)p\,(z_1) + u(z_2)p(z_2), \end{aligned}$$

where

$$\begin{aligned} u\,(z_0) &= 0, \\ u(z_1) &= u_1, \\ u(z_2) &= u_2, \end{aligned}$$

giving the desired representation.

This is true in general, as stated in the next theorem:

Theorem 2 *A relation $\succeq$ on P can be represented by a von Neumann-Morgenstern utility function $u : Z \to R$ as in (1) if and only if $\succeq$ satisfies Axioms 1-3. Moreover, u and $\tilde{u}$ represent the same preference relation if and only if $\tilde{u} = au + b$ for some $a > 0$ and $b \in \mathbb{R}$.*

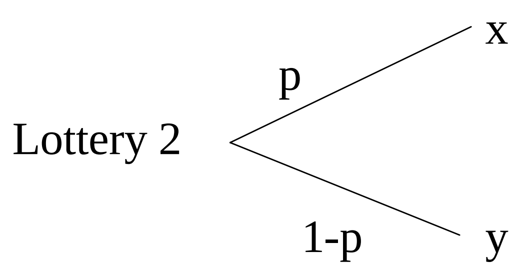
By the last statement in our theorem, this representation is "unique up to affine transformations". That is, an agent's preferences do not change when we change his von Neumann-Morgenstern (VNM) utility function by multiplying it with a positive number, or adding a constant to it; but they do change when we transform it through a non-linear transformation. In this sense, this representation is "cardinal". Recall that, in ordinal representation, the preferences wouldn't change even if the transformation were non-linear, so long as it was increasing. For instance, under certainty, $v = \sqrt{u}$ and u would represent the same preference relation, while (when there is uncertainty) the VNM utility function $v = \sqrt{u}$ represents a very different set of preferences on the lotteries than those are represented by u. Because, in cardinal representation, the curvature of the function also matters, measuring the agent's attitudes towards risk.

3 Attitudes Towards Risk

Suppose individual A has utility function u_A. How do we determine whether he dislikes risk or not?

The answer lies in the cardinality of the function u.

Let us first define a *fair* gamble, as a lottery that has expected value equal to 0. For instance, lottery 2 below is a fair gamble if and only if $px + (1-p)y = 0$.



We define an agent as *Risk-Neutral* if and only if he is indifferent between accepting and rejecting all fair gambles. Thus, an agent with utility function u is risk neutral if and only if

$$E(u(\text{lottery } 2)) = pu(x) + (1-p)u(y) = u(0)$$

for all p, x, and y.

This can only be true for all p, x, and y if and only if the agent is maximizing the expected value, that is, $u(x) = ax + b$. Therefore, we need the utility function to be linear.

Therefore, an agent is risk-neutral if and only if he has a linear Von-Neumann-Morgenstern utility function.

An agent is *strictly risk-averse* if and only if he rejects *all* fair gambles:

$$E(u(\text{lottery } 2)) < u(0)$$
$$pu(x) + (1-p)u(y) < u(px + (1-p)y) \equiv u(0)$$

Now, recall that a function $g(\cdot)$ is strictly concave if and only if we have

$$g(\lambda x + (1-\lambda)y) > \lambda g(x) + (1-\lambda)g(y)$$

for all $\lambda \in (0,1)$. Therefore, strict risk-aversion is equivalent to having a strictly concave utility function. We will call an agent *risk-averse* iff he has a *concave* utility function, i.e., $u(\lambda x + (1-\lambda)y) > \lambda u(x) + (1-\lambda)u(y)$ for each x, y, and λ.

Similarly, an agent is said to be (strictly) risk seeking iff he has a (strictly) convex utility function.

Consider Figure 5. The cord AB is the utility difference that this risk-averse agent would lose by taking the gamble that gives W_1 with probability p and W_2 with probability $1-p$. BC is the maximum amount that she would pay in order to avoid to take the gamble. Suppose W_2 is her wealth level and $W_2 - W_1$ is the value of her house and p is the probability that the house burns down. Thus in the absense of fire insurance this individual will have utility given by EU(gamble), which is lower than the utility of the expected value of the gamble.

3.1 Risk sharing

Consider an agent with utility function $u : x \mapsto \sqrt{x}$. He has a (risky) asset that gives \$100 with probability 1/2 and gives \$0 with probability 1/2. The expected utility of our agent from this asset is $EU_0 = \frac{1}{2}\sqrt{0} + \frac{1}{2}\sqrt{100} = 5$. Now consider another agent who is identical to our agent, in the sense that he has the same utility function and an asset that pays \$100 with probability 1/2 and gives \$0 with probability 1/2. We assume throughout that what an asset pays is statistically independent from what the other asset pays. Imagine that our agents form a mutual fund by pooling their assets, each agent owning half of the mutual fund. This mutual fund gives \$200 the probability 1/4 (when both assets yield high dividends), \$100 with probability 1/2 (when only one on the assets gives high dividend), and gives \$0 with probability 1/4 (when both assets yield low dividends). Thus, each agent's share in the mutual fund yields \$100 with probability

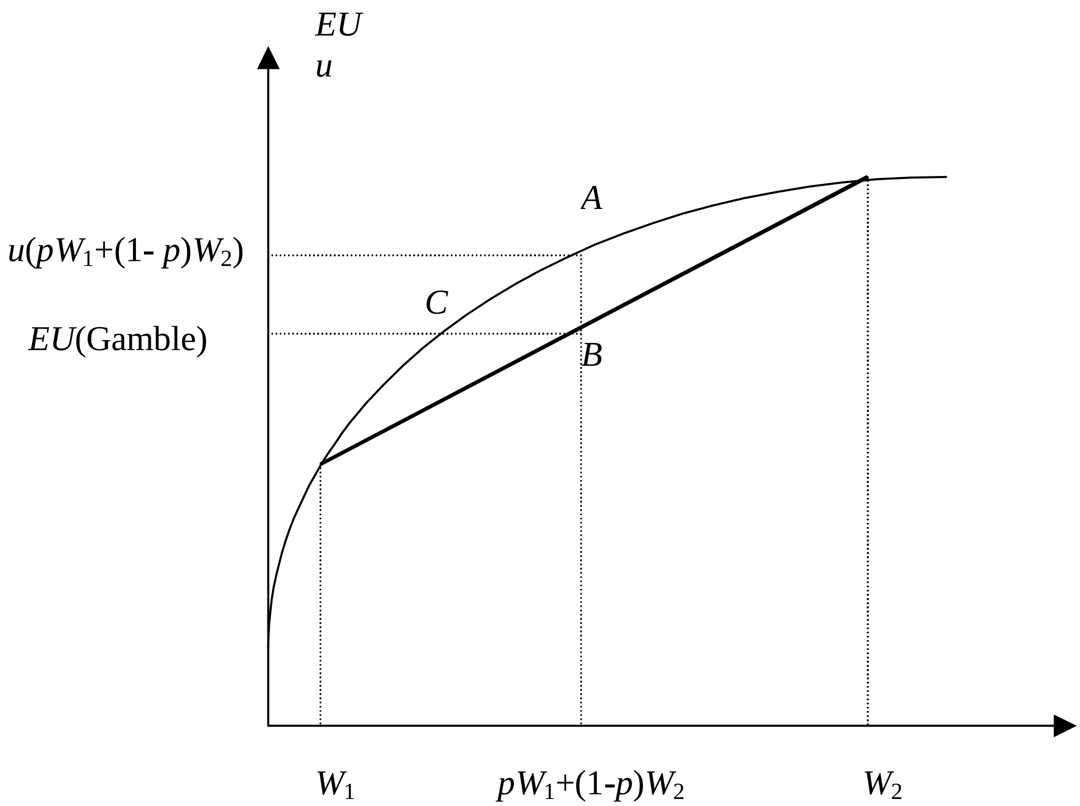


Figure 5:

1/4, \$50 with probability 1/2, and \$0 with probability 1/4. Therefore, his expected utility from the share in this mutual fund is $EU_S = \frac{1}{4}\sqrt{100} + \frac{1}{2}\sqrt{50} + \frac{1}{4}\sqrt{0} = 6.0355$. This is clearly larger than his expected utility from his own asset, therefore our agents gain from sharing the risk in their assets.

3.2 Insurance

Imagine a world where in addition to one of the agents above (with utility function $u : x \mapsto \sqrt{x}$ and a risky asset that gives \$100 with probability 1/2 and gives \$0 with probability 1/2), we have a risk-neutral agent with lots of money. We call this new agent the insurance company. The insurance company can insure the agent's asset, by giving him \$100 if his asset happens to yield \$0. How much premium, P, our risk averse agent would be willing to pay to get this insurance? [A premium is an amount that is to be paid to insurance company regardless of the outcome.]

If the risk-averse agent pays premium P and buys the insurance his wealth will be \$100 $- P$ for sure. If he does not, then his wealth will be \$100 with probability 1/2 and \$0 with probability 1/2. Therefore, he will be willing to pay P in order to get the insurance iff

$$u(100 - P) \geq \frac{1}{2}u(0) + \frac{1}{2}u(100)$$

i.e., iff

$$\sqrt{100 - P} \geq \frac{1}{2}\sqrt{0} + \frac{1}{2}\sqrt{100}$$

iff

$$P \leq 100 - 25 = 75.$$

On the other hand, if the insurance company sells the insurance for premium P, it will get P for sure and pay \$100 with probability 1/2. Therefore it is willing to take the deal iff

$$P \geq \frac{1}{2}100 = 50.$$

Therefore, both parties would gain, if the insurance company insures the asset for a premium $P \in (50, 75)$, a deal both parties are willing to accept.

Exercise 3 *Now consider the case that we have two identical risk-averse agents as above, and the insurance company. Insurance company is to charge the same premium*

P for each agent, and the risk-averse agents have an option of forming a mutual fund. What is the range of premiums that are acceptable to all parties?

14.12 Game Theory Lecture Notes*
Lectures 3-6

Muhamet Yildiz†

In these lectures, we will formally define the games and solution concepts, and discuss the assumptions behind these solution concepts.

In previous lectures we described a theory of decision-making under uncertainty. The second ingredient of the games is what each player knows. The knowledge is defined as an operator on the propositions satisfying the following properties:

1. if I know X, X must be true;
2. if I know X, I know that I know X;
3. if I don't know X, I know that I don't know X;
4. if I know something, I know all its logical implications.

We say that X is common knowledge if everyone knows X, and everyone knows that everyone knows X, and everyone knows that everyone knows that everyone knows X, ad infinitum.

1 Representations of games

The games can be represented in two forms:

*These notes are somewhat incomplete — they do not include some of the topics covered in the class. I will add the notes for these topics soon.

†Some parts of these notes are based on the notes by Professor Daron Acemoglu, who taught this course before.

1. The normal (strategic) form,

2. The extensive form.

1.1 Normal form

Definition 1 *(Normal form) An n-player game is any list* $G = (S_1, \ldots, S_n; u_1, \ldots, u_n)$*, where, for each* $i \in N = \{1, \ldots, n\}$*,* S_i *is the set of all strategies that are available to player* i*, and* $u_i : S_1 \times \ldots \times S_n \to \mathbb{R}$ *is player* i*'s von Neumann-Morgenstern utility function.*

Notice that a player's utility depends not only on his own strategy but also on the strategies played by other players. Moreover, u_i is a von Neumann-Morgenstern utility function so that player i tries to maximize the expected value of u_i (where the expected values are computed with respect to his own beliefs). We will say that player i is rational iff he tries to maximize the expected value of u_i (given his beliefs).[1]

It is also assumed that it is common knowledge that the players are $N = \{1, \ldots, n\}$, that the set of strategies available to each player i is S_i, and that each i tries to maximize expected value of u_i given his beliefs.

When there are only 2 players, we can represent the (normal form) game by a bimatrix (i.e., by two matrices):

1\2	left	right
up	0,2	1,1
down	4,1	3,2

Here, Player 1 has strategies up and down, and 2 has the strategies left and right. In each box the first number is 1's payoff and the second one is 2's (e.g., u_1 (up,left) $= 0$, u_2 (up,left) $= 2$.)

1.2 Extensive form

The extensive form contains all the information about a game, by defining who moves when, what each player knows when he moves, what moves are available to him, and

[1] We have also made another very strong "rationality" assumption in defining knowledge, by assuming that, if I know something, then I know all its logical consequences.

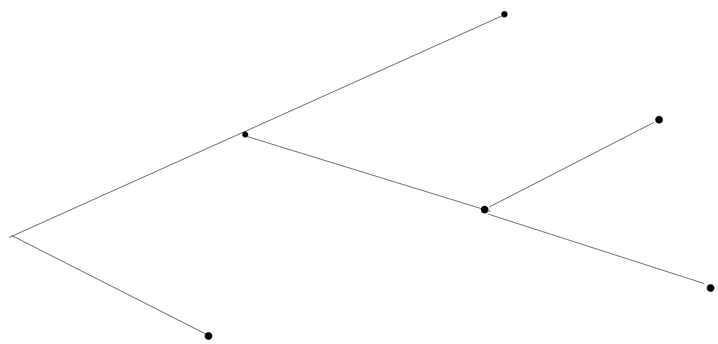
where each move leads to, etc., (whereas the normal form is more of a 'summary' representation). We first introduce some formalisms.

Definition 2 *A* <u>tree</u> *is a set of nodes and directed edges connecting these nodes such that*

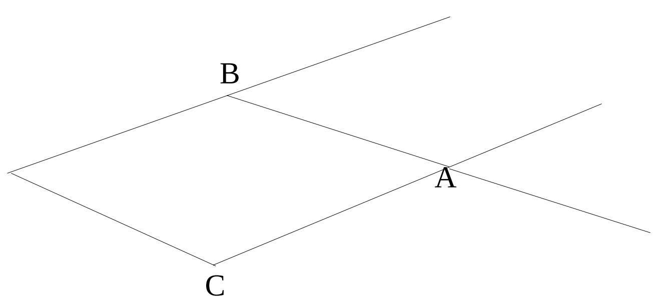
1. *for each node, there is at most one incoming edge;*

2. *for any two nodes, there is a unique path that connect these two nodes.*

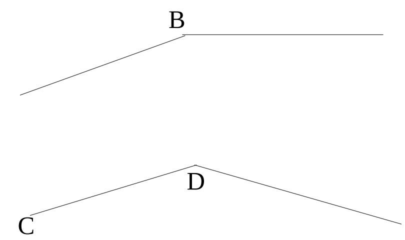
Imagine the branches of a tree arising from the trunk. For example,



is a tree. On the other hand,



is not a tree because there are two alternative paths through which point A can be reached (via B and via C).



is not a tree either since A and B are not connected to C and D.

Definition 3 ***(Extensive form)*** *A* Game *consists of a set of players, a tree, an allocation of each node of the tree (except the end nodes) to a player, an informational partition, and payoffs for each player at each end node.*

The set of players will include the agents taking part in the game. However, in many games there is room for chance, e.g. the throw of dice in backgammon or the card draws in poker. More broadly, we need to consider the "chance" whenever there is uncertainty about some relevant fact. To represent these possibilities we introduce a fictional player: Nature. There is no payoff for Nature at end nodes, and every time a node is allocated to Nature, a probability distribution over the branches that follow needs to be specified, e.g., Tail with probability of 1/2 and Head with probability of 1/2.

An *information set* is a collection of points (nodes) $\{n_1, \ldots, n_k\}$ such that

1. the same player i is to move at each of these nodes;

2. the same moves are available at each of these nodes.

Here the player i, who is to move at the information set, is asusumed to be unable to distinguish between the points in the information set, but able to distinguish between the points outside the information set from those in it. For instance, consider the following game:

Here, Player 2 knows that Player 1 has taken action T or B and not action X; but Player 2 cannot know for sure whether 1 has taken T or B. The following picture means the same thing:

An *information partition* is an allocation of each node of the tree (except the starting and end-nodes) to an information set.

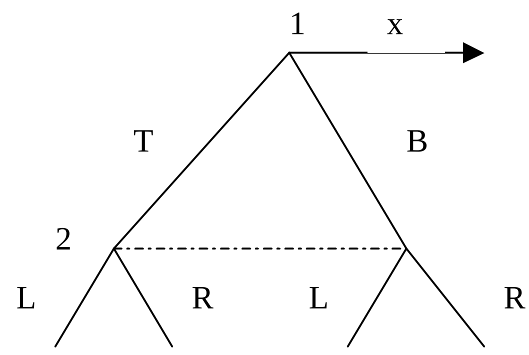


Figure 1:

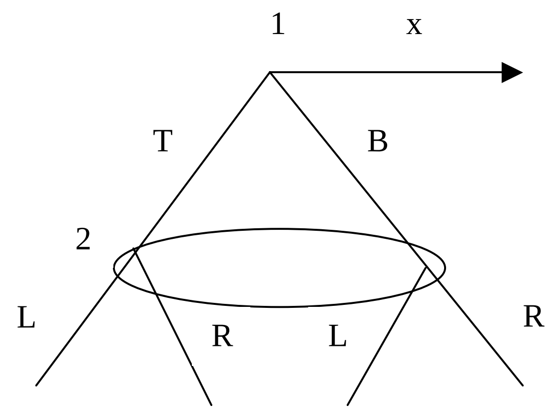


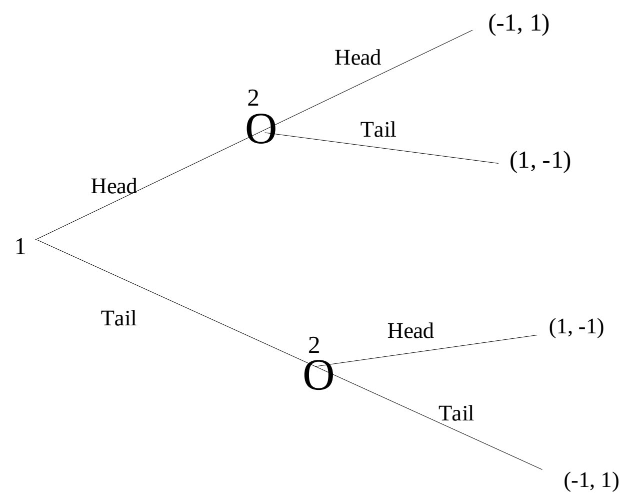
Figure 2:

To sum up: at any node, we know: which player is to move, which moves are available to the player, and which information set contains the node, summarizing the player's information at the node. Of course, if two nodes are in the same information set, the available moves in these nodes must be the same, for otherwise the player could distinguish the nodes by the available choices. Again, all these are assumed to be common knowledge. For instance, in the game above, player 1 knows that, if player 1 takes X, player 2 will know this, but if he takes T or B, player 2 will not know which of these two actions has been taken.

Definition 4 A strategy *of a player* is a complete contingent-plan *determining which action he will take at each information set he is to move (including the information sets* *that will not be reached according to this strategy).*

Let us now consider some examples:

Game 1: Matching Pennies with Perfect Information



The tree consists of 7 nodes. The first one is allocated to player 1, and the next two to player 2. The four end-nodes have payoffs attached to them. Since there are two players, payoff vectors have two elements. The first number is the payoff of player 1 and the second is the payoff of player 2. These payoffs are <u>von Neumann-Morgenstern</u> utilities so that we can take expectations over them and calculate <u>expected utilities</u>.

The informational partition is very simple; all nodes are in their own information set. In other words, all information sets are <u>singletons</u> (have only 1 element). This implies that there is no uncertainty regarding the previous play (history) in the game. At this point recall that in a tree, each node is reached through a unique path. Therefore, if all information sets are singletons, a player can construct the <u>history</u> of the game perfectly. For instance in this game, player 2 knows whether player 1 chose Head or Tail. And player 1 knows that when he plays Head or Tail, Player 2 will know what player 1 has played. (Games in which all information sets are singletons are called *games of perfect information*.)

In this game, the set of strategies for player 1 is {Head, Tail}. A strategy of player 2 determines what to do depending on what player 1 does. So, his strategies are:

HH = Head if 1 plays Head, and Head if 1 plays Tail;

HT = Head if 1 plays Head, and Tail if 1 plays Tail;

TH = Tail if 1 plays Head, and Head if 1 plays Tail;

TT = Tail if 1 plays Head, and Tail if 1 plays Tail.

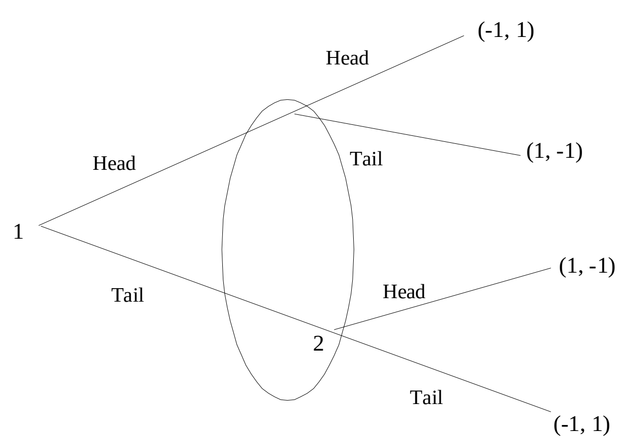
What are the payoffs generated by each strategy pair? If player 1 plays Head and 2 plays HH, then the outcome is [1 chooses Head and 2 chooses Head] and thus the payoffs are (-1,1). If player 1 plays Head and 2 plays HT, the outcome is the same, hence the payoffs are (-1,1). If 1 plays Tail and 2 plays HT, then the outcome is [1 chooses Tail and 2 chooses Tail] and thus the payoffs are once again (-1,1). However, if 1 plays Tail and 2 plays HH, then the outcome is [1 chooses Tail and 2 chooses Head] and thus the payoffs are (1,-1). One can compute the payoffs for the other strategy pairs similarly.

Therefore, the normal or the strategic form game corresponding to this game is

	HH	HT	TH	TT
Head	-1,1	-1,1	1,-1	1,-1
Tail	1,-1	-1,1	1,-1	-1,1

Information sets are very important! To see this, consider the following game.

Game 2: Matching Pennies with Imperfect Information

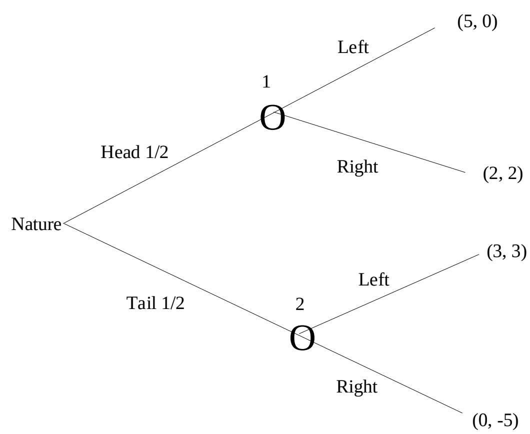


Games 1 and 2 appear very similar but in fact they correspond to two very different situations. In Game 2, when she moves, player 2 does not know whether 1 chose Head or Tail. This is a game of imperfect information (That is, some of the information sets contain more than one node.)

The strategies for player 1 are again Head and Tail. This time player 2 has also only two strategies: Head and Tail (as he does not know what 1 has played). The normal form representation for this game will be:

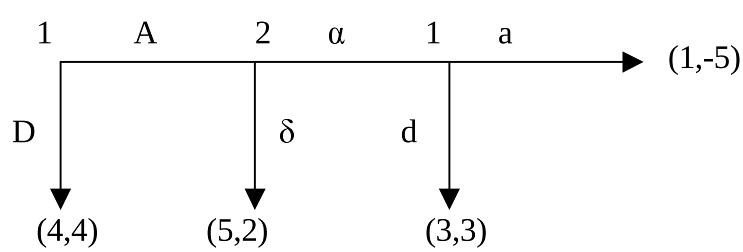
1\2	Head	Tail
Head	-1,1	1,-1
Tail	1,-1	-1,1

Game 3: A Game with Nature:



Here, we toss a fair coin, where the probability of Head is 1/2. If Head comes up, Player 1 chooses between Left and Right; if Tail comes up, Player 2 chooses between Left and Right.

Exercise 5 *What is the normal-form representation for the following game:*



Can you find another extensive-form game that has the same normal-form representation?

[Hint: For each extensive-form game, there is only one normal-form representation (up to a renaming of the strategies), but a normal-form game typically has more than one extensive-form representation.]

In many cases a player may not be able to guess exactly which strategies the other players play. In order to cover these situations we introduce the mixed strategies:

Definition 6 *A* mixed strategy *of a player is a probability distribution over the set of his strategies.*

If player i has strategies $S_i = \{s_{i1}, s_{i2}, \ldots, s_{ik}\}$, then a mixed strategy σ_i for player i is a function on S_i such that $0 \leq \sigma_i(s_{ij}) \leq 1$ and $\sigma_i(s_{i1}) + \sigma_i(s_{i2}) + \cdots + \sigma_i(s_{ik}) = 1$. Here σ_i represents other players' beliefs about which strategy i would play.

2 How to play?

We will now describe the most common "solution concepts" for normal-form games. We will first describe the concept of "dominant strategy equilibrium," which is implied by the rationality of the players. We then discuss "rationalizability" which corresponds to the common knowledge of rationality, and finally we discuss Nash Equilibrium, which is related to the mutual knowledge of players' conjectures about the other players' actions.

2.1 Dominant-strategy equilibrium

Let us use the notation s_{-i} to mean the list of strategies s_j played by all the players j other than i, i.e.,

$$s_{-i} = (s_1, \ldots s_{i-1}, s_{i+1}, \ldots s_n).$$

Definition 7 *A strategy* s_i^* strictly dominates s_i *if and only if*

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i}.$$

That is, no matter what the other players play, playing s_i^* is strictly better than playing s_i for player i. In that case, if i is rational, he would never play the strictly dominated strategy s_i.[2]

A mixed strategy σ_i dominates a strategy s_i in a similar way: σ_i *strictly dominates* s_i if and only if

$$\sigma_i(s_{i1})u_i(s_{i1}, s_{-i}) + \sigma_i(s_{i2})u_i(s_{i2}, s_{-i}) + \cdots \sigma_i(s_{ik})u_i(s_{ik}, s_{-i}) > u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i}.$$

A rational player i will never play a strategy s_i iff s_i is dominated by a (mixed or pure) strategy.

Similarly, we can define weak dominance.

Definition 8 *A strategy* s_i^* weakly dominates s_i *if and only if*

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i}$$

and

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i})$$

for some $s_{-i} \in S_{-i}$.

That is, no matter what the other players play, playing s_i^* is at least as good as playing s_i, and there are some contingencies in which playing s_i^* is strictly better than s_i. In that case, if rational, i would play s_i only if he believes that these contingencies will never occur. If he is cautious in the sense that he assigns some positive probability for each contingency, he will not play s_i.

Definition 9 *A strategy* s_i^d *is a* (weakly) dominant strategy *for player i if and only if* s_i^d *weakly dominates all the other strategies of player* i. *A strategy* s_i^d *is a* strictly dominant strategy *for player i if and only if* s_i^d *strictly dominates all the other strategies of player* i.

If i is rational, and has a strictly dominant strategy s_i^d, then he will not play any other strategy. If he has a weakly dominant strategy and cautious, then he will not play other strategies.

[2] That is, there is no belief under which he would play s_i. Can you prove this?

Example:

1\2	work hard	shirk
hire	2,2	1,3
don't hire	0,0	0,0

In this game, player 1 (firm) has a strictly dominant strategy which is to "hire." Player 2 has only a weakly dominated strategy. If players are rational, and in addition player 2 is cautious, then we expect player 1 to "hire", and player 2 to "shirk":[3]

1\2	work hard	shirk
hire	2,2 $\Longrightarrow$	1,3
don't hire	0,0 $\Uparrow$	0,0 $\Uparrow$

Definition 10 *A strategy profile* $s^d = (s_1^d, s_2^d, ...s_N^d)$ *is a* dominant strategy equilibrium*, if and only if* s_i^d *is a dominant strategy for each player* i*.*

As an example consider the Prisoner's Dilemma.

1\2	confess	don't confess
confess	-5,-5	0,-6
don't confess	-6,0	-1,-1

"Confess" is a strictly dominant strategy for both players, therefore ("confess", "confess") is a dominant strategy equilibrium.

1\2	confess	don't confess
confess	-5,-5	$\Longleftarrow$ 0,-6
don't confess	-6,0 $\Uparrow$	$\Longleftarrow$-1,-1 $\Uparrow$

[N.B.: An equilibrium is a strategy combination, not the outcome (-5,-5), this is simply the "equilibrium outcome"].

This game also illustrates another important point. Although there is an outcome that is better for both players, don't confess, don't confess, they both end up confessing. That is, individuals' maximization may yield a Pareto dominated outcome, -5, -5, rather

[3] This is the only outcome, provided that each player is rational and player 2 knows that player 1 is rational. Can you show this?

than -1,-1. That is why this type of analysis is often called "non-cooperative game theory" as opposed to co-operative game theory, which would pick the mutually beneficial outcome here. Generally, there is no presumption that the equilibrium outcome will correspond to what is best for the society (or the players). There will be many "market failures" in situations of conflict, and situations of conflict are the ones game theory focuses on.

Example: (second-price auction) We have an object to be sold through an auction. There are two buyers. The value of the object for any buyer i is v_i, which is known by the buyer i. Each buyer i submits a bid b_i in a sealed envelope, simultaneously. Then, we open the envelopes, the agent i^* who submits the highest bid

$$b_{i^*} = \max\{b_1, b_2\}$$

gets the object and pays the second highest bid (which is b_j with $j \neq i^*$). (If two or more buyers submit the highest bid, we select one of them by a coin toss.)

Formally the game is defined by the player set $N = \{1, 2\}$, the strategies b_i, and the payoffs

$$u_i(b_1, b_2) = \begin{cases} v_i - b_j & \text{if } b_i > b_j \\ (v_i - b_j)/2 & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

where $i \neq j$.

In this game, bidding his true valuation v_i is a dominant strategy for each player i. To see this, consider strategy of bidding some other value $b'_i \neq v_i$ for any i. We want to show that b'_i is weakly dominated by bidding v_i. Consider the case $b'_i < v_i$. If the other player bids some $b_j < b'_i$, player i would get $v_i - b_j$ under both strategies b'_i and v_i. If the other player bids some $b_j \geq v_i$, player i would get 0 under both strategies b'_i and v_i. But if $b_j = b'_i$, bidding v_i yields $v_i - b_j > 0$, while b'_i yields only $(v_i - b_j)/2$. Likewise, if $b'_i < b_j < v_i$, bidding v_i yields $v_i - b_j > 0$, while b'_i yields only 0. Therefore, bidding v_i dominates b'_i. The case $b'_i > v_i$ is similar, except for when $b'_i > b_j > v_i$, bidding v_i yields 0, while b'_i yields negative payoff $v_i - b_j < 0$. Therefore, bidding v_i is dominant strategy for each player i.

Exercise 11 *Extend this to the n-buyer case.*

When it exists, the dominant strategy equilibrium has an obvious attraction. In that case, the rationality of players implies that the dominant strategy equilibrium will be played. However, it does not exist in general. The following game, the Battle of the Sexes, is supposed to represent a timid first date (though there are other games from animal behavior that deserve this title much more). Both the man and the woman want to be together rather than go alone. However, being timid, they do not make a firm date. Each is hoping to find the other either at the opera or the ballet. While the woman prefers the ballet, the man prefers the opera.

Man\Woman	opera	ballet
opera	1,4	0,0
ballet	0,0	4,1

Clearly, no player has a dominant strategy:

Man\Woman	opera	ballet
opera	1,4	$\Longleftarrow$ $\Downarrow$ 0,0
ballet	0,0 $\Uparrow$ $\Longrightarrow$	4,1

2.2 Rationalizability or Iterative elimination of strictly dominated strategies

Consider the following Extended Prisoner's Dilemma game:

1\2	confess	don't confess	run away
confess	-5,-5	0,-6	-5,-10
don't confess	-6,0	-1,-1	0,-10
run away	-10,-6	-10,0	-10,-10

In this game, no agent has any dominant strategy, but there exists a dominated strategy: "run away" is strictly dominated by "confess" (both for 1 and 2). Now consider 2's problem. She knows 1 is "rational," therefore she can predict that 1 will not choose "run away," thus she can eliminate "run away" and consider the smaller game

1\2	confess	don't confess	run away
confess	-5,-5	0,-6	-5,-10
don't confess	-6,0	-1,-1	0,-10

where we have eliminated "run away" because it was strictly dominated; the column player reasons that the row player would never choose it.

In this smaller game, 2 has a dominant strategy which is to "confess." That is, if 2 is rational and knows that 1 is rational, she will play "confess."

In the original game "don't confess" did better against "run away," thus "confess" was not a dominant strategy. However, 1 playing "run away" cannot be *rationalized* because it is a dominated strategy. This leads to the *Elimination of Strictly Dominated Strategies*. What happens if we "Iteratively Eliminate Strictly Dominated" strategies? That is, we eliminate a strictly dominated strategy, and then look for another strictly dominated strategy in the reduced game. We stop when we can no longer find a strictly dominated strategy. Clearly, if it is common knowledge that players are rational, they will play only the strategies that survive this iteratively elimination of strictly dominated strategies. Therefore, we call such strategies *rationalizable*. **Caution**: we do eliminate the strategies that are dominated by some mixed strategies!

In the above example, the set of rationalizable strategies is once again "confess," "confess."

At this point you should stop and apply this method to the Cournot duopoly!! (See Gibbons.) Also, make sure that you can generate the rationality assumption at each elimination. For instance, in the game above, player 2 knows that player 1 is rational and hence he will not "run away;" and since she is also rational, she will play only "confess," for the "confess" is the only best response for any belief of player 2 that assigns 0 probability to that player 1 "runs away."

The problem is there may be too many rationalizable strategies. Consider the Matching Pannies game:

1\2	Head	Tail
Head	-1,1	1,-1
Tail	1,-1	-1,1

Here, the set of rationalizable strategies contains {Head,Tail} for both players. If 1 believes that 2 will play Head, he will play Tail and if 2 believes that 1 will play Tail, he will play Tail. Thus, the strategy-pair (Head,Tail) is rationalizable. But note that the beliefs of 1 and 2 are not congruent.

The set of rationalizable strategies is in general very large. In contrast, the concept of dominant strategy equilibrium is too restrictive: usually it does not exist.

The reason for existence of too many rationalizable strategies is that we do not restrict players' conjectures to be 'consistent' with what the others are actually doing. For instance, in the rationalizable strategy (Head, Tail), player 2 plays Tail by conjecturing that Player 1 will play Tail, while Player 1 actually plays Head. We consider another concept — Nash Equilibrium (henceforth NE), which assumes mutual knowledge of conjectures, yielding consistency.

2.3 Nash Equilibrium

Consider the battle of the sexes

Man\Woman	opera	ballet
opera	1,4	0,0
ballet	0,0	4,1

In this game, there is no dominant strategy. But suppose W is playing opera. Then, the best thing M can do is to play opera, too. Thus opera is a best-response for M against opera. Similarly, opera is a best-response for W against opera. Thus, at (opera, opera), neither party wants to take a different action. This is a Nash Equilibrium.

More formally:

Definition 12 *For any player i, a strategy s_i^{BR} is a* best response *to s_{-i} if and only if*

$$u_i(s_i^{BR}, s_{-i}) \geq u_i(s_i, s_{-i}), \forall s_i \in S_i$$

This definition is identical to that of a dominant strategy except that it is not for all $s_{-i} \in S_{-i}$ but for a specific strategy s_{-i}. If it were true for all s_{-i}, then S_i^{BR} would also be a dominant strategy, which is a stronger requirement than being a best response against some strategy s_{-i}.

Definition 13 *A strategy profile* $(s_1^{NE}, ...s_N^{NE})$ *is a* Nash Equilibrium *if and only if* s_i^{NE} *is a best-response to* $s_{-i}^{NE} = (s_1^{NE}, ...s_{i-1}^{NE}, s_{i+1}^{NE}, ...s_N^{NE})$ *for each* i. *That is, for all* i, *we have that*

$$U_i(s_i^{NE}, s_{-i}^{NE}) \geq U_i(s_i, s_{-i}^{NE}) \quad \forall s_i \in S_i.$$

In other words, no player would have an incentive to deviate, if he knew which strategies the other players play.

If a strategy profile is a dominant strategy equilibrium, then it is also a NE, but the reverse is not true. For instance, in the Battle of the Sexes, (O,O) is a NE and B-B is an NE but neither are dominant strategy equilibria. Furthermore, a dominant strategy equilibrium is unique, but as the Battle of the Sexes shows, NE is not unique in general.

At this point you should stop, and compute the Nash equilibrium in Cournot Duopoly game!! Why does Nash equilibrium coincide with the rationalizable strategies. In general: Are all rationalizable strategies Nash equilibria? Are all Nash equilibria rationalizable? You should also compute the Nash equilibrium in Cournot oligopoly, Bertrand duopoly and in the commons problem.

The definition above covers only the pure strategies. We can define the Nash equilibrium for mixed strategies by changing the pure strategies with the mixed strategies. Again given the mixed strategy of the others, each agent maximizes his expected payoff over his own (mixed) strategies.[4]

Example Consider the Battle of the Sexes again where we located two pure strategy equilibria. In addition to the pure strategy equilibria, there is a mixed strategy equilibrium.

Man\Woman	opera	ballet
opera	1,4	0,0
ballet	0,0	4,1

Let's write q for the probability that M goes to opera; with probability $1-q$, he goes to ballet. If we write p for the probability that W goes to opera, we can compute her

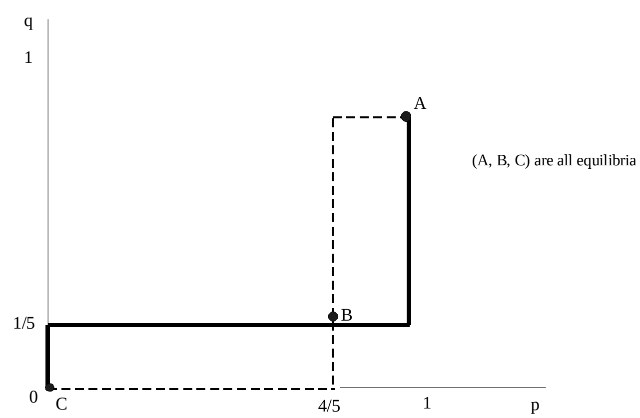
[4]In terms of beliefs, this correspondes to the requirement that, if i assigns positive probability to the event that j may play a particular pure strategy s_j, then s_j must be a best response given j's beliefs.

expected utility from this as

$$
\begin{aligned}
U_2(p;q) &= pqu_2(\text{opera,opera}) + p(1-q)u_2(\text{ballet,opera}) \\
&\quad + (1-p)qu_2(\text{opera,ballet}) + (1-p)(1-q)u_2(\text{ballet,ballet}) \\
&= p[qu_2(\text{opera,opera}) + (1-q)u_2(\text{ballet,opera})] \\
&\quad + (1-p)[qu_2(\text{opera,ballet}) + (1-q)u_2(\text{ballet,ballet})] \\
&= p[q4 + (1-q)0] + (1-p)[0q + 1(1-q)] \\
&= p[4q] + (1-p)[1-q].
\end{aligned}
$$

Note that the term $[4q]$ multiplied with p is her expected utility from going to opera, and the term multiplied with $(1-p)$ is her expected utility from going to ballet. $U_2(p;q)$ is strictly increasing with p if $4q > 1-q$ (i.e., $q > 1/5$); it is strictly decreasing with p if $4q < 1-q$, and is constant if $4q = 1-q$. In that case, W's best response is $p = 1$ of $q > 1/5$, $p = 0$ if $q < 1/5$, and p is any number in $[0,1]$ if $q = 1/5$. In other words, W would choose opera if her expected utility from opera is higher, ballet if her expected utility from ballet is higher, and can choose any of opera or ballet if she is indifferent between these two.

Similarly we compute that $q = 1$ is best response if $p > 4/5$; $q = 0$ is best response if $p < 4/5$; and any q can be best response if $p = 4/5$. We plot the best responses in the following graph.



The Nash equilibria are where these best responses intersect. There is one at (0,0), when they both go to ballet, one at (1,1), when they both go to opera, and there is one at (4/5,1/5), when W goes to opera with probability 4/5, and M goes to opera with probability 1/5.

*Note how we compute the mixed strategy equilibrium (for 2*x*2 games). We choose 1's probabilities so that 2 is indifferent between his strategies, and we choose 2's probabilities so that 1 is indifferent.*

14.12 Game Theory Lecture Notes*
Lectures 6-8

Muhamet Yildiz

In these lectures we analyze dynamic games (with complete information). We first analyze the perfect information games, where each information set is singleton, and develop the notion of backward induction. Then, considering more general dynamic games, we will introduce the concept of the subgame perfection. We explain these concepts on economic problems, most of which can be found in Gibbons.

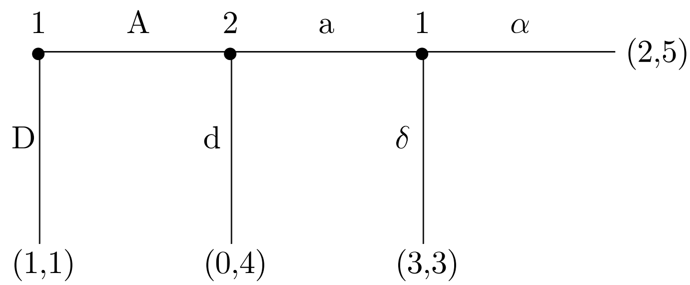
1 Backward induction

The concept of backward induction corresponds to the assumption that it is common knowledge that each player will act rationally at each node where he moves – even if his rationality would imply that such a node will not be reached. Mechanically, it is computed as follows. Consider a finite horizon perfect information game. Consider any node that comes just before terminal nodes, that is, after each move stemming from this node, the game ends. If the player who moves at this node acts rationally, he will choose the best move for himself. Hence, we select one of the moves that give this player the highest payoff. Assigning the payoff vector associated with this move to the node at hand, we delete all the moves stemming from this node so that we have a shorter game, where our node is a terminal node. Repeat this procedure until we reach the origin.

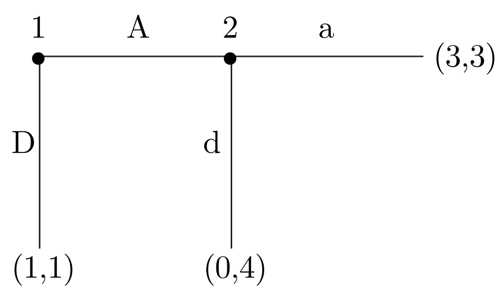
Example: consider the following well-known game, called as *the centipedes game*. This game illustrates the situation where it is mutually beneficial for all players to stay

*These notes do not include all the topics that will be covered in the class. I will add those topics later.

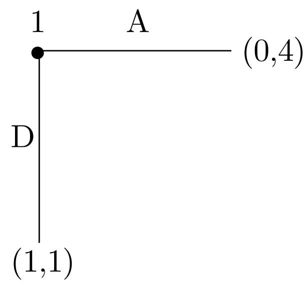
in a relation, while a player would like to exit the relation, if she knows that the other player will exit in the next day.



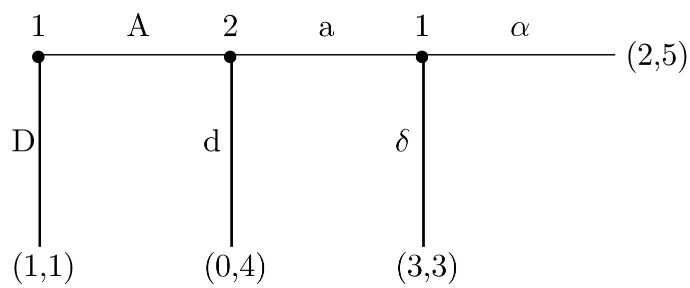
In the third day, player 1 moves, choosing between going across (α) or down (δ). If he goes across, he would get 2; if he goes down, he will get 3. Hence, we reckon that he will go down. Therefore, we reduce the game as follows:



In the second day, player 2 moves, choosing between going across (a) or down (d). If she goes across, she will get 3; if she goes down, she will get 4. Hence, we reckon that she will go down. Therefore, we reduce the game further as follows:



Now, player 1 gets 0 if he goes across (A), and gets 1 if he goes down (D). Therefore, he goes down. The equilibrium that we have constructed is as follows:



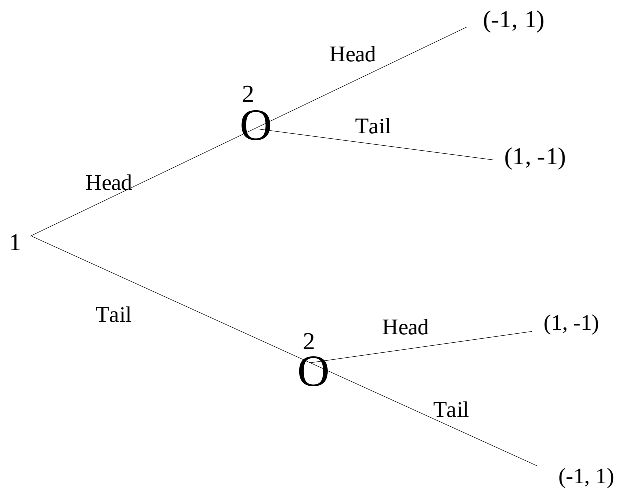
That is, at each node, the player who is to move goes down, exiting the relation.

Let's go over the assumptions that we have made in constructing our equilibrium. We assumed that player 1 will act rationally at the last date, when we reckoned that he goes down. When we reckoned that player 2 goes down in the second day, we assumed that player 2 assumes that player 1 will act rationally on the third day, and also assumed that she is rational, too. On the first day, player 1 anticipates all these. That is, he is assumed to know that player 2 is rational, and that she will keep believing that player 1 will act rationally on the third day.

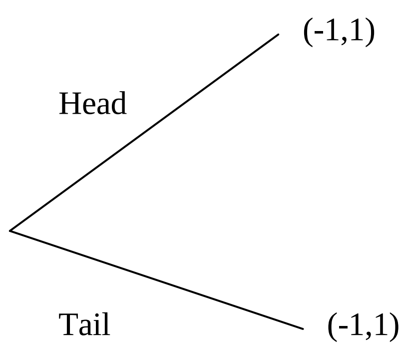
This example also illustrates another notion associated with backward induction – commitment (or the lack of commitment). Note that the outcomes on the third day (i.e., (3,3) and (2,5)) are both strictly better than the equilibrium outcome (1,0). But they cannot reach these outcomes, because player 2 cannot commit to go across, whence player 1 exits the relation in the first day. There is also a further commitment problem in this example. If player 1 where able to commit to go across on the third day, player

2 would definitely go across on the second day, whence player 1 would go across on the first. Of course, player 1 cannot commit to go across on the third day, and the game ends in the first day, yielding the low payoffs (1,0).

As another example, let us apply backward induction to the Matching Pennies with Perfect Information:



If player 1 chooses Head, player 2 will Head; and if 1 chooses Tail, player 2 will prefer Tail, too. Hence, the game is reduced to



In that case, Player 1 will be indifferent between Head and Tail, choosing any of these two option or any randomization between these two acts will give us an equilibrium with backward induction.

At this point, you should stop and study the Stackelberg duopoly in Gibbons. You should also check that there is also a Nash equilibrium of this game in which the follower produces the Cournot quantity irrespective of what the leader produces, and the leader produces the Cournot quantity. Of course, this is not consistent with backward induction: when the follower knows that the leader has produced the Stackelberg quantity, he will change his mind and produce a lower quantity, the quantity that is computed during the backward induction. For this reason, we say that this Nash equilibrium is based on a non-credible threat (of the follower).

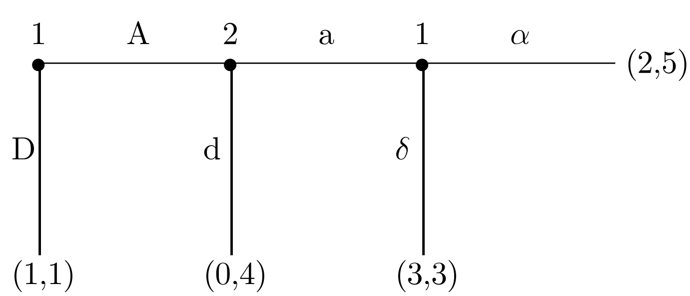
Backward induction is a powerful solution concept with some intuitive appeal. Unfortunately, we cannot apply it beyond perfect information games with a finite horizon. Its intuition, however, can be extended beyond these games through subgame perfection.

2 Subgame perfection

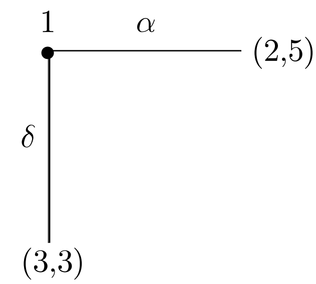
A main property of backward induction is that, when we confine ourselves to a subgame of the game, the equilibrium computed using backward induction remains to be an equilibrium (computed again via backward induction) of the subgame. Subgame perfection generalizes this notion to general dynamic games:

Definition 1 *A Nash equilibrium is said to be* subgame perfect *if an only if it is a Nash equilibrium in every subgame of the game.*

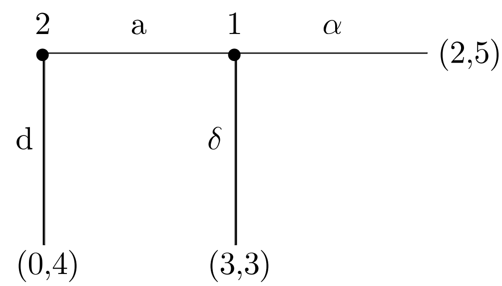
What is a subgame? In any given game, there may be some smaller games embedded; we call each such embedded game a subgame. Consider, for instance, the centipedes game (where the equilibrium is drawn in thick lines):



This game has three subgames. Here is one subgame:



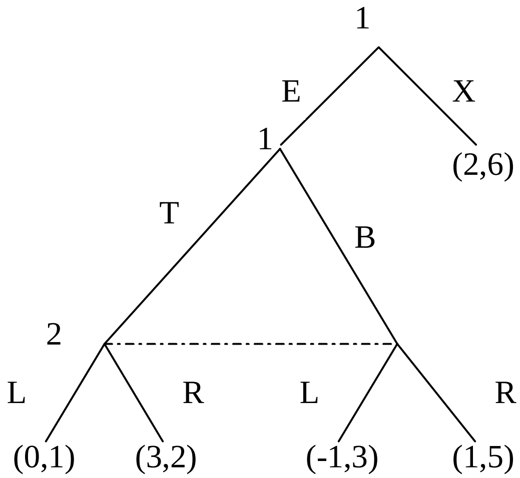
This is another subgame:



And the third subgame is the game itself. We call the first two subgames (excluding the game itself) proper. Note that, in each subgame, the equilibrium computed via backward induction remains to be an equilibrium of the subgame.

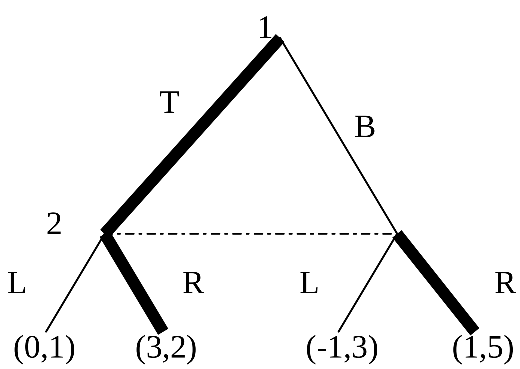
Now consider the matching penny game with perfect information. In this game, we have three subgames: one after player 1 chooses Head, one after player 1 chooses Tail, and the game itself. Again, the equilibrium computed through backward induction is a Nash equilibrium at each subgame.

Now consider the following game.

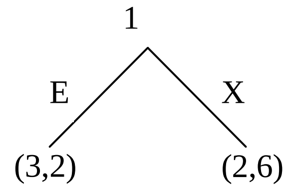


We cannot apply backward induction in this game, because it is not a perfect information game. But we can compute the subgame perfect equilibrium. This game has two subgames: one starts after player 1 plays E; the second one is the game itself. We compute the subgame perfect equilibria as follows. We first compute a Nash equilibrium of the subgame, then fixing the equilibrium actions as they are (in this subgame), and taking the equilibrium payoffs in this subgame as the payoffs for entering in the subgame, we compute a Nash equilibrium in the remaining game.

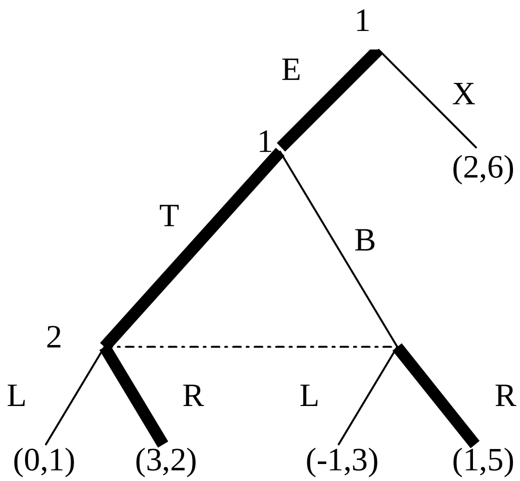
The subgame has only one Nash equilibrium, as T dominates B: Player 1 plays T and 2 plays R, yielding the payoff vector (3,2).



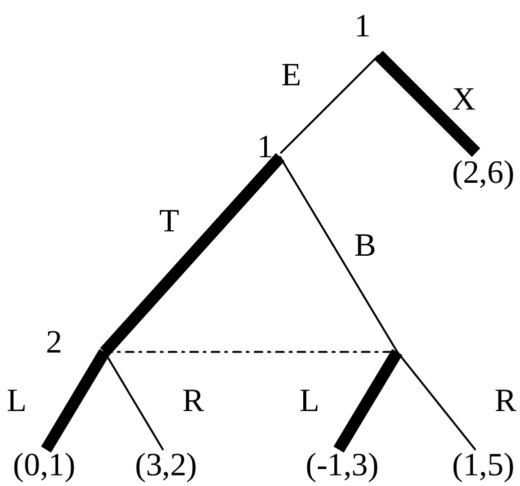
Given this, the remaining game is



where player 1 chooses E. Thus, the subgame-perfect equilibrium is as follows.

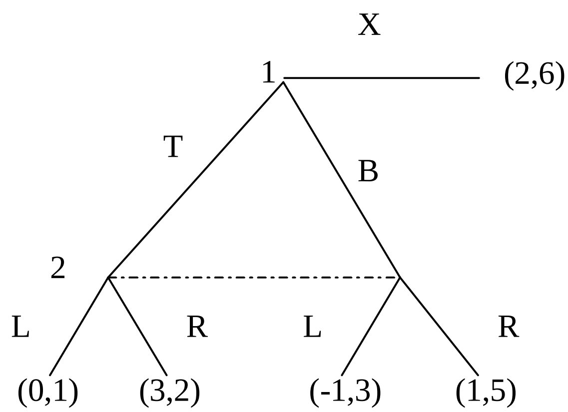


Note that there are other Nash Equilibria; one of them is depicted below.

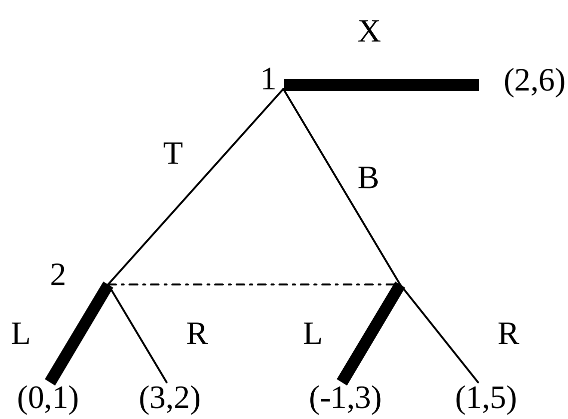


You should be able to check that this is a Nash equilibrium. But it is not subgame perfect, for, in the proper subgame, 2 plays a strictly dominated strategy.

Now, consider the following game, which is essentially the same game as above, with a slight difference that here player 1 makes his choices at once:



Note that the only subgame of this game is itself, hence any Nash equilibrium is subgame perfect. In particular, the non-subgame-perfect Nash equilibrium of the game above is subgame perfect. In the new game it takes the following form:



At this point you should stop reading and study "tariffs and imperfect international competition".

3 Sequential Bargaining

Imagine that two players own a dollar, which they can use only after they decide how to divide it. Each player is risk-neutral and discounts the future exponentially. That is, if a player gets x dollar at day t, his payoff is $\delta^t x$ for some $\delta \in (0,1)$. The set of all feasible divisions is $D = \left\{(x,y) \in [0,1]^2 \,|x+y \leq 1\right\}$. Consider the following scenario. In the first day player one makes an offer $(x_1, y_1) \in D$. Then, knowing what has been offered, player 2 accepts or rejects the offer. If he accepts the offer, the offer is implemented, yielding payoffs (x_1, y_1). If he rejects the offer, then they wait until the next day, when player 2 makes an offer $(x_2, y_2) \in D$. Now, knowing what player 2 has offered, player 1 accepts or rejects the offer. If player 1 accepts the offer, the offer is implemented, yielding payoffs $(\delta x_2, \delta y_2)$. If player two rejects the offer, then the game ends, when they lose the dollar and get payoffs (0,0).

Let us analyze this game. On the second day, if player 1 rejects the offer, he gets 0. Hence, he accepts any offer that gives him more than 0, and he is indifferent between accepting and rejecting any offer that gives him 0. Assume that he accepts the offer (0,1).[1] Then, player 2 would offer (0,1), which is the best player 2 can get. Therefore, if they do not agree on the first day, on the second day, player 2 takes the entire dollar, leaving player 1 nothing. The value of taking the dollar on the next day for player 2 is δ. Hence, on the first day, player 2 will accept any offer that gives him more than δ, will reject any offer that gives him less than δ, and he is indifferent between accepting and rejecting any offer that gives him δ. As above, assume that player 2 accepts the offer $(1-\delta, \delta)$. In that case, player 1 will offer $(1-\delta, \delta)$, which will be accepted. For any division that gives player 1 more than $1-\delta$ will give player 2 less than δ, and will be rejected.

Now, consider the game in which the game above is repeated n times. That is, if they have not yet reached an agreement by the end of the second day, on the third day, player 1 makes an offer $(x_3, y_3) \in D$. Then, knowing what has been offered, player 2 accepts or rejects the offer. If he accepts the offer, the offer is implemented, yielding payoffs $\left(\delta^2 x_3, \delta^2 y_3\right)$. If he rejects the offer, then they wait until the next day, when player 2 makes an offer $(x_4, y_4) \in D$. Now, knowing what player 2 has offered, player 1 accepts or rejects the offer. If player 1 accepts the offer, the offer is implemented, yielding payoffs $\left(\delta^3 x_4, \delta^3 y_4\right)$. If player two rejects the offer, then they go to the 5th day... And this goes on like this until the end of day $2n$. If they have not yet agreed at the end of that day,

[1] In fact, player 1 must accept (0,1) in equilibrium. For, if he doesn't accept (0,1), the best response of player 2 will be empty, inconsistent with an equilibrium. (Any offer $(\epsilon, 1-\epsilon)$ of player 2 will be accepted. But for any offer $(\epsilon, 1-\epsilon)$, there is a better offer $(\epsilon/2, 1-\epsilon/2)$, which will also be accepted.)

the game ends, when they lose the dollar and get payoffs (0,0).

The subgame perfect equilibrium will be as follows. At any day $t = 2n - 2k$ (k is a non-negative integer), player 1 accepts any offer (x, y) with

$$x \geq \frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}$$

and will reject any offer (x, y) with

$$x < \frac{\delta\left(1-\delta^{2k}\right)}{1+\delta};$$

and player 2 offers

$$(x_t, y_t) = \left(\frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}, 1 - \frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}\right) \equiv \left(\frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}, \frac{1+\delta^{2k+1}}{1+\delta}\right).$$

And at any day $t - 1 = 2n - 2k - 1$, player 2 accepts an offer (x, y) iff

$$y \geq \frac{\delta\left(1+\delta^{2k+1}\right)}{1+\delta};$$

and Player 1 will offer

$$(x_{t-1}, y_{t-1}) = \left(1 - \frac{\delta\left(1+\delta^{2k+1}\right)}{1+\delta}, \frac{\delta\left(1+\delta^{2k+1}\right)}{1+\delta}\right) \equiv \left(\frac{1-\delta^{2k+2}}{1+\delta}, \frac{\delta\left(1+\delta^{2k+1}\right)}{1+\delta}\right).$$

We can prove this is the equilibrium given by backward induction using mathematical induction on k. (That is, we first prove that it is true for $k = 0$; then assuming that it is true for some $k - 1$, we prove that it is true for k.)

Proof. Note that for $k = 0$, we have the last two periods, identical to the 2-period example we analyzed above. Putting $k = 0$, we can easily check that the behavior described here is the same as the equilibrium behavior in the 2-period game. Now, assume that, for some $k - 1$ the equilibrium is as described above. That is, at the beginning of date $t + 1 := 2n - 2(k-1) - 1 = 2n - 2k + 1$, player 1 offers

$$(x_{t+1}, y_{t+1}) = \left(\frac{1-\delta^{2(k-1)+2}}{1+\delta}, \frac{\delta\left(1+\delta^{2(k-1)+1}\right)}{1+\delta}\right) = \left(\frac{1-\delta^{2k}}{1+\delta}, \frac{\delta\left(1+\delta^{2k-1}\right)}{1+\delta}\right);$$

and his offer is accepted. At date $t = 2n - 2k$, player one accepts an offer iff the offer is at least as good as having $\frac{1-\delta^{2k}}{1+\delta}$ in the next day, which is worth $\frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}$. Therefore, he will accept an offer (x, y) iff

$$x \geq \frac{\delta\left(1-\delta^{2k}\right)}{1+\delta};$$

as we have described above. In that case, the best player 2 can do is to offer

$$(x_t, y_t) = \left(\frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}, 1 - \frac{\delta\left(1-\delta^{2k}\right)}{1+\delta} \right) = \left(\frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}, \frac{1+\delta^{2k+1}}{1+\delta} \right).$$

For any offer that gives 2 more than y_t will be rejected in which case player 2 will get

$$\delta y_{t+1} = \frac{\delta^2\left(1+\delta^{2k-1}\right)}{1+\delta} < y_t.$$

That is, at t player 2 offers (x_t, y_t); and it is accepted. In that case, at $t-1$, player 2 will accept an offer (x, y) iff

$$y \geq \delta y_t = \frac{\delta\left(1+\delta^{2k+1}\right)}{1+\delta}.$$

In that case, at $t-1$, player 1 will offer

$$(x_{t-1}, y_{t-1}) \equiv (1-\delta y_t, \delta y_t) = \left(\frac{1-\delta^{2k+2}}{1+\delta}, \frac{\delta\left(1+\delta^{2k+1}\right)}{1+\delta} \right),$$

completing the proof. ■

Now, let $n \to \infty$. At any odd date t, player 1 will offer

$$(x_t^\infty, y_t^\infty) = \lim_{k\to\infty} \left(\frac{1-\delta^{2k+2}}{1+\delta}, \frac{\delta\left(1+\delta^{2k+1}\right)}{1+\delta} \right) = \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta} \right);$$

and any even date t player 2 will offer

$$(x_t^\infty, y_t^\infty) = \lim_{k\to\infty} \left(\frac{\delta\left(1-\delta^{2k}\right)}{1+\delta}, \frac{1+\delta^{2k+1}}{1+\delta} \right) = \left(\frac{\delta}{1+\delta}, \frac{1}{1+\delta} \right);$$

and the offers are barely accepted.

14.12 Game Theory Lecture Notes

Lectures13-14

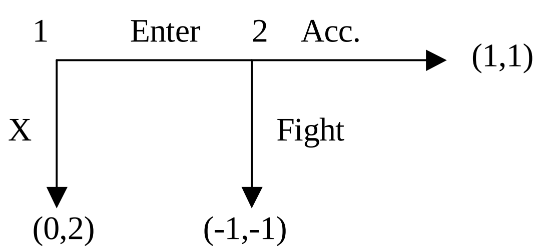
Muhamet Yildiz

1 Repeated Games

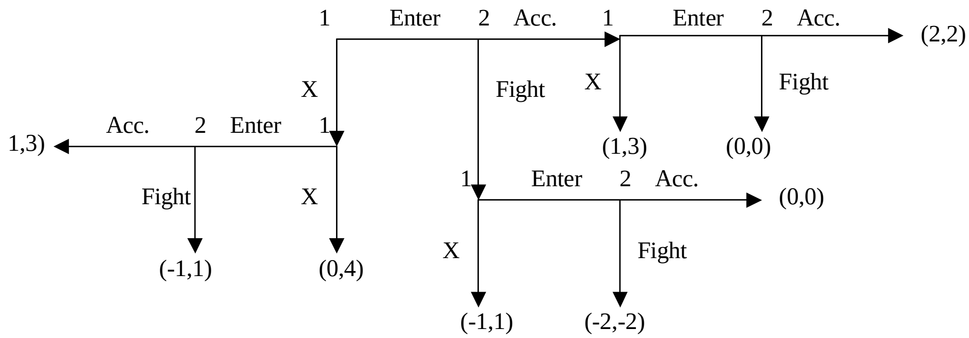
In these notes, we'll discuss the repeated games, the games where a particular smaller game is repeated; the small game is called the stage game. The stage game is repeated regardless of what has been played in the previous games. For our analysis, it is important whether the game is repeated finitely or infinitely many times, and whether the players observe what each player has played in each previous game.

1.1 Finitely repeated games with observable actions

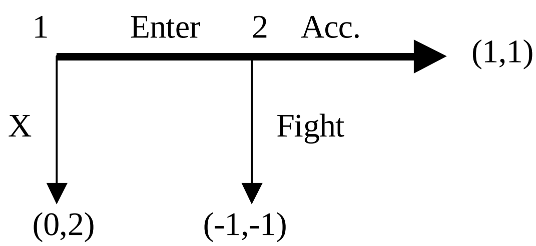
We will first consider the games where a stage game is repeated finitely many times, and at the beginning of each repetition each player recalls what each player has played in each previous play. Consider the following entry deterrence game, where an entrant (1) decides whether to enter a market or not, and the incumbent (2) decides whether to fight or accommodate the entrant if he enters.



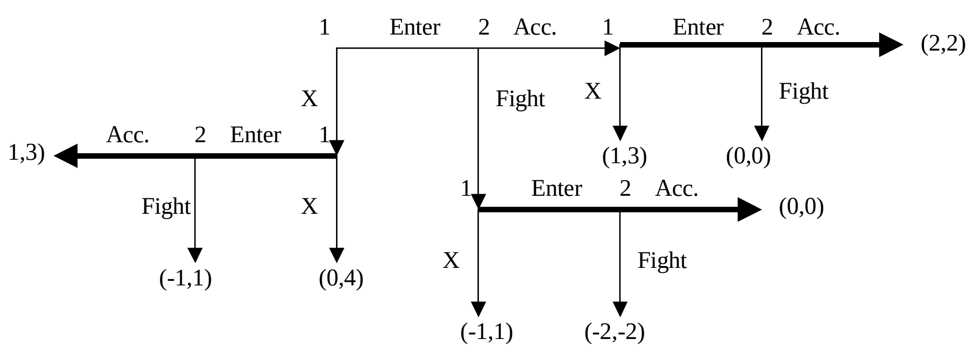
Consider the game where this entry deterrence game is repeated twice, and all the previous actions are observed. Assume that a player simply cares about the sum of his payoffs at the stage games. This game is depicted in the following figure.



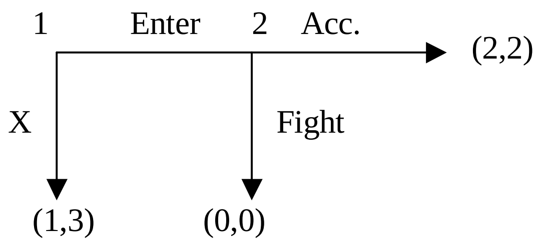
Note that after the each outcome of the first play, the entry deterrence game is played again –where the payoff from the first play is added to each outcome. Since a player's preferences over the lotteries do not change when we add a number to his utility function, each of the three games played on the second "day" is the same as the stage game (namely, the entry deterrence game above). The stage game has a unique subgame perfect equilibrium, where the incumbent accommodates the entrant, anticipating this, the entrant enters the market.



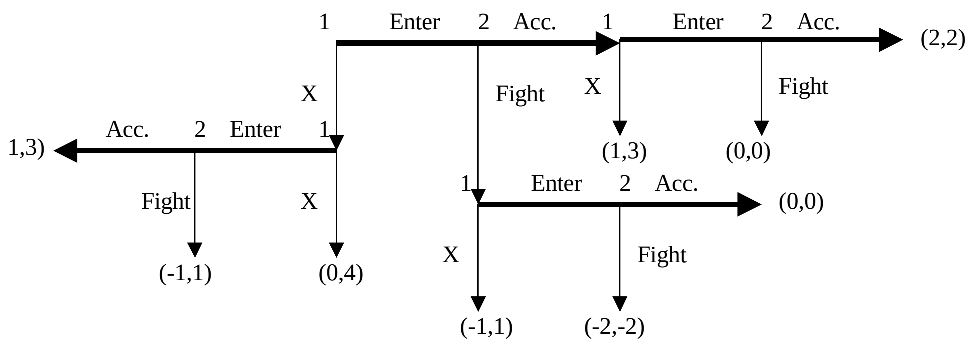
In that case, each of the three games played on the second day has only this equilibrium as its subgame perfect equilibrium. This is depicted in the following.



Using backward induction, we therefore reduce the game to the following.



Notice that we simply added the unique subgame perfect equilibrium payoff of 1 from the second day to each payoff in the stage game. Again, adding a constant to a player's payoffs does not change the game, and hence the reduced game possesses the subgame perfect equilibrium of the stage game as its unique subgame perfect equilibrium. Therefore, the unique subgame perfect equilibrium is as depicted below.



This can be generalized. That is, given any finitely repeated game with observable actions, if the stage game has a unique subgame perfect equilibrium, then the repeated game has a unique subgame perfect equilibrium, where the subgame perfect equilibrium of the stage game is player at each day.

If the stage game has more than one equilibrium, then in the repeated game we may have some subgame perfect equilibria where, in some stages, players play some actions that are not played in any subgame perfect equilibria of the stage game. For the equilibrium to be played on the second day can be conditioned to the play on the first day, in which case the " reduced game" for the first day is no longer the same as the stage game, and thus may obtain some different equilibria. **To see this, see Gibbons.**

1.2 Infinitely repeated games with observed actions

Now we consider the infinitely repeated games where all the previous moves are common knowledge at the beginning of each stage. In an infinitely repeated game, we cannot simply add the payoffs of each stage, as the sum becomes infinite. For these games, we will confine ourselves to the case where players maximize the discounted sum of the payoffs from the stage games. The *present value* of any given payoff stream $\pi = (\pi_0, \pi_1, \ldots, \pi_t, \ldots)$ is computed by

$$PV(\pi; \delta) = \sum_{t=0}^{\infty} \delta^t \pi_t = \pi_0 + \delta\pi_1 + \cdots + \delta^t \pi_t + \cdots,$$

where $\delta \in (0, 1)$ is the *discount factor.* By the *average value*, we simply mean

$$(1 - \delta) PV(\pi; \delta) \equiv (1 - \delta) \sum_{t=0}^{\infty} \delta^t \pi_t.$$

Note that, when we have a constant payoff stream (i.e., $\pi_0 = \pi_1 = \cdots = \pi_t = \cdots$), the average value is simply the stage payoff (namely, π_0). Note that the present and the average values can be computed with respect to the current date. That is, given any t, the present value at t is

$$PV_t(\pi; \delta) = \sum_{s=t}^{\infty} \delta^{s-t} \pi_s = \pi_t + \delta\pi_{t+1} + \cdots + \delta^k \pi_{t+k} + \cdots.$$

Clearly,

$$PV(\pi; \delta) = \pi_0 + \delta\pi_1 + \cdots + \delta^{t-1}\pi_{t-1} + \delta^t PV_t(\pi; \delta).$$

Hence, the analysis does not change whether one uses PV or PV_t, but using PV_t is simpler.

The main property of infinitely repeated games is that the set of equilibria becomes very large as players get more patients, i.e., $\delta \to 1$. given any payoff vector that gives each player more than some Nash equilibrium outcome of the stage game, for sufficiently large values of δ, there exists some subgame perfect equilibrium that yields the payoff vector at hand as the average value of the payoff stream. This fact is called the folk theorem. **See Gibbons for details.**

In these games, to check whether a strategy profile $s = (s_1, s_2, \ldots, s_n)$ is a subgame perfect equilibrium, we use the *single-deviation principle*, defined as follows.[1] Take any formation set, where some player i is to move, and play a strategy a^* of the stage game according to the strategy profile s. Assume that the information set is reached, each player $j \neq i$ sticks to his strategy s_j in the remaining game, and player i will stick to his strategy s_i in the remaining game except for the information set at hand. given all these, we check whether the player has an incentive to deviate to some action a' at the information set (rather than playing a^*).

[1] Note that a strategy profile s_i is an infinite sequence $s_i = (a_0, a_1, \ldots, a_t, \ldots)$ of functions a_t determining which "strategy of the stage game" to be played at t depending on which actions each player has taken in the previous plays of the stage game.

[Note that all players, including player i, are assumed to stick to this strategy profile in the remaining game.] The single-deviation principle states that if there is no information set the player has an incentive to deviate in this sense, then the strategy profile is a subgame perfect equilibrium.

Let us analyze the infinitely repeated version of the entry deterrence game. Consider the following strategy profile. At any given stage, the entrant enters the market if an only if the incumbent has accommodated the entrant sometimes in the past. The incumbent accommodates the entrant if an only if he has accommodated the entrant before. (This is a switching strategy, where initially incumbent fights whenever there is an entry and the entrant never enters. If the incumbent happens to accommodate an entrant, they switch to the new regime where the entrant enters the market no matter what the incumbent does after the switching, and incumbent always accommodates the entrant.) For large values of δ, this an equilibrium.

To check whether this is an equilibrium, we use the single-deviation principle. We first take a date t and any history (at t) where incumbent has accommodated the entrants. According to the strategy of the incumbent, he will always accommodate the entrant in the remaining game; and the entrant will always enter the market (again according to his own strategy). Thus, the continuation value of incumbent (i.e., the present value of the equilibrium payoff-stream of the incumbent) at $t + 1$ is

$$V_A = 1 + \delta + \delta^2 + \cdots = 1/(1-\delta).$$

If he accomodates at t, his present value (at t) will be $1+\delta V_A$; and if he fights his present value will be $-1+\delta V_A$. Therefore, the incumbent has no incentive to fight, rather than accomodating as stipulated by his strategy. The entrants continuation value at $t+1$ will also be independent of what happens at t, hence the entrant will enter (whence he gets $1 + \delta$[His present value at $t + 1$]) rather than deviating (whence he gets $0 + \delta$[His present value at $t + 1$]).

Now consider a history at some date t where the incumbent has never accommodated the entrant before. Consider the incumbent's information set. If he accommodates the entrant, his continuation value at $t + 1$ will be $V_A = 1/(1-\delta)$, whence his continuation value at t will be $1 + \delta V_A = 1 + \delta/(1-\delta)$. If he fights, however, according to the strategy profile, he will never accommodate any entrants in the future, and the entrant will never enter, in which case the incumbent will get the constant payoff stream of 2, whose present value at t+ 1 is $2/(1-\delta)$. hence, in this case, his continuation value at t will be $-1 + \delta \cdot 2/(1-\delta)$. therefore, the incumbent will not have any incentive to deviate (and accommodate the entrant) if and only if

$$-1+\delta\cdot 2/\left(1-\delta\right)\geq 1+\delta/\left(1-\delta\right),$$

which is true if and only if

$$\delta\geq 2/3.$$

When this condition holds, the incumbent do not have an incentive to deviate in such histories. now, if the entrant enters the market, incumbent will fight, and the entrant will never enter in the future, in which case his continuation value will be -1. If he does not enter, his continuation value is 0. Therefore, he will not have any incentive to enter, either. Since we have covered, all possible histories, by the single-deviation principle, this strategy profile is a subgame perfect equilibrium if and only if $\delta\geq 2/3$.

Now, study the cooperation in the prisoners' dilemma, implicit collusion in a Cournot duopoly, and other examples in Gibbons.

In the entry deterrence example above, the equilibrium that we have described will remain to be a subgame perfect equilibrium if the incumbent where facing a different entrant at each date, so long as the previous play of the game is always common knowledge. This kind of games can be utilized in order to understand the problem of reputation of a "long-run" player. **For the details, study Kreps (A course in Microeconomic Theory, pp 531-535).**

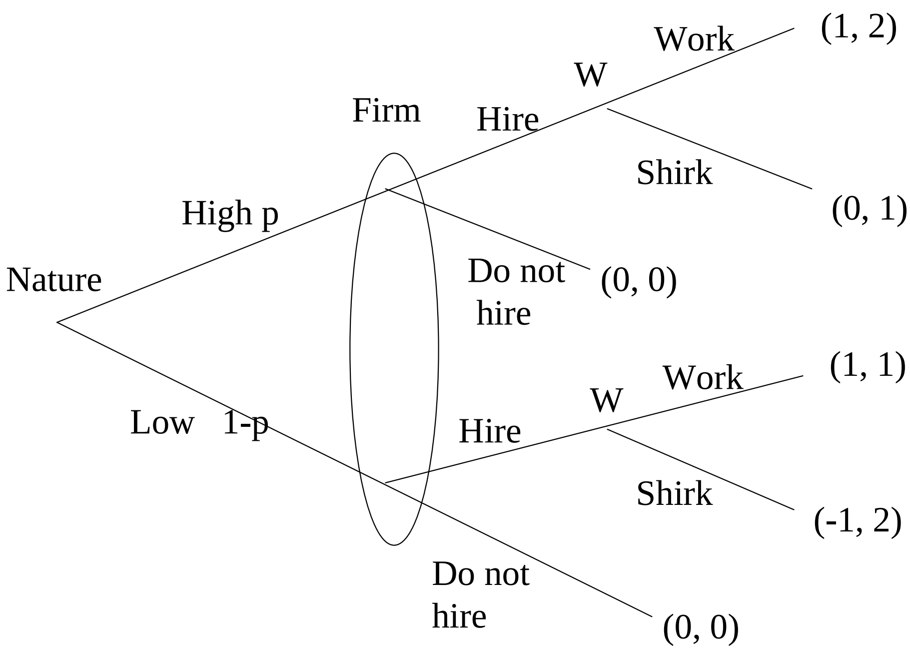
14.12 Game Theory Lecture Notes
Lectures 15-18

Muhamet Yildiz*

1 Static Games with Incomplete Information

An incomplete information game is a game where a party knows something that some other party does not know. For instance, a player may not know another player's utility function, while the player himself knows his utility function. Such situations are modeled through games where Nature moves and some players can distinguish certain moves of nature while some others cannot.

Example: Firm hiring a worker, the worker can be high or low ability and the firm does not know which.



*These notes are based on the notes by Professor Daron Acemoglu, who taught this course in previous years.

In this game W knows whether he is of high (who will work) or low (who will shirk) ability worker, while the Firm does not know this; the Firm believes that the worker is of high ability with probability p. And all these are common knowledge. We call a player's private information as his "type". For instance, here W has two types – high and low – while Firm has only one type (as everything firm knows is common knowledge). Note that we represent the incomplete information game in an extensive form very similar to imperfect information games.

Formally, a static game with incomplete information is as follows. First, Nature chooses some $t = (t_1, t_2, \ldots, t_n) \in T$, where each $t \in T$ is selected with probability $p(t)$. Here, $t_i \in T_i$ is the type of player $i \in N = \{1, 2, \ldots, n\}$. Then, each player observes his own type, but not the others'. finally, players simultaneously choose their actions, each player knowing his own type. We write $a = (a_1, a_2, \ldots, a_2) \in A$ for any list of actions taken by all the players, where $a_i \in A_i$ is the action taken by player i. Such a static gamewith incomplete information is denoted by (N, T, A, p).

Notice that a strategy of that player determines which action he will take at each information set of his, represented by some $t_i \in T_i$. That is, a strategy of a player i is a function $s_i : T_i \to A_i$ –from his types to his actions. for instance, in example above, W has four strategies, such as (Work,Work) –meaning that he will work regardless nature chooses high or low– (Work, Shirk), (Shirk, Work), and (Shirk, Shirk).

Any Nash equilibrium of such a game is called Bayesian Nash equilibrium. For instance, for $p > 1/2$, a Bayesian Nash equilibrium of the game above is as follows. Worker chooses to Work if he is of high ability, and chooses to Shirk if he is of low ability; and the firm Hires him.

Notice that there is another Nash equilibrium, where the worker chooses to Shirk regardless of his type and the firm doesn't hire him. Since the game has no proper subgame, the latter equilibrium is subgame perfect, even though it clearly relies on an "incredible" threat. this is a very common problem in games with incomplete information, motivating a more refined equilibrium concept we'll discuss in our next lectures.

It's very important to note that players' types may be "correlated", meaning that a player "updates" his beliefs about the other players' type when he learns his own type. Since he knows his type when he takes his action, he maximizes his expected utility with respect to the new beliefs he came to after "updating" his beliefs. We assume that he updates his beliefs using Bayes' Rule.

Bayes' Rule Let A and B be two events, then probability that A occurs conditional on B occurring is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$

where $P(A \cap B)$ is the probability that A and B occur simultaneously and $P(B)$: the (unconditional) probability that B occurs.

In static games of incomplete information, the application of Bayes' Rule will often be trivial, but as we move to study dynamic games of incomplete information, the importance of Bayes' Rule will increase.

Let $p_i(t'_{-i}|t_i)$ denote i's belief that the types of all other players is $t'_{-i} = (t'_1, t'_2, ..., t'_{i-1}, t'_{i+1}, \ldots, t'_n)$ given that his type is $t_{i.}$ [We may need to use Bayes' Rule if types across players are 'correlated'. But if they are independent, then life is simpler; players do not update their beliefs.]

With this formalism, a strategy profiles $s^* = (s_1^*, ..., s_n^*)$ is a Bayesian Nash Equilibrium in an n-person static game of incomplete information if and only if for each player i and type $t_i \in T_i$,

$$s_i^*(t_i^1) \in \arg\max_{a_i} \sum u_i \left[s_i^*(t_i), ..., a_i, ..., s_N^*(t_N)\right] \times p_i(t'_{-i}|t_i)$$

where u_i is the utility of player i and a_i denotes action. That is, for each player i each possible type, the action chosen is *optimal* given the *conditional beliefs* of that type against the optimal strategies of all other players.

The remaining notes are about the applications and very sketchy; for the details see Gibbons.

Example: Cournot with Incomplete Information. We have a linear demand function:

$$P(Q) = a - Q$$

where $Q = q_1 + q_2$ is the total demand, and constant marginal costs. The marginal cost of Firm 1 is c_1 and common knowledge, while Firm 2's marginal cost is its private information. Its marginal cost (type) can be either c_H with probaility θ, or c_L with probability $1 - \theta$. Each firm maximizes its expected profit.

How to find the Bayesian Nash Equilibrium?

Firm 2 has two possible types; and different actions will be chosen for the two different types.

$\{q_2(c_L), q_2(c_H)\}$

Suppose firm 2 is type high. Then, given the quantity q_1^* chosen by player, its problem is

$$\max_{q_2}(P - c_H)q_2 = [a - q_1 - q_2 - c_H]\, q_2.$$

Hence,

$$q_2^*(c_H) = \frac{a - q_1^* - c_H}{2} \tag{*}$$

Similarly, suppose firm 2 is low type:

$$\max_{q_2} [a - q_1^* - q_2 - c_L]\, q_2,$$

hence

$$q_2^*(c_L) = \frac{a - q_1^* - c_L}{2}. \tag{**}$$

Important Remark: The same level of q_1 in both cases. Why??

Firm 1's problem:

$$\max_{q_1} \theta [a - q_1 - q_2^*(c_H) - c]\, q_1 + (1-\theta)[a - q_1 - q_2^*(c_L) - c]\, q_1$$

$$q_1^* = \frac{\theta [a - q_2^*(c_H) - c] + (1-\theta)[a - q_2^*(c_L) - c]}{2} \tag{***}$$

Solve *, **, and *** for $q_1^*, q_2^*(c_L), q_2^*(c_H)$.

$$q_2^*(c_H) = \frac{a - 2c_H + c}{3} + \frac{(1-\theta)(c_H - c_L)}{6}$$

$$q_2^*(c_L) = \frac{a - 2c_L + c}{3} - \frac{\theta(c_H - c_L)}{6}$$

$$q_1^* = \frac{a - 2c + \theta c_H + (1-\theta)c_L}{3}$$

Harsaryi's Justification for Mixed Strategies Consider the game

	O	F
O	$2+t_1, 1$	$0, 0$
F	$0, 0$	$1, 2+t_2$

where t_1, t_2 private information of players 1 and 2, respectively, and are independent drawn from uniform distribution over $[-\epsilon, 2\epsilon]$. Check that

$$s_1^*(t_1) = \begin{cases} \text{O} & \text{if } t_1 \geq 0 \\ \text{F} & \text{otherwise,} \end{cases}$$

and

$$s_2^*(t_2) = \begin{cases} \text{F} & \text{if } t_2 \geq 0 \\ \text{O} & \text{otherwise} \end{cases}$$

form a Bayesian Nash equilibrium. Note that player 1 plays 0 with probability 2/3 and player 2 plays F with probability 2/3. As $\epsilon \longrightarrow 0$ (i.e., as uncertainty disappears), this strategy profile converges to a mixed strategy equilibrium in which player 1 plays 0 with probability 2/3 and player 2 plays F with probability 2/3. See Gibbons for details.

Auctions Two bidders for a unique good.

v_i : valuation of bidder i.

Let us assume that v_i's are drawn independently from a uniform distribution over $[0, 1]$. v_i is player i's private information. The game takes the form of both bidders submitting a bid, then the highest bidder wins and pays her bid.

Let b_i be player i's bid.

$$v_i(b_1, b_2, v_1, v_2) = v_i - b_i \text{ if } b_i > b_j$$
$$\frac{v_i - b_i}{2} \text{ if } b_i = b_j$$
$$0 \text{ if } b_i < b_j$$

$$\max_{b_i}(v_i - b_i)Prob\{b_i > b_j(v_j)|\text{given beliefs of player i}) + \frac{1}{2}(v_i - b_i)Prob\{b_i = b_j(v_j)|...)$$

Let us first conjecture the form of the equilibrium: Conjecture: Symmetric and linear equilibrium

$$b = a + cv.$$

Then, $\frac{1}{2}(v_i - b_i)Prob\{b_i = b_j(v_j)|...) = 0$. Hence,

$$\max_{b_i}(v_i - b_i)Prob\{b_i \geq a + cv_j\} =$$
$$(v_i - b_i)Prob\{v_j \leq \frac{b_i - a}{c}\} = (v_i - b_i) \cdot \frac{(b_i - a)}{c}$$

FOC:

$$\begin{aligned} b_i &= \frac{v_i + a}{2} \qquad \text{if} \quad v_i \geq a \\ &= a \qquad \text{if} \quad v_i < a \end{aligned} \tag{1}$$

The best response b_i can be a linear strategy only if $a = 0$. Thus,

$$b_i = \frac{1}{2}v_i.$$

Double Auction Simultaneously, Seller names P_s and Buyer names P_b. If $P_b < P_s$, then no trade; if $P_b \geq P_s$ trade at price $p = \frac{P_b + P_s}{2}$.

Valuations are private information:

V_b uniform over $(0, 1)$

V_s uniform over $(0, 1)$ and independent from V_b

Strategies $P_b(V_b)$ and $P_s(V_s)$.

The buyer's problem is

$$\max_{P_b} E\left[V_b - \frac{P_b + P_s(V_s)}{2} : P_b \geq P_s(V_s)\right] =$$
$$\max_{P_b}\left[V_b - \frac{P_b + E[P_s(V_s)|P_b \geq P_s(V_s)]}{2}\right] \times Prob\{P_b \geq P_s(V_s)\}$$

where $E[P_s(V_s)|P_b \geq P_s(V_s)]$ is the expected seller bid *conditional* on P_b being greater than $P_s(V_s)$.

Similarly, the seller's problem is

$$\max_{P_s} E\left[\frac{P_s + P_b(V_b)}{2} - V_s : P_b(V_b) \geq P_s\right]] \quad =$$

$$\max_{P_s}\left[\frac{P_s + E[P_b(V_b)|P_b(V_b) \geq P_s}{2}\right] - V_s] \times Prob\{P_b(V_b) \quad \geq \quad P_s\}$$

Equilibrium is where $P_s(V_j)$ is a best response to $P_b(V_b)$ while $P_b(V_b)$ is a best response to $P_s(V_s)$.

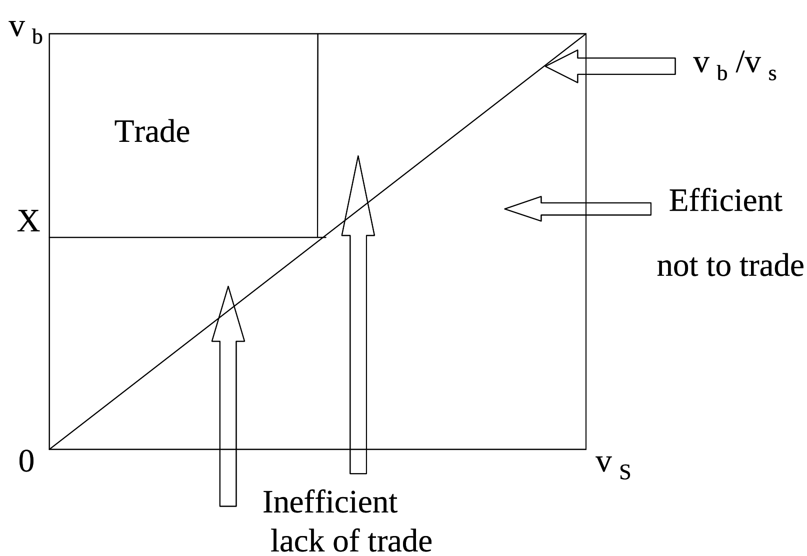
There are many Bayesian Nash Equilibria. Here is one.

$P_s = X$ if $V_s \leq X$

$P_b = X$ if $V_b \geq X$.

An equilibrium with "fixed" price.

Why is this an equilibrium? Because given $P_s = X$ if $V_s \leq X$, the buyer does not want to trade with $V_b < X$ and with $V_b > X, P_b = X$ is optimal.



Now construct an equilibrium with linear strategies:

$$p_b = a_b + c_b v_b$$

$$p_s = a_s + c_s v_s,$$

where a_b, a_s, c_b, and c_s are to be determined. Note that $p_b \geq p_s(v_s) = a_s + c_s v_s$ iff

$$v_s \leq \frac{p_b - a_s}{c_s}.$$

Likewise, $p_s \leq p_b(v_b) = a_b + c_b v_b$ iff

$$v_b \geq \frac{p_s - a_b}{c_b}.$$

Then, the buyer's problem is[1]

$$\begin{aligned}
&\max_{p_b} E\left[v_b - \frac{p_b + p_s(v_s)}{2} : p_b \geq p_s(v_s)\right] \\
&= \max_{p_b} \int_0^{\frac{p_b - a_s}{c_s}} \left[v_b - \frac{p_b + p_s(v_s)}{2}\right] dv_s \\
&= \max_{p_b} \int_0^{\frac{p_b - a_s}{c_s}} \left[v_b - \frac{p_b + a_s + c_s v_s}{2}\right] dv_s \\
&= \max_{p_b} \frac{p_b - a_s}{c_s}\left(v_b - \frac{p_b + a_s}{2}\right) - \frac{c_s}{2}\int_0^{\frac{p_b - a_s}{c_s}} v_s dv_s \\
&= \max_{p_b} \frac{p_b - a_s}{c_s}\left(v_b - \frac{p_b + a_s}{2}\right) - \frac{c_s}{4}\left(\frac{p_b - a_s}{c_s}\right)^2 \\
&= \max_{p_b} \frac{p_b - a_s}{c_s}\left(v_b - \frac{p_b + a_s}{2} - \frac{p_b - a_s}{4}\right) \\
&= \max_{p_b} \frac{p_b - a_s}{c_s}\left(v_b - \frac{3p_b + a_s}{4}\right).
\end{aligned}$$

F.O.C.:

$$\frac{1}{c_s}\left(v_b - \frac{3p_b + a_s}{4}\right) - \frac{3(p_b - a_s)}{4c_s} = 0$$

i.e.,

$$p_b = \frac{2}{3}v_b + \frac{1}{3}a_s. \tag{2}$$

Similarly, the seller's problem is

[1] There is somewhat simpler way in to get the same outcome; see Gibbons.

$$\max_{p_s} E\left[\frac{p_s+p_b(v_b)}{2} - v_s : p_b(v_b) \geq p_s\right] = \max_{p_s} \int_{\frac{p_s-a_b}{c_b}}^{1} \left[\frac{p_s+a_b+c_b v_b}{2} - v_s\right] dv_b$$

$$= \max_{p_s} \left(1-\frac{p_s-a_b}{c_b}\right)\left[\frac{p_s+a_b}{2} - v_s\right] + \frac{c_b}{2}\int_{\frac{p_s-a_b}{c_b}}^{1} v_b dv_b$$

$$= \max_{p_s} \left(1-\frac{p_s-a_b}{c_b}\right)\left[\frac{p_s+a_b}{2} - v_s\right] + \frac{c_b}{4}\left(1-\left(\frac{p_s-a_b}{c_b}\right)^2\right)$$

$$= \max_{p_s} \left(1-\frac{p_s-a_b}{c_b}\right)\left[\frac{p_s+a_b}{2} - v_s + \frac{c_b}{4} + \frac{p_s-a_b}{4}\right]$$

$$= \max_{p_s} \left(1-\frac{p_s-a_b}{c_b}\right)\left[\frac{3p_s+a_b}{4} - v_s + \frac{c_b}{4}\right]$$

F.O.C.

$$-\frac{1}{c_b}\left[\frac{3p_s+a_b}{4} - v_s + \frac{1}{4}\right] + \frac{3}{4}\left(1-\frac{p_s-a_b}{c_b}\right) = 0$$

i.e.,

$$-\left[\frac{3p_s+a_b}{4} - v_s + \frac{c_b}{4}\right] + \frac{3}{4}\left(c_b-(p_s-a_b)\right) = 0,$$

i.e.,

$$\frac{3p_s}{2} = -\frac{a_b}{4} + v_s - \frac{c_b}{4} + \frac{3}{4}(c_b+a_b) = v_s + \frac{a_b+c_b}{2}$$

i.e.,

$$p_s = \frac{2}{3}v_s + \frac{a_b+c_b}{3}. \tag{3}$$

By (2), $a_b = a_s/3$, and by (3), $a_s = \frac{a_b}{3} + \frac{2}{9}$. Hence, $9a_s = a_s + 2$, thus $a_s = 1/4$. Therefore, $a_b = 1/12$. The equilibrium is

$$p_b = \frac{2}{3}v_b + \frac{1}{12} \tag{4}$$

$$p_s = \frac{2}{3}v_s + \frac{1}{4}. \tag{5}$$

14.12 Review Notes 1

Kenichi Amaya*

September 14, 2001

1 Why do we learn game theory?

Economic agents' (e.g., consumers, firms, government) behavior is described as a maximization problem (utility maximization, profit maximization). For example, consider a firm's profit maximization problem.

Perfect competition Under perfect competition, a firm takes the price as given, and choose the quantity to sell.

$$\max_q pq - c(q)$$
$$\Rightarrow \text{FOC} \;\; p = c'(q) = MC.$$

Monopoly A monopolist choose the price, and the quantity to sell is determined by the demand curve $D(p)$.

$$\max_p pD(p) - c(D(p))$$
$$\Rightarrow \text{FOC} \;\; pD'(p) + D(p) = c'(D(p))D(p).$$

In the examples above, the firm is the only decision maker and everything other than the choice variables (q in perfect competition and q in monopoly) are exogenously given. Therefore we can solve for the optimal value of the choice variable.

Consider the Cournot Duopoly. Firm 1 chooses the quantity to sell q_1, and Firm 2 chooses q_2. Let $P(q)$ be the inverse demand curve, where $q = q_1 + q_2$. Firm 1 maximizes its profit:

$$\max_{q_1} P(q_1 + q_2)q_1 - c_1(q_1).$$

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Firm 2 maximizes its profit:

$$\max_{q_2} P(q_1 + q_2)q_2 - c_2(q_2).$$

To solve for the optimal q_1, we need to know what q_2 is. But to know the optimal q_2, we need to know what q_1 is. This goes forever and we can never solve it mathematically. We need some theory to solve it: *Game theory.*

Game theory is a tool to analyze people's behavior under *strategic environment:* An agent's payoff depends not only on his decisions but also other agents' decisions.

Other examples includes:

- Employer — Employee (Wage scheme — Effort)
- Government — Firm (Regulation — Business strategy)
- U.S. Government — Japanese government (Trade policy)

2 Decision making under uncertainty

In the environment of games, economic agents often faces uncertainty. They must choose among alternatives which gives different probability distribution over outcomes. We often call outcomes of lotteries *prizes* We call these alternatives *lotteries.* For example:

Lottery 1 gives you beer with probability $\frac{1}{2}$ and chocolate with probability $\frac{1}{2}$.

Lottery 2 gives you beer with probability $\frac{1}{3}$ and coffee with probability $\frac{2}{3}$.

We want to have a utility function $U(\cdot)$ such that

$$U(\textit{lottery } 1) \geq U(\textit{lottery } 2)$$

if and only if the agent prefers lottery 1 to lottery 2, i.e.,

$$\textit{lottery } 1 \succeq \textit{lottery } 2.$$

The basic theory of choice implies there exists such a utility function if $\succeq$ is a *preference relationship*, i.e., it is *complete* and *transitive*

Completeness For all $p, q \in P$, $p \succeq q$ or $q \succeq p$.

Transitivity If $p \succeq q$ and $q \succeq r$, then $p \succeq r$.

But thinking directly about this function U is too messy! It is more convenient if we have a function u such that

$$U(lottery\ 1) = \frac{1}{2}u(beer) + \frac{1}{2}u(chocolate),$$
$$U(lottery\ 2) = \frac{1}{3}u(beer) + \frac{2}{3}u(coffee).$$

But can we really find such an convenient function u? Actually, it is known that we can find u if the agent's preference $\succeq$ over lotteries satisfies the following condition, in addition to completeness and transitivity.

Independence For any $p, q, r \in P$, and any $a \in (0, 1]$,

$$ap + (1-a)r \succeq aq + (1-a)r \Leftrightarrow p \succeq q.$$

Continuity For any $p, q, r \in P$, if $p \succ r$, then there exist $a, b \in (0, 1)$ such that

$$ap + (1-p)r \succ q \succ bp + (1-b)r.$$

Furthermore, if a von-Neuman Morgenstern utility function u represents $\succeq$, then its affine transformation $u' = au + b$ where $a > 0$ also represents $\succeq$.

3 Attitudes toward risk

Suppose now the prizes of lotteries are money. Lottery A gives you 50 dollars for sure (a degenerated lottery). Lottery B gives you 100 dollars with probability $\frac{1}{2}$ and 0 with probability $\frac{1}{2}$. Which do you prefer? The two lotteries give you the same expected amount of money, 50 dollars, so your choice reflects whether you like risk or not.

If you prefer Lottery A, then we can say you are risk averse. Your vNM utility function satisfies

$$u(50) > \frac{1}{2}u(0) + \frac{1}{2}u(100).$$

This is true if your vNM utility function is concave ($u'' < 0$ if u is twice differentiable). In general, if your vNM utility function is concave, you always prefer getting the avarage for sure to getting some risky lottery.

In contrast, if your vNM utility function is concave, you always prefer getting some risky lottery to getting the avarage for sure. In this case, you are called risk loving.

If your vNM utility function is linear, you are indifferent between getting some risky lottery and getting the avarage for sure. In this case, you are called risk neutral.

4 Modelling games

When we write down a game which represents some economic environment, the payoffs of the game are vNM utilities of the players, not the actual money amount received. In this way, we can assume the players maximize their expected payoffs. The only case where the payoffs of a player is equal to the actual money amount received is the case where the player is risk neutral. Actually, in many textbook modelling we find the players are *assumed* to be risk neutral and therefore maximize expected revenue.

14.12 Review Notes
Dynamic games with incomplete information

Kenichi Amaya*

November 30, 2001

1 Perfect Bayesian equilibrium

- $s = (s_1, \cdots, s_n)$: *strategy profile.*
- A *belief* at an information set is a probability distribution over decision nodes in the information set.
- A *belief system* μ is the collection of beliefs at all information sets.

Definition A pair of strategy profile and a belief system (s, μ) is a Perfect Bayesian (Nash) equilibrium (PBE) if

1. Given their beliefs, the players' strategies must be *sequentially rational.* That is, at each information set the action taken by the player with the move (and the player's subsequent strategy) must be optimal given the player's belief at that information set and the other players' subsequent strategies.
2. Beliefs are determined by Bayes' rule and the players' equilibrium strategies wherever possible.

Note 1: When you are asked to describe a PBE, you need to show not only the strategy profile but also the belief system. (However, you usually don't need to describe the belief at a singleton information set because it is obvious).

Note 2: PBE is a stronger concept than subgame perfect Nash equilibrium. Therefore, the strayegy profile of a PBE is a subgame perfect Nash equilibrium.

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2 How do we find PBE?

Here is an abstract procedure of looking for PBE.

1. Using sequential rationality and Bayes rule, determine strategies and beliefs wherever possible. If you can determine strategies and beliefs everywhere, that's it.
2. When you can't do more, make an assumption about a strategy at any information set. Then, using sequential rationality and Bayes rule, determine strategies and beliefs wherever possible.
3. You may be able to determine strategies and beliefs everywhere, i.e., to find an equilibrium. Then, change your assumption and look for another equilibrium.
4. You may reach a contradiction. In this case, the assumption was wrong. Change your assumption.
5. When you can't do more only with the assumption you made, make a further assumption.

The idea we use in finding a mixed strategy Nash equilibrium applies here too: The mixing probability must be such that the other player is indifferent between the strategies he is mixing.

14.12 Economic Applications of Game Theory

Professor: Muhamet Yildiz

Lecture: MW 2:30-4:00 @4-153?

Office Hours: M 4-5:30 @E52-251a

TA: Kenichi Amaya

F 10,3 @E51-85

Office Hours:TBA

Web: http://web.mit.edu/14.12/www/

Name of the game

Game Theory = Multi-person decision theory

- The outcome is determined by the actions independently taken by multiple decision makers.
- Strategic interaction.
  - Need to understand what the others will do
  - … what the others think that you will do
  - …

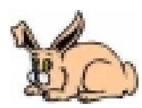
Hawk-Dove game

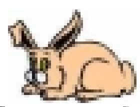
		
	$\left(\frac{V-c}{2},\frac{V-c}{2}\right)$	$(V,0)$
	$(0,V)$	$(V/2,V/2)$

Chicken

		
	(-1,-1)	(1,0)
	(0,1)	(1/2,1/2)

Stag Hunt




(2,2)	(4,0)
(0,4)	(6,6)

Quiz Problem

- Without discussing with anyone, each student is to write down a real number x_i between 0 and 100 on a paper and submit it to the TA.
- The TA will then compute the average

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

- The students who submit the number that is closest to $\bar{x}/3$ will share 100 points equally; the others will get 0.

14.12 Game Theory

Lecture 2: Decision Theory

Muhamet Yildiz

Road Map

1. Basic Concepts (Alternatives, preferences,…)
2. Ordinal representation of preferences
3. Cardinal representation – Expected utility theory
4. Applications: Risk sharing and Insurance
5. **Quiz**

Basic Concepts: Alternatives

- Agent chooses between the alternatives
- X = The set of all alternatives
- Alternatives are
  - Mutually exclusive, and
  - Exhaustive
- **Example:** Options = {Tea, Coffee}
  X = {T, C, TC, NT} where
  T= Tea, C = Coffee, TC = Tea and Coffee,
  NT = Neither Tea nor Coffee

Basic Concepts: Preferences

- TeX
- TeX –ordinal representation

Examples

- Define a relation among the students in this class by
  - x T y iff x is at least as tall as y;
  - x M y iff x's final grade in 14.04 is at least as high as y's final grade;
  - x H y iff x and y went to the same high school;
  - X Y y iff x is strictly younger than y;
  - x S y iff x is as old as y;

Exercises

- Imagine a group of students sitting around a round table. Define a relation R, by writing x R y iff x sits to the right of y. Can you represent R by a utility function?
- Consider a relation $\succeq$ among positive real numbers represented by u with $u(x) = x^2$.

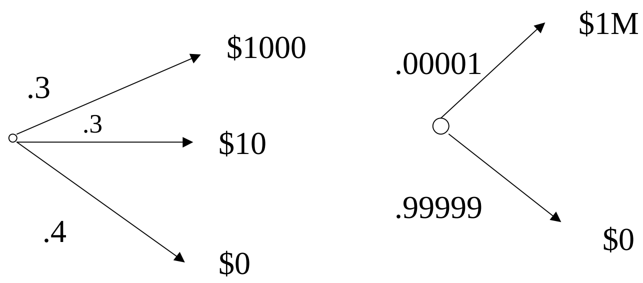
  Can this relation be represented by $u^*(x) = \sqrt{x}$?

  What about $u^{**}(x) = 1/x$?

- TeX – OR Theorem
- TeX – Cardinal representation
- VNM Axioms
- Theorem

A Lottery

Two Lotteries



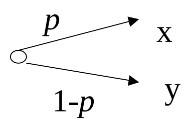
Exercise

- Consider an agent with VNM utility function u with $u(x) = x^2$.

  Can his preferences be represented by VNM utility function $u^*(x) = \sqrt{x}$?

  What about $u^{**}(x) = 1/x$?

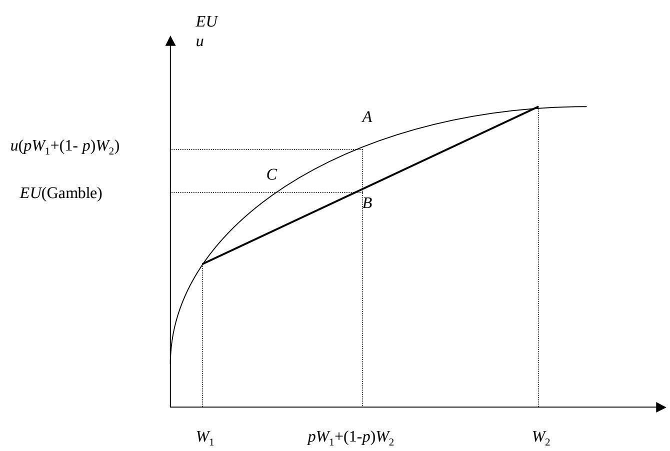
Attitudes towards Risk

- A fair gamble: 

$px+(1-p)y = 0.$

- An agent is said to be *risk neutral* iff he is indifferent towards all fair gambles. He is said to be *(strictly) risk averse* iff he never wants to take any fair gamble, and (strictly) *risk seeking* iff he always wants to take fair gambles.

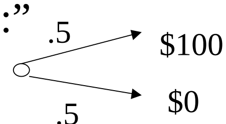
A utility function



- An agent is risk-neutral iff he has a linear utility function, i.e., $u(x) = ax + b$.
- An agent is risk-averse iff his utility function is concave.
- An agent is risk-seeking iff his utility function is convex.

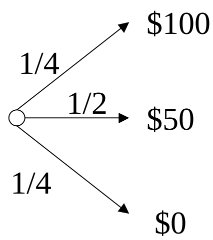
Risk Sharing

- Two agents, each having a utility function u with $u(x) = \sqrt{x}$ and an "asset:"



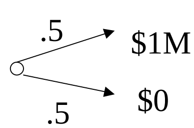
- For each agent, the value of the asset is
- Assume that the value of assets are independently distributed.

- If they form a mutual fund so that each agent owns half of each asset, each gets



Insurance

- We have an agent with $u(x)=\sqrt{x}$ and



- And a risk-neutral insurance company with lots of money, selling full insurance for "premium" P.

Lecture 3
Decision Theory/Game Theory

14.12 Game Theory

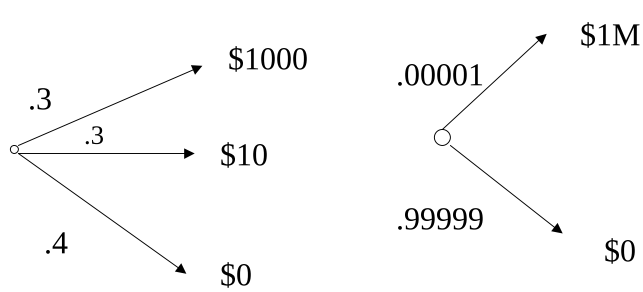
Muhamet Yildiz

Road Map

1. Basic Concepts (Alternatives, preferences,…)
2. Ordinal representation of preferences
3. Cardinal representation – Expected utility theory
4. Applications: Risk sharing and Insurance
5. **Quiz**
6. Representation of games in strategic and extensive forms
7. **Quiz?**

A Lottery

Two Lotteries



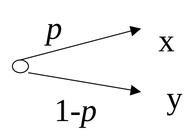
Exercise

- Consider an agent with VNM utility function *u* with $u(x) = x^2$.

  Can his preferences be represented by VNM utility function $u^*(x) = \sqrt{x}$?

  What about $u^{**}(x) = 1/x$?

Attitudes towards Risk

- A fair gamble: 

  $px + (1-p)y = 0$.

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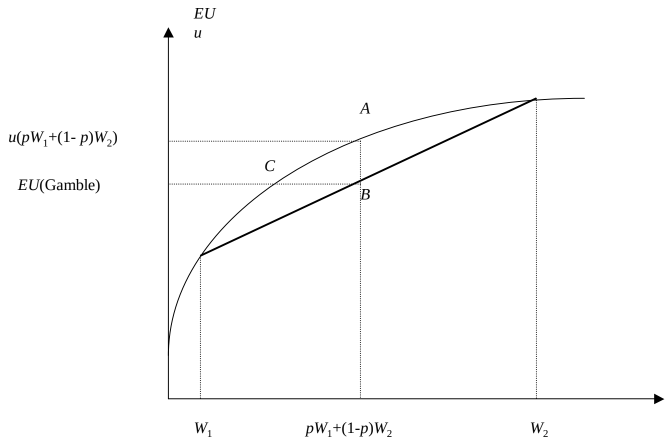
A utility function



- An agent is risk-neutral iff he has a linear utility function, i.e., $u(x) = ax + b$.
- An agent is risk-averse iff his utility function is concave.
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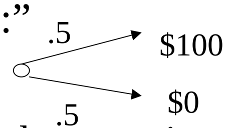
- Two agents, each having a utility function *u* with $u(x) = \sqrt{x}$ and an "asset:"



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- Assume that the value of assets are independently distributed.

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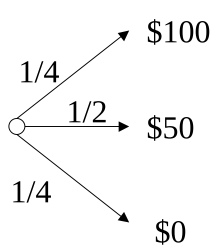
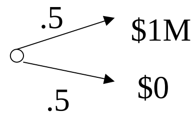


Insurance

- We have an agent with $u(x) = \sqrt{x}$ and



- And a risk-neutral insurance company with lots of money, selling full insurance for "premium" P.

Quiz Problem

- Without discussing with anyone, each student is to write down a real number x_i between 0 and 100 on a paper and submit it to the TA.
- The TA will then compute the average

$$\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

- The students who submit the number that is closest to $\overline{x}/3$ will share 100 points equally; the others will get 0.

Multi-person Decision Theory

- Who are the players?
- Who has which options?
- Who knows what?
- Who gets how much?

Knowledge

1. If I know something, it must be true.
2. If I know x, then I know that I know x.
3. If I don't know x, then I know that I don't know x.
4. If I know something, I know all its logical implications.

Common Knowledge: x is common knowledge iff

•Each player knows x

•Each player knows that each player knows x

• Each player knows that each player knows that each player knows x

•Each player knows that each player knows that each player knows that each player knows x

•… ad infinitum

Representations of games

Normal-form representation

Definition (Normal form): A game is any list

$$G = (S_1, \ldots, S_n; u_1, \ldots, u_n)$$

where, for each $i \in N = \{1,2,\ldots,n\}$,

- S_i is the set of all strategies available to i,
- $u_i : S_1 \times \cdots \times S_n \to \Re$ is the VNM utility function of player i.

Assumption: G is common knowledge.

Definition: A player i is rational iff he tries to maximize the expected value of u_i given his beliefs.

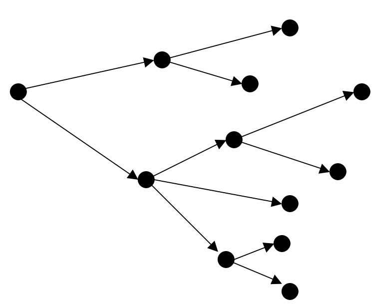
Chicken

		
	(-1,-1)	(1,0)
	(0,1)	(1/2,1/2)

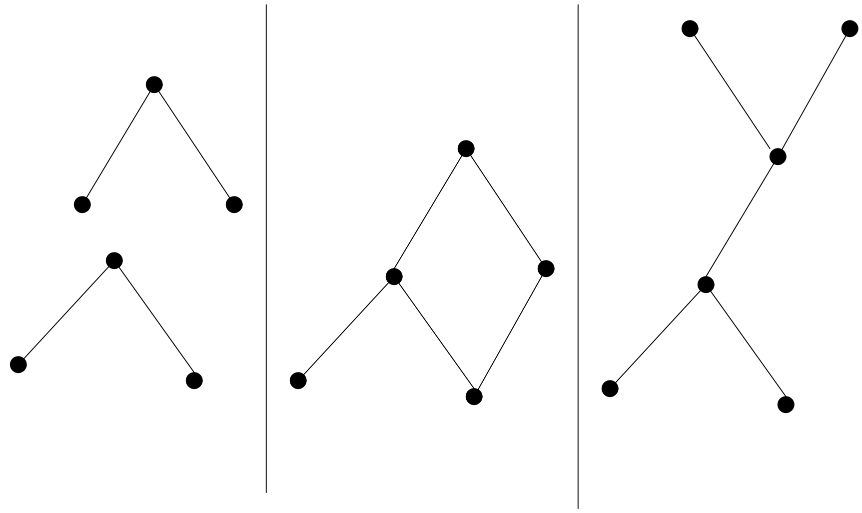
Extensive-form representation

Definition: A **tree** is a set of nodes connected with directed arcs such that

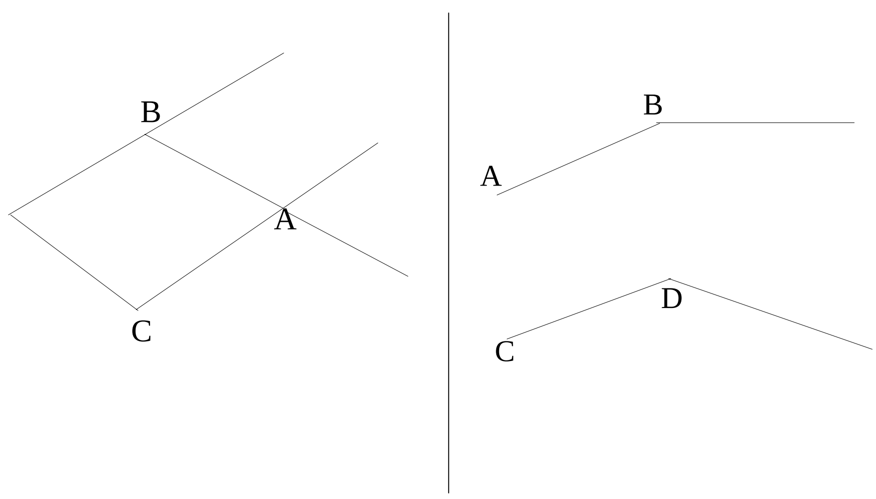
1. For each node, there is at most one incoming arc;
2. each node can be reached through a unique path;



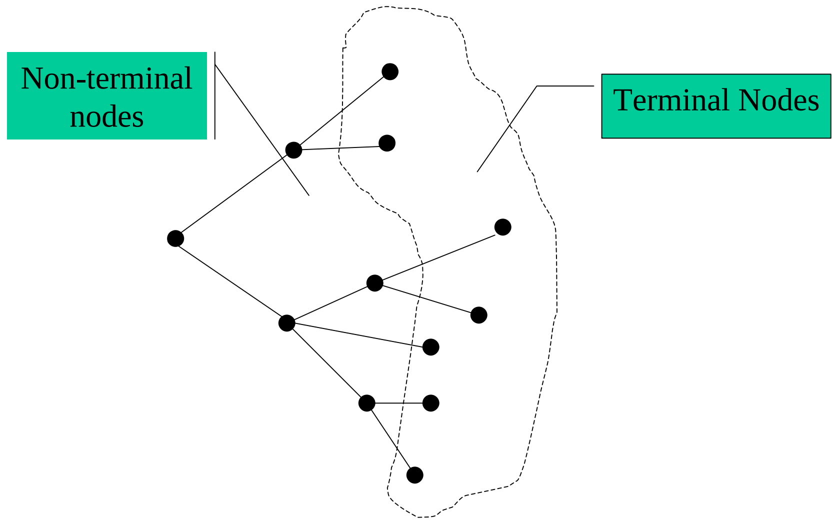
A tree?



A tree??



A tree



Extensive form – definition

Definition: A game consists of

– a set of players

– a tree

– an allocation of each non-terminal node to a player

– an informational partition (to be made precise)

– a payoff for each player at each terminal node.

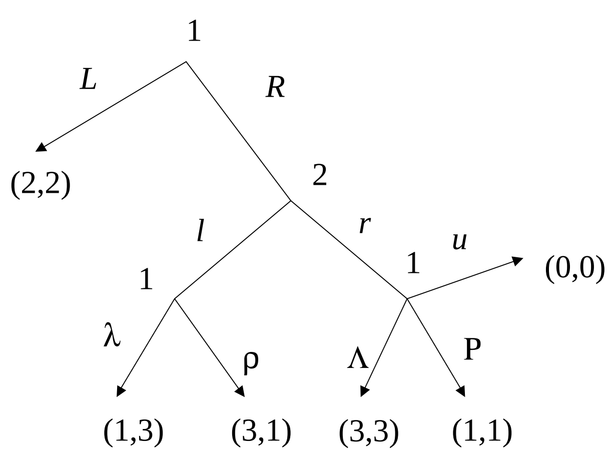
Information set

An **information set** is a collection of nodes such that

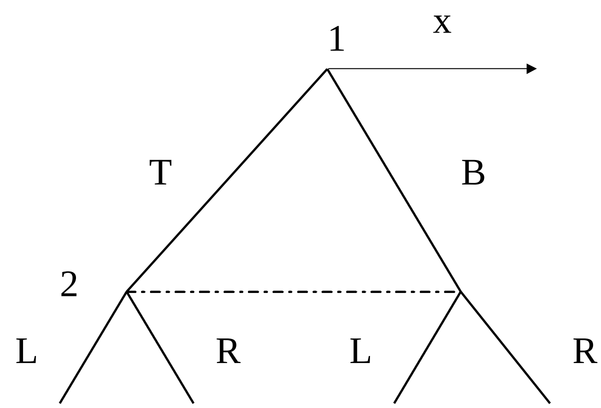
1. The same player is to move at each of these nodes;
2. The same moves are available at each of these nodes.

An **informational partition** is an allocation of each non-terminal node of the tree to an information set.

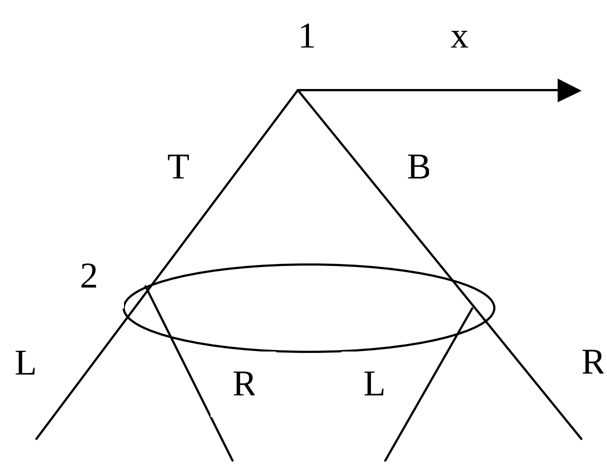
A game



Another Game



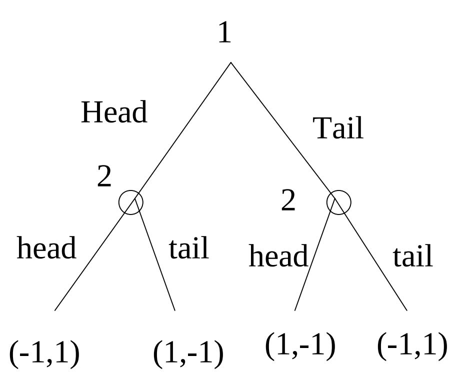
The same game



Strategy

A **strategy** of a player **is a complete contingent-plan**, determining which action he will take at each information set he is to move (including the information sets that will not be reached according to this strategy).

Matching pennies with perfect information



2's Strategies:

HH = Head if 1 plays Head,
Head if 1 plays Tail;

HT = Head if 1 plays Head,
Tail if 1 plays Tail;

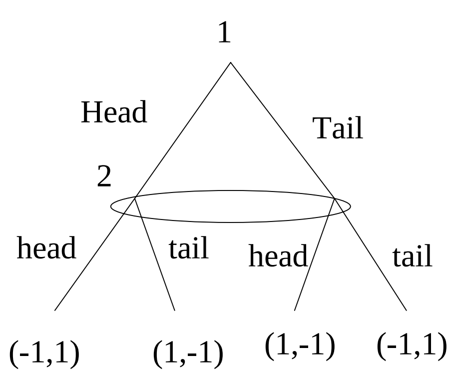
TH = Tail if 1 plays Head,
Head if 1 plays Tail;

TT = Tail if 1 plays Head,
Tail if 1 plays Tail.

Matching pennies with perfect information

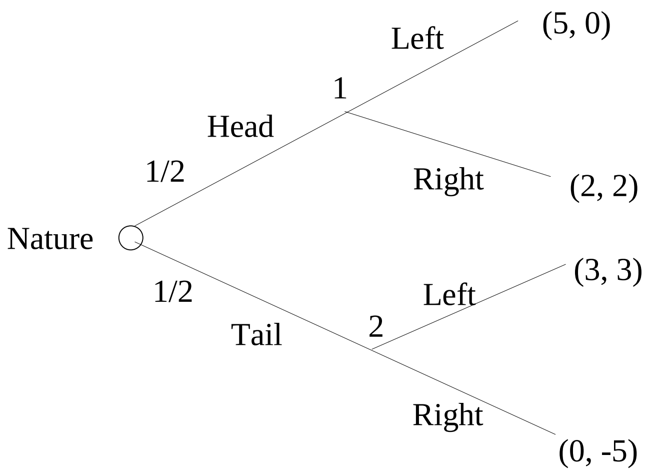
1 \ 2	HH	HT	TH	TT
Head				
Tail				

Matching pennies with Imperfect information

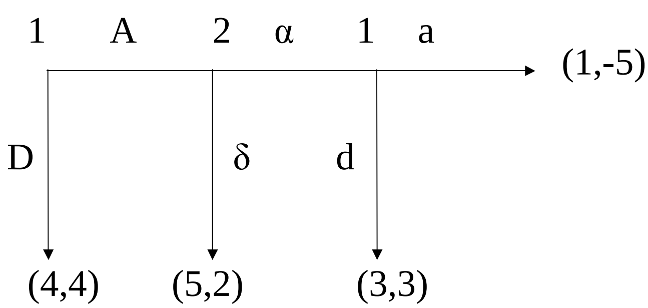


1 \ 2	Head	Tail
Head	(-1,1)	(1,-1)
Tail	(1,-1)	(-1,1)

A game with nature



A centipede game



Lecture 4
Representation of Games & Rationalizability

14.12 Game Theory

Muhamet Yildiz

Road Map

1. Representation of games in strategic and extensive forms
2. Dominance
3. Dominant-strategy equilibrium
4. Rationalizability

Normal-form representation

Definition (Normal form): A game is any list

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where, for each $i \in N = \{1,2,\ldots,n\}$,

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Assumption: G is common knowledge.

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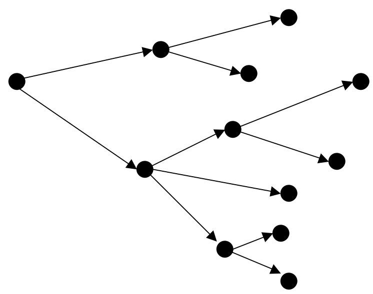
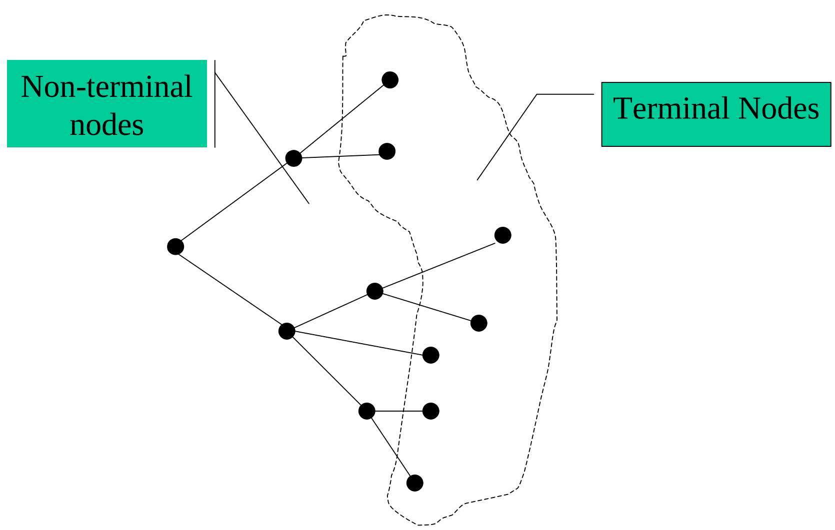


A tree



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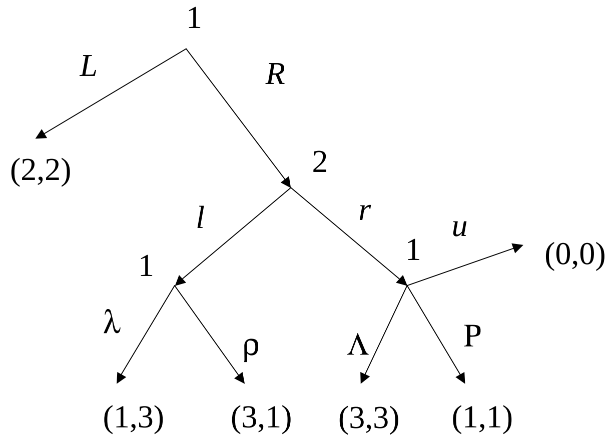
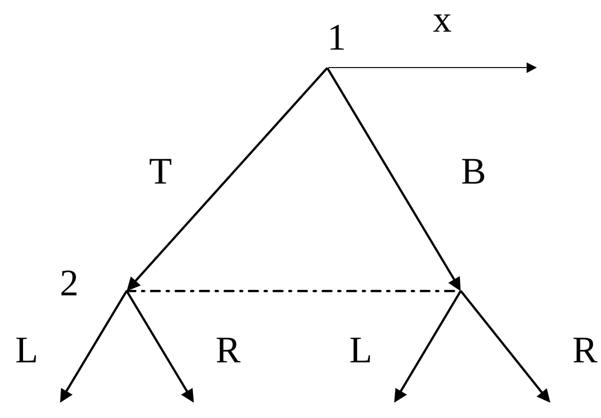
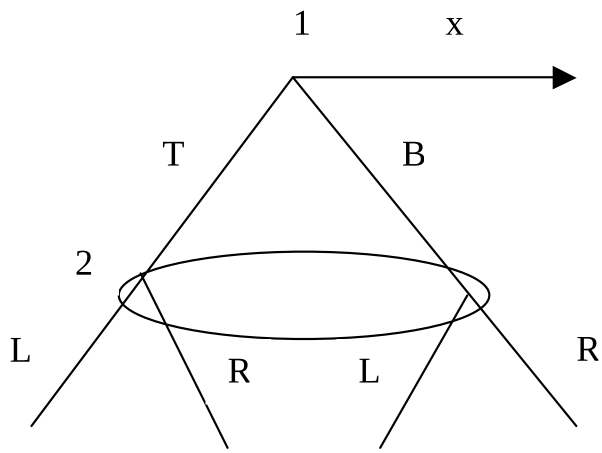
A game



Another Game



The same game



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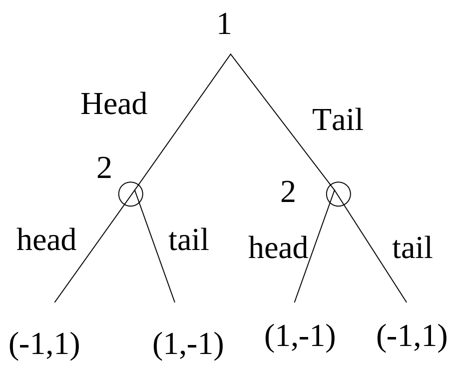
Matching pennies with perfect information



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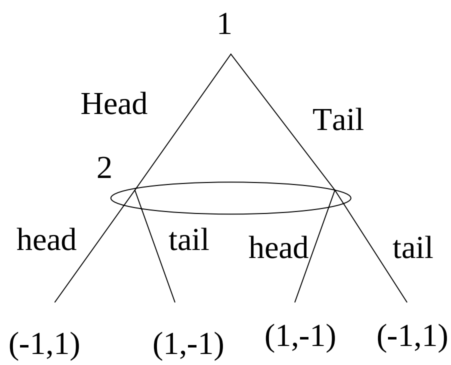
Tail if 1 plays Tail.

Matching pennies with perfect information

1 \ 2	HH	HT	TH	TT
Head				
Tail				

Matching pennies with Imperfect information



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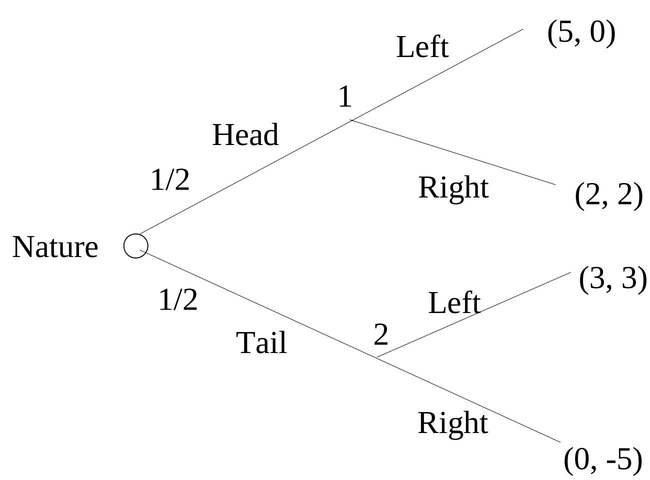
Tail	(1,-1)	(-1,1)

A game with nature



Mixed Strategy

Definition: A **mixed strategy** of a player is a probability distribution over the set of his strategies.

Pure strategies: $S_i = \{s_{i1}, s_{i2}, \ldots, s_{ik}\}$
A mixed strategy: $\sigma_i: S \rightarrow [0,1]$ s.t.

$$\sigma_i(s_{i1}) + \sigma_i(s_{i2}) + \ldots + \sigma_i(s_{ik}) = 1.$$

If the other players play $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$, then the expected utility of playing σ_i is

$$\sigma_i(s_{i1})u_i(s_{i1}, s_{-i}) + \sigma_i(s_{i2})\, u_i(s_{i2}, s_{-i}) + \ldots + \sigma_i(s_{ik})\, u_i(s_{ik}, s_{-i}).$$

How to play

Dominance

$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$

Definition: A pure strategy s_i^* **strictly dominates** s_i if and only if

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_{-i}.$$

A mixed strategy σ_i^* **strictly dominates** s_i iff

$$\sigma_i(s_{i1})u_i(s_{i1}, s_{-i}) + \cdots + \sigma_i(s_{ik})u_i(s_{ik}, s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_i$$

A rational player never plays a strictly dominated strategy.

Prisoners' Dilemma

1 \ 2	Cooperate	Defect
Cooperate	(5,5)	(0,6)
Defect	(6,0)	(1,1)

A game

1 \ 2	L	m	R
T	(3,0)	(1,1)	(0,3)
M	(1,0)	(0,10)	(1,0)
B	(0,3)	(1,1)	(3,0)

Weak Dominance

Definition: A pure strategy s_i* weakly **dominates** s_i if and only if

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \forall s_{-i}.$$

and at least one of the inequalities is strict. A mixed strategy σ_i* **weakly dominates** s_i iff

$$\sigma_i(s_{i1})u_i(s_{i1}, s_{-i}) + \cdots + \sigma_i(s_{ik})u_i(s_{ik}, s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_i$$

and at least one of the inequalities is strict.

If a player is rational and cautious (i.e., he assigns positive probability to each of his opponents' strategies), then he will not play a weakly dominated strategy.

Dominant-strategy equilibrium

Definition: A strategy s_i* is a **dominant strategy** iff s_i* **weakly dominates** every other strategy s_i.

Definition: A strategy profile s* is a dominant-strategy equilibrium iff s_i* is a dominant strategy for each player i.

If there is a dominant strategy, then it will be played, so long as the players are …

Prisoners' Dilemma

1 \ 2	Cooperate	Defect
Cooperate	(5,5)	(0,6)
Defect	(6,0)	(1,1)

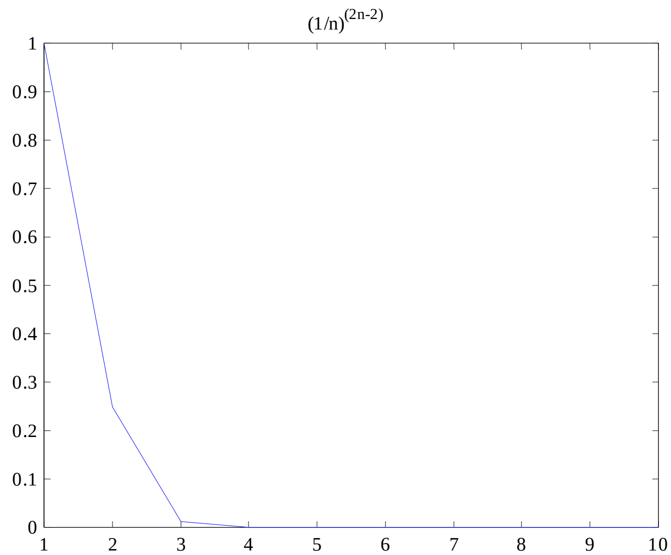
Second-price auction

- $N = \{1,2\}$ buyers;
- The value of the house for buyer i is v_i;
- Each buyer i simultaneously bids b_i;
- i* with $b_{i*} = \max b_i$ gets the house and pays the second highest bid

$$p = \max_{j \neq i} b_j.$$

Question

What is the probability that an nxn game has a dominant strategy equilibrium given that the payoffs are independently drawn from the same (continuous) distribution on [0,1]?



A game

1 \ 2	L	m	R
T	(3,0)	(1,1)	(0,3)
M	(1,0)	(0,10)	(1,0)
B	(0,3)	(1,1)	(3,0)

Assume: Players are rational and player 2 knows that 1 is rational.

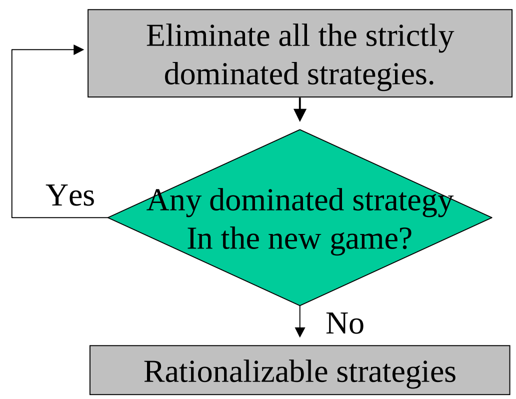
1 is rational and 2 knows this:

	L	m	R
T	(3,0)	(1,1)	(0,3)
B	(0,3)	(1,1)	(3,0)

And 2 is rational:

	L	R
T	(3,0)	(0,3)
B	(0,3)	(3,0)

Rationalizability



The play is rationalizable, provided that …

Simplified price-competition

Firm 1 \ Firm 2	High	Medium	Low
High	6,6	0,10	0,8
Medium	10,0	5,5	0,8
Low	8,0	8,0	4,4

Dutta

A strategy profile is rationalizable when …

- Each player’s strategy is consistent with his rationality, i.e., maximizes his payoff with respect to a conjecture about other players’ strategies;
- These conjectures are consistent with the other players’ rationality, i.e., if i conjectures that j will play s_j with positive probability, then s_j maximizes j’s payoff with respect to a conjecture of j about other players’ strategies;
- These conjectures are also consistent with the other players’ rationality, i.e., …
- Ad infinitum

Lecture 5
Nash equilibrium & Applications

14.12 Game Theory

Muhamet Yildiz

Road Map

1. Rationalizability – summary
2. Nash Equilibrium
3. Cournot Competition
   1. Rationalizability in Cournot Duopoly
4. Bertrand Competition
5. Commons Problem
6. **Quiz**
7. Mixed-strategy Nash equilibrium

Dominant-strategy equilibrium

$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$

Definition: s_i* **strictly dominates** s_i iff

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_{-i};$$

s_i* **weakly dominates** s_i iff $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \forall s_{-i}$ and at least one of the inequalities is strict.

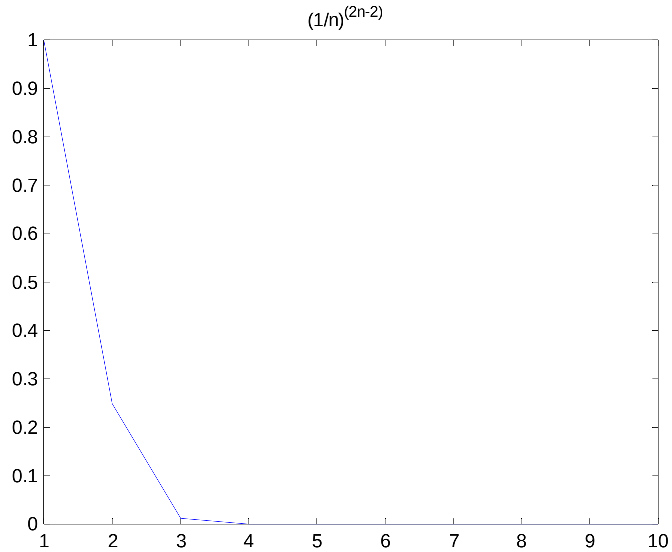
Definition: A strategy s_i* is a **dominant strategy** iff s_i* **weakly dominates** every other strategy s_i.

Definition: A strategy profile s* is a **dominant-strategy equilibrium** iff s_i* is a dominant strategy for each player i.

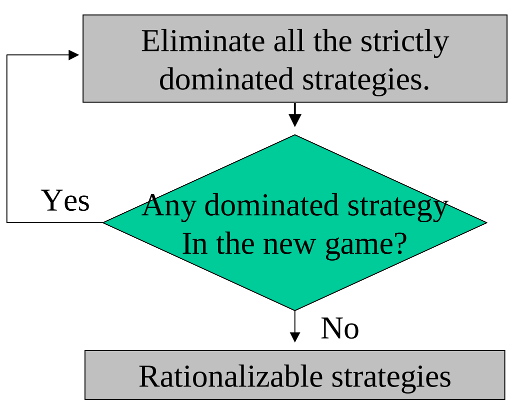
Examples: Prisoners' Dilemma; Second-Price auction.

Question

What is the probability that an nxn game has a dominant strategy equilibrium given that the payoffs are independently drawn from the same (continuous) distribution on [0,1]?



Rationalizability



The play is rationalizable, provided that …

Simplified price-competition

Firm 1 \ Firm 2	High	Medium	Low
High	6,6	0,10	0,8
Medium	10,0	5,5	0,8
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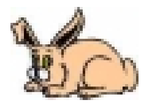
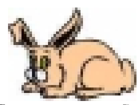
Dutta

A strategy profile is rationalizable when …

- Each player's strategy is consistent with his rationality, i.e., maximizes his payoff with respect to a conjecture about other players' strategies;
- These conjectures are consistent with the other players' rationality, i.e., if i conjectures that j will play s_j with positive probability, then s_j maximizes j's payoff with respect to a conjecture of j about other players' strategies;
- These conjectures are also consistent with the other players' rationality, i.e., …
- Ad infinitum

Stag Hunt

		
	(2,2)	(4,0)
	(0,4)	(6,6)

A summary

- If players are rational (and cautious), then they play the dominant-strategy equilibrium whenever it exists
  - But, typically, it does not exist
- If it is common knowledge that players are rational, then they will play a rationalizable strategy-profile
  - Typically, there are too many rationalizable strategies
- Now, a stronger assumption: The players are rational and their conjectures are mutually known.

Nash Equilibrium

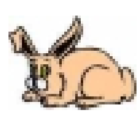
Definition: A strategy-profile $s^* = (s_1^*,\ldots,s_n^*)$ is a **Nash Equilibrium** iff, for each player i, and for each strategy s_i, we have

$$u_i(s_1^*,\ldots,s_{i-1}^*, s_i^*, s_{i+1}^*,\ldots,s_n^*)$$
$$\geq u_i(s_1^*,\ldots,s_{i-1}^*, s_i, s_{i+1}^*,\ldots,s_n^*),$$

i.e., no player has any incentive to deviate if he knows what the others play.

??If players are rational, and their conjectures about what the others play are mutually known, then they must be playing a Nash equilibrium.

Stag Hunt




(2,2)	(4,0)
(0,4)	(6,6)

Economic Applications

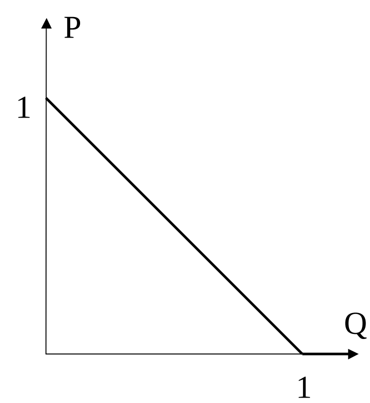
1. Cournot (quantity) Competition
   1. Nash Equilibrium in Cournot duopoly
   2. Nash Equilibrium in Cournot oligopoly
   3. Rationalizability in Cournot duopoly
2. Bertrand (price) Competition
3. Commons Problem

Cournot Oligopoly

- $N = \{1,2,\ldots,n\}$ firms;
- Simultaneously, each firm i produces q_i units of a good at marginal cost c,
- and sells the good at price

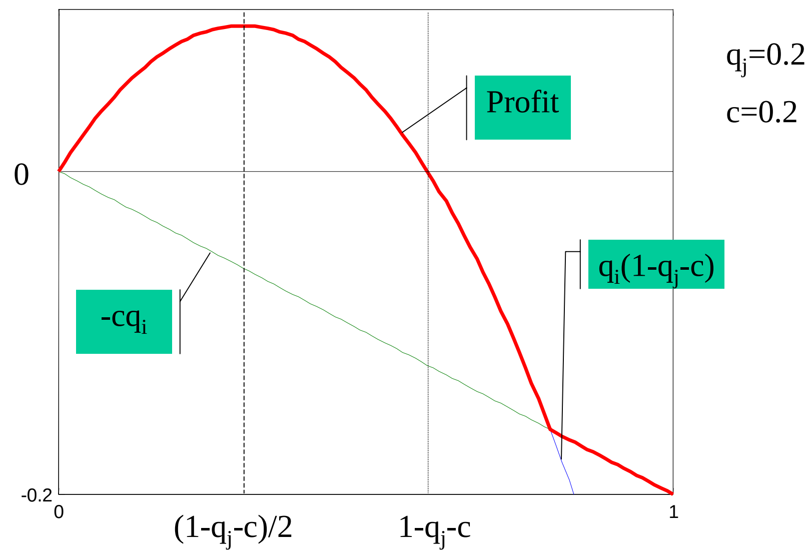
  $$P = \max\{0,1-Q\}$$

  where $Q = q_1+\ldots+q_n$.
- Game = $(S_1,\ldots,S_n; \pi_1,\ldots,\pi_n)$ where $S_i = [0,\infty)$,



$$\pi_i(q_1,\ldots,q_n) = \begin{cases} q_i[1-(q_1+\ldots+q_n)-c] & \text{if } q_1+\ldots+qn < 1, \\ -q_i c & \text{otherwise.} \end{cases}$$

Cournot Duopoly -- profit

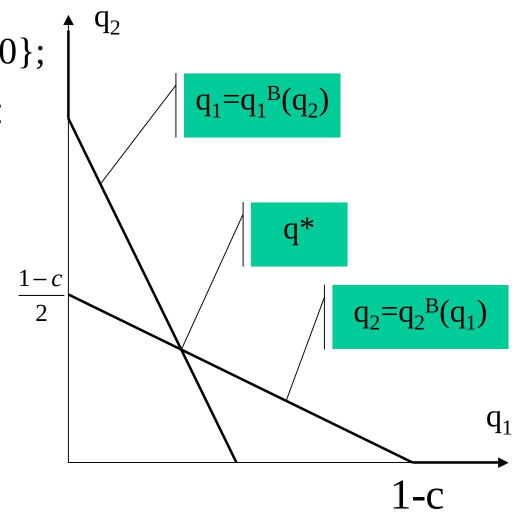


C-D – best responses

- $q_i^B(q_j) = \max\{(1-q_j-c)/2, 0\}$;
- Nash Equilibrium q*:

  $q_1^* = (1-q_2^*-c)/2;$

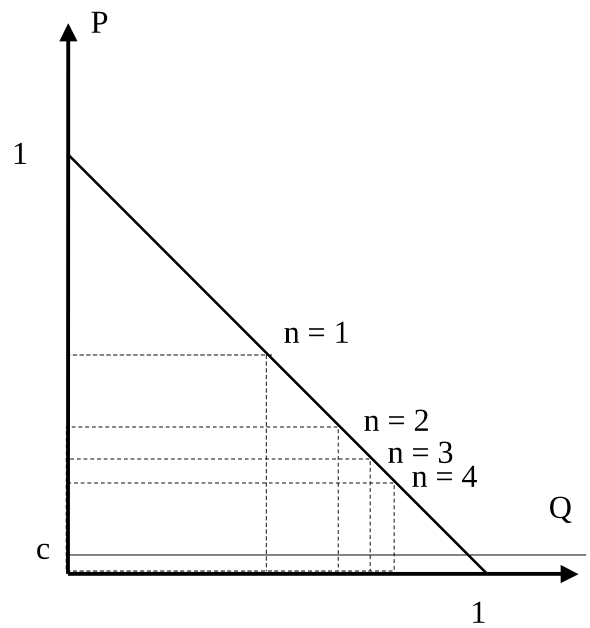
  $q_2^* = (1-q_1^*-c)/2;$
- $q_1^* = q_2^* = (1-c)/3$



Cournot Oligopoly --Equilibrium

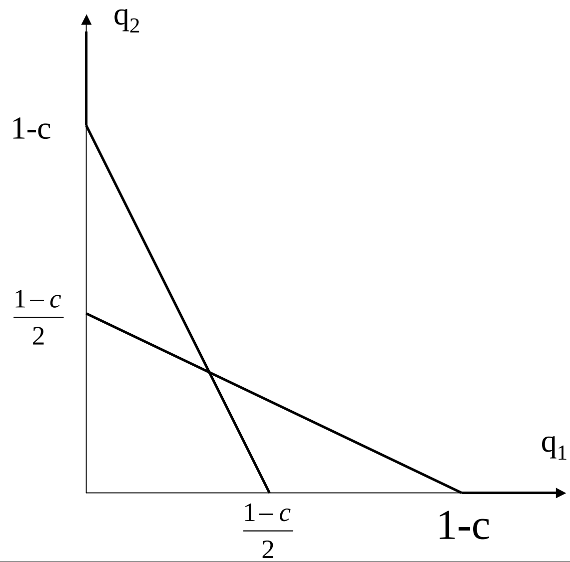
- $q > 1-c$ is strictly dominated, so $q \le 1-c$.
- $\pi_i(q_1,\ldots,q_n) = q_i[1-(q_1+\ldots+q_n)-c]$ for each i.
- FOC: $$\left.\frac{\partial \pi_i(q_1,\ldots,q_n)}{\partial q_i}\right|_{q=q^*} = \left.\frac{\partial[q_i(1-q_1-\cdots-q_n-c)]}{\partial q_i}\right|_{q=q^*}$$
$$= (1-q_1^*-\cdots-q_n^*-c)-q_i^* = 0.$$
- That is, $$\begin{aligned} 2q_1^*+q_2^*+\cdots+q_n^* &= 1-c \\ q_1^*+2q_2^*+\cdots+q_n^* &= 1-c \\ &\vdots \\ q_1^*+q_2^*+\cdots+nq_n^* &= 1-c \end{aligned}$$
- Therefore, $q_1^*=\ldots=q_n^*=(1-c)/(n+1)$.

Cournot oligopoly – comparative statics



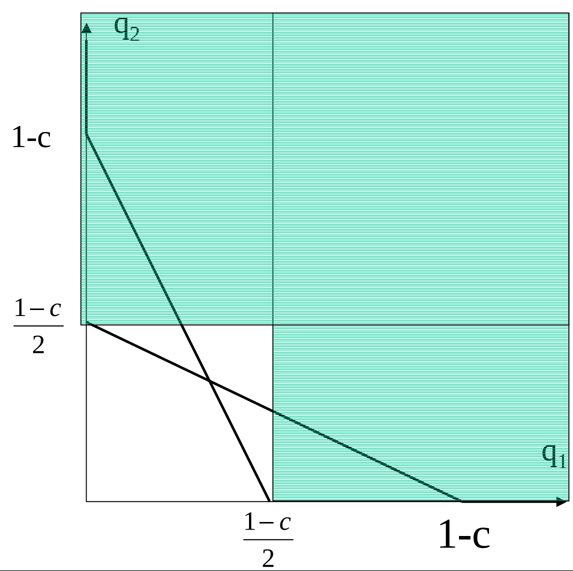
Rationalizability in Cournot Duopoly

Assume that players are rational.



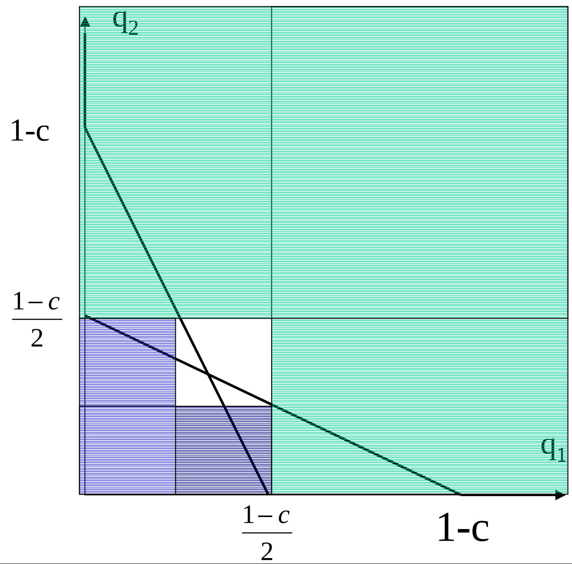
Players are rational:

Assume that players know this.



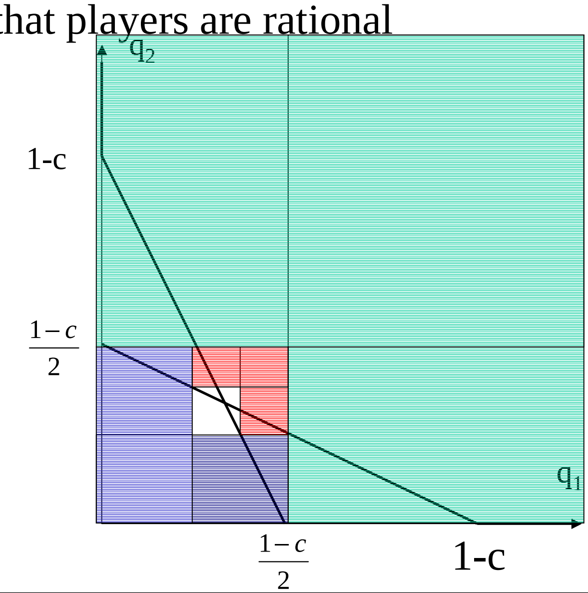
Players are rational and know that players are rational

Assume that players know this.



Players are rational; players know that players are rational; players know that players know that players are rational

Assume that players know this.



Rationalizability in Cournot duopoly

- If i knows that $q_j \leq q$, then $q_i \geq (1\text{-}c\text{-}q)/2$.
- If i knows that $q_j \geq q$, then $q_i \leq (1\text{-}c\text{-}q)/2$.
- We know that $q_j \geq q^0 = 0$.
- Then, $q_i \leq q^1 = (1\text{-}c\text{-}q^0)/2 = (1\text{-}c)/2$ for each i;
- Then, $q_i \geq q^2 = (1\text{-}c\text{-}q^1)/2 = (1\text{-}c)(1\text{-}1/2)/2$ for each i;
- …
- Then, $q^n \leq q_i \leq q^{n+1}$ or $q^{n+1} \leq q_i \leq q^n$ where
  $q^{n+1} = (1\text{-}c\text{-}q^n)/2 = (1\text{-}c)(1\text{-}1/2+1/4\text{-}\ldots+(\text{-}1/2)^n)/2$.
- As $n \rightarrow \infty$, $q^n \rightarrow (1\text{-}c)/3$.

Bertrand (price) competition

- $N = \{1,2\}$ firms.
- Simultaneously, each firm i sets a price p_i;
- If $p_i < p_j$, firm i sells $Q = \max\{1 - p_i, 0\}$ unit at price p_i; the other firm gets 0.
- If $p_1 = p_2$, each firm sells Q/2 units at price p_1, where $Q = \max\{1 - p_1, 0\}$.
- The marginal cost is 0.

$$\pi_1(p_1, p_2) = \begin{cases} p_1(1-p_1) & \text{if } p_1 < p_2 \\ p_1(1-p_1)/2 & \text{if } p_1 = p_2 \\ 0 & \text{otherwise.} \end{cases}$$

Bertrand duopoly -- Equilibrium

Theorem: The only Nash equilibrium in the "Bertrand game" is $p^* = (0,0)$.

Proof:

1. $p^*=(0,0)$ is an equilibrium.
2. If $p = (p_1,p_2)$ is an equilibrium, then $p = p^*$.
   1. If $p = (p_1,p_2)$ is an equilibrium, then $p_1 = p_2$..
      - If $p_i > p_j = 0$, for sufficiently small $\varepsilon>0$, $p_j' = \varepsilon$ is a better response to p_i for j. If $p_i > p_j > 0$, $p_i' = p_j$ is a better response for i.
   2. Given any equilibrium $p = (p_1,p_2)$ with $p_1 = p_2$, $p = p^*$.
      - If $p_1 = p_2>0$, for sufficiently small $\varepsilon>0$, $p_j' = p_j - \varepsilon$ is a better response to p_j for i.

Commons Problem

- $N = \{1,2,\ldots,n\}$ players, each with unlimited money;
- Simultaneously, each player i contributes $x_i \geq 0$ to produce $y = x_1+\ldots x_n$ unit of some public good, yielding payoff

$$U_i(x_i,y) = y^{1/2} - x_i.$$

Quiz

Each student i is to submit a real number x_i. We will pair the students randomly. For each pair (i,j), if $x_i \neq x_j$, the student who submits the number that is closer to $(x_i+x_j)/4$ gets 100; the other student gets 20. If $x_i = x_j$, then each of i and j gets 50.

Lectures 6-7

Nash Equilibrium
&
Backward Induction

14.12 Game Theory

Muhamet Yildiz

Road Map

1. Bertrand Competition
2. Commons Problem
3. Mixed-strategy Nash equilibrium
4. Bertrand competition with costly search
5. Backward Induction
6. Stackelberg Competition
7. Sequential Bargaining

Bertrand (price) competition

- N = {1,2} firms.
- Simultaneously, each firm i sets a price p_i;
- If $p_i < p_j$, firm i sells Q = max{1 – p_i,0} unit at price p_i; the other firm gets 0.
- If $p_1 = p_2$, each firm sells Q/2 units at price p_1, where Q = max{1 – p_1,0}.
- The marginal cost is 0.

$$\pi_1(p_1, p_2) = \begin{cases} p_1(1-p_1) & \text{if } p_1 < p_2 \\ p_1(1-p_1)/2 & \text{if } p_1 = p_2 \\ 0 & \text{otherwise.} \end{cases}$$

Bertrand duopoly -- Equilibrium

Theorem: The only Nash equilibrium in the “Bertrand game” is p* = (0,0).

Proof:

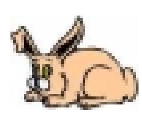
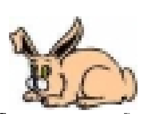
1. p*=(0,0) is an equilibrium.
2. If p = (p_1,p_2) is an equilibrium, then p = p*.
   1. If p = (p_1,p_2) is an equilibrium, then $p_1 = p_2$...
      - If $p_i > p_j = 0$, for sufficiently small ε>0, $p_j' = ε$ is a better response to p_i for j. If $p_i > p_j > 0$, $p_i' = p_j$ is a better response for i.
   2. Given any equilibrium p = (p_1,p_2) with $p_1 = p_2$, p = p*.
      - If $p_1 = p_2 > 0$, for sufficiently small ε>0, $p_j' = p_j - ε$ is a better response to p_j for i.

Commons Problem

- N = {1,2,…,n} players, each with unlimited money;
- Simultaneously, each player i contributes $x_i \geq 0$ to produce $y = x_1 + \ldots x_n$ unit of some public good, yielding payoff

$$U_i(x_i, y) = y^{1/2} - x_i.$$

Stag Hunt

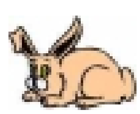
		
	(2,2)	(4,0)
	(0,4)	(5,5)

Equilibrium in Mixed Strategies

What is a strategy?
- A complete contingent-plan of a player.
- What the others think the player might do under various contingency.

What do we mean by a mixed strategy?
- The player is randomly choosing his pure strategies.
- The other players are not certain about what he will do.

Stag Hunt




(2,2)	(4,0)
(0,4)	(5,5)

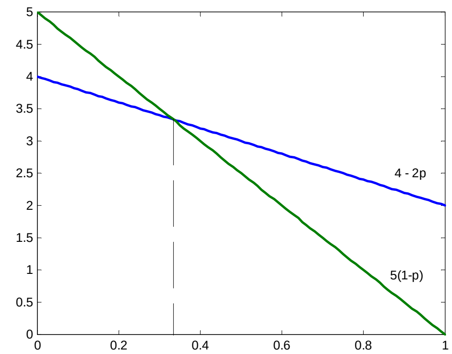
Mixed-strategy equilibrium in Stag-Hunt game

- Assume: Player 2 thinks that, with probability p, Player 1 targets for Rabbit. What is the best probability q she wants to play Rabbit?
- His payoff from targeting Rabbit:

  $U_2(R;p) = 2p + 4(1-p)$
  $= 4-2p.$
- From Stag:

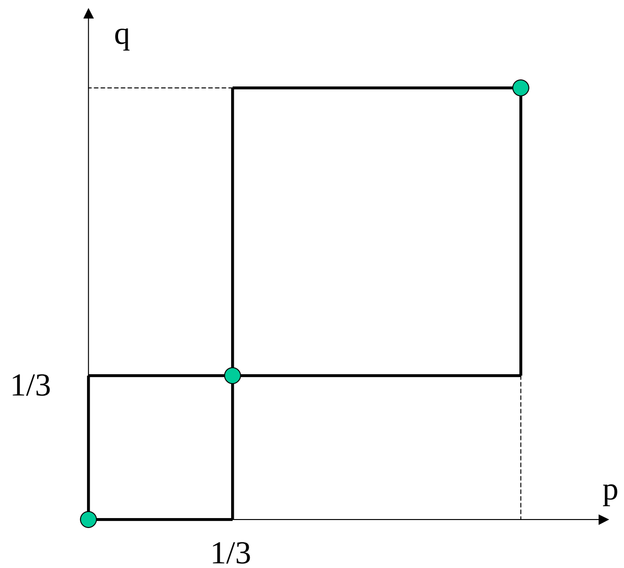
  $U_2(R;p) = 5(1-p)$
- She is indifferent iff

  $4-2p = 5(1-p)$ iff $p = 1/3$.



$$q^{BR}(p) = \begin{cases} 0 & \text{if } p < 1/3 \\ q \in [0,1] & \text{if } p = 1/3 \\ 1 & \text{if } p > 1/3 \end{cases}$$

Best responses in Stag-Hunt game



Bertrand Competition with costly search

- $N = \{F1, F2, B\}$; F1, F2 are firms; B is buyer
- B needs 1 unit of good, worth 6;
- Firms sell the good; Marginal cost = 0.
- Possible prices $P = \{1,5\}$.
- Buyer can check the prices with a small cost $c > 0$.

Game:

1. Each firm i chooses price p_i;
2. B decides whether to check the prices;
3. (Given) If he checks the prices, and $p_1 \neq p_2$, he buys the cheaper one; otherwise, he buys from any of the firm with probability ½.

Bertrand Competition with costly search

F1 \ F2	High	Low
High		
Low		

Check

F1 \ F2	High	Low
High		
Low		

Don’t Check

Mixed-strategy equilibrium

- Symmetric equilibrium: Each firm charges "High" with probability q;
- Buyer Checks with probability r.
- U(check;q) = $q^2 1 + (1-q^2)5 - c = 5 - 4q^2 - c$;
- U(Don't;q) = $q1 + (1-q)5 = 5 - 4q$;
- Indifference: $4q(1-q) = c$; i.e.,
- U(high;q,r) = $0.5(1-r(1-q))5$;
- U(low;q,r) = $qr1 + 0.5(1-qr)$
- Indifference = $r = 4/(5-4q)$.

Dynamic Games of Perfect Information & Backward Induction

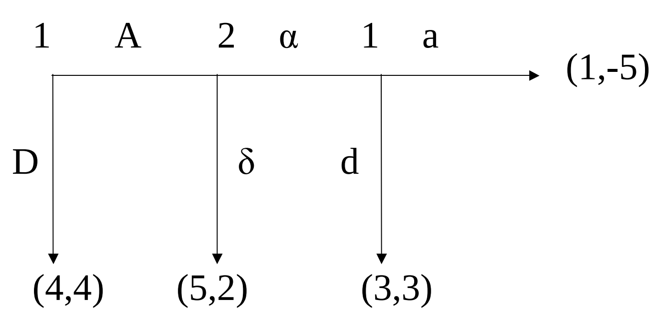
Definitions

Perfect-Information game is a game in which all the information sets are singleton.

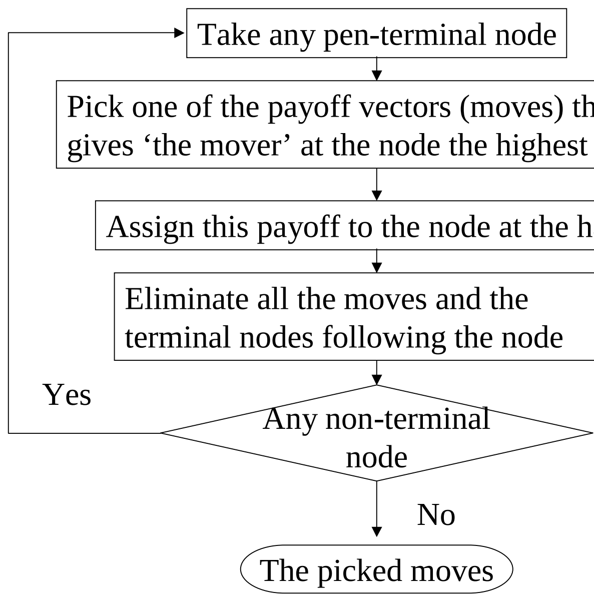
Sequential Rationality: A player is sequentially rational iff, at each node he is to move, he maximizes his expected utility conditional on that he is at the node – even if this node is precluded by his own strategy.

In a finite game of perfect information, the common knowledge of sequential rationality gives **“Backward Induction”** outcome.

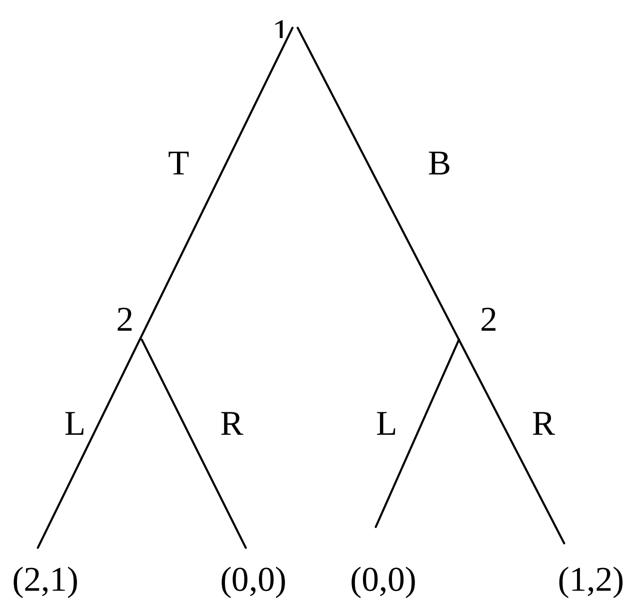
A centipede game



Backward Induction



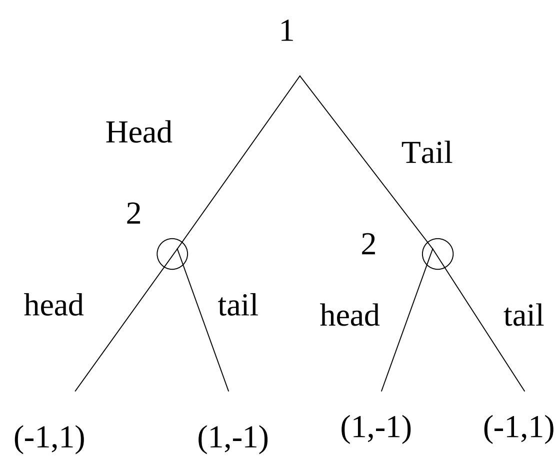
Battle of The Sexes with perfect information



Note

- There are Nash equilibria that are different from the Backward Induction outcome.
- Backward Induction always yields a Nash Equilibrium.
- That is, Sequential rationality is stronger than rationality.

Matching Pennies (wpi)



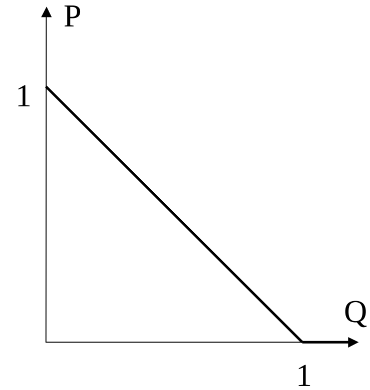
Stackelberg Duopoly

Game:

N = {1,2} firms w MC = 0;

1. Firm 1 produces q_1 units
2. Observing q_1, Firm 2 produces q_2 units
3. Each sells the good at price

   $P = \max\{0,1-(q_1+q_2)\}$.



$\pi_i(q_1, q_2) = q_i[1-(q_1+q_2)]$ if $q_1+ q_2 < 1$,

0 otherwise.

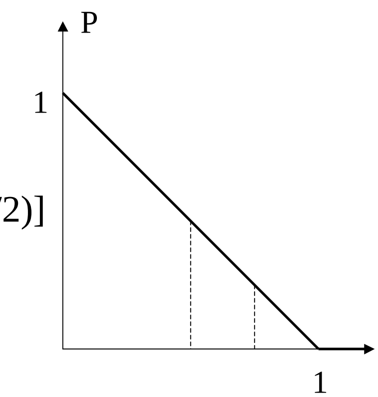
"Stackelberg equilibrium"

- If $q_1 > 1$, $q_2^*(q_1) = 0$.
- If $q_1 \leq 1$, $q_2^*(q_1) = (1-q_1)/2$.
- Given the function q_2^*, if $q_1 \leq 1$

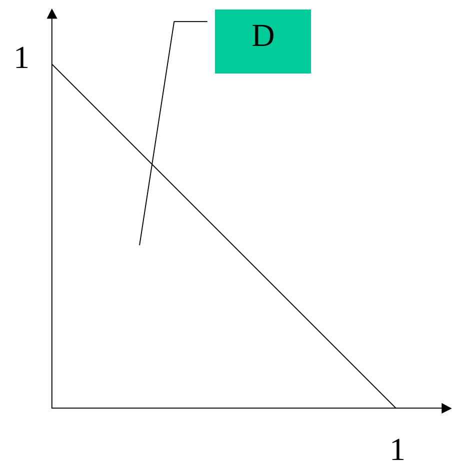
  $\pi_1(q_1;q_2^*(q_1)) = q_1[1-(q_1+ (1-q_1)/2)]$

  $= q_1 (1-q_1)/2$;

  0 otherwise.
- $q_1^* = ½$.
- $q_2^*(q_1^*) = ¼$.



Sequential Bargaining



- N = {1,2}
- X = feasible expected-utility pairs (x,y ∈ X)
- $U_i(x,t) = \delta_i^t x_i$
- d = (0,0) ∈ D disagreement payoffs

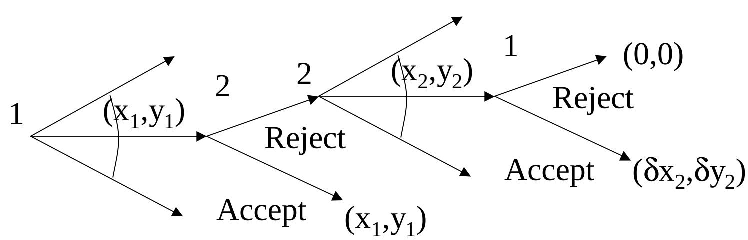
Timeline – 2 period

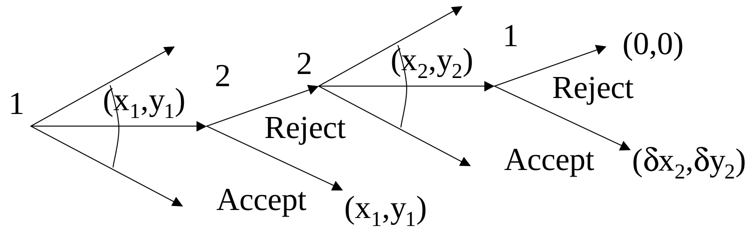
At t = 1,

- Player 1 offers some (x_1,y_1),
- Player 2 Accept or Rejects the offer
- If the offer is Accepted, the game ends yielding (x_1,y_1),
- Otherwise, we proceed to date 2.

At t = 2,

- Player 2 offers some (x_2,y_2),
- Player 1 Accept or Rejects the offer
- If the offer is Accepted, the game ends yielding payoff $\delta(x_2,y_2)$.
- Otherwise, the game end yielding d = (0,0).





At t = 2,
•Accept iff $y_2 \geq 0$.
•Offer (0,1).
At t = 1,
•Accept iff $x_2 \geq \delta$.
•Offer $(1-\delta,\delta)$.

Timeline – 2n period

T = {1,2,…,2n-1,2n}

If t is odd,

- Player 1 offers some (x_t,y_t),
- Player 2 Accept or Rejects the offer
- If the offer is Accepted, the game ends yielding $\delta^t(x_t,y_t)$,
- Otherwise, we proceed to date t+1.

If t is even

- Player 2 offers some (x_t,y_t),
- Player 1 Accept or Rejects the offer
- If the offer is Accepted, the game ends yielding payoff (x_t,y_t),
- Otherwise, we proceed to date t+1, except at t = 2n, when the game end yielding d = (0,0).

Equilibrium

- Scientific Word

Lectures 8

Subgame-perfect Equilibrium
&
Applications

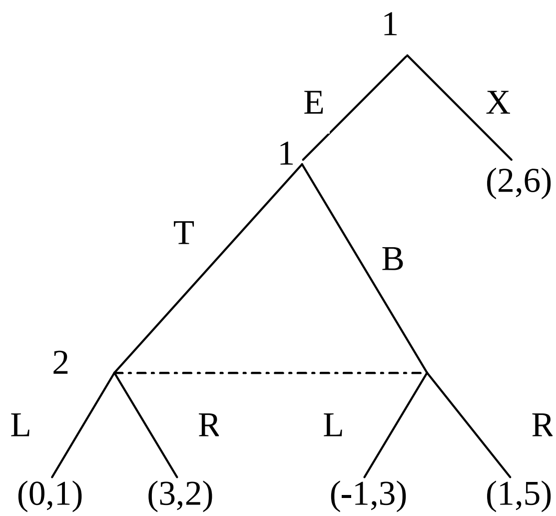
14.12 Game Theory

Muhamet Yildiz

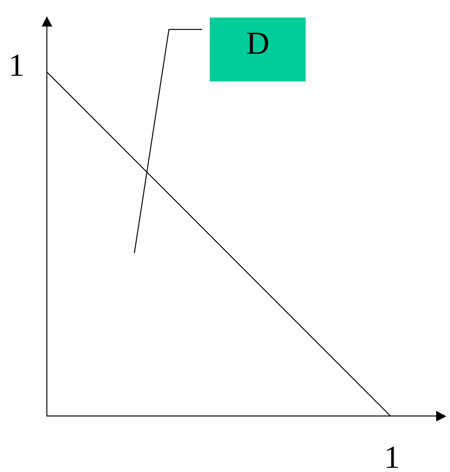
Road Map

1. Subgame-perfect Equilibrium
    1. Motivation
    2. What is a subgame?
    3. Definition
    4. Example
2. Applications
    1. Bank Runs
    2. Tariffs & Intra-industry trade
3. Quiz

A game



Sequential Bargaining



- N = {1,2}
- X = feasible expected-utility pairs ($x,y \in X$)
- $U_i(x,t) = \delta_i^t x_i$
- $d = (0,0) \in D$ disagreement payoffs

Timeline – ∞ period

$T = \{1,2,\ldots, n-1,n,\ldots\}$

If t is odd,

- Player 1 offers some (x_t,y_t),
- Player 2 Accept or Rejects the offer
- If the offer is Accepted, the game ends yielding $\delta^t(x_t,y_t)$,
- Otherwise, we proceed to date t+1.

If t is even

- Player 2 offers some (x_t,y_t),
- Player 1 Accept or Rejects the offer
- If the offer is Accepted, the game ends yielding payoff (x_t,y_t),
- Otherwise, we proceed to date t+1.

Backward induction

- Can be applied only in perfect information games of finite horizon.

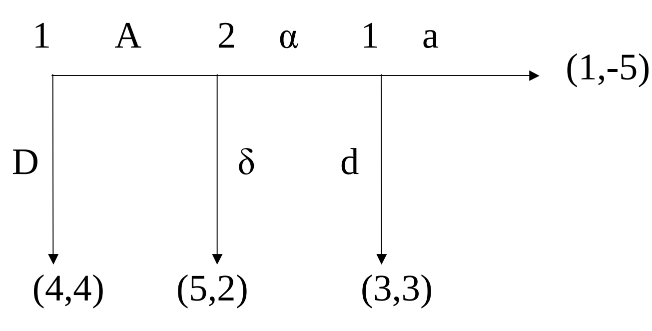
How can we extend this notion to infinite horizon games, or to games with imperfect information?

A subgame

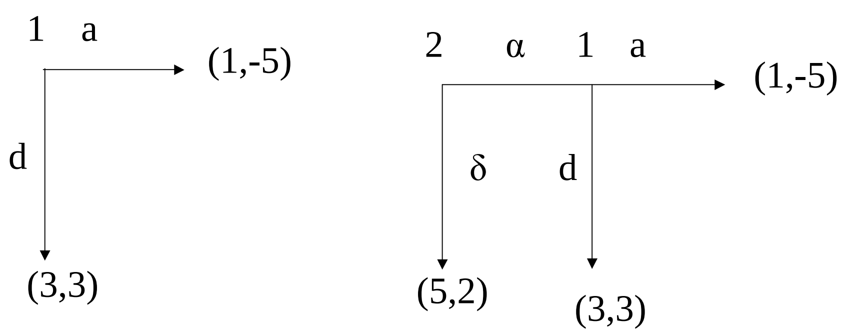
A *subgame* is part of a game that can be considered as a game itself.

- It must have a unique starting point;
- It must contain all the nodes that follow the starting node;
- If a node is in a subgame, the entire information set that contains the node must be in the subgame.

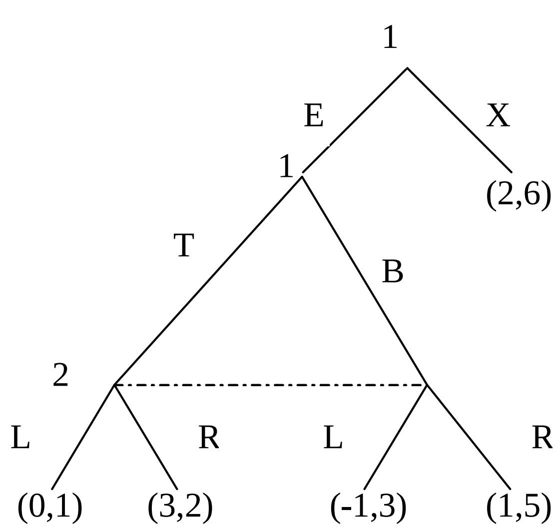
A game



And its subgames



A game

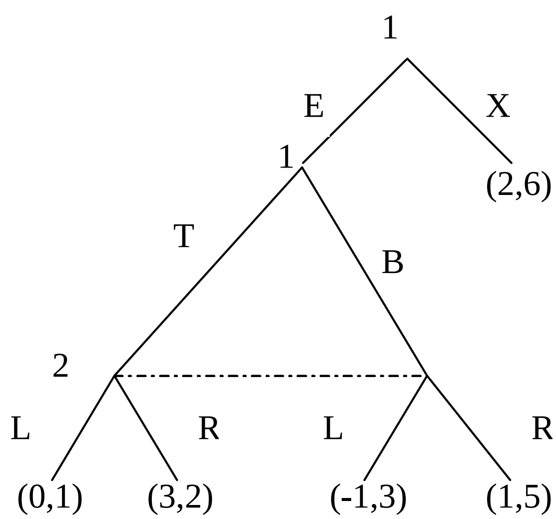


Definitions

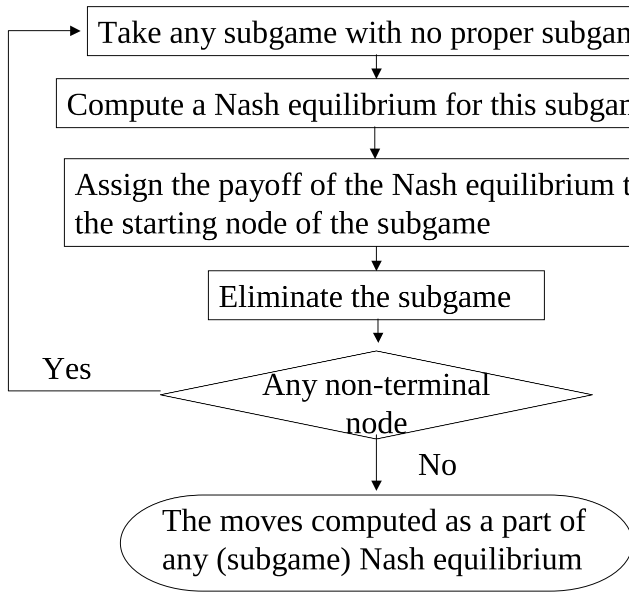
A *substrategy* is the restriction of a strategy to a subgame.

A subgame-perfect Nash equilibrium is a Nash equilibrium whose substrategy profile is a Nash equilibrium at each subgame.

Example



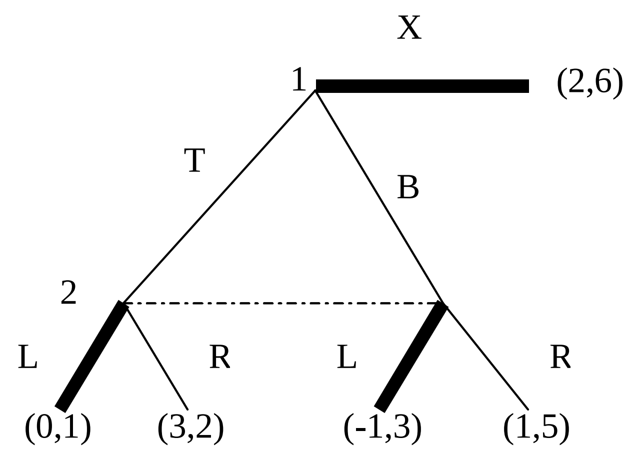
A “Backward-Induction-like” method



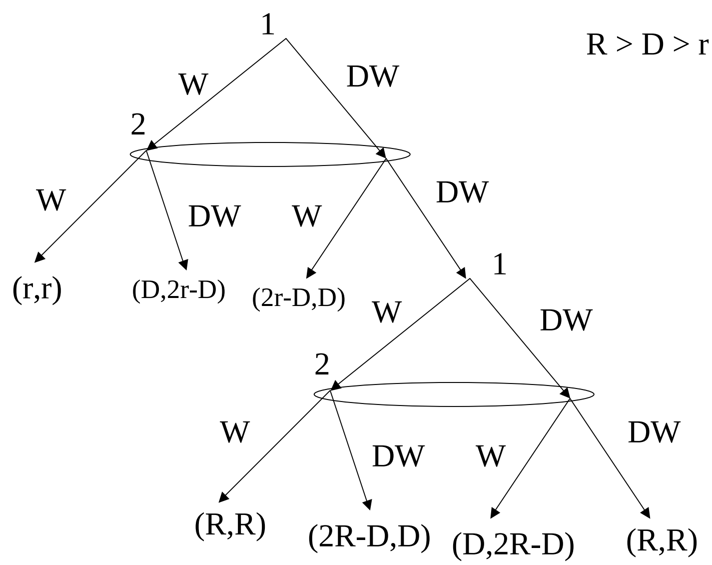
Theorem

In a finite, perfect-information game, the set of subgame-perfect equilibria is the set of strategy profiles that are computed via backward induction.

A subgame-perfect equilibrium?



Bank Run



R > D > r > D/2

Lectures 9

Applications of Subgame-perfect Equilibrium
&
Forward Induction

14.12 Game Theory

Muhamet Yildiz

Road Map

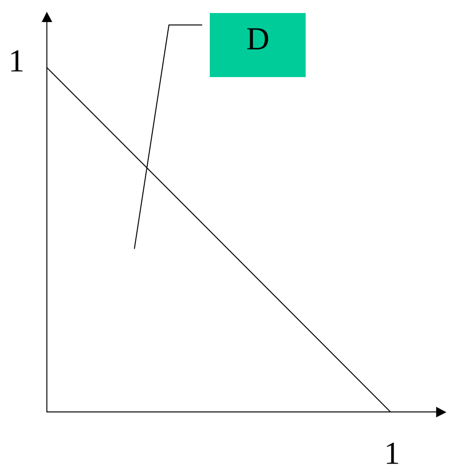
1. Applications
   1. Tariffs & Intra-industry trade
   2. Infinite horizon bargaining – Single-deviation principle
2. Forward Induction – Examples
3. Finitely Repeated Games
4. Quiz

Single-Deviation principle

Definition: An extensive-form game is *continuous at infinity* iff, given any $\varepsilon > 0$, there exists some t such that, for any two path whose first t arcs are the same, the payoff difference of each player is less than ε.

Theorem: Let G be a game that is continuous at infinity. A strategy profile $s = (s_1,s_2,\ldots,s_n)$ is a subgame-perfect equilibrium of G iff, at any information set, where a player i moves, given the other players strategies and given i's moves at the other information sets, player i cannot increase his conditional payoff at the information set by deviating from his strategy at the information set.

Sequential Bargaining



- N = {1,2}
- D = feasible expected-utility pairs ($x,y \in D$)
- $U_i(x,t) = \delta_i^t x_i$
- $d = (0,0) \in D$ disagreement payoffs

Timeline – ∞ period

T = {1,2,…, n-1,n,…}

If t is odd,

- Player 1 offers some (x_t,y_t),
- Player 2 Accept or Rejects the offer
- If the offer is Accepted, the game ends yielding $\delta^t(x_t,y_t)$,
- Otherwise, we proceed to date t+1.

If t is even

- Player 2 offers some (x_t,y_t),
- Player 1 Accept or Rejects the offer
- If the offer is Accepted, the game ends yielding payoff $\delta^t(x_t,y_t)$,
- Otherwise, we proceed to date t+1.

SPE of ∞-period bargaining

Theorem: At any t, proposer offers the other player $\delta/(1+\delta)$, keeping himself $1/(1+\delta)$, while the other player accept an offer iff he gets $\delta/(1+\delta)$.

"Proof:" Single-deviation principle: Take any date t, at which i offers, j accepts/rejects. According to the strategies in the continuation game, at t+1, j will get $1/(1+\delta)$. Hence, j accepts an offer iff she gets at least $\delta/(1+\delta)$. i must offer $\delta/(1+\delta)$.

Forward Induction

Strong belief in rationality: At any history of the game, each agent is assumed to be rational if possible. (That is, if there are two strategies s and s' of a player i that are consistent with a history of play, and if s is strictly dominated but s' is not, at this history no player j believes that i plays s.)

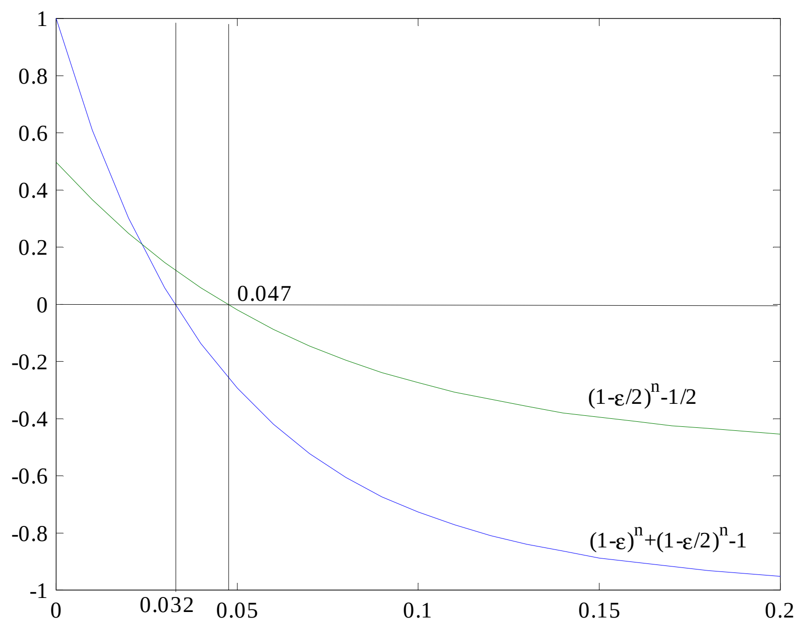
Table for the bidding game

$U_i = 20(2+2\min_j \text{bid}_j - \text{bid}_i)$

bid \ min	1	2	3
1	60	-	-
2	40	80	-
3	20	60	100

Nash equilibria of bidding game

- 3 equilibria: s^1 = everybody plays 1; s^2 = everybody plays 2; s^3 = everybody plays 3.
- Assume each player trembles with probability $\varepsilon < 1/2$, and plays each unintended strategy w.p. $\varepsilon/2$, e.g., w.p. $\varepsilon/2$, he thinks that such other equilibrium is to be played.
  - s^3 is an equilibrium iff
  - s^2 is an equilibrium iff
  - s^1 is an equilibrium iff



Bidding game with entry fee

Each player is first to decide whether to play the bidding game (E or X); if he plays, he is to pay a fee p > 60.

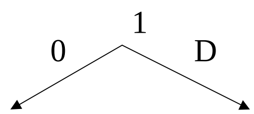
min / Bid	1	2	3
1	60	-	-
2	40	80	-
3	20	60	100

For each m =1,2,3, ∃SPE: (m,m,m) is played in the bidding game, and players play the game iff $20(2+m) \geq p$.

Forward induction: when $20(2+m) < p$, (Em) is strictly dominated by (Xk). After E, no player will assign positive probability to min bid ≤ m. FI-Equilibria: (Em,Em,Em) where $20(2+m) \geq p$.

What if an auction before the bidding game?

Burning Money



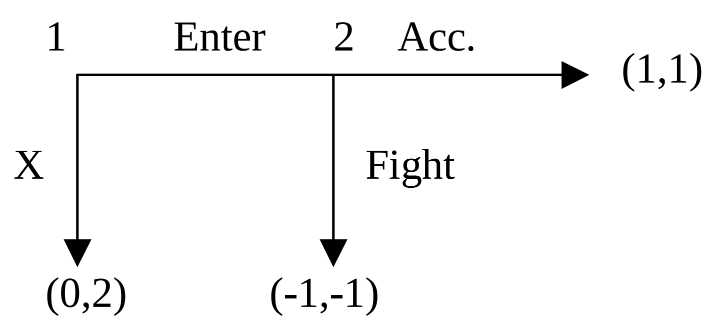
	B	S
B	3,1	0,0
S	0,0	1,3

	B	S
B	2,1	-1,0
S	-1,0	0,3

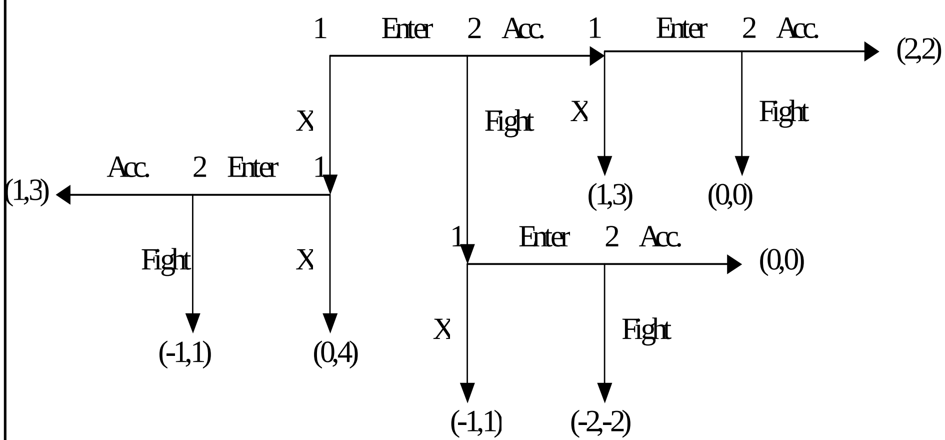
	BB	BS	SB	SS
0B				
0S				
DB				
DS				
O	T	H	E	R

Repeated Games

Entry deterrence



Entry deterrence, repeated twice



Lectures 10 -11

Repeated Games

14.12 Game Theory

Muhamet Yildiz

Road Map

1. Forward Induction – Examples
2. Finitely Repeated Games with observable actions
    1. Entry-Deterrence/Chain-store paradox
    2. Repeated Prisoners' Dilemma
    3. A general result
    4. When there are multiple equilibria
3. Infinitely repeated games with observable actions
    1. Discounting / Present value
    2. Examples
    3. The Folk Theorem
    4. Repeated Prisoners' Dilemma, revisited –tit for tat
    5. Repeated Cournot oligopoly
4. Infinitely repeated games with unobservable actions

Forward Induction

Strong belief in rationality: At any history of the game, each agent is assumed to be rational if possible. (That is, if there are two strategies s and s’ of a player i that are consistent with a history of play, and if s is strictly dominated but s’ is not, at this history no player j believes that i plays s.)

Bidding game with entry fee

Each player is first to decide whether to play the bidding game (E or X); if he plays, he is to pay a fee $p > 60$.

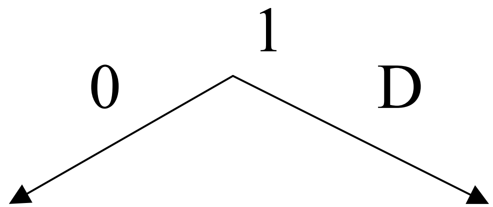
Bid \ min	1	2	3
1	60	-	-
2	40	80	-
3	20	60	100

For each $m = 1,2,3$, $\exists$SPE: (m,m,m) is played in the bidding game, and players play the game iff $20(2+m) \geq p$.

Forward induction: when $20(2+m) < p$, (Em) is strictly dominated by (Xk). After E, no player will assign positive probability to min bid $\leq m$. FI-Equilibria: (Em,Em,Em) where $20(2+m) \geq p$.

What if an auction before the bidding game?

Burning Money



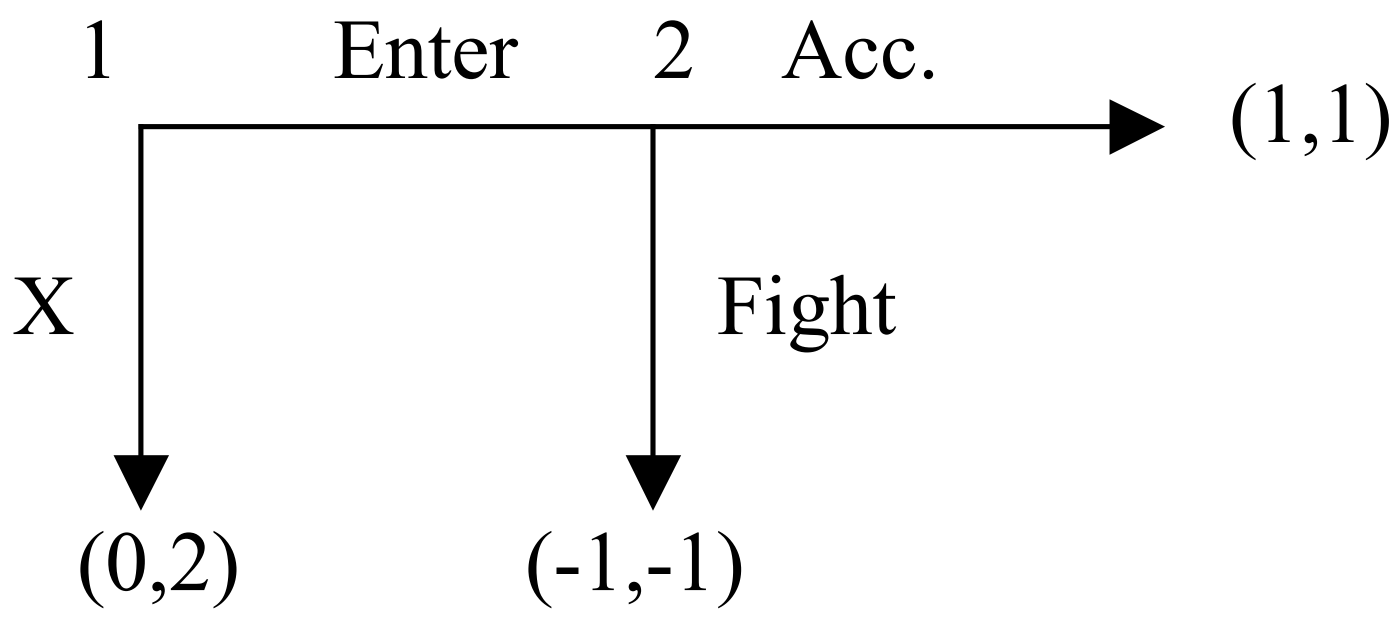
	B	S
B	3,1	0,0
S	0,0	1,3

	B	S
B	2,1	-1,0
S	-1,0	0,3

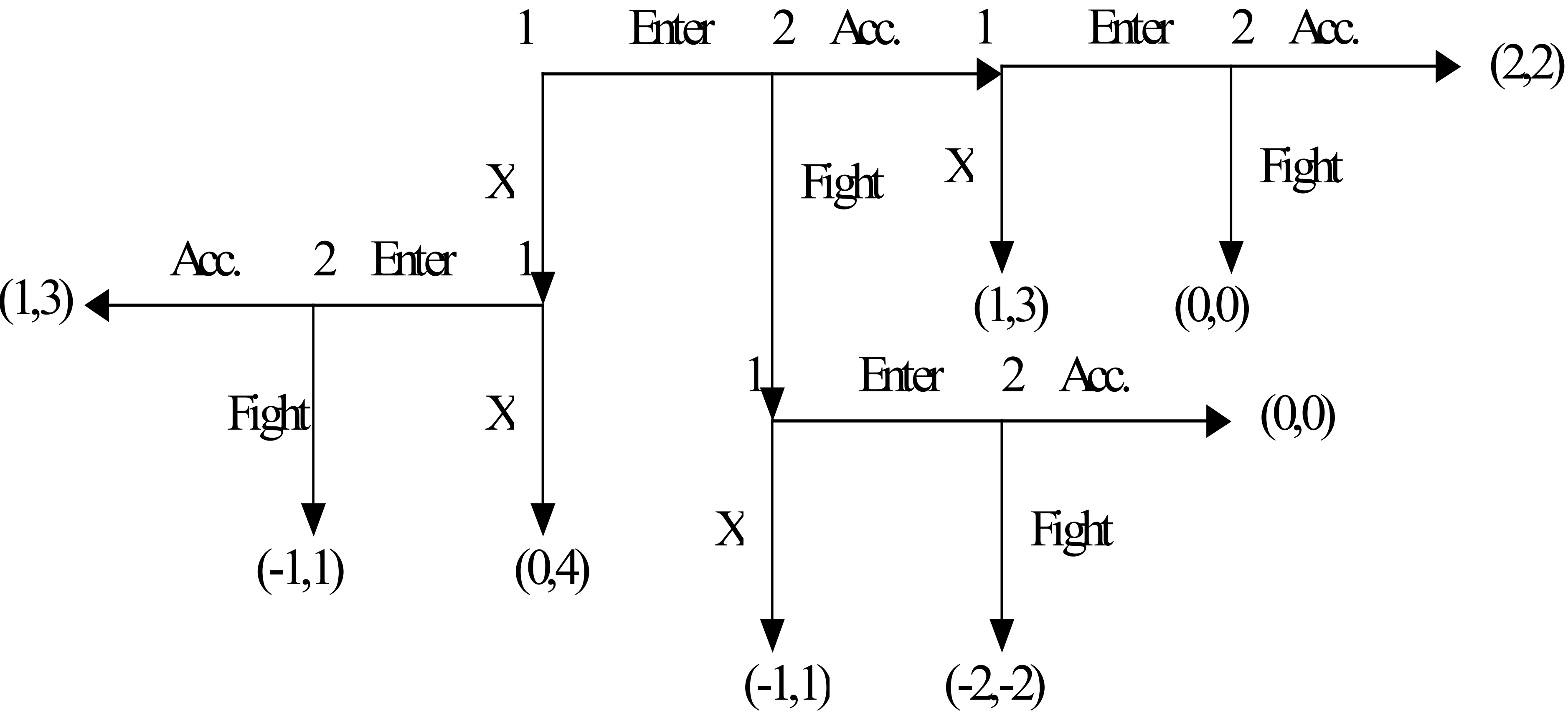
	BB	BS	SB	SS
0B				
0S				
DB				
DS				
O	T	H	E	R

Repeated Games

Entry deterrence



Entry deterrence, repeated twice, many times



What would happen if repeated n times?

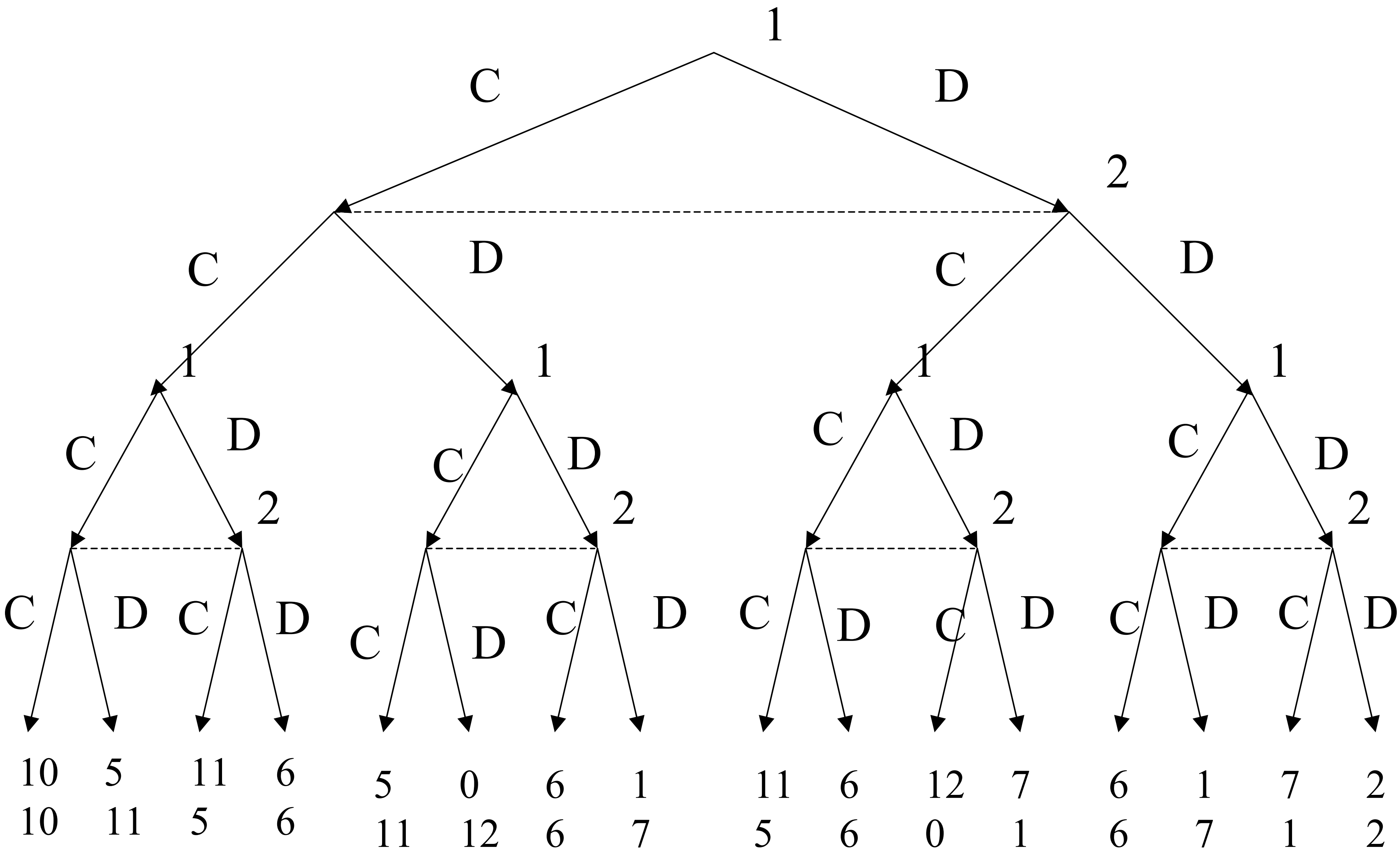
Prisoners' Dilemma, repeated twice, many times

- Two dates $T = \{0,1\}$;
- At each date the prisoners' dilemma is played:

	C	D
C	5,5	0,6
D	6,0	1,1

- At the beginning of 1 players observe the strategies at 0. Payoffs= sum of stage payoffs.

Twice-repeated PD



What would happen if T = {0,1,2,…,n}?

A general result

- G = “stage game” = a finite game
- T = {0,1,…,n}
- At each t in T, G is played, and players remember which actions taken before t;
- Payoffs = Sum of payoffs in the stage game.
- Call this game G(T).

Theorem: If G has a unique subgame-perfect equilibrium s*, G(T) has a unique subgame-perfect equilibrium, in which s* is played at each stage.

With multiple equilibria

T = {0,1}

1 \ 2	L	M2	R
T	1,1	5,0	0,0
M1	0,5	4,4	0,0
B	0,0	0,0	3,3

s* =

- At t = 0, each i play Mi;
- At t = 1, play (B,R) if (M1,M2) at t = 0, play (T,L) otherwise.

	L	M2	R
T	2,2	6,1	1,1
M1	1,6	7,7	1,1
B	1,1	1,1	4,4

Infinitely repeated Games with observable actions

- T = {0,1,2,…,t,…}
- G = "stage game" = a finite game
- At each t in T, G is played, and players remember which actions taken before t;
- Payoffs = Discounted sum of payoffs in the stage game.
- Call this game G(T).

Definitions

The *Present Value* of a given payoff stream $\pi = (\pi_0, \pi_1, \ldots, \pi_t, \ldots)$ is

$$PV(\pi;\delta) = \Sigma^{\infty}_{t=1}\, \delta^t \pi_t = \pi_0 + \delta\pi_1 + \ldots + \delta^t \pi_t + \ldots$$

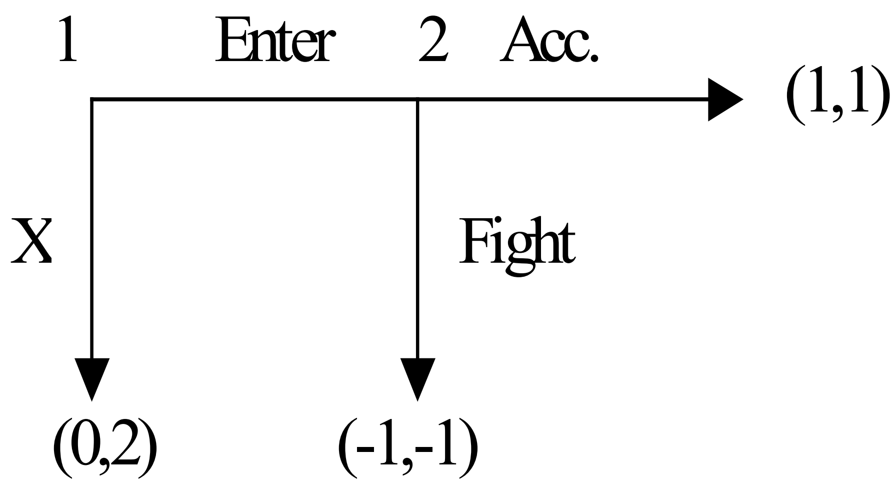
The *Average Value* of a given payoff stream π is

$$(1-\delta)PV(\pi;\delta) = (1-\delta)\Sigma^{\infty}_{t=1}\, \delta^t \pi_t$$

The *Present Value* of a given payoff stream π at t is

$$PV_t(\pi;\delta) = \Sigma^{\infty}_{s=t}\, \delta^{s-t}\, \pi_s = \pi_t + \delta\pi_{t+1} + \ldots + \delta^s \pi_{t+s} + \ldots$$

Infinite-period entry deterrence



Strategy of Entrant:

Enter iff
Accomodated before.

Strategy of Incumbent:

Accommodate iff
accomodated before.

Incumbent:

- $V(\text{Acc.}) = V_A = 1/(1-\delta)$;
- $V(\text{Fight}) = V_F = 2/(1-\delta)$;
- Case 1: Accommodated before.
  - Fight => $-1 + \delta V_A$
  - Acc. => $1 + \delta V_A$.
- Case 2: Not Accommodated
  - Fight => $-1 + \delta V_F$
  - Acc. => $1 + \delta V_A$
  - Fight $\Leftrightarrow -1 + \delta V_F \geq 1 + \delta V_A$

  $\Leftrightarrow V_F - V_A = 1/(1-\delta) \geq 2/\delta$

  $\Leftrightarrow \delta \geq 2/3$.

Entrant:

- Accommodated
  - Enter => $1+V_{AE}$
  - X => $0 + V_{AE}$
- Not Acc.
  - Enter => $-1+V_{FE}$
  - X => $0 + V_{FE}$

Infinitely-repeated PD

	C	D
C	5,5	0,6
D	6,0	1,1

A Grimm Strategy: Defect iff someone defected before.

- $V_D = 1/(1-\delta)$;
- $V_C = 5/(1-\delta) = 5V_D$;
- Defected before (easy)
- Not defected
  - D =>
  - C =>
  - C ⇔

Tit for Tat

- Start with C; thereafter, play what the other player played in the previous round.
- Is (Tit-for-tat,Tit-for-tat) a SPE?

- **Modified:** Start with C; if any player plays D when the previous play is (C,C), play D in the next period, then switch back to C.

Folk Theorem

Definition: A payoff vector $v = (v_1, v_2, \ldots, v_n)$ is feasible iff v is a convex combination of some pure-strategy payoff-vectors, i.e.,

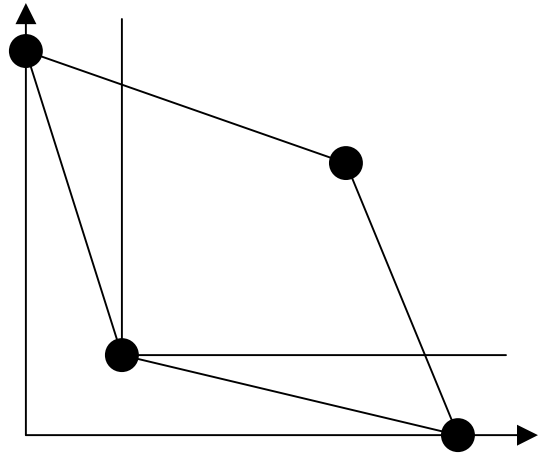
$$v = p_1 u(a^1) + p_2 u(a^2) + \ldots + p_k u(a^k),$$

where $p_1 + p_2 + \ldots + p_k = 1$, and $u(a^j)$ is the payoff vector at strategy profile a^j of the stage game.

Theorem: Let $x = (x_1, x_2, \ldots, x_n)$ be s feasible payoff vector, and $e = (e_1, e_2, \ldots, e_n)$ be a payoff vector at some equilibrium of the stage game such that $x_i > e_i$ for each i. Then, there exist $\underline{\delta} < 1$ and a strategy profile s such that s yields x as the expected average-payoff vector and is a SPE whenever $\delta > \underline{\delta}$.

Folk Theorem in PD

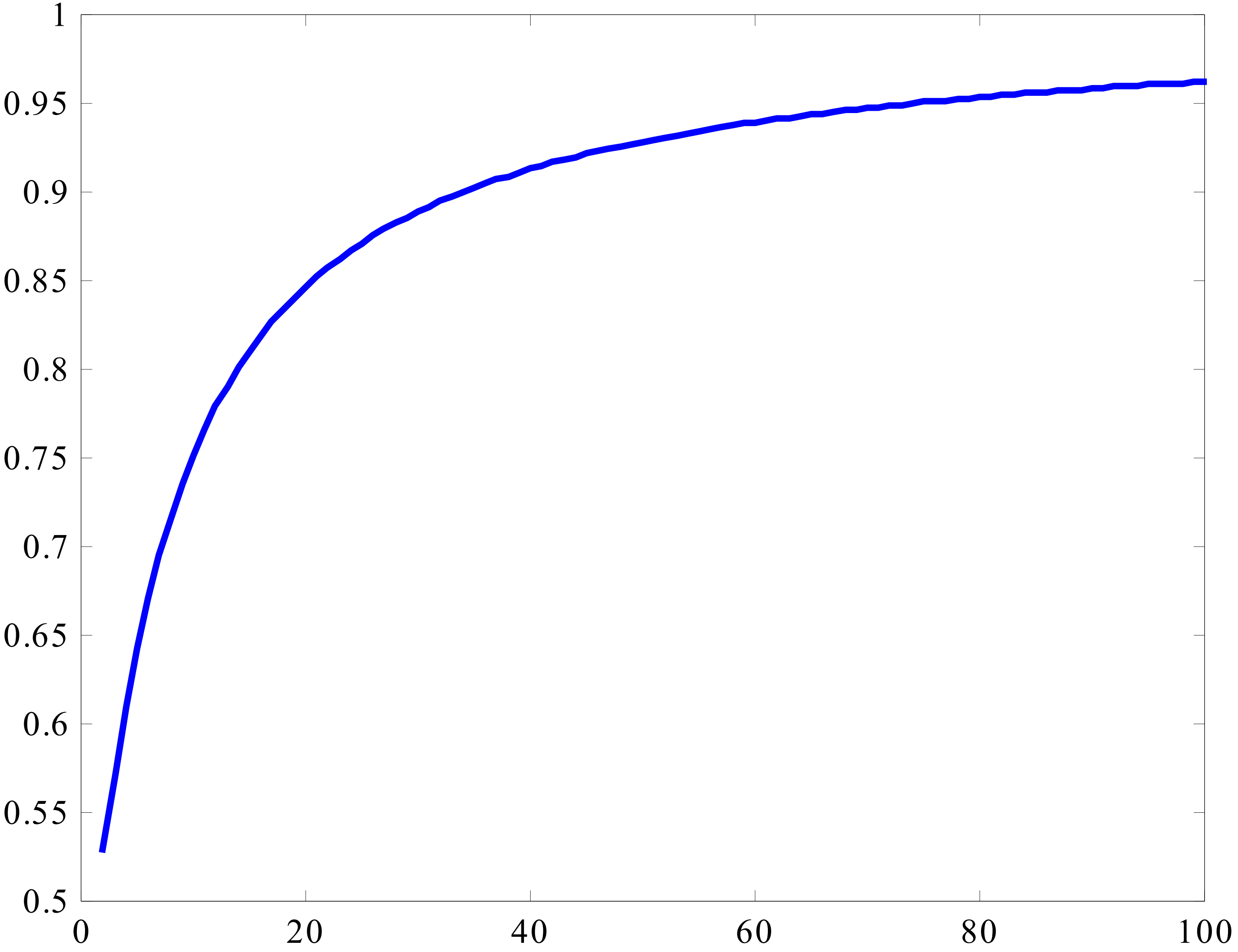
	C	D
C	5,5	0,6
D	6,0	1,1

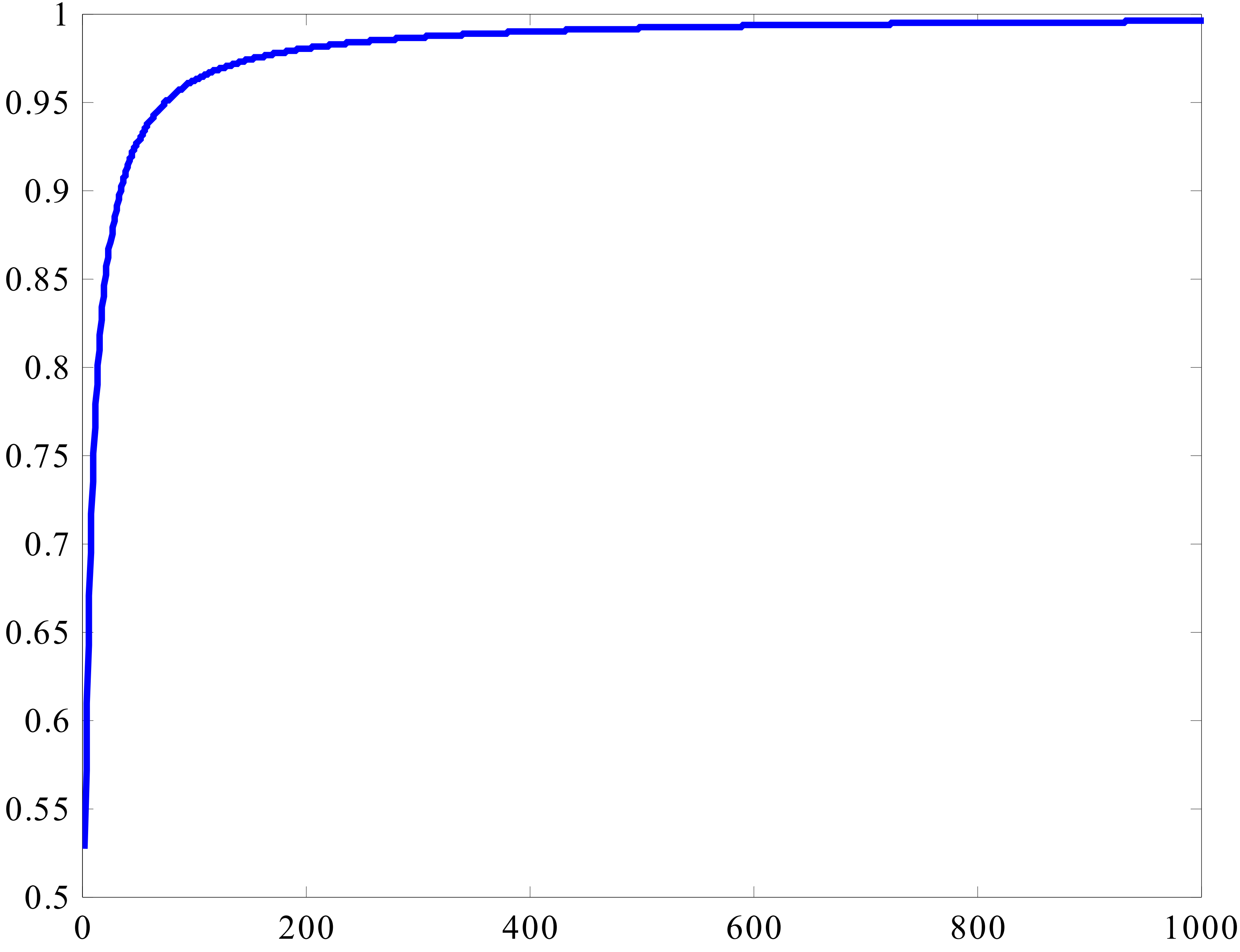


- A SPE with PV (1.1,1.1)?
  - With PV (1.1,5)?
  - With PV (6,0)?
  - With PV (5.9,0.1)?

Infinitely-repeated Cournot oligopoly

- N firms, MC = 0; P = max{1-Q,0};
- Strategy: Each is to produce q = 1/(2n); if any firm defects produce q = 1/(1+n) forever.
- $V_C =$
- $V_D =$
- $V(D|C) =$
- Equilibrium $\Leftrightarrow$


1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
0
20
40
60
80
100



1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
0
200
400
600
800
1000

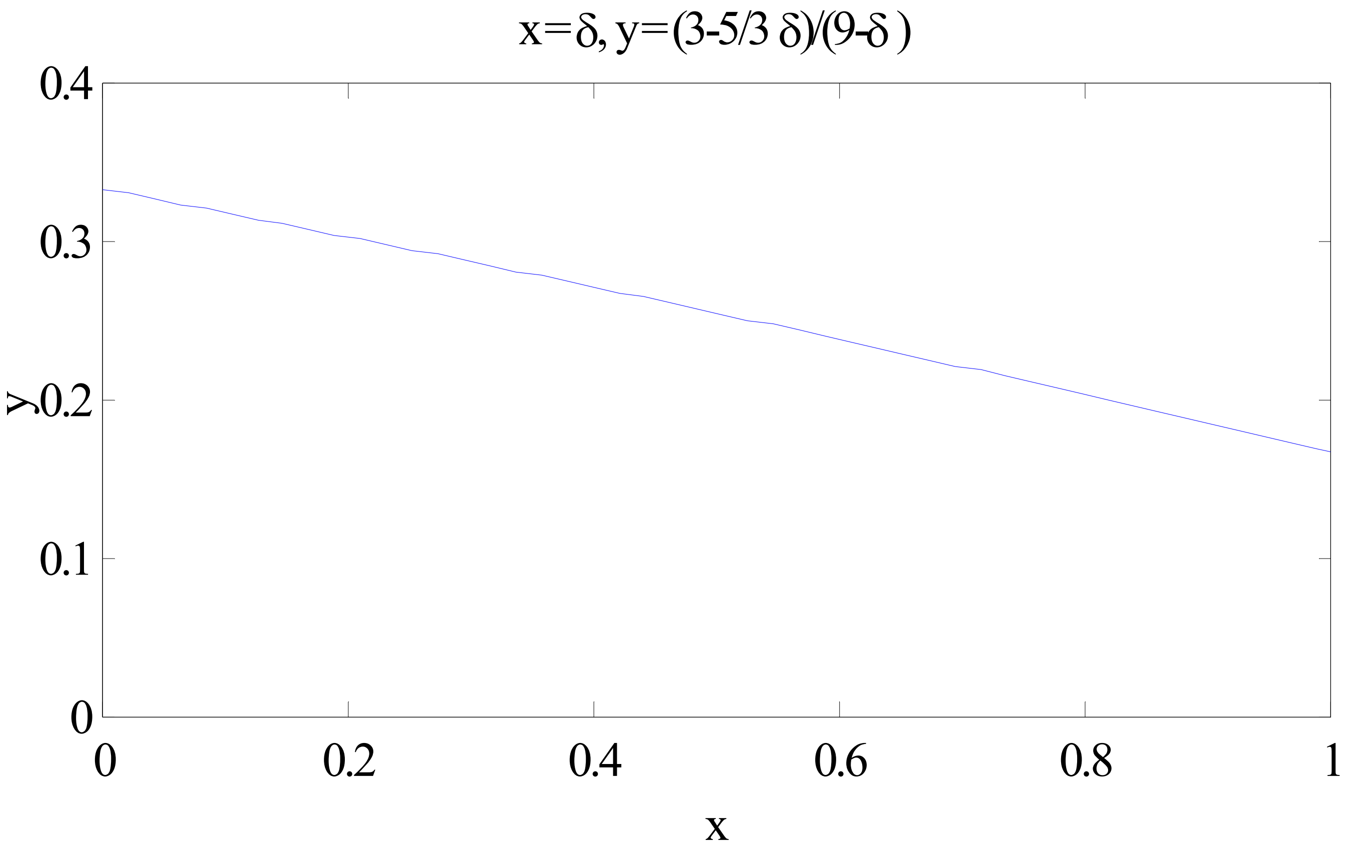
IRCD (n=2)

- Strategy: Each firm is to produce q*; if any one deviates, each produce 1/(n+1) thereafter.
- $V_C = q^*(1-2q^*)/(1-\delta)$;
- $V_D = 1/(9(1-\delta))$;
- $V_{D|C} = \max q(1-q^*-q) + \delta V_D = (1-q^*)^2/4 + \dfrac{\delta}{9(1-\delta)}$
- Equilibrium iff

$$q^*(1-2q^*) \geq (1-\delta)(1-q^*)^2/4 + \delta/9$$

- $\Leftrightarrow$

$$q^* \geq \frac{9-5\delta}{3(9-\delta)}$$



x=δ, y=(3-5/3 δ)/(9-δ)
0.4
0.3
0.2
0.1
0
y
0
0.2
0.4
0.6
0.8
1
x

Carrot and Stick

Produce ¼ at the beginning; at ant $t > 0$, produce ¼ if both produced ¼ or both produced x at t-1; otherwise, produce x.

Two Phase: Cartel & Punishment

$V_C = 1/8(1-\delta)$. $V_x = x(1-2x) + \delta V_C$.

$V_{D|C} = \max q(1-1/4-q) + \delta V_X = (3/8)^2 + \delta V_X$

$V_{D|x} = \max q(1-x-q) + \delta V_X = (1-x)^2/4 + \delta V_X$

$V_C \geq V_{D|C} \Leftrightarrow V_C \geq (3/8)^2 + \delta^2 V_C + \delta\, x(1-2x)$

$\Leftrightarrow (1-\delta^2)\, V_C - (3/8)^2 \geq \delta\, x(1-2x) \Leftrightarrow (1+\delta)/8 - (3/8)^2 \geq \delta\, x(1-2x)$

$V_X \geq V_{D|C} \Leftrightarrow (1-\delta)V_x \geq (1-x)^2/4 \Leftrightarrow (1-\delta)(x(1-2x) + \delta/8(1-\delta)) \geq (1-x)^2/4$
$\Leftrightarrow (1-\delta)x(1-2x) + \delta/8 \geq (1-x)^2/4$

$$2x^2 - x + 1/8 - 9/64\delta \geq 0$$

$$(9/4-2\delta)x^2 - (3-2\delta)x + \delta/8(1-\delta) \leq 0$$

Lectures 12-13

Incomplete Information
Static Case

14.12 Game Theory

Muhamet Yildiz

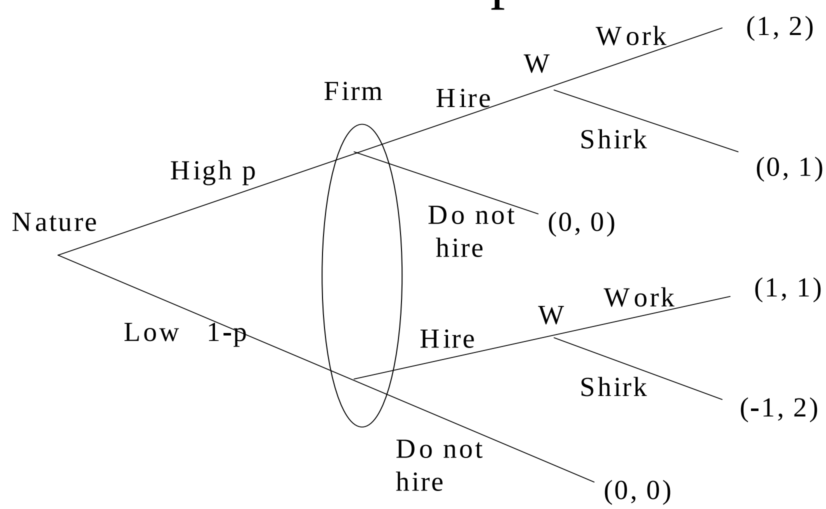
Road Map

1. Examples
2. Bayes' rule
3. Definitions
   1. Bayesian Game
   2. Bayesian Nash Equilibrium
4. Mixed strategies, revisited
5. Economic Applications
   1. Cournot Duopoly
   2. Auctions
   3. Double Auction

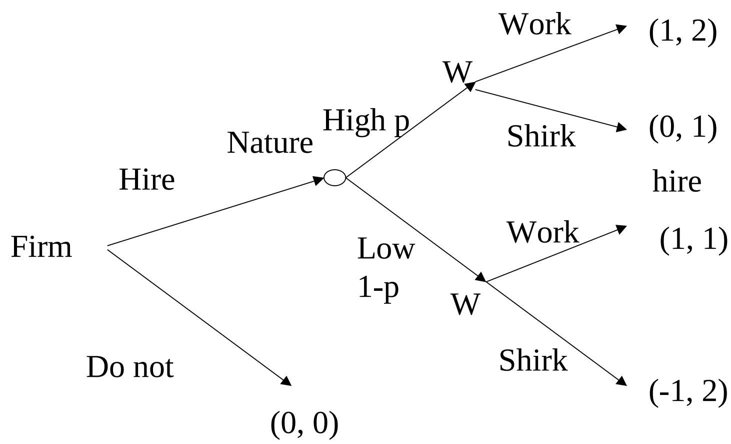
Incomplete information

We have incomplete (or asymmetric) information if one player knows something (relevant) that some other player does not know.

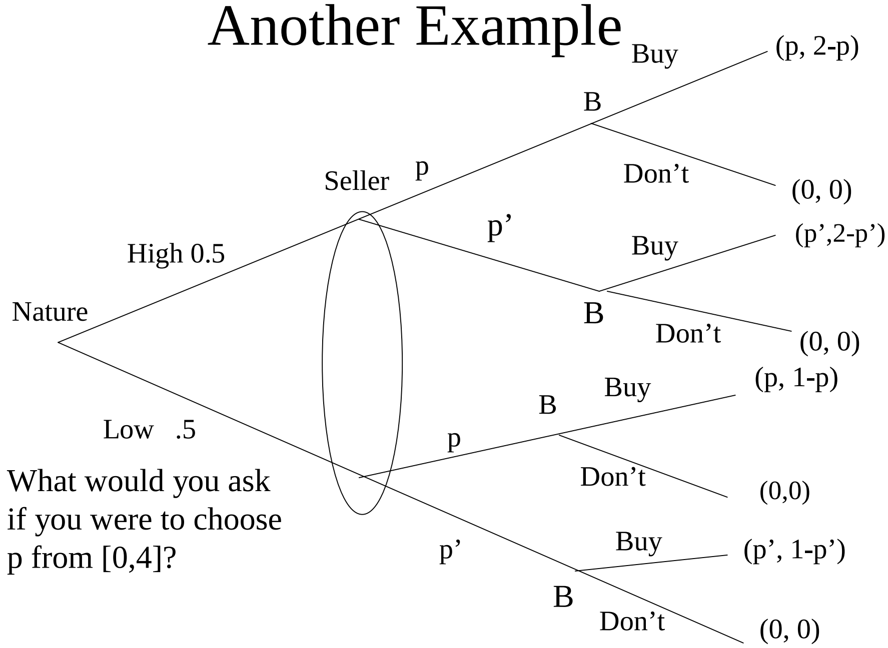
An Example



The same example

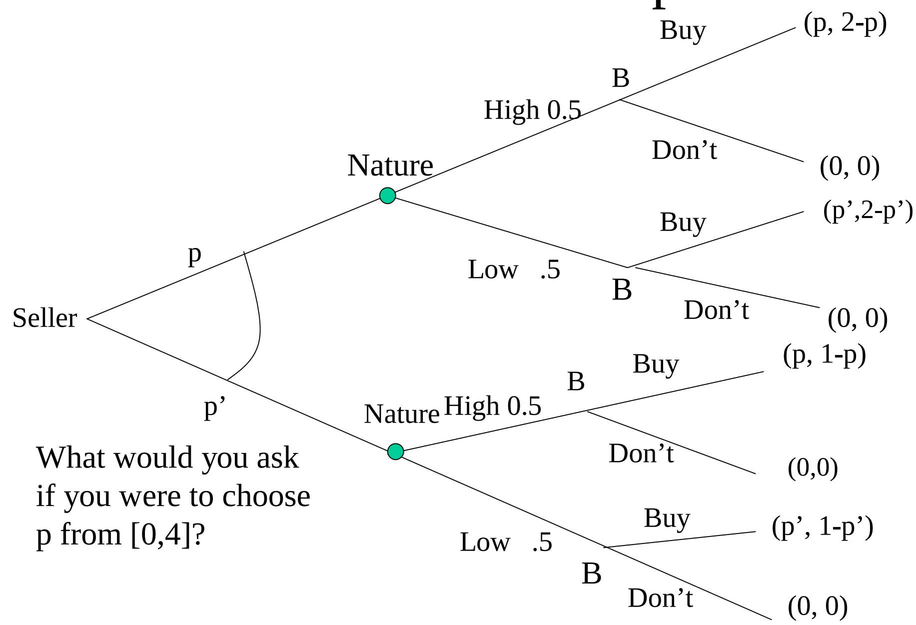


Another Example



What would you ask if you were to choose p from [0,4]?

Same “Another Example”

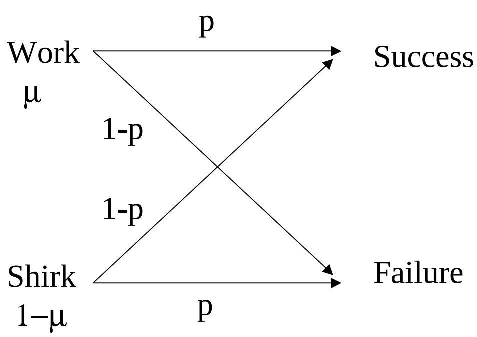


What would you ask if you were to choose p from [0,4]?

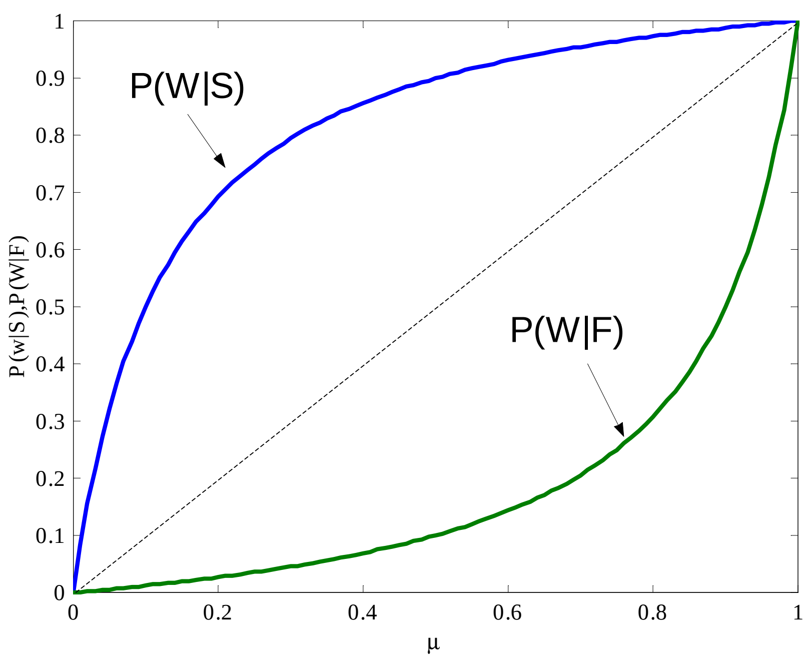
Bayes’ Rule

- $\text{Prob}(A|B) = \dfrac{\text{Prob}(A \text{ and } B)}{\text{Prob}(B)}$
- $\text{Prob}(A \text{ and } B) = \text{Prob}(A|B)\text{Prob}(B) = \text{Prob}(B|A)\text{Prob}(A)$
- $\text{Prob}(A|B) = \dfrac{\text{Prob}(B|A)\text{Prob}(A)}{\text{Prob}(B)}$

Example



- Prob(Work|Success) = $\mu p/[\mu p + (1-\mu)(1\text{-}p)]$
- Prob(Work|Failure) = $(1\text{-}\mu)p/[\mu(1-p) + (1-\mu)p]$



Bayesian Game (Normal Form)

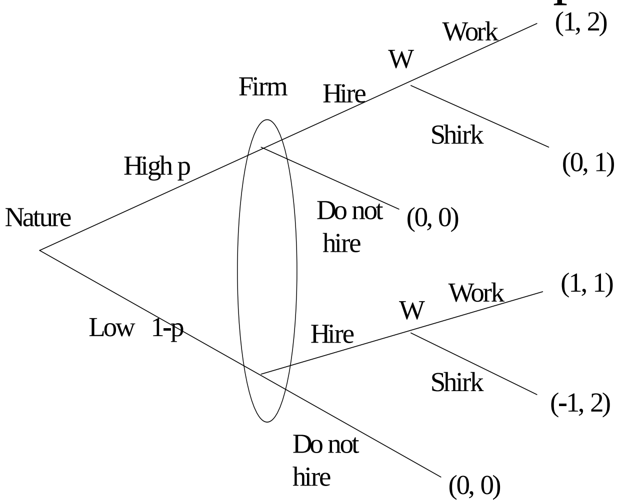
A Bayesian game is a list

$G = \{A_1,\ldots,A_n;T_1,\ldots,T_n;p_1,\ldots,p_n;u_1,\ldots,u_n\}$

where

- A_i is the action space of i (a_i in A_i)
- T_i is the type space of i (t_i)
- $p_i(t_{-i}|t_i)$ is i's belief about the other players
- $u_i(a_1,\ldots,a_n;t_1,\ldots,t_n)$ is i's payoff.

An Example



$T_{Firm}=\{t_f\}$;
$T_W = \{High,Low\}$
$A_{Firm} = \{Hire, Don't\}$
$A_W = \{Work,Shirk\}$
$p_F(High) = p$
$p_F(Low) = 1\text{-}p$

Bayesian Nash equilibrium

A Bayesian Nash equilibrium is a Nash equilibrium of a Bayesian game.

Given any Bayesian game G = $\{A_1,\ldots,A_n;T_1,\ldots,T_n;p_1,\ldots,p_n;u_1,\ldots,u_n\}$

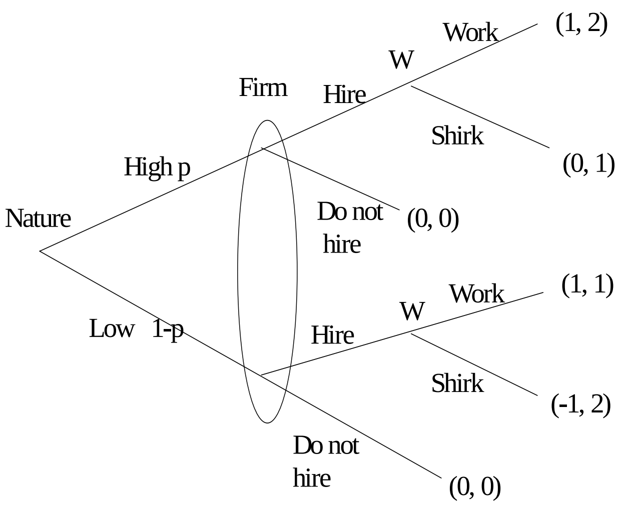
a **strategy** of a player i in a is any function $s_i{:}T_i \rightarrow A_i$;

A strategy profile $s^* = (s_1^*,\ldots, s_1^*)$ is a **Bayesian Nash equilibrium** iff $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i\left(s_1^*(t_1),\ldots, s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}),\ldots, s_n^*(t_n); t\right) p_i(t_{-i} \mid t_i)$$

i.e., s_i^* is a best response to s_{-i}^*.

An Example

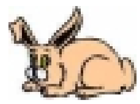


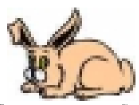
$T_{Firm}=\{t_f\}$;
T_W = {High,Low}
A_{Firm} = {Hire, Don't}
A_W = {Work,Shirk}
p_F(High) = p >1/2
p_F(Low) = 1-p

s_F* = Hire,
s_F* (High) = Work
s_F* (Low) = Shirk

Another equilibrium?

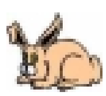
Stag Hunt, Mixed Strategy




(2,2)	(4,0)
(0,4)	(6,6)

Mixed Strategies





2+t,2+v	4+t,0
0,4+v	6,6

$U_1(R|t) = t + 2q + 4(1-q) = t + 4 - 2q$
$U_1(S|t) = 6(1-q);$
$U_1(R|t) > U_1(S|t) \Leftrightarrow t + 4 - 2q > 6(1-q)$
$\Leftrightarrow t > 6 - 6q + 2q - 4 = 2 - 4q = 0.$

- t and v are iid with uniform distribution on $[-\varepsilon,\varepsilon]$.
- t and v are privately known by 1 and 2, respectively, i.e., are types of 1 and 2, respectively.
- Pure strategy:
  - $s_1(t)$ = Rabbit iff $t > 0$;
  - $s_2(v)$ = Rabbit iff $t > 0$.
- $p = \text{Prob}(s_1(t)=\text{Rabbit}|v) = \text{Prob}(t > 0) = 1/2.$
- $q = \text{Prob}(s_2(v)=\text{Rabbit}|t) = 1/2.$

Economic Applications with Incomplete Information

14.12 Game Theory
Muhamet Yildiz

Road Map

1. Cournot duopoly with Incomplete Information
2. A first price auction
3. A double auction
4. Quiz

1 Cournot duopoly with Incomplete Information

- Demand:
$$P(Q) = a - Q$$
where $Q = q_1 + q_2$.

- The marginal cost of Firm 1 = c; common knowledge.

- Firm 2's marginal cost:
$$\begin{array}{ll} c_H & \text{with probaility } \theta, \\ c_L & \text{with probaility } 1-\theta; \end{array}$$
its private information.

- Each firm maximizes its expected profit.

1.1 Bayesian Nash Equilibrium

Firm 2 of high type:

$\max_{q_2}(P - c_H)q_2 = \max_{q_2}\left[a - q_1^* - q_2 - c_H\right]q_2.$

$$q_2^*(c_H) = \frac{a - q_1^* - c_H}{2} \qquad (*)$$

Firm 2 of low type:

$$\max_{q_2}\left[a - q_1^* - q_2 - c_L\right]q_2,$$

$$q_2^*(c_L) = \frac{a - q_1^* - c_L}{2}. \qquad (**)$$

Firm 1:

$$\max_{q_1}\theta\left[a - q_1 - q_2^*(c_H) - c\right]q_1 + (1-\theta)\left[a - q_1 - q_2^*(c_L) - c\right]q_1$$

$$q_1^* = \frac{\theta\left[a - q_2^*(c_H) - c\right] + (1-\theta)\left[a - q_2^*(c_L) - c\right]}{2}$$

(***)

Solve *, **, and *** for $q_1^*, q_2^*(c_L), q_2^*(c_H)$.

$$\begin{pmatrix} q_1^* \\ q_2^*(c_H) \\ q_2^*(c_L) \end{pmatrix} = \begin{bmatrix} 2 & \theta & 1-\theta \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}^{-1} \begin{pmatrix} a-c \\ a-c_H \\ a-c_L \end{pmatrix};$$

$$\begin{aligned} q_2^*(c_H) &= \frac{a-2c_H+c}{3} + \frac{(1-\theta)(c_H-c_L)}{6} \\ q_2^*(c_L) &= \frac{a-2c_L+c}{3} - \frac{\theta(c_H-c_L)}{6} \\ q_1^* &= \frac{a-2c+\theta c_H+(1-\theta)c_L}{3} \end{aligned}$$

2 A First-price Auction

- One object, two bidders
- $v_i =$ Valuation of bidder i; iid with uniform distribution over $[0, 1]$.
- Simultaneously, each bidder i submits a bid b_i, then the highest bidder wins the object and pays her bid.
- The payoffs:

$$u_i(b_1, b_2, v_1, v_2) = \begin{cases} v_i - b_i & \text{if } b_i > b_j, \\ \frac{v_i - b_i}{2} & \text{if } b_i = b_j, \\ 0 & \text{if } b_i < b_j. \end{cases}$$

- Objective of i:

$$\max_{b_i} E\left[u_i(b_1, b_2, v_1, v_2)\right]$$

where

$$\begin{aligned} E\left[u_i\right] = (v_i - b_i) \Pr\{b_i > b_j(v_j)\} \\ + \frac{1}{2}(v_i - b_i) \Pr\{b_i = b_j(v_j)\} \end{aligned}$$

2.1 Symmetric, linear equilibrium

$$b_j = a + cv_j.$$

Then,

$$\Pr\{b_i = b_j(v_j)\} = 0.$$

Equilibrium condition: $a \leq b_i \leq v_i$.

Hence,

$$\begin{aligned} E[u_i] &= (v_i - b_i)\Pr\{b_i \geq a + cv_j\} \\ &= (v_i - b_i)\Pr\{v_j \leq \frac{b_i - a}{c}\} \\ &= (v_i - b_i) \cdot \frac{b_i - a}{c}. \end{aligned}$$

FOC:

$$b_i = \begin{cases} \frac{v_i + a}{2} & \text{if } v_i \geq a \\ a & \text{if } v_i < a. \end{cases} \tag{1}$$

Therefore,

$$b_i = \frac{1}{2}v_i.$$

2.2 Any symmetric equilibrium

$$b_i = b(v_i)$$

Hence,

$$\begin{aligned} E[u_i] &= (v_i - b_i)\Pr\{b_i \geq b(v_j)\} \\ &= (v_i - b_i)\Pr\{v_j \leq b^{-1}(b_i)\} \\ &= (v_i - b_i)b^{-1}(b_i). \end{aligned}$$

FOC ($\partial/\partial b_i = 0$):

$$\begin{aligned} -b^{-1}(b_i) + (v_i - b_i)\frac{db^{-1}(b_i)}{db_i} &= 0 \\ -v_i + (v_i - b(v_i))\frac{1}{b'(v_i)} &= 0 \\ b'(v_i)\,v_i + b(v_i) &= v_i \\ \frac{d[b(v_i)\,v_i]}{dv_i} &= v_i \end{aligned}$$

$$\begin{aligned} b(v_i)\,v_i &= v_i^2/2 + const. \\ b(v_i) &= v_i/2 + const/v_i. \end{aligned}$$

$b(0) = 0$, => cons = 0.

$$b(v_i) = v_i/2.$$

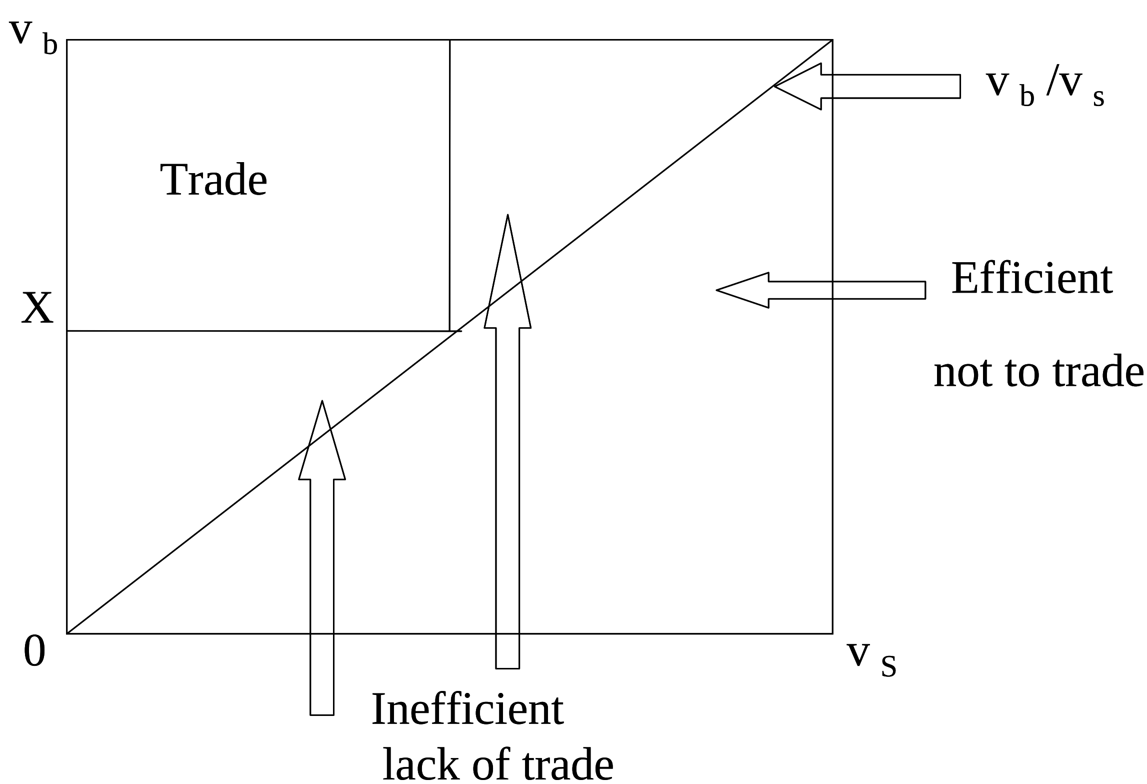
3 Double Auction

1. Simultaneously, Seller names p_s and Buyer names p_b.
   a. If $p_b < p_s$, then no trade;
   b. if $p_b \geq p_s$, trade at price $p = \frac{p_b + p_s}{2}$.
2. Valuations are private information: v_b, v_s iid w/ uniform on $[0, 1]$.
3. Payoffs:
$$u_b = \begin{cases} v_b - \frac{p_b+p_s}{2} & \text{if } p_b \geq p_s \\ 0 & \text{otherwise} \end{cases}$$
$$u_s = \begin{cases} \frac{p_b+p_s}{2} - v_s & \text{if } p_b \geq p_s \\ 0 & \text{otherwise} \end{cases}$$
4. The buyer's problem:
$$\max_{p_b} E\left[v_b - \frac{p_b + p_s(v_s)}{2} : p_b \geq p_s(v_s)\right].$$
5. The seller's problem:
$$\max_{p_s} E\left[\frac{p_s + p_b(v_b)}{2} - v_s : p_b(v_b) \geq p_s\right].$$

An Equilibrium:

$$p_b = \begin{cases} X & \text{if } v_b \geq X \\ 0 & \text{otherwise} \end{cases},$$

$$p_s = \begin{cases} X & \text{if } v_s \leq X \\ 1 & \text{otherwise} \end{cases}.$$



Equilibrium with linear strategies:

$$
\begin{aligned}
p_b &= a_b + c_b v_b \\
p_s &= a_s + c_s v_s .
\end{aligned}
$$

$$p_b \geq p_s (v_s) = a_s + c_s v_s \iff v_s \leq \frac{p_b - a_s}{c_s}.$$

$$p_s \leq p_b (v_b) = a_b + c_b v_b \iff v_b \geq \frac{p_s - a_b}{c_b}.$$

$$
\begin{aligned}
E\left[u_b\right] &= E\left[v_b - \frac{p_b + p_s(v_s)}{2} : p_b \geq p_s(v_s)\right] \\
&= \int_0^{\frac{p_b - a_s}{c_s}} \left[v_b - \frac{p_b + p_s(v_s)}{2}\right] dv_s \\
&= \int_0^{\frac{p_b - a_s}{c_s}} \left[v_b - \frac{p_b + a_s + c_s v_s}{2}\right] dv_s \\
&= \frac{p_b - a_s}{c_s}\left(v_b - \frac{p_b + a_s}{2}\right) - \frac{c_s}{2}\int_0^{\frac{p_b - a_s}{c_s}} v_s dv_s \\
&= \frac{p_b - a_s}{c_s}\left(v_b - \frac{p_b + a_s}{2}\right) - \frac{c_s}{4}\left(\frac{p_b - a_s}{c_s}\right)^2 \\
&= \frac{p_b - a_s}{c_s}\left(v_b - \frac{p_b + a_s}{2} - \frac{p_b - a_s}{4}\right) \\
&= \frac{p_b - a_s}{c_s}\left(v_b - \frac{3p_b + a_s}{4}\right).
\end{aligned}
$$

F.O.C. ($\max_{p_b} E\left[u_b\right]$):

$$\frac{1}{c_s}\left(v_b - \frac{3p_b + a_s}{4}\right) - \frac{3(p_b - a_s)}{4c_s} = 0$$

i.e.,

$$p_b = \frac{2}{3}v_b + \frac{1}{3}a_s. \qquad (2)$$

Similarly,

$$
\begin{aligned}
E\left[u_s\right] &= E\left[\frac{p_s + p_b(v_b)}{2} - v_s : p_b(v_b) \geq p_s\right] \\
&= \int_{\frac{p_s - a_b}{c_b}}^{1} \left[\frac{p_s + a_b + c_b v_b}{2} - v_s\right] dv_b \\
&= \left(1 - \frac{p_s - a_b}{c_b}\right) \left[\frac{p_s + a_b}{2} - v_s\right] + \frac{c_b}{2} \int_{\frac{p_s - a_b}{c_b}}^{1} v_b dv_b \\
&= \left(1 - \frac{p_s - a_b}{c_b}\right) \left[\frac{p_s + a_b}{2} - v_s\right] \\
&\quad + \frac{c_b}{4} \left(1 - \left(\frac{p_s - a_b}{c_b}\right)^2\right) \\
&= \left(1 - \frac{p_s - a_b}{c_b}\right) \left[\frac{p_s + a_b}{2} - v_s + \frac{c_b}{4} + \frac{p_s - a_b}{4}\right] \\
&= \left(1 - \frac{p_s - a_b}{c_b}\right) \left[\frac{3p_s + a_b}{4} - v_s + \frac{c_b}{4}\right]
\end{aligned}
$$

F.O.C. ($\max_{p_s} E[u_s]$):

$$-\frac{1}{c_b}\left[\frac{3p_s + a_b}{4} - v_s + \frac{1}{4}\right] + \frac{3}{4}\left(1 - \frac{p_s - a_b}{c_b}\right) = 0$$

$$-\left[\frac{3p_s + a_b}{4} - v_s + \frac{c_b}{4}\right] + \frac{3}{4}\left(c_b - (p_s - a_b)\right) = 0,$$

$$\frac{3p_s}{2} = -\frac{a_b}{4} + v_s - \frac{c_b}{4} + \frac{3}{4}(c_b + a_b) = v_s + \frac{a_b + c_b}{2}$$

i.e.,

$$p_s = \frac{2}{3}v_s + \frac{a_b + c_b}{3}. \tag{3}$$

- $a_b = a_s/3$
- $a_s = a_b/3 + 2/9$
- Hence, $9a_s = a_s + 2$,
- $a_s = 1/4$; $a_b = 1/12$.

$$p_b = \frac{2}{3}v_b + \frac{1}{12} \tag{4}$$

$$p_s = \frac{2}{3}v_s + \frac{1}{4}. \tag{5}$$

We have trade iff

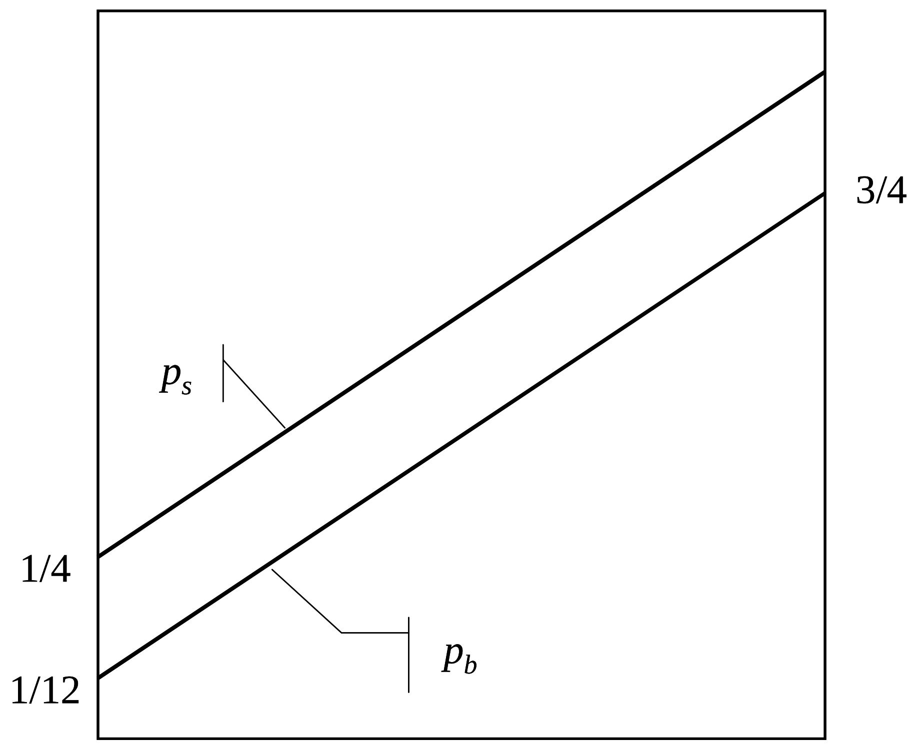
$$p_b \geq p_s$$

iff

$$\frac{2}{3}v_b + \frac{1}{12} \geq \frac{2}{3}v_s + \frac{1}{4}$$

iff

$$v_b - v_s \geq \frac{3}{2}\left(\frac{1}{4} - \frac{1}{12}\right) = \frac{3}{2}\frac{1}{6} = \frac{1}{4}.$$



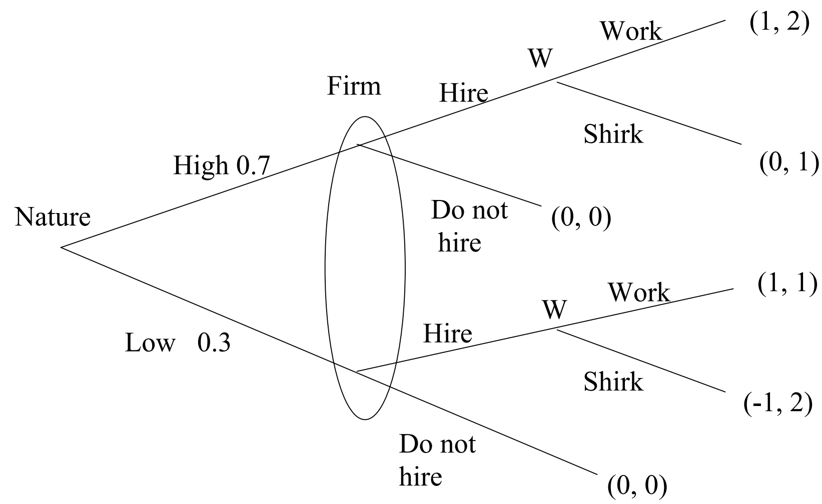
Lectures 15-18
Dynamic Games with Incomplete Information

14.12 Game Theory
Muhamet Yildiz

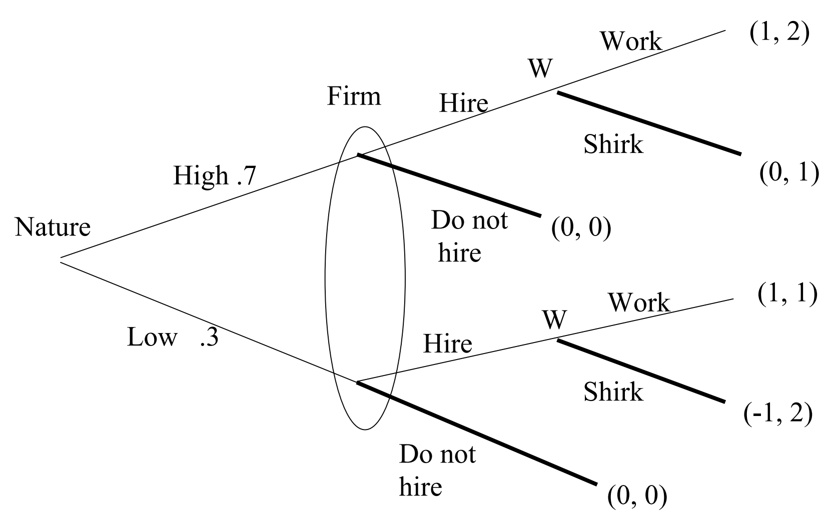
Road Map

1. Examples
2. Sequential Rationality
3. Perfect Bayesian Nash Equilibrium
4. Economic Applications
   1. Sequential Bargaining with incomplete information
   2. Reputation

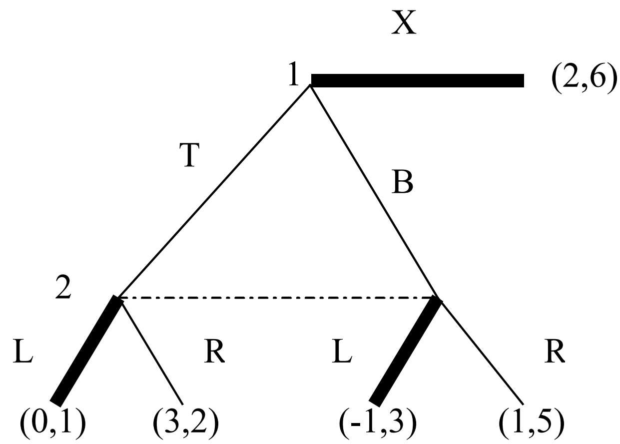
An Example



What is wrong with this equilibrium?

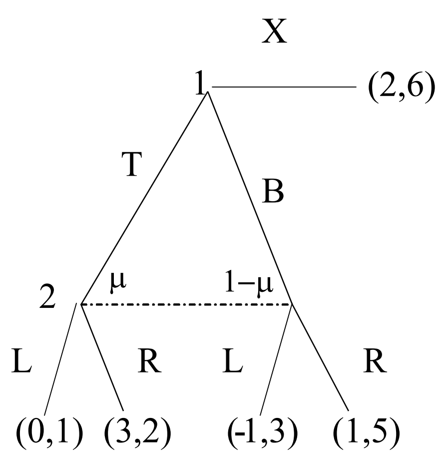


What is wrong with this equilibrium?



Beliefs

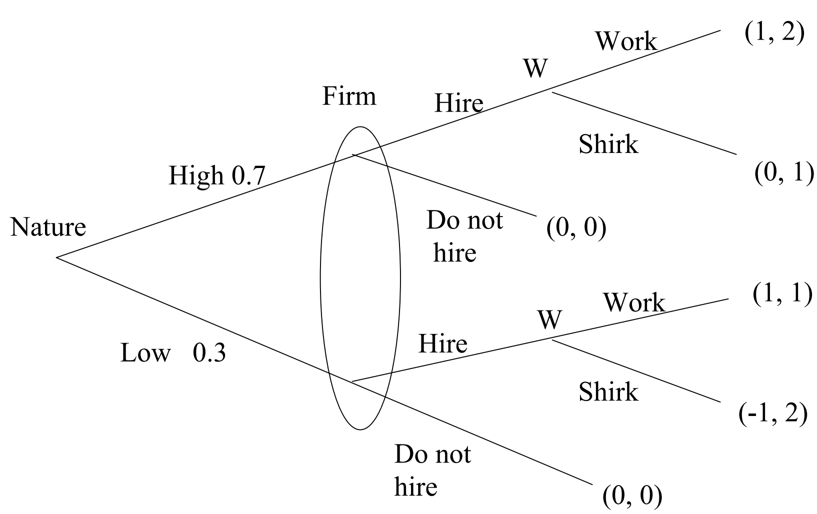
- Beliefs of an agent at a given information set is a probability distribution on the information set.
- For each information set, we must specify the beliefs of the agent who moves at that information set.



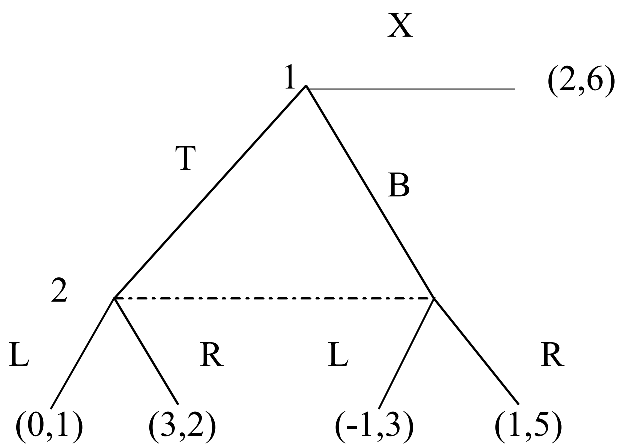
Sequential Rationality

A player is said to be **sequentially rational** iff, at each information set he is to move, <u>he maximizes his expected utility given his beliefs at the information set</u> (and given that he is at the information set) – even if this information set is precluded by his own strategy.

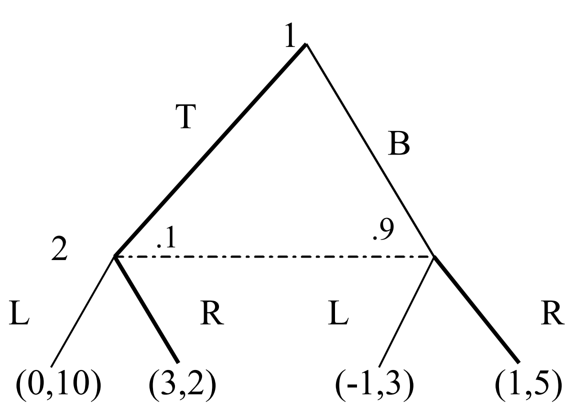
An Example



Another example



Example

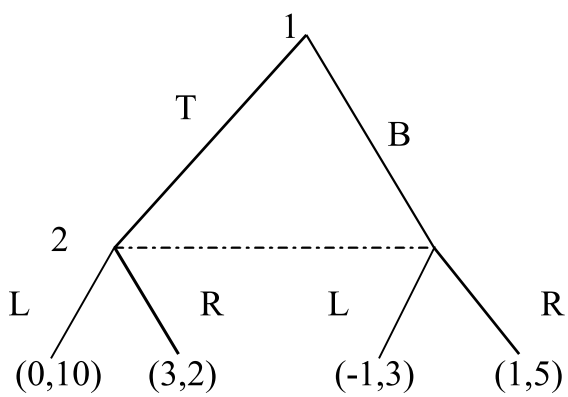


"Consistency"

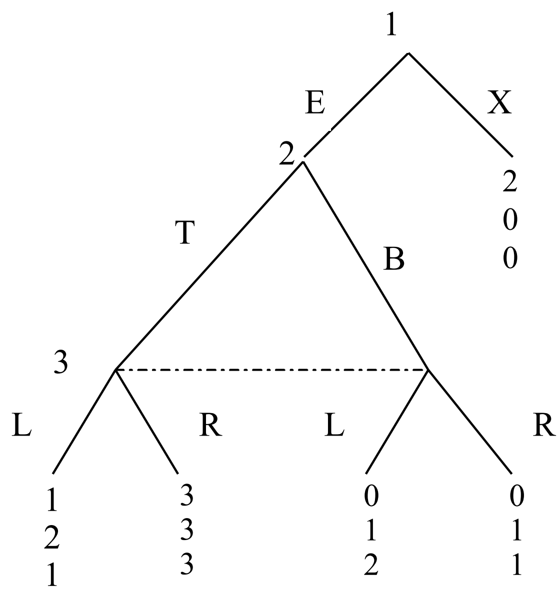
Definition: Given any (possibly mixed) strategy profile s, an information set is said to be **on the path of play** iff the information set is reached with positive probability if players stick to s.

Definition: Given any strategy profile s and any information set I on the path of play of s, a player's beliefs at I is said to be **consistent** with s iff the beliefs are derived using the Bayes' rule and s.

Example



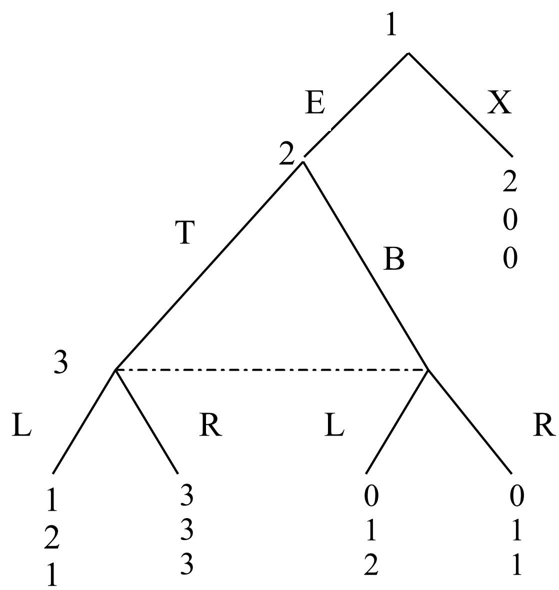
Example



"Consistency"

- Given s and an information set I, even if I is off the path of play, the beliefs must be derived using the Bayes' rule and s "whenever possible," e.g., if players tremble with very small probability so that I is on the path, the beliefs must be very close to the ones derived using the Bayes' rule and s.

Example



Perfect Bayesian Nash Equilibrium

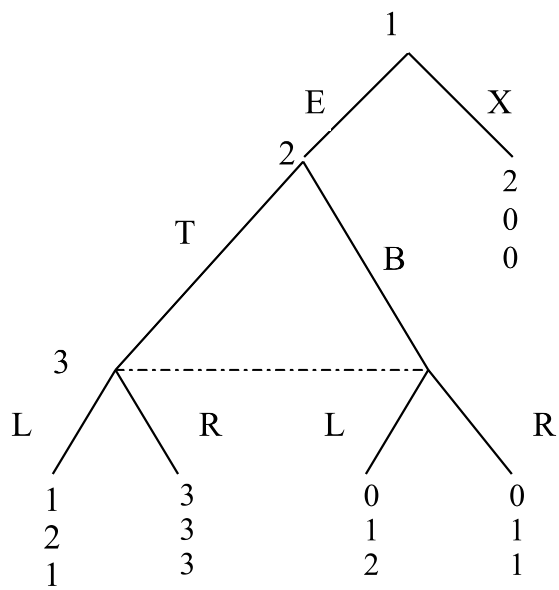
A Perfect Bayesian Nash Equilibrium is a pair (s,b) of strategy profile and a set of beliefs such that

1. Strategy profile s is sequentially rational given beliefs b, and
2. Beliefs b are consistent with s.

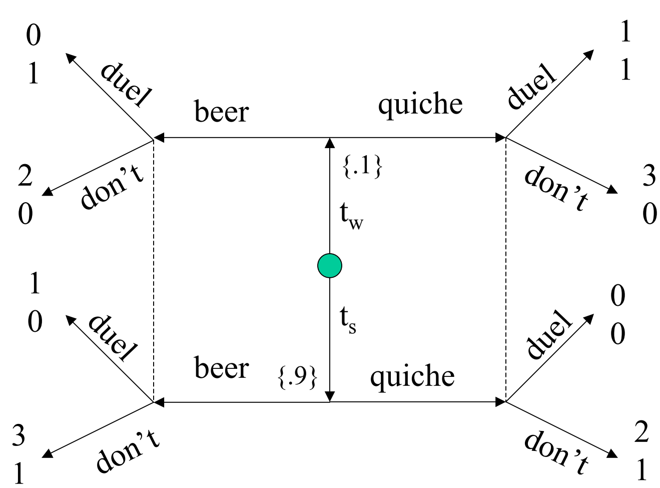
Nash $\longrightarrow$ Subgame-perfect

Bayesian Nash $\longrightarrow$ Perfect Bayesian

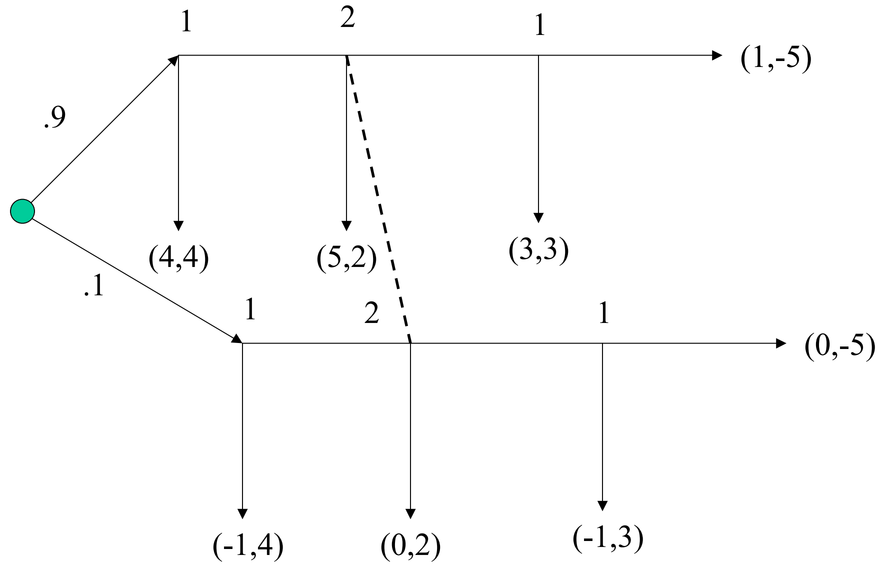
Example



Beer – Quiche



Example

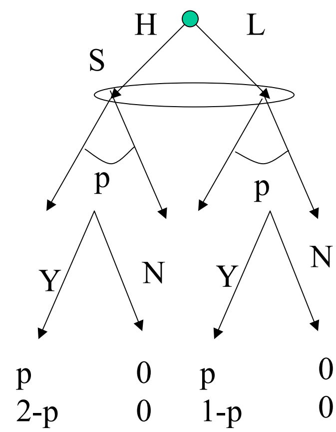


Sequential Bargaining

1. 1-period bargaining – 2 types
2. 2-period bargaining – 2 types
3. 1-period bargaining – continuum
4. 2-period bargaining – continuum

Sequential bargaining 1-p

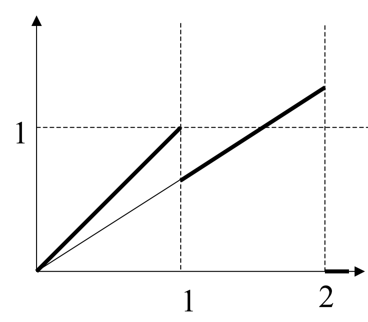
- A seller S with valuation 0
- A buyer B with valuation v;
  - B knows v, S does not
  - v = 2 with probability π
  - = 1 with probability $1-\pi$
- S sets a price $p \geq 0$;
- B either
  - buys, yielding (p,v-p)
  - or does not, yielding (0,0).



Solution

1. B buys iff $v \geq p$;
   1. If $p \leq 1$, both types buy: S gets p.
   2. If $1 < p \leq 2$, only H-type buys: S gets πp.
   3. If $p > 2$, no one buys.
2. S offers

- 1 if $\pi < ½$,
- 2 if $\pi > ½$.



Sequential bargaining 2-period

- A seller S with valuation 0
- A buyer B with valuation v;
  - B knows v, S does not
  - v = 2 with probability π
  - = 1 with probability $1-\pi$

1. At t = 0, S sets a price $p_0 \geq 0$;
2. B either
   - buys, yielding $(p_0, v-p_0)$
   - or does not, then
3. At t = 1, S sets a price $p_0 \geq 0$;
4. B either
   - buys, yielding $(\delta p_0, \delta(v-p_0))$
   - or does not, yielding (0,0)

Solution, 2-period

1. Let $\mu = \Pr(v = 2|\text{history at } t=1)$.
2. At t = 1, buy iff $v \geq p$;
3. If $\mu > ½$, $p_1 = 2$
4. If $\mu < ½$, $p_1 = 1$.
5. If $\mu = ½$, mix between 1 and 2.
6. B with v=1 buys at t=0 if $p_0 \leq 1$.
7. If $p_0 > 1$, $\mu = \Pr(v = 2|p_0, t=1) \leq \pi$.

Solution, cont. $\pi < 1/2$

1. $\mu = \Pr(v = 2|p_0, t=1) \leq \pi < 1/2$.
2. At $t = 1$, buy iff $v \geq p$;
3. $p_1 = 1$.
4. B with $v=2$ buys at $t=0$ if

$$(2-p_0) \geq \delta(2-1) = \delta \Leftrightarrow p_0 \leq 2-\delta.$$

5. $p_0 = 1$:

$$\pi(2-\delta) + (1-\pi)\delta = 2\pi(1-\delta) + \delta < 1-\delta+\delta = 1.$$

Solution, cont. $\pi > 1/2$

- If $v=2$ is buying at $p_0 > 2-\delta$, then
  - $\mu = \Pr(v = 2|p_0 > 2-\delta, t=1) = 0$;
  - $p_1 = 1$;
  - $v = 2$ should not buy at $p_0 > 2-\delta$.
- If $v=2$ is not buying at $2 > p_0 > 2-\delta$, then
  - $\mu = \Pr(v = 2|p_0 > 2-\delta, t=1) = \pi > 1/2$;
  - $p_1 = 2$;
  - $v = 2$ should buy at $2 > p_0 > 2-\delta$.
- No pure-strategy equilibrium.

Mixed-strategy equilibrium, $\pi > 1/2$

1. For $p_0 > 2-\delta$, $\mu(p_0) = ½$;
2. $\beta(p_0) = 1 - Pr(v=2 \text{ buys at } p_0)$

$$\mu = \frac{\beta(p_0)\pi}{\beta(p_0)\pi + (1-\pi)} = \frac{1}{2} \Leftrightarrow \beta(p_0)\pi = 1-\pi \Leftrightarrow \beta(p_0) = \frac{1-\pi}{\pi}.$$

3. $v = 2$ is indifferent towards buying at p_0:

$$2 - p_0 = \delta\gamma(p_0) \Leftrightarrow \gamma(p_0) = (2 - p_0)/\delta$$

where $\gamma(p_0) = Pr(p_1=1|p_0)$.

Sequential bargaining, v in [0,1]

- 1 period:
  - B buys at p iff $v \geq p$;
  - S gets $U(p) = p\,Pr(v \geq p)$;
  - v in $[0,a]$ => $U(p) = p(a-p)/a$;
  - $p = a/2$.

Sequential bargaining, v in [0,1]

- 2 periods: (p_0, p_1)
  - At t = 0, B buys at p_0 iff $v \geq a(p_0)$;
  - $p_1 = a(p_0)/2$;
  - Type $a(p_0)$ is indifferent:

$$a(p_0) - p_0 = \delta(a(p_0) - p_1) = \delta a(p_0)/2$$

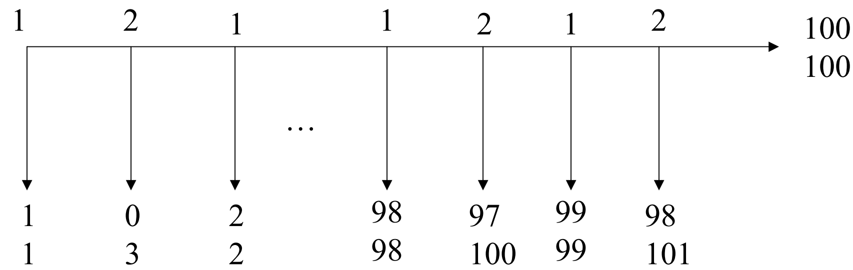
$$\Leftrightarrow a(p_0) = p_0/(1-\delta/2)$$

  - S gets $\left(1 - \frac{p_0}{1-\delta/2}\right)p_0 + \left(\frac{p_0}{2-\delta}\right)^2$
  - FOC:

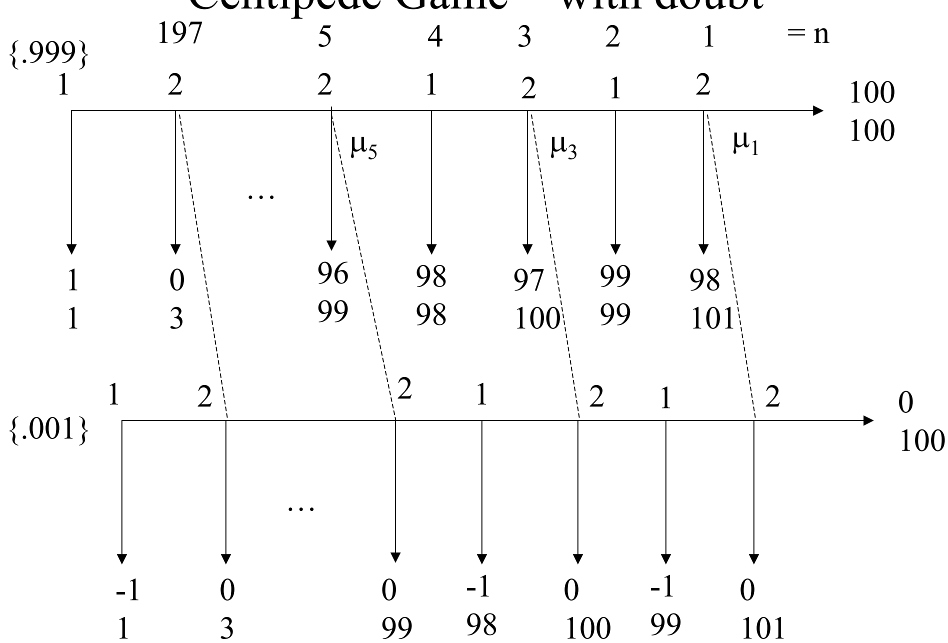
$$1 - \frac{2p_0}{1-\delta/2} + \frac{2p_0}{2-\delta} = 0 \Rightarrow p_0 = \frac{(1-\delta/2)^2}{2(1-3\delta/4)}$$

Reputation

Centipede Game



Centipede Game – with doubt



Facts about the Centipede

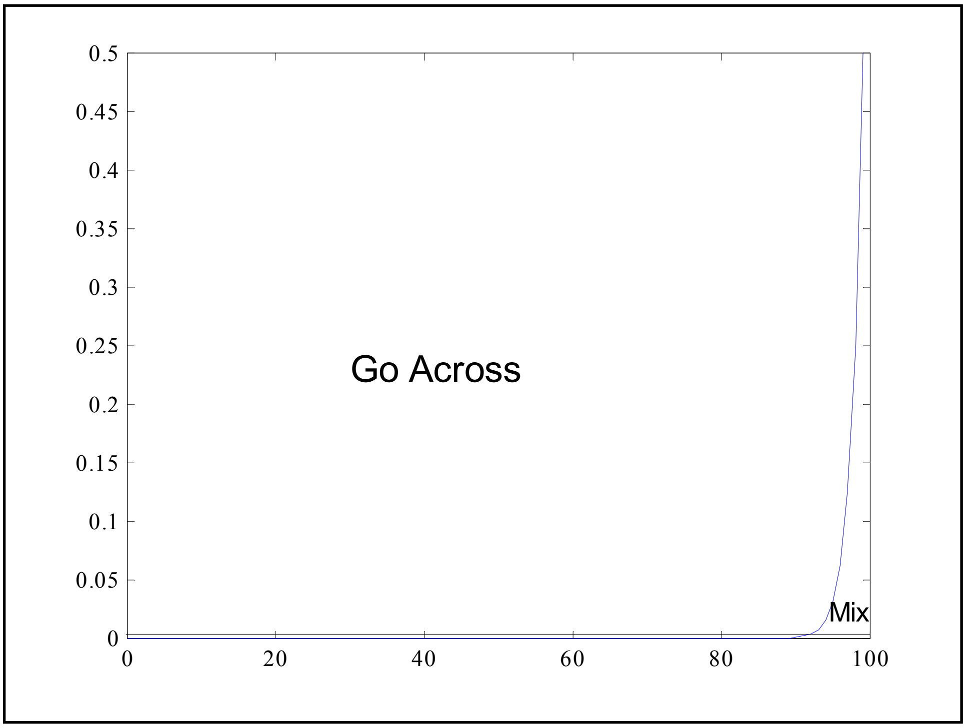
- Every information set of 2 is reached with positive probability.
- 2 always goes across with positive probability.
- If 2 strictly prefers to go across at n, then
  - she must strictly prefer to go across at n+1,
  - her posterior at n is her prior.
- For any n > 2, 1 goes across with positive probability. If 1 goes across w/p 1 at n, then 2's posterior at n-1 is her prior.

If 2's payoff at any n is x and 2 is mixing, then

$$x = \mu_n(x+1) + (1-\mu_n)[(x-1)p_n + (1-p_n)(x+1)]$$
$$= \mu_n(x+1) + (1-\mu_n)[(x+1) - 2p_n]$$
$$= x+1 - 2p_n(1-\mu_n)$$
$$\Leftrightarrow (1-\mu_n)\, p_n = 1/2$$

$$\mu_{n-1} = \frac{\mu_n}{\mu_n + (1-\mu_n)(1-p_n)} = \frac{\mu_n}{\mu_n + (1-\mu_n) - p_n(1-\mu_n)} = 2\mu_n$$

$$\mu_n = \frac{\mu_{n-1}}{2}$$


Go Across
Mix
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
20
40
60
80
100

Lecture 19-21

14.12 Game Theory
Muhamet Yildiz

Road Map

- Market for Lemons (Adverse Selection)
- Insurance Market (Screening)
- Signaling – theory
- Job Market Signaling

Example for Adverse selection

- A buyer and a seller, who owns an object.
- The value of the object is
  - v for the seller,
  - v+b for the buyer.
- v is uniformly distributed on [0,1]. Seller knows v. b > 0 is a known constant.
- Buyers offers a price p; seller decides on whether to sell.

Solution

- Seller sells at p iff p ≥ v;
- Buyer's payoff:

$$U(p) = \int_0^p (v + b - p)dv = \frac{1}{2}p^2 + p(b - p) = p\left(b - \frac{p}{2}\right)$$

- Buyer offers p = 2b.

Example for Adverse selection

- A buyer and a seller, who owns an object.
- The value of the object is
  - v for the seller,
  - 3v/2 for the buyer.
- v is uniformly distributed on [0,1]. Seller knows v.
- Buyers offers a price p; seller decides on whether to sell.

Solution

- Seller sells at p iff p ≥ v;
- Buyer's payoff:

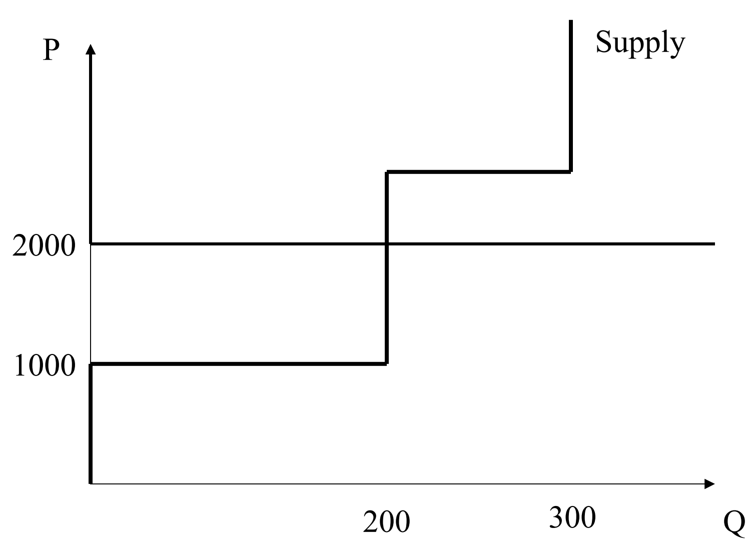
$$U(p)=\int_0^p\left(\frac{3}{2}v-p\right)dv=p\left(\frac{3}{2}\frac{p}{2}-p\right)=-p^2/4$$

- Buyer offers p = 0.

Market for Lemons

- Two types of cars: Lemons and Peaches.
  - A Peach is worth \$2500 to seller, \$3000 to buyer;
  - A lemon is worth \$1000 to seller, \$2000 to buyer;
- Each seller knows whether his car is a Peach or Lemon; buyers cannot tell.
- There are 200 Lemons and 100 Peaches.
- Equilibrium: a market clearing price p.

Demand/Supply – Market for Lemons



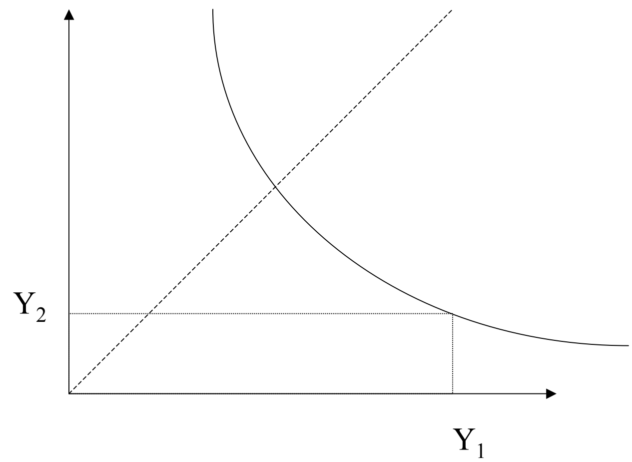
Quiz

- The students are grouped in pairs;
- In each pair, one is assigned to be seller, the other one is buyer. The buyer is given a valuation v, an integer between 20 and 40 with uniform distribution.
  - Buyer offers a price p;
  - seller accepts or rejects;
  - If seller accepts, seller gets p, buyer gets 20+v-p.
  - If seller rejects, w/probability 0.1 each gets 20; w/p 0.95 we proceed to next date, and
  - Seller offers a price p;
  - If buyer accepts, seller gets p, buyer gets 20+v-p; otherwise each gets 20.

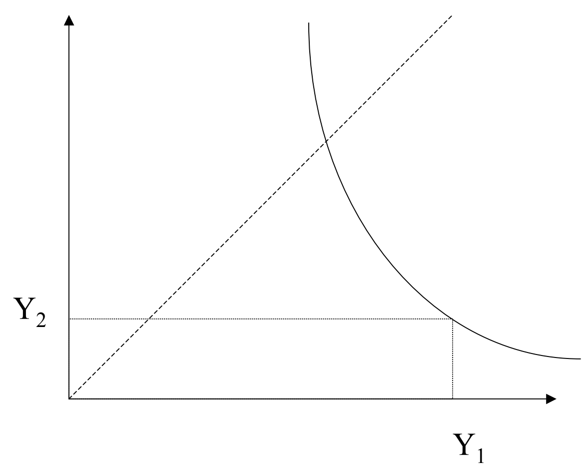
Insurance with adverse selection

- Two states: s_1, s_2.
- A risk averse agent with
  - risky endowment (Y_1,Y_2) where $Y_1 > Y_2$, and
  - Utility function u.
- Two types:
  - High risk: $U(y_1,y_2) = (1-\pi_H)u(y_1) + \pi_H u(y_2)$
  - Low risk: $U(y_1,y_2) = (1-\pi_L)u(y_1) + \pi_L u(y_2)$, where $\pi_H > \pi_L$.
- A risk neutral insurance company offers a menu $((y_{1H},y_{2H}), (y_{1L},y_{2L}))$ of insurance policies, and the risk averse agent either chooses one of the policies in the menu or rejects the offer.
- Agent knows his type, the company does not.

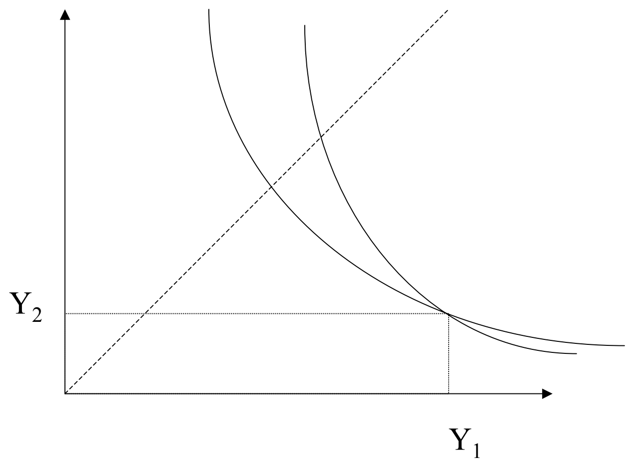
High-risk Agent



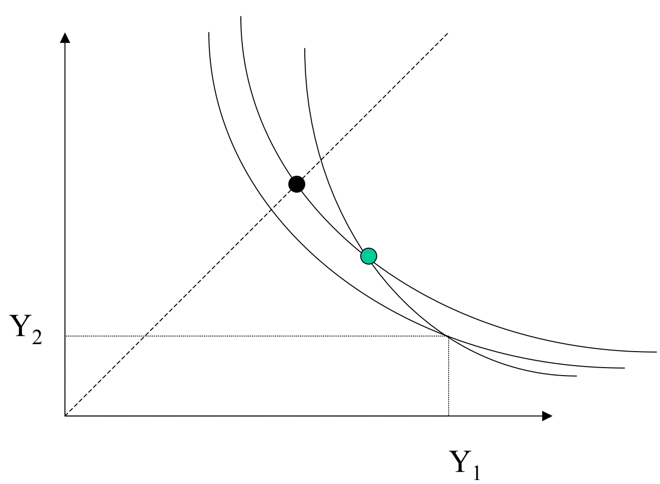
Low-risk Agent



High- and Low-risk Agent

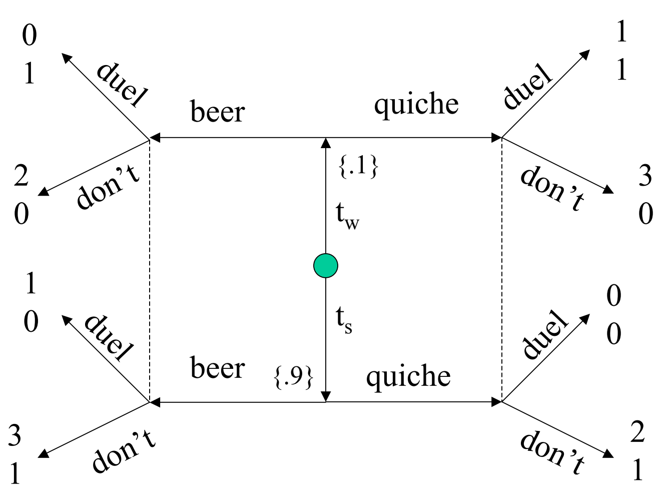


Optimal menu



Signaling Games

Beer – Quiche

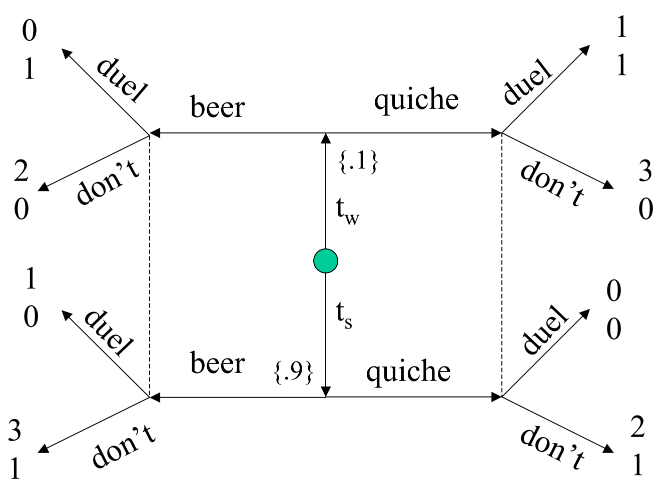


Signaling Game -- Definition

- Two Players: (S)ender, (R)eceiver

1. Nature selects a type t_i from $T = \{t_1,\ldots,t_I\}$ with probability $p(t_i)$;
2. Sender observes t_i, and then chooses a message m_j from $M = \{m_1,\ldots,m_I\}$;
3. Receiver observes m_j (but not t_i), and then chooses an action a_k from $A = \{a_1,\ldots,a_K\}$;
4. Payoffs are $U_S(t_i,m_j,a_k)$ and $U_S(t_i,m_j,a_k)$.

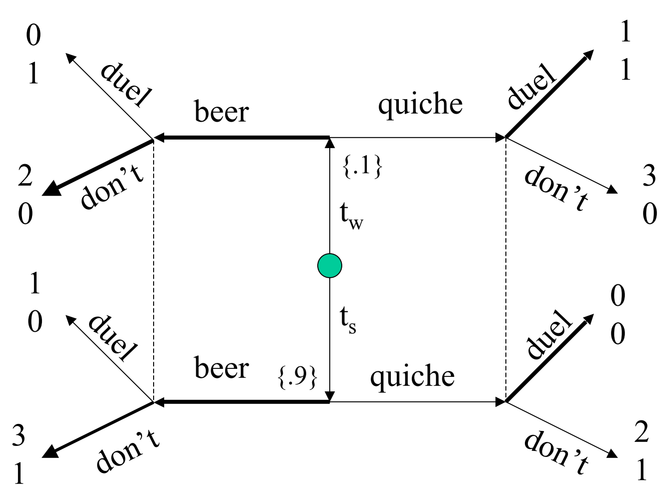
Beer – Quiche



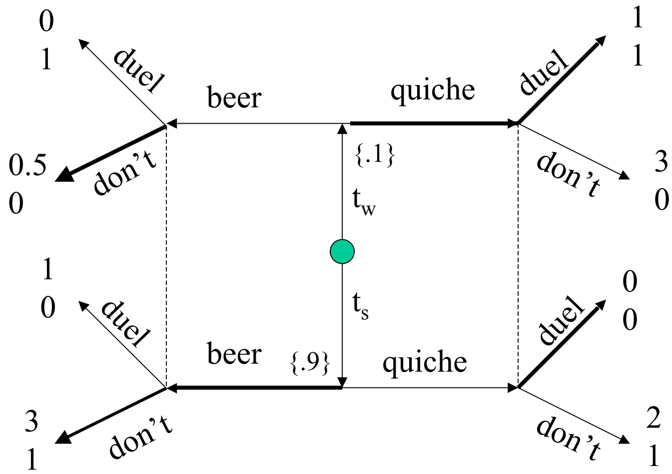
Types of Equilibria

- A **pooling equilibrium** is an equilibrium in which all types of sender send the same message.
- A **separating equilibrium** is an equilibrium in which all types of sender send different messages.
- A **partially separating/pooling equilibrium** is an equilibrium in which some types of sender send the same message, while some others sends some other messages.

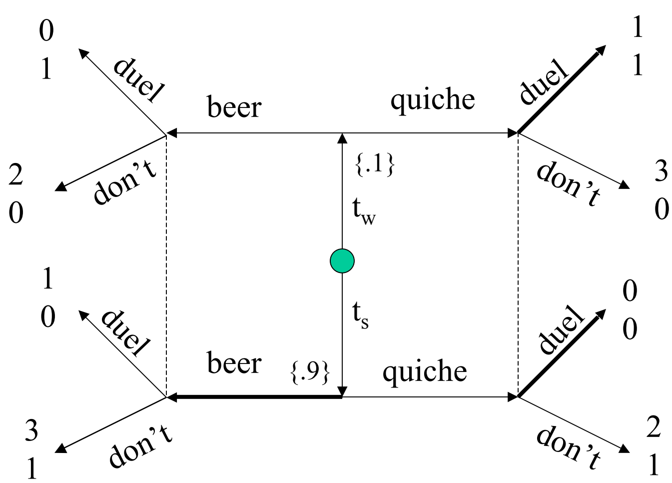
A Pooling equilibrium



A Separating equilibrium



A Mixed equilibrium



Job Market Signaling

Model

- A worker
  - with ability $t = H$ or $t = L$ (his private information)
    - $\Pr(t = H) = q$,
  - obtains an observable education level e,
  - incurring cost $c(t,e)$ where $c(H,e) < c(L,e)$, and
  - finds a job with wage $w(e)$, where he
  - produces $y(t,e)$.
- Firms compete for the worker: in equilibrium,

$$w(e) = \mu(H|e)y(H,e) + (1 - \mu(H|e))y(L,e).$$

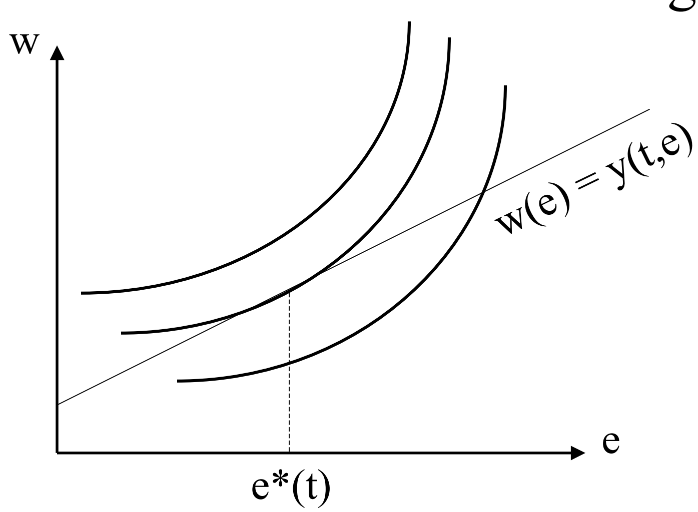
Equilibrium

$(e_H, e_L, w(e), \mu(H|e))$ where

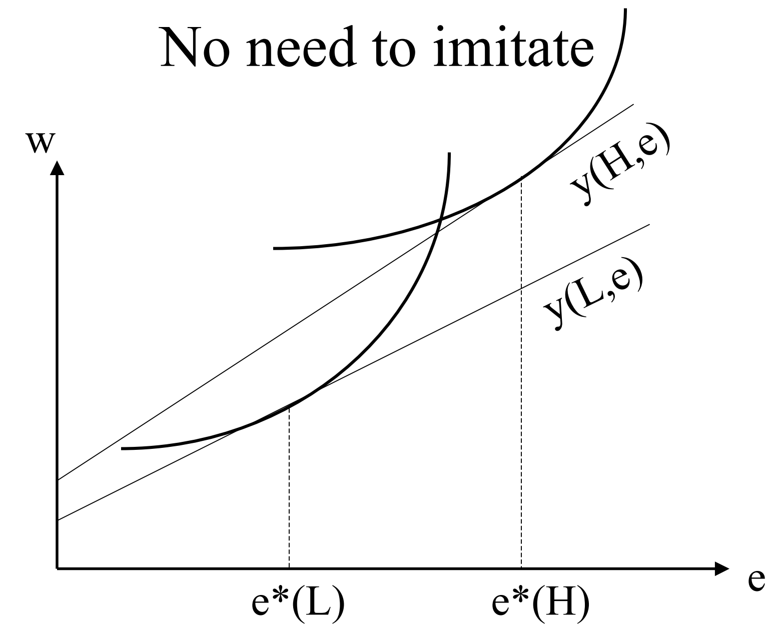
- $e_t = \text{argmax}_e\ w(e) - c(t,e)$ for each t;
- $w(e) = \mu(H|e)y(H,e) + (1-\mu(H|e))y(L,e)$;
- $$\mu(H|e) = \frac{q\Pr(e_H = e)}{q\Pr(e_H = e) + (1\text{-}q)\Pr(e_L = e)}$$

  whenever well-defined.

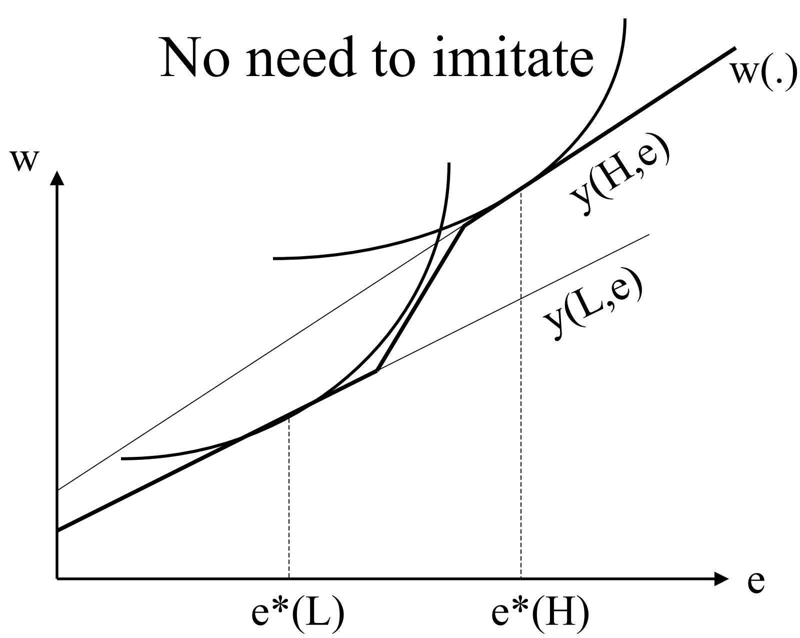
If t were common knowledge



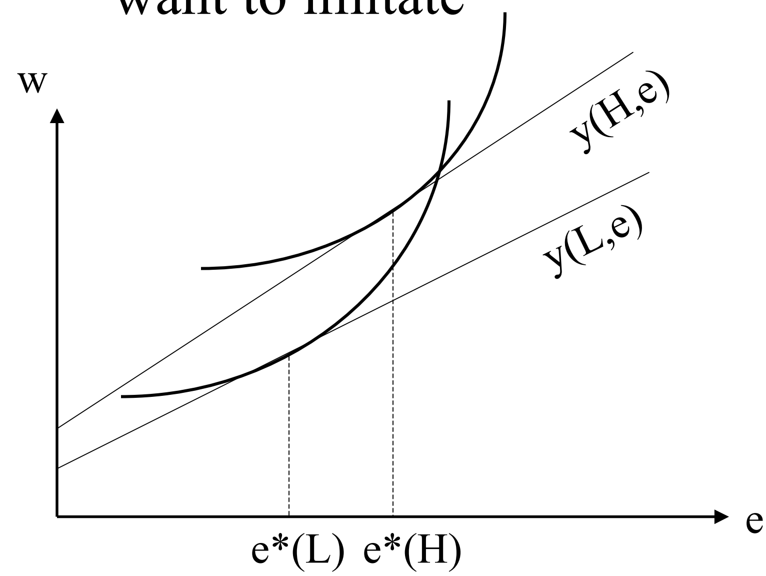
No need to imitate



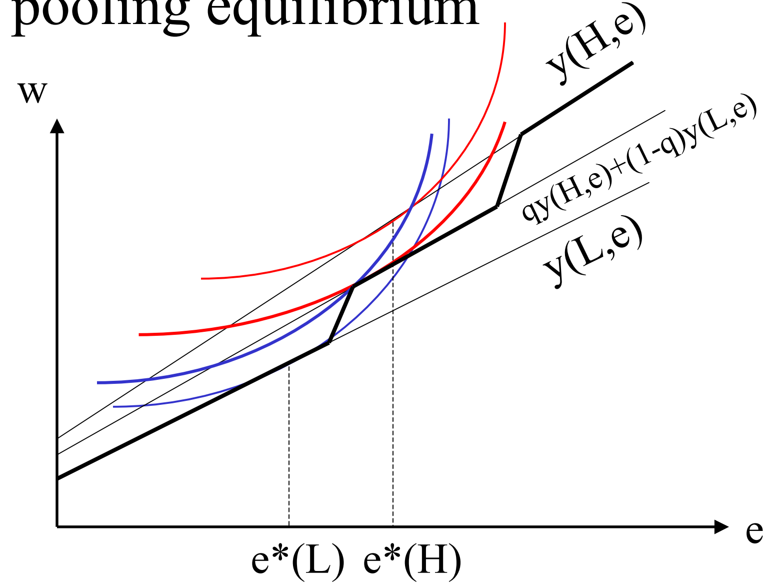
No need to imitate



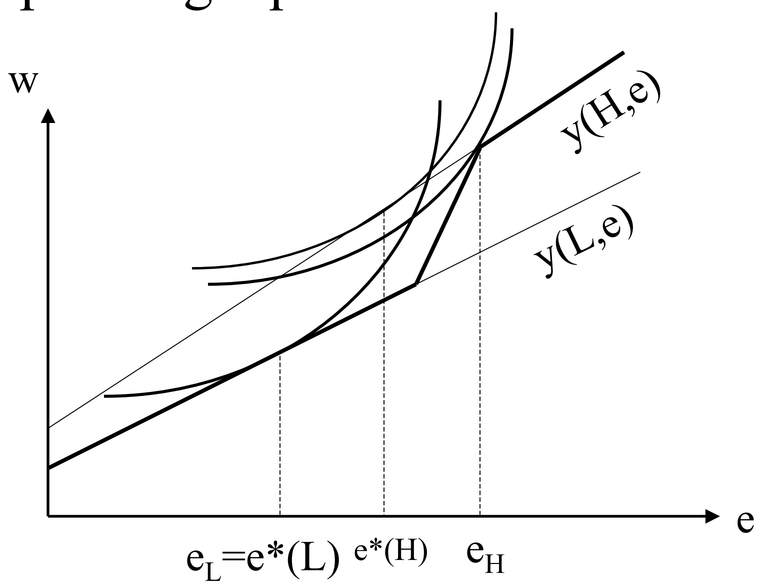
want to imitate



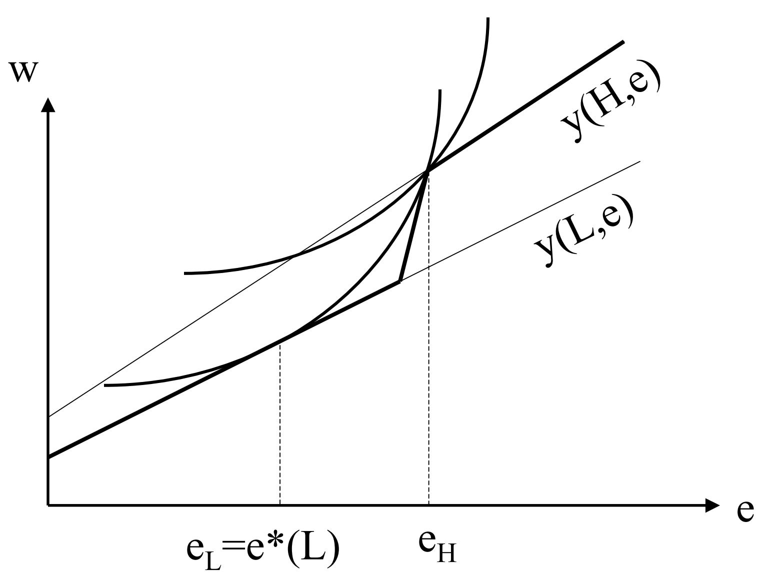
A pooling equilibrium



A separating equilibrium



An intuitive separating equilibrium



14.12 Game Theory
Prof. Muhamet Yildiz
TA: Kenichi Amaya
Fall 2001

Homework 1

Due on 9/26/2001 (in class)

1. Take $X = \mathbb{R}$, the set of real numbers, as the set of alternatives. Define a relation $\succeq$ on X by
$$x \succeq y \iff x \geq y - 1/2 \qquad \text{for all } x, y \in X.$$
   (a) Is $\succeq$ a preference relation? (Provide a proof.)

   (b) Define the relations $\succ$ and $\sim$ by
$$x \succ y \iff [x \succeq y \text{ and } y \not\succeq x]$$
   and
$$x \sim y \iff [x \succeq y \text{ and } y \succeq x],$$
   respectively. Is $\succ$ transitive? Is $\sim$ transitive? Prove your claims.

   (c) Would $\succeq$ be a preference relation if we had $X = \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of all natural numbers?

2. We have two dates: 0 and 1. We have a security that pays a single dividend, at date 1. The dividend may be either \$100, or \$50, or \$0, each with probability 1/3. Finally, we have a risk-neutral agent with a lot of money. (The agent will learn the amount of the dividend at the beginning of date 1.)

   (a) An agent is asked to decide whether to buy the security or not at date 0. If he decides to buy, he needs to pay for the security only at date 1 (not immediately at date 0). What is the highest price π_S at which the risk-neutral agent is willing to buy this security?

   (b) Now consider an "option" that gives the holder the right (but not obligation) to buy this security at a strike price K at date 1 — after the agent learns the amount of the dividend. If the agent buys this option, what would be the agent's utility as a function of the amount of the dividend?

   (c) An agent is asked to decide whether to buy this option or not at date 0. If he decides to buy, he needs to pay for the option only at date 1 (not immediately at date 0). What is the highest price π_O at which the risk-neutral agent is willing to buy this option?

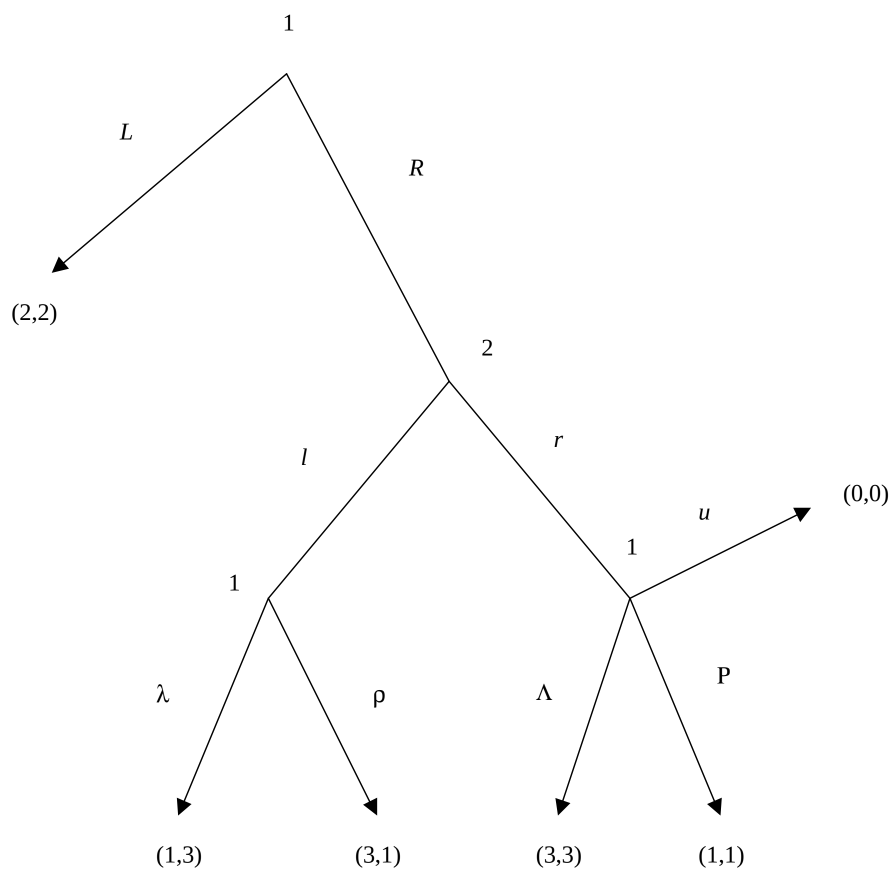
3. Construct a 2-player game with the following property. Player 1 has strategies s, s', and s'' such that neither s nor s' strictly dominates s'', but the mixed strategy that assigns probability 1/2 to each of the strategies s and s' strictly dominates s''.

4. Compute the set of rationalizable strategies in the following game that is played in a class of n students where $n \geq 2$: Without discussing with anyone, each student i is to write down a real number $x_i \in [0, 100]$ on a paper and submit it to the TA. The TA will then compute the average

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

of these numbers. The students who submit the number that is closest to $\bar{x}/3$ will share the total payoff of 100, while the other students get 0. Everything described above is common knowledge. (Bonus: would the answer change if the students did not know n, but it were common knowledge that $n \geq 2$?)

5. Consider the following game in extensive form.



(a) Write this game in the strategic form.

(b) What are the strategies that survive the *iterative elimination of weakly-dominated strategies* in the following order: first eliminate all weakly-dominated strategies of player 1; then, eliminate all the strategies of player 2 that are weakly dominated in the remaining game; then, eliminate all the strategies of player 1 that are weakly dominated in the remaining game, and so on?

14.12 Solutions for Homework 1

Kenichi Amaya[1]

September 28, 2001

Question 1

(a)

Remember that a binary relation $\succeq$ is a *preference relation* if it is *complete* and *transitive*

Completeness For all $x, y \in X$, $x \succeq y$ or $y \succeq x$.

Transitivity If $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

The relation defined here is complete, but it is not transitive. We can show this by a counterexample. Suppose $x = 0$, $y = .5$, and $z = 1$. Then, $x \succeq y$ because $x = 0 = y - \frac{1}{2}$, and $y \succeq z$ because $y = \frac{1}{2} = z - \frac{1}{2}$, but we don't have $x \succeq z$ because $x = 0 < \frac{1}{2} = z - \frac{1}{2}$. Therefore, $\succeq$ is not a preference relation.

(b)

$$
\begin{aligned}
& x \succ y \\
\Leftrightarrow x \geq y - 1/2 \ \text{ and } \ & y < x - 1/2 \\
\Leftrightarrow & x > y + 1/2
\end{aligned}
$$

Suppose $x \succ y$ and $y \succ z$. Then, $x > y + 1/2$ and $y > z + 1/2$. This implies that

$$x > y + 1/2 > (z + 1/2) + 1/2 > z + 1/2.$$

Therefore $\succ$ is transitive.

We can show that $\sim$ is not transitive by using the same counterexample used in (a). Suppose $x = 0$, $y = .5$, and $z = 1$. Since $x = 0 = y - \frac{1}{2}$, $x \succeq y$ and $y \succeq x$, therefore $x \sim y$. Since $y = \frac{1}{2} = z - \frac{1}{2}$, $y \succeq z$ and $z \succeq y$, therefore $y \sim z$. However, we don't have $x \succeq z$ because $x = 0 < \frac{1}{2} = z - \frac{1}{2}$. Therefore $x \sim z$ is not satisfied.

(c)

If we had $x, y \in X = \{0, 1, 2, \cdots\}$,

$$x \succeq y \Leftrightarrow x \geq y - 1/2 \Leftrightarrow x \geq y.$$

Then both completeness and transitivity are satisfied, so $\succeq$ is a preference relation.

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Question 2

Since the agent is risk neutral, his von-Neuman Morgenstern utility function is represented by the amount of money he receives (or any affine transformation of it).

(a)

If he doesn't buy the security, the amount of money he has at date 1 is 0 for sure, so the expected utility is 0. If he buys the security at price p, the amount of money he has at date 1 is $100 - p$, $50 - p$, or $-p$, each with probability 1/3, and thus the expected utility is

$$\frac{1}{3}(100 - p) + \frac{1}{3}(50 - p) + \frac{1}{3}(-p).$$

He wants to buy this security if and only if this expected utility is higher than or equal to 0, the expected utility from not buying.

$$\frac{1}{3}(100 - p) + \frac{1}{3}(50 - p) + \frac{1}{3}(-p) \geq 0 \Leftrightarrow p \leq 50.$$

Therefore $\pi_S = 50$.

(b)

If the agent exercises the option, he will pay K and receive the dividend d, so he will receive net payment of $d - K$. If he doesn't exercise the option, he receives 0. Therefore he will excercise if and only if $d - K \geq 0$. His utility is represented by

$$u(d) = \max\{0, d - K\} - p,$$

where p is the price of the option.

(c)

Case 1 $K > 100$

In this case, the agent will never exercise the option, so he ends up with receiving 0 no matter what the dividend is. Therefore he wants to pay for this option no more than 0.

Case 2 $50 < K \leq 100$

If $d = 100$, the agent will exercise the option and receive net payment of $100 - K$. However, if $d = 0$ or 50, he won't exercise the option and receive 0. Therefore, he wants to buy this option if

$$\begin{aligned} \frac{1}{3}(100 - K - p) + \frac{2}{3}(-p) &\geq 0 \\ \Leftrightarrow p &\leq \frac{1}{3}(100 - K). \end{aligned}$$

Case 3 $0 \leq K \leq 50$

If $d = 50$ or 100, the agent will exercise the option and receive net payment of $d - K$. However, if $d = 0$, he won't exercise the option and receive 0. Therefore, he wants to buy this option if

$$\frac{1}{3}(100 - K - p) + \frac{1}{3}(50 - K - p) + \frac{1}{3}(-p) \geq 0$$
$$\Leftrightarrow p \leq 50 - \frac{2}{3}K.$$

To summarise,

$$\pi_O = \begin{cases} 0 & if \quad K > 100 \\ \frac{1}{3}(100 - K) & if \quad 50 < K \leq 100 \\ 50 - \frac{2}{3}K & if \quad 0 \leq K \leq 50 \end{cases}$$

Question 3

Here is an example.

	L	R
s	4,0	0,0
s'	0,0	6,0
s"	1,1	1,1

The mixed strategy $\frac{1}{2}s + \frac{1}{2}s'$ gives player 1 expected utility of 2 if player 2 plays l and expected utility of 3 if player 2 plays r.

Question 4

The intuition tells us only 0 would be selected, but actually, all strategies except 100 are rationalizable! The reason is that a player's objective is just to win the game, i.e., to name closer number to one third of the average than everyone else, and not to name exactly one third of the average. For example, if everybody else is naming 100, you can win by naming any number below 100, even 99 wins. The proof is given in the answer 1.

But if, not like this game, a player's objective is to name exactly one third of the average, our first intuition works. The proof for this case is given in the answer 2.

Answer 1

If $0 \leq x < 100$, x is the best response to all other players' playing x', where $x < x' < 100$ and x' is close enough to x. To see this, if a player plays x and all others plays x', $\bar{x}/3 < x$ (because x' is close enough to x), therefore she is the only winner. Since she can do no better than being the only winner, x is the best response.

For any $0 \leq x < 100$, we can find a sequence $x < x_1 < x_2 < \cdots < 100$ such that x is a best response to everyone else's playing x_1, x_1 is a best response to everyone else's playing x_2, and so on.

Naming 100 is not a best response to anything. The only case a player wins is when everyone is naming 100, but even in this case, she can be the only winner by naming a smaller number.

To conclude, x is rationalizable if and only if $0 \leq x < 100$.

Answer 2

Suppose the award is $100 - |x - \frac{\bar{x}}{3}|$, so that a player always want to name exactly $\frac{\bar{x}}{3}$.

Given that $0 \leq x_i \leq 100$ for all i, $\frac{\bar{x}}{3}$ always lies between 0 and $33.33\cdots$. So $x > 33.33\cdots$ is never best response, i.e., it is not rationalizable.

As long as everyone is naming $0 \leq x_i \leq 33.33\cdots$, $\frac{\bar{x}}{3}$ always lies between 0 and $11.11\cdots$. So $x > 11.11\cdots$ is never best response, i.e., it is not rationalizable.

Repeating this procedure forever, any number greater than 0 is not rationalizable.

If everyone else is naming 0, the best response is to name 0 as well. Therefore 0 is rationalizable.

To conclude, $x = 0$ is the only rationalizable strategy.

Bonus: In the discussion above, the number of players played no role at all.

Question 5
(a)

	l	r
$L\lambda\Lambda$	2,2	2,2
$L\lambda P$	2,2	2,2
$L\lambda u$	2,2	2,2
$L\rho\Lambda$	2,2	2,2
$L\rho P$	2,2	2,2
$L\rho u$	2,2	2,2
$R\lambda\Lambda$	1,3	3,3
$R\lambda P$	1,3	1,1
$R\lambda u$	1,3	0,0
$R\rho\Lambda$	3,1	3,3
$R\rho P$	3,1	1,1
$R\rho u$	3,1	0,0

(b)

First, $R\rho\Lambda$ weakly dominates all other strategies of player 1. We eliminate them and then we have the following game.

	l	r
$R\rho\Lambda$	3,1	3,3

Now, l is weakly dominated by r, so we eliminate l and we have

	r
$R\rho\Lambda$	3,3

14.12 Game Theory
Prof. Muhamet Yildiz
TA: Kenichi Amaya
Fall 2001

Homework 2

Due on 10/3/2001 (in class)

1. Consider the following game:

1\2	L	R
T	$(1,1)$	$(1,0)$
B	$(0,1)$	$(0,10000)$

   (a) Compute the rationalizable strategies.

   (b) Now assume that players can tremble: when a player intends to play a strategy s, with probability $\epsilon = 0.001$, nature switch to the other strategy s', when s' is played. For instance, if player 2 plays L (or intends to play L), with probability ϵ L is played, with probability $1-\epsilon$, R is played. Compute the rationalizable strategies for this new game.

   (c) Discuss your results (briefly).

2. Compute all the Nash equilibria of the following game.

	L	M	R
A	$(3,1)$	$(0,0)$	$(1,0)$
B	$(0,0)$	$(1,3)$	$(1,1)$
C	$(1,1)$	$(0,1)$	$(0,10)$

3. Compute the pure-strategy Nash equilibria in the following linear Cournot oligopoly for arbitrary n firms: each firm has marginal cost $c > 0$ and a fixed cost $F > 0$, which it needs to incur only if it produces a positive amount; the inverse-demand function is given by $P(Q) = \max\{1 - Q, 0\}$, where Q is the total supply.

4. A group of n students go to a restaurant. It is common knowledge that each student will simultaneously choose his own meal, but all students will share the total bill equally. If a student gets a meal of price p and contributes x towards paying the bill, his payoff will be $\sqrt{p} - x$. Compute the Nash equilibrium. Discuss the limiting cases $n = 1$ and $n \to \infty$.

14.12 Solutions for Homework 2

Kenichi Amaya[1]

October 5, 2001

Question 1

(a)

	L	R
T	1,1	1,0
B	0,1	0,10000

Fact: The set of rationalizable strategies is same as the set of strategies which survives iterated elimination of strictly dominated strategies.

Step 1: For player 1, B is strictly dominated by T. Now we have

	L	R
T	1,1	1,0

Step 2: For player 2, R is strictly dominated by L. Now we have

	L
T	1,1

Therefore, the rationalizable strategies are T for player 1 and L for player 2.

(b)

There was a typo in the problem. If player 2 intends to play L, L is played with probability $1-\epsilon$, and R is played with probability ϵ. Of course, if you interpreted the problem differently, you will get credit as long as you are solving the problem consistently with your interpretation.

We can write a new game which represents this situation as following: We look for the players' expected payoffs as functions of the intended play. If player 1 intends to play T and player intends to play L, what are the expected payoffs? (T,L) is realized with probability $(1-\epsilon)^2$, (T,R) and (B,L) are realized each with probability $\epsilon(1-\epsilon)$, and (B,R) is realized with probability ϵ^2. Therefore, the expected payoff of player 1 is

$$(1-\epsilon)^2 \cdot 1 + \epsilon(1-\epsilon) \cdot 1 + \epsilon(1-\epsilon) \cdot 0 + \epsilon^2 \cdot 0 = 1-\epsilon = .999.$$

In the same way, player 2's expected payoff is

$$(1-\epsilon)^2 \cdot 1 + \epsilon(1-\epsilon) \cdot 0 + \epsilon(1-\epsilon) \cdot 1 + \epsilon^2 \cdot 10000 = (1-\epsilon) + \epsilon^2 \cdot 10000 = 1.009$$

Therefore we must have

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	L	R
T	.999, 1.009	
B		

In the same way, we can calculate expected payoffs for the remaining of the payoff matrix and we have

	L	R
T	.999, 1.009	.999, 9.991
B	.001, 10.999	.001, 9980.01

For player 1, B is strictly dominated by T, and for player 2, L is strictly dominated by R. Now we have

	R
T	.999, 9.991

Therefore, the rationalizable strategies are T for player 1 and R for player 2.

(c)

For player 1, the possibility that opponent trembles doesn't affect her optimal choice because T gives higher payoff than B no matter what the opponent is doing.

For player 2, in the case of (a), the large payoff differnce between (B,L) and (B,R) didn't matter because he was sure B is never played. However, when we introduce trembles, this large payoff difference matters with positive probability even though player 2 knows that player 1 never intends to play B. In this situation, the payoff difference between (T,L) and (T,R) is so small that it is outweighed by the payoff differnce between (B,L) and (B,R).

Question 2

Fact All the Nash equilibrium strategies are rationalizable, and therefore survives iterated elimination of strictly dominated strategies. When we look for Nash equilibria, we can first eliminate strictly dominated strategies and then find Nash equilibria of the game left.

	L	M	R
A	3,1	0,0	1,0
B	0,0	1,3	1,1
C	1,1	0,1	0,10

Step 1 For player 1, C is strictly dominated by $\frac{1}{2}A + \frac{1}{2}B$. Now we have

	L	M	R
A	3,1	0,0	1,0
B	0,0	1,3	1,1

Step 2 For player 2, R is stricly dominated by $\frac{1}{2}L + \frac{1}{2}M$. Now we have

	L	M
A	3,1	0,0
B	0,0	1,3

Step 3 In the game left, (A,L) and (B,M) are pure strategy Nash equilibria, because players are taking best response to each other.

Step 4 We want to find the mixed strategy equilibrium. Let $(pA + (1 - p)B, qL + (1 - q)M)$ be the equilibrium. First, player 1 must be indifferent between playing A and B, which implies

$$3q = 1(1 - q),$$

where the LHS is the expected payoff of playing A and the RHS is that of B, when player 2 is playing the equilibrium strategy. This implies

$$q = \frac{1}{4}.$$

Next, player 2 must be indifferent between playing L and M, implying

$$\begin{aligned} 1p &= (1 - p)3 \\ p &= \frac{3}{4}. \end{aligned}$$

To conclude, the Nash equilibria are; (A,L), (B,M), and $(\frac{3}{4}A+\frac{1}{4}B, \frac{1}{4}L+\frac{3}{4}M)$.

Question 3

This question is hard!

Notice that in the Cournot model with fixed cost, there can be equilibria where not every firm is producing. For example, if there are two firms, there can be an equilibrium where one firm is producing the monopoly quiantity and the other firm is producing nothing. (Of course, whether this is actually a Nash equilibrium or not depends on the cost and demand structure.) To see why this can be an equilibium, consider each firm's best response. If the opponent firm is producing the monopoly quantity, the revenue from selling some amount can not cover the fixed cost, if the fixed cost is large enough, and thus producing nothing is the best response. On the other hand, if the opponent is producing nothing, you can get a revenue equal to the monopoly profit, which is higher than the fixed cost.

It is hard to characterize all the equilibria, so we limit attention to pure strategy equilibria which are symmetric in the sense that all the producing firms are choosing the same quantity. (Keep in mind that there might be other equilibria! Actually, it is not very hard to show that all pure strategy equilibria are symmetric. This would be a good exercise.)

First, consider the equilibrium where all firms are producing q^*. What we need is that q^* is a best response to all other firms choosing q^*. The profit of firm i when it is producing q_i and all other firms are producing q^* is

$$\pi_i(q_i, q^*) = (\max\{1 - (q_i + (1-n)q^*), 0\} - c)q_i - F. \tag{1}$$

In the equilibrium all firms must be getting positive revenue (because there is fixed cost), so it must be $1 - (q_i + (n-1)q^*) > 0$. Therefore

$$\pi_i(q_i, q^*) = (1 - (q_i + (n-1)q^*) - c)q_i - F. \tag{2}$$

Taking first order condition,

$$q_i = \frac{1 - (n-1)q^* - c}{2}. \tag{3}$$

In the symmetric equilibium, we must have $q_i = q^*$, thus

$$q^* = \frac{1-c}{n+1}. \tag{4}$$

For this quantity to be actually construct equilibrium, each firm must be getting nonnegative profit. Plugging this into (2),

$$\pi_i = \left(\frac{1-c}{n+1}\right)^2 - F. \tag{5}$$

Therefore there exists an equilibrium where all firms are producing $q^* = \frac{1-c}{n+1}$ if and only if $(\frac{1-c}{n+1})^2 \geq F$ and $c \leq 1$.

Next consider an equilibrium where only $k < n$ firms are producing q^*. We need to check incentives of firms which are producing and incentives of firms which are not producing.

For a firm which is producing, $q_i = q^*$ must be a best response to $k-1$ of other firms producing q^* and the rest producing zero. The firm's profit when producing q_i is

$$\pi_i(q_i, q^*) = (\max\{1 - (q_i + (1-k)q^*), 0\} - c)q_i - F. \tag{6}$$

Notice this is same as equation (1), n being replaced by k. Therefore the same argument as above holds.

$$q^* = \frac{1-c}{k+1}, \tag{7}$$

For this to be an equilibrium, we need $(\frac{1-c}{k+1})^2 \geq F$ and $c \leq 1$.

However, this is not a sufficient condition. It must also be true that for non-producing firms, not producing is actually the best response. From the point of view of a non-producing firm i, k other firms are producing q^*. By the same argument above, if this firm does choose to produce, the optimal amount is

$$q_i^* = \frac{1 - kq^* - c}{2}, \tag{8}$$

and the payoff is

$$\pi_i = \left(\frac{1 - kq^* - c}{2}\right)^2 - F. \tag{9}$$

If this profit is positive, the firm wants to produce q_i^* rather than producing nothing. Therefore, we need $\left(\frac{1-kq^*-c}{2}\right)^2 \leq F$, i.e.,

$$\left(\frac{1 - c}{2(k+1)}\right)^2 \leq F. \tag{10}$$

Finally, consider an equilibrium where no firm produces. This is the case where $c \geq 1$ or where even the monopoly profit is less than the fixed cost. That is,

$$\left(\frac{1 - c}{2}\right)^2 \leq F. \tag{11}$$

Question 4

Each player $i = 1, \cdots, n$ chooses his price of meal p_i. The payoff of player i is

$$\begin{aligned} u_i &= \sqrt{p_i} - \frac{\sum_{j=1}^n p_j}{n} \\ &= \sqrt{p_i} - \frac{p_i}{n} - \frac{\sum_{j \neq i} p_j}{n}. \end{aligned}$$

Taking first order condition,

$$\begin{aligned} \frac{1}{2\sqrt{p_i}} &= \frac{1}{n} \\ p_i &= \frac{n^2}{4}. \end{aligned}$$

This gives the best response, but actually this optimum does not depend on what other players are doing, which means that it is the dominant strategy.

To conclude, the Nash equilibrium is

$$p_i = \frac{n^2}{4} \;\; \forall i.$$

Notice if $n = 1$ then $p = \frac{1}{2}$. This is the efficient choice, because there is no externality. If $n \to \infty$, then $p_i \to \infty$. This is because every marginal increase in the price of the meal will splitted by all the people and the marginal increase of the payment of the player herself converges to 0.

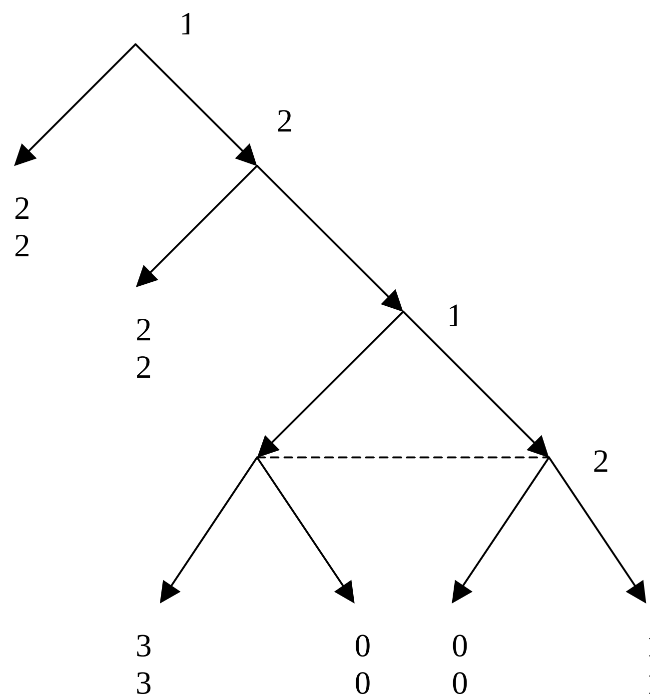
14.12 Game Theory
Fall 2001
Problem Set 3
Due on 10/29

1. Compute the pure-strategy subgame-perfect equilibria in the following linear Cournot oligopoly for arbitrary n firms:

   (1) each firm simultaneously decided whether to enter the market; if a firm enters the market, it pays a fixed cost $F > 0$;

   (2) observing which firms entered the market, each firm i produces quantity q_i at zero marginal cost; the inverse-demand function is given by $P(Q) = \max\{1 - Q, 0\}$, where Q is the total supply.

2. Consider the following two-person game:



   (a) Compute all pure-strategy subgame-perfect equilibriua?

   (b) Which of these equilibria are consistent with the commom knowledge of strong belief in rationality?

3. Consider an infinitly repeated version of the Bertand duoploy game. Market demand is given by $Q = 1 - P$, and each firm has constant marginal cost equal to 0. Consumers buy from the firm with the lowest price; and if there are more than one firm charging the lowest price, demand is shared by such firms equally.

(a) What is the minimum δ for which the monopoly price can be supported as a subgame perfect equilibrium?

(b) What is the minimum δ when there are n firms?

4. Consider the following "repeated" Stackelberg duopoly, where a long-run firm plays against many short-run firms, each of which is in the market only for one date, while the long-run firm remains in the market throughout the game. At each date t, first, the short run firm sets its quantity x_t; then, knowing x_t, the long-run firm sets its quantity y_t; and each sells his good at price $p_t = 1 - (x_t + y_t)$. The marginal costs are all 0. The short-run firm maximizes its profit, which incurs at t. The long-run firm maximizes the present value of its profit stream where the discount rate is $\delta = 0.99$. At the beginning of each date, the actions taken previously are all common knowledge.

   (a) What is the subgame perfect equilibrium if there are only finitely many dates, i.e., $t \in \{0, 1, \ldots, T\}$.

   (b) Now consider the infinitely repeated game. Find a subgame perfect equilibrium, where $x_t = 1/4$ and $y_t = 1/2$ at each t on the path of equilibrium play, namely in the contingencies that happen with positive probability given the strategies.

   (c) Can you find a subgame perfect equilibrium, where $x_t = y_t = 1/4$ for each t on the path of equilibrium play?

14.12 Solutions for Homework 3

Kenichi Amaya[1]

November 2, 2001

Question 1

We solve this problem by backward induction. First, we solve for the quantity choice (stage 2) given the entrance decision, and then we solve for the entrance decision (stage 1).

Stage 2 Suppose m firms has entered in stage 1. Assuming $1 - Q \geq 0$ in equilibrium (we check this is actually true later), each firm i maximizes

$$\begin{aligned} \pi_i &= q_i(1 - Q) \\ &= q_i(1 - \sum_{j \neq i} q_j - q_i). \end{aligned}$$

Notice, since the fixed cost F was payed in stage 1 already, i.e., it is already sunk, the firm doesn't care about it any more. Taking first order condition,

$$q_i = \frac{1 - \sum_{j \neq i} q_j}{2}.$$

Assuming the equilibrium is symmetric (q_i is same for all i),

$$q_i = \frac{1 - (m-1)q_i}{2},$$

implying

$$q_i = \frac{1}{m+1}.$$

(It is not hard to prove the equilibrium is actually symmetric.) Now we can confirm that our assumption $1 - Q \geq 0$ is actually satisfied because

$$Q = mq_i = \frac{m}{m+1} < 1.$$

The equilibrium payoff of each firm is

$$\pi = \frac{1}{(m+1)^2}.$$

Stage 1 Let $\pi(m)$ be the profit (gained in stage 2) of each firm when m firms are entering. As we solved above,

$$\pi(m) = \frac{1}{(m+1)^2}.$$

There are three possible classes of equilibria:

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1. All n firms enter.
2. No firm enters
3. k firms enter, where $0 < k < n$.

The first class of equilibrium exists if it is better to enter and gain $\pi(n)-F$ than to exit and gain 0. That is,

$$\pi(n) - F \geq 0,$$
$$\frac{1}{(n+1)^2} \geq F.$$

The second class of equilibrium exists if it is better to exit and get 0 than to enter to be the only entering firm and gain $\pi(1) - F$. That is,

$$\pi(1) - F < 0,$$
$$\frac{1}{4} \leq F.$$

To see when the third class of equilibrium exists, we need to check the incentives of the firms which are entering and the firms which are not entering.

Let's consider the incentive of an entering firm first. It gets $\pi(k) - F$ if it enters, and gets 0 if it exits. Therefore, there is no incentive to deviate if

$$\pi(k) - F \geq 0,$$
$$\frac{1}{(k+1)^2} \geq F.$$

Next consider the incentive of an exiting firm. It gets 0 if it exits. If it enters, it gets $\pi(k+1) - F$ because k other firms are also entering. Therefore, there is no incentive to deviate if

$$\pi(k+1) - F \leq 0,$$
$$\frac{1}{(k+2)^2} \leq F.$$

Notice, the argument above says nothing about which k firms enter. If the condition above is satisfied, any selection of k firms constitutes an equilibrium.

To summarize, let's formally describe the equilibrium strategies. Remember, we need to describe all the contingent plans for stage 2, even for the information sets which are not achieved on the equilibrium path.

The equilibrium strategies are as the following.

1.
   - All firms enter in stage 1.
   - If m firms enter in stage 1, each entering firm produces $\frac{1}{m+1}$ in stage 2.

   The equilibrium payoff is $\frac{1}{(n+1)^2} - F$ for every firm. This equilibrium exists if and only if $\frac{1}{(n+1)^2} \geq F$.

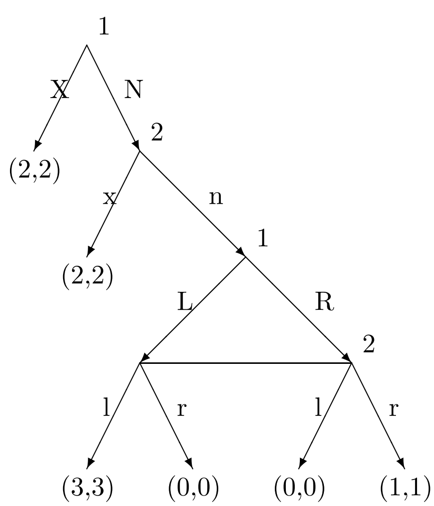
2.
   - No firm enter in stage 1.
   - If m firms enter in stage 1, each entering firm produces $\frac{1}{m+1}$ in stage 2.

   The equilibrium payoff is 0 for every firm. This equilibrium exists if and only if $\frac{1}{4} \leq F$.

3.
   - k firms enter in stage 1, where $0 < k < n$.
   - If m firms enter in stage 1, each entering firm produces $\frac{1}{m+1}$ in stage 2.

   The equilibrium payoff is $\frac{1}{(k+1)^2} - F$ for every entering firm, and 0 for every exiting firm. This equilibrium exists if and only if $\frac{1}{(k+1)^2} \geq F$ and $\frac{1}{(k+2)^2} \leq F$.

Question 2
(a)



We can solve for subgame perfect equilibria by backward induction. This game has three subgames: the whole game itself, a subgame starting from player

2's first decision node, and a subgame starting from player 1's second decision node.

First we look at the subgame starting from player 1's second decision node. This subgame is equivalent to the following nomal form game:

	l	r
L	3,3	0,0
R	0,0	1,1

Obviously, pure strategy equilibria are (L,l) and (R,r).

If (L,l) is played in this subgame, player 2 chooses n at his first decision node, because he gets payoff of 2 by playing x and 3 by playing n. Given this, player 1 chooses N at her first decision node because she gets payoff of 2 by playing X and 3 by playing N. Therefore, player 1's strategy (N,L) and player 2's strategy (n,l) constitute a subgame perfect Nash equilibrium.

If (R,r) is played in the subgame starting from player 1's second decision node, player 2 chooses x at his first decision node, because he gets payoff of 2 by playing x and 1 by playing n. Given this, player 1 is indifferent between X and N at her first decision node because she gets payoff of 2. Therefore, player 1's strategy (X,R) and player 2's strategy (x,r) constitute a subgame perfect Nash equilibrium, and player 1's strategy (N,r) and player 2's strategy (x,r) constitute another subgame perfect Nash equilibrium.

To conclude, there are three subgame perfect Nash equilibria:

1. Player 1: (N,L), Player 2: (n,l)
2. Player 1: (X,R), Player 2: (x,r)
3. Player 1: (N,R), Player 2: (x,r)

(b)

Let's apply the forward induction argument here.

Suppose player 1's second decision node is reached. This means that player 2 has chosen N at his first decision node, i.e., his strategy is either (n,l) or (n,r) (or any mixture of them). Should player 1 think that player 2 is choosing (n,r)? Player 2 can get payoff of 0 or 1 by playing (n,r), whereas he can get 2 for sure by playing (x,l) or (x,r). In other words, (n,r) is a strictly dominated strategy. If player 1 knows player 2 is rational, she must know that there is no reason player 2 play (n,r). Therefore, given player 2 has played N at his first decision node, player 1 must conclude player 2 is choosing (X,l), and choose the best response to it, namely L. To conclude, in any subgame perfect equilibrium which is consistent with common knowlegde of rationality, player 1's choice at his second decision node must be L.

Therefore, only the first equilibrium (N,L), (n,l) is consistent.

Question 3
(a)

The monopoly price is $p^m = \frac{1}{2}$. Construct the following "trigger strategy".

- Start with playing p^m, and play p^m if everybody has played p^m all the time.
- Play 0 if anybody has played something different from p^m at least once.

By the single deviation principle, it is sufficient to check incentives to deviate only once, in order to check when this is actually a subgabe perfect Nash equilibrium.

Firstly, check the incentive when nobody has deviated from p^m before. If a firm plays p^m, as suggested by the strategy, then everyone will be playing p^m all the time in future, and a firm's profit per period is

$$\frac{p^m(1-p^m)}{2} = \frac{1}{8},$$

and the present discounted sum of the profit sequence from today is

$$\frac{1}{8} + \delta\frac{1}{8} + \delta^2\frac{1}{8} + \cdots = \frac{1}{1-\delta}\frac{1}{8}.$$

If the firm is to deviate, the best it can do today is to set a price slightly less than p^m and get the whole monopoly profit

$$p^m(1-p^m) = \frac{1}{4}.$$

If the firm does deviate, everyone will be playing 0 from the next period, yielding per period payoff of zero. Therefore, the present discounted sum of the profit sequence from today is

$$\frac{1}{4} + \delta 0 + \delta^2 0 + \cdots = \frac{1}{4}.$$

The firm has no incentive to deviate if and only if

$$\frac{1}{1-\delta}\frac{1}{8} \geq \frac{1}{4}$$

$$\delta \geq \frac{1}{2}.$$

Secondly, check the incentive when somebody has ever deviated from p^m before. In this case, a firm's action today doesn't affect it's future payoff because everyone will be playing 0 in future no matter what happens today. Therefore it is sufficient to check if the firm can increase the present period profit by deviating. If it plays $p = 0$, as suggested by the strategy, it gets a payoff of 0.

If it deviates and plays $p > 0$, it gets a payoff of 0 again because all consumers buy from the other firm which is choosing $p = 0$. Therefore it can't benefit from deviating.

To conclude, the trigger strategy constitutes a subgame perfect Nash equilibrium if $\delta \geq \frac{1}{2}$.

(b)

When there are n firms, we can use the trigger strategy constructed in part (a) again. As we did in part (a), we check incentives to deviate before and after any deviation.

Firstly, check the incentive when nobody has deviated from p^m before. If a firm plays p^m, as suggested by the strategy, then everyone will be playing p^m all the time in future, and a firm's profit per period is

$$\frac{p^m(1-p^m)}{n} = \frac{1}{4n},$$

and the present discounted sum of the profit sequence from today is

$$\frac{1}{4n} + \delta\frac{1}{4n} + \delta^2\frac{1}{4n} + \cdots = \frac{1}{1-\delta}\frac{1}{4n}.$$

If the firm is to deviate, the best it can do today is to set a price slightly less than p^m and get the whole monopoly profit

$$p^m(1-p^m) = \frac{1}{4}.$$

If the firm does deviate, everyone will be playing 0 from the next period, yielding per period payoff of zero. Therefore, the present discounted sum of the profit sequence from today is

$$\frac{1}{4} + \delta 0 + \delta^2 0 + \cdots = \frac{1}{4}.$$

The firm has no incentive to deviate if and only if

$$\begin{aligned} \frac{1}{1-\delta}\frac{1}{4n} &\geq \frac{1}{4} \\ \delta &\geq \frac{n-1}{n}. \end{aligned}$$

To check the incentive when somebody has ever deviated from p^m before, the same argument as in part (a) holds.

To conclude, the trigger strategy constitutes a subgame perfect Nash equilibrium if $\delta \geq \frac{n-1}{n}$.

Question 4
(a)

To begin with, let's find the subgame perfect Nash equilibrium of the stage game. We do this by backward induction. The long run firm, observing the short run firm's quantity x_t, chooses its quantity y_t to maximize its profit

$$\pi_t^L = y_t(1 - (x_t + y_t)).$$

Solving first order condition, the optimum is

$$y_t^*(x_t) = \frac{1 - x_t}{2}.$$

The short run firm chooses its quantity x_t to maximize its profit

$$\pi_t^S = x_t(1 - (x_t + y_t)),$$

knowing that if it choose x_t, the long run firm reacts with

$$y_t^*(x_t) = \frac{1 - x_t}{2}.$$

Therefore, the short run firm's objective function can be rewritten as a function of x_t;

$$\pi_t^S = x_t(1 - (x_t + \frac{1 - x_t}{2})).$$

Solving first order condition, the optimum is

$$x_t^* = \frac{1}{2}.$$

Therefore, the subgame perfect Nash equilibrium of the stage game is

$$x_t^* = \frac{1}{2}, \;\; y_t^*(x_t) = \frac{1 - x_t}{2}.$$

Now let's solve for the subgame perfect Nash equilibrium of the finitely repeated game. Since it is finite, we can use backward induction. At the last period, $t = T$, the players don't care about future and they concern only about the payoff of that period. Therefore they must play the subgame perfect Nash equilibrium of the stage game, regardless of what happened in the past. At time $t = T - 1$, the players know that the actions today doesn't affect tomorrow's outcome, so they concern only about the payoff of that period. Therefore they must play the subgame perfect Nash equilibrium of the stage game, regardless of what happened in the past. We can repeat the same argument until we reach the very first period.

Therefore, the subgame perfect Nash equilibrium of the repeated game is

$$x_t^* = \frac{1}{2}, \;\; y_t^*(x_t) = \frac{1 - x_t}{2} \;\text{ for all }\; t,$$

regardless of the history.

(b)

Construct the following trigger strategy.

Long rum firm

- Start with playing the following strategy:

$$(*)\ \ y_t(x_t) = \begin{cases} 1/2 & \text{if} \quad x_t \leq 1/2 \\ 1 - x_t & \text{if} \quad x_t \geq 1/2 \end{cases}$$

Keep playing this strategy as long it has not deviated from it.

- Play $y_t^*(x_t) = \frac{1-x_t}{2}$ if it has deviated from (*) at least once.

Short run firms

- Play $x_t = 1/4$ if the long run firm has not deviated from (*) before.
- Play $x_t = \frac{1}{2}$ if the long run firm has deviated from (*) at least once.

To see this is actually a subgame perfect Nash equilibrium, let's check incentives to deviate.

Firstly, consider the long run firm's incentive when it has never deviated from (*) before.

Case 1: $x_t \leq 1/2$ If it follows the strategy and choose $y_t = 1/2$, the present period profit of the long run firm is

$$\frac{1}{2}(1 - (\frac{1}{2} + x_t)).$$

Starting from the next period, the outcome will be $x_t = 1/4$ and $y_t = 1/2$ every period, and the long run firm's per period profit is 1/8 and therefore the present discounted value of the profit sequence is

$$\frac{1}{2}(1 - (\frac{1}{2} + x_t)) + \frac{\delta}{8(1-\delta)}.$$

If it is to deviate, the best it can do today is to play

$$y_t = \frac{1 - x_t}{2}$$

and get payoff of

$$\frac{(1 - x_t)^2}{4}.$$

However, starting from next period, the outcome will be $x_t = 1/2$ and $y_t = 1/4$ and the long run firm's per period profit is 1/16. Therefore the present discounted value of the profit sequence is

$$\frac{(1 - x_t)^2}{4} + \frac{\delta}{16(1-\delta)}.$$

If $\delta = 0.99$,

$$\frac{1}{2}(1-(\frac{1}{2}+x_t)) + \frac{\delta}{8(1-\delta)} > \frac{(1-x_t)^2}{4} + \frac{\delta}{16(1-\delta)},$$

(check it!), and therefore it is better to follow the equilibrium strategy than to deviate.

Case 2: $x_t \geq 1/2$ If it follows the strategy, $p_t = 0$ and today's payoff is 0, and the outcome will be $x_t = 1/4$ and $y_t = 1/2$ every period, starting the next period. The long run firm's per period profit is 1/8 and therefore the present discounted value of the profit sequence is

$$\frac{\delta}{8(1-\delta)}.$$

If it is to deviate, the best it can do today is to play $y_t = \frac{1-x_t}{2}$ and get payoff of $\frac{(1-x_t)^2}{4}$. However, starting from next period, the outcome will be $x_t = 1/2$ and $y_t = 1/4$ and the long run firm's per period profit is 1/16. Therefore the present discounted value of the profit sequence is

$$\frac{(1-x_t)^2}{4} + \frac{\delta}{16(1-\delta)}.$$

This value is the largest when $x_t = 1/2$ and is equal to

$$\frac{1}{16} + \frac{\delta}{16(1-\delta)}.$$

If $\delta = 0.99$, it is better to follow the equilibrium strategy than to deviate (check it!).

Secondly, consider the long run firm's incentive when it has deviated from (*) before. According to the strategy strategy profile, future outcomes don't depend on today's behavior. Therefore the long run firm cares only about its payoff today. Actually, by following the strategy, it is taking best response to the short run firm.

Finally, consider the short run firm's incentives. Since they never care about future payoff, it must be playing a best response to the long run firm's strategy, which is actually true. (Check it!)

(c)

In the equilibrium we saw in part (b), the per period profit on the equilibrium path were 1/8 for the long run firm and 1/16 for the short run firms.

If there is a subgame perfect Nash equilibrium where $x_t = y_t - 1/4$ on th eequilibrium path, then the per period profit on the equilibrium path are 1/8 for the long run firm and 1/8 for the short run firms.

Construct the following trigger strategy, which is different from (b) only in (*) where $x_t = 1/4$:

Long rum firm

- Start with playing the following strategy:

$$(*)\ \ y_t(x_t) = \begin{cases} 1/4 & \text{if} \quad x_t = 1/4 \\ 1/2 & \text{if} \quad x_t \leq 1/2 \ \text{ and } \ x_t \neq 1/4 \\ 1 - x_t & \text{if} \quad x_t \geq 1/2 \end{cases}$$

  Keep playing this strategy as long it has not deviated from it.

- Play $y_t^*(x_t) = \frac{1-x_t}{2}$ if it has deviated from (*) at least once.

Short run firms

- Play $x_t = 1/4$ if the long run firm has not deviated from (*) before.
- Play $x_t = \frac{1}{2}$ if the long run firm has deviated from (*) at least once.

The incentive of the long run firm when it is supposed to play (*) and $x_t = 1/4$ is satisfied because it gets current payoff of 1/8, which is same as part (b), and what it can get by deviating is same as in part (b). The incentive problem of the long run firm is same as in (b).

The incentives of the short run firms are also satisfied because the payoff from following the strategy is larger than part (b) and payoff when deviating is same as in (b).

14.12 Game Theory
Fall 2001
Problem Set 4
Due on 11/7

1. Consider the following variant of the Battle of the Sexes: player 1 is unsure whether player 2 prefers to go out with her or prefers to avoid her, while player 2 knows player 1's preferences. Specifically, suppose player 1 thinks that with probability $\frac{1}{2}$ the game is

	B	S
B	$2,1$	$0,0$
S	$0,0$	$1,2$

   and with probability $\frac{1}{2}$ it is

	B	S
B	$2,0$	$0,2$
S	$0,1$	$1,0$

   Player 2 knows which game is being played.

   (a) Model this as a Bayesian game; that is, write down the action spaces, type spaces, prior beliefs, and von Neuman-Morgenstern utilities.

   (b) Show that player 1 playing B and player 2 playing B in the top game and S in the bottom game is a Bayesian Nash Equilibrium.

2. Gibbons prob. 3.3

3. Gibbons prob. 3.5

4. Gibbons prob. 3.6

14.12 Solutions for Homework 4

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November 9, 2001

Question 1

(a)

Action spaces $A_1 = A_2 = \{B, S\}$.

Type spaces $T_1 = \{\alpha\}$, $T_2 = \{\beta_1, \beta_2\}$. Since player 1 has no private information, we can model it so that her type can take only one value. Player 2 knows that the game above is played when his type is β_1 and he knows that the game below is played when his type is β_2.

Belief Player i's belief $\mu_i(t_j|t_i)$ is the probability that player j's type is t_j conditional on that player i's type is t_i. In this model, since it is assumed that the types are independent,

$$\mu_1(\beta_1|\alpha) = \mu_1(\beta_2|\alpha) = \frac{1}{2},$$
$$\mu_2(\alpha|\beta_1) = \mu_2(\alpha|\beta_2) = 1.$$

vNM utility function $u_i(a_1, a_2, t_1, t_2)$ is the vNM utility when player 1's action is a_1, player 2's action is a_2, player 1's type is t_1 and player 2's type is t_2.

$$u_1(B, B, \alpha, \beta_1) = 2, \quad u_2(B, B, \alpha, \beta_1) = 1,$$
$$u_1(B, S, \alpha, \beta_1) = 0, \quad u_2(B, S, \alpha, \beta_1) = 0,$$
$$u_1(S, B, \alpha, \beta_1) = 0, \quad u_2(S, B, \alpha, \beta_1) = 0,$$
$$u_1(S, S, \alpha, \beta_1) = 1, \quad u_2(S, S, \alpha, \beta_1) = 2,$$
$$u_1(B, B, \alpha, \beta_2) = 2, \quad u_2(B, B, \alpha, \beta_2) = 0,$$
$$u_1(B, S, \alpha, \beta_2) = 0, \quad u_2(B, S, \alpha, \beta_2) = 2,$$
$$u_1(S, B, \alpha, \beta_2) = 0, \quad u_2(S, B, \alpha, \beta_2) = 1,$$
$$u_1(S, S, \alpha, \beta_2) = 1, \quad u_2(S, S, \alpha, \beta_2) = 0.$$

(b)

First consider player 1's incentive. Since she doesn't know the game which is being played, she maximizes her expected payoff.

If she plays B, with probability 1/2 the top game is being played and player 2 chooses B and thus she gets a payoff of 2, and with probability 1/2 the bottom game is being played and player 2 chooses S and thus she gets a payoff of 0.

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Therefore her expected payoff is 1. If she plays S, with probability 1/2 the top game is being played and player 2 chooses B and thus she gets a payoff of 0, and with probability 1/2 the bottom game is being played and player 2 chooses S and thus she gets a payoff of 1. Therefore her expected payoff is 1/2.

Therefore B is actually player 1's best response against player 2's strategy.

Next consider player 2's incentive. When he knows that the top game is being played, B is the best response given player 1 is choosing B. When he knows that the bottom game is being played, S is the best response given player 1 is choosing B.

Therefore choosing B when the top game is being played and choosing S when the bottom game is being played is actually player 2's best response against player 1's strategy.

Since both players are taking a best response to each other, the strategy profile constitutes a Bayesian Nash equilibrium.

Question 2

Each player's action is the choice of price. A price can take any nonnegative real number. Therefore the action space is R_+ for both players.

Player i's type is her private information. In this model, b_i is player i's type. b_i is either b_H or b_L. Therefore, the type space of each player is $\{b_H, b_L\}$.

Player i's belief $\mu_i(b_j|b_i)$ is the probability that player j's type is b_j conditional on that player i's type is b_i. In this model, since it is assumed that the types are independent,

$$\mu_i(b_j|b_i) = \begin{cases} \theta & \text{if} \quad b_j = b_H \\ 1-\theta & \text{if} \quad b_j = b_L \end{cases} .$$

(von Neuman Morgenstern) utility in this model is the profit of each player (assuming the firms are risk neutral) as a function of the actions and types of both players:

$$u_i(p_i, p_j, b_i, b_j) = p_i(a - p_i - b_i p_j).$$

Player i's strategy specifies what action to take for any realization of her type. In this model, it is a two dimensional vector $(p_i(b_H), p_i(b_L))$, where $p_i(b_H)$ is the price when its type is b_H and $p_i(b_L)$ is the price when its type is b_L. The strategy space is R_+^2 for each i.

A strategy profile $\{(p_1^*(b_H), p_1^*(b_L)), (p_2^*(b_H), p_2^*(b_L))\}$ constitutes a Bayesian Nash equilibrium if each $p_i^*(b_i)$ is a best response, i.e., a maximizer of player i's expected payoff, conditional on that her type is b_i and the opponent is choosing strategy $(p_j^*(b_H), p_j^*(b_L))$. That is:

$$\begin{aligned} p_1^*(b_H) &= \text{argmax}_{p_1} \theta p_1(a - p_1 - b_H p_2^*(b_H)) + (1-\theta) p_1(a - p_1 - b_H p_2^*(b_L)), \\ p_1^*(b_L) &= \text{argmax}_{p_1} \theta p_1(a - p_1 - b_L p_2^*(b_H)) + (1-\theta) p_1(a - p_1 - b_L p_2^*(b_L)), \\ p_2^*(b_H) &= \text{argmax}_{p_2} \theta p_2(a - p_2 - b_H p_1^*(b_H)) + (1-\theta) p_2(a - p_2 - b_H p_1^*(b_L)), \\ p_2^*(b_L) &= \text{argmax}_{p_2} \theta p_2(a - p_2 - b_L p_1^*(b_H)) + (1-\theta) p_2(a - p_2 - b_L p_1^*(b_L)). \end{aligned}$$

Taking first order conditions,

$$p_1^*(b_H) = \frac{a - b_H(\theta p_2^*(b_H) + (1-\theta)p_2^*(b_L))}{2},$$
$$p_1^*(b_L) = \frac{a - b_L(\theta p_2^*(b_H) + (1-\theta)p_2^*(b_L))}{2},$$
$$p_2^*(b_H) = \frac{a - b_H(\theta p_1^*(b_H) + (1-\theta)p_1^*(b_L))}{2},$$
$$p_2^*(b_L) = \frac{a - b_L(\theta p_1^*(b_H) + (1-\theta)p_1^*(b_L))}{2}.$$

Since the game is symmetric, let's look for a symmetric equilibrium where $p_1^*(b_H) = p_2^*(b_H) = p_H^*$ and $p_1^*(b_L) = p_2^*(b_L) = p_L^*$. Then the conditions are reduced to

$$p_H^* = \frac{a - b_H(\theta p_H^* + (1-\theta)p_L^*)}{2},$$
$$p_L^* = \frac{a - b_L(\theta p_H^* + (1-\theta)p_L^*)}{2}.$$

Solving these, we get

$$p_H^* = \frac{a}{2}\left(1 - \frac{b_H}{2 + \theta b_H + (1-\theta)b_L}\right),$$
$$p_L^* = \frac{a}{2}\left(1 - \frac{b_L}{2 + \theta b_H + (1-\theta)b_L}\right).$$

Question 3

Consider the following incomplete information game.

	H	T
H	$1+\epsilon\alpha,\ -1+\epsilon\beta$	$-1+\epsilon\alpha,\ 1$
T	$-1,\ 1+\epsilon\beta$	$1,\ -1$

Here, α is player 1's type, β is player 2's type, α and β are independent draw from uniform distribution in the interval $[0,1]$, and $\epsilon > 0$ is a constant.

The interpretation is like this: Player 1 (2) can get an extra payoff of $\epsilon\alpha$ ($\epsilon\beta$) by playing head, in addition to the payoff generated from the original game. This extra payoff is her private information.

Notice, as we converge ϵ to 0, the game converges to the original game.

From the construction, player 1 (2) is more tempted to play H if she has a higher value of α (β). Observing this, try to find the pure strategy Bayesian Nash equilibrium of the following form.

- Player 1 chooses H if $\alpha \geq \bar{\alpha}$ and chooses L if $\alpha \leq \bar{\alpha}$, where $\bar{\alpha} \in [0,1]$ is a constant.

- Player 2 chooses H if $\beta \geq \bar{\beta}$ and chooses L if $\beta \leq \bar{\beta}$, where $\bar{\beta} \in [0, 1]$ is a constant.

This strategy profile is actually a Bayesian Nash equilibrium if player 1 with cutoff type $\bar{\alpha}$ and player 2 with cutoff type $\bar{\beta}$ are indifferent between choosing H and T.

According to the strategy, the probability that player 2 plays H is $1-\bar{\beta}$, and the probability that player 2 plays L is $\bar{\beta}$. Therefore, if player 1 of type $\bar{\alpha}$ plays H, her expected payoff is

$$1(1-\bar{\beta}) + (-1)\bar{\beta} + \epsilon\bar{\alpha},$$

and if she plays T, she gets

$$(-1)(1-\bar{\beta}) + 1\bar{\beta}.$$

Therefore, we need

$$1(1-\bar{\beta}) + (-1)\bar{\beta} + \epsilon\bar{\alpha} = (-1)(1-\bar{\beta}) + 1\bar{\beta}. \tag{1}$$

We do the same thing for player 2 with type $\bar{\beta}$. According to the strategy, the probability that player 1 plays H is $1-\bar{\alpha}$, and the probability that player 1 plays L is $\bar{\alpha}$. Therefore, if player 2 of type $\bar{\beta}$ plays H, his expected payoff is

$$(-1)(1-\bar{\alpha}) + 1\bar{\alpha} + \epsilon\bar{\beta},$$

and if she plays T, she gets

$$1(1-\bar{\alpha}) + (-1)\bar{\alpha}.$$

Therefore, we need

$$(-1)(1-\bar{\alpha}) + 1\bar{\alpha} + \epsilon\bar{\beta} = 1(1-\bar{\alpha}) + (-1)\bar{\alpha}. \tag{2}$$

Solving (1) and (2), we get

$$\bar{\alpha} = \frac{8-2\epsilon}{16+\epsilon^2}, \tag{3}$$

$$\bar{\beta} = \frac{8+2\epsilon}{16+\epsilon^2}. \tag{4}$$

Now, as we converge ϵ to 0, i.e., as we converge the game to the complete information game, both $\bar{\alpha}$ and $\bar{\beta}$ converge to 1/2. Therefore, in the limit, both players are choosing H and T with probability 1/2, which is equivalent to the mixed strategy equilibrium of the complete information game.

Question 4

Let $i = 1, \cdots, n$ be the index of bidders, v_i be bidder i's valuation of the good, and b_i be player i's bid. We denote player i's strategy by a function $x_i(v_i)$, meaning that player i bids $b_i = x_i(v_i)$ when her valuation is v_i.

We want to show that the strategy profile

$$x_i(v_i) = \frac{(n-1)v_i}{n} \text{ for all } i$$

constitutes a Bayesian Nash equilibrium. Since the game is symmetric and strategy profile is symmetric, it is sufficient to check one player's incentive because every player is facing the same incentive problem.

We are going to show that if player i's valuation is v_i and all other players are taking the strategy

$$x_j(v_j) = \frac{(n-1)v_j}{n},$$

then the bid which maximizes her expected payoff is

$$b_i = x_i(v_i) = \frac{(n-1)v_i}{n}.$$

First consider the probability of winning the auction by bidding b_i. She wins if and only if all other players' bids are less than b_i, i.e.,

$$x_j(v_j) = \frac{(n-1)v_j}{n} \leq b_i \text{ for all } j \neq i.$$

This is equivalent to

$$v_j \leq \frac{nb_i}{n-1} \text{ for all } j \neq i.$$

Since

$$Prob(v_j \leq \frac{nb_i}{n-1}) = \frac{nb_i}{n-1}$$

for all j because v_j is uniformly distributed in [0.1],

$$\begin{aligned} Prob(win) = Prob(v_j \leq \frac{nb_i}{n-1} \text{ for all } j \neq i) \\ = Prob(v_j \leq \frac{nb_i}{n-1})^{n-1} \\ = (\frac{nb_i}{n-1})^{n-1}. \end{aligned}$$

Therefore, the expected payoff from bidding b_i is

$$U_i = (v_i - b_i) Prob(win) = (v_i - b_i)(\frac{nb_i}{n-1})^{n-1}.$$

Taking first order condition,

$$\frac{\partial U_i}{\partial b_i} = (v_i - b_i)(n-1)\Big(\frac{nb_i}{n-1}\Big)^{n-2}\frac{n}{n-1} - \Big(\frac{nb_i}{n-1}\Big)^{n-1} = 0,$$
$$(v_i - b_i)n = \frac{nb_i}{n-1},$$
$$b_i = \frac{(n-1)v_i}{n}.$$

Therefore, the strategy of player i

$$b_i = x_i(v_i) = \frac{(n-1)v_i}{n}$$

is actually the best response to other player's playing

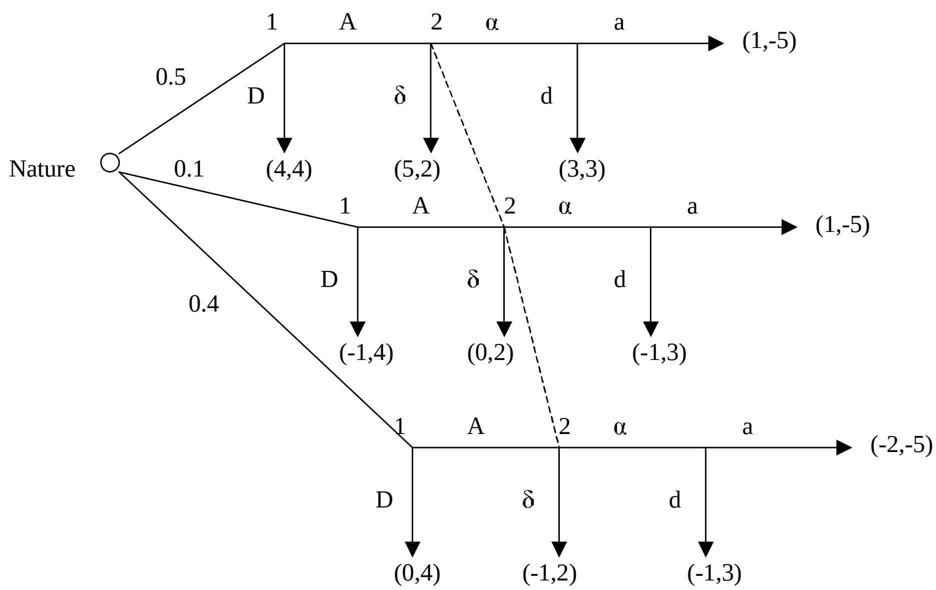
$$x_j(v_j) = \frac{(n-1)v_j}{n}.$$

Q.E.D.

14.12 Game Theory
Fall 2001
Problem Set 5
Due on 12/7

This is an optional homework. Therefore, if you return this, two of the worst homeworks will be dropped, otherwise only one. Due to the MIT regulation, this is due on Friday, but as usual later submissions can be accepted depending on how late they are.

1. Compute all the perfect Bayesian equilibria of the following game.



2. Consider the entry deterrence game, where an Entrant decides whether to enter the market; if he enters the Incumbent decides whether to Fight or Accommodate. We consider a game where Incumbent's payoff from the Fight is private information, the entry deterrence game is repeated twice and the discount rate is $\delta = 0.9$. The payoff vectors for the stage game are (0,2) if the Entrant does not enter, $(-1, a)$ if he enters and the Incumbent Fights; and (1,1) if he enters and the Incumbent accommodates, where the first entry in each parenthesis is the payoff for the entrant. Here, a can be either -1 or 2, and is privately known by the Incumbent. Entrants believes that $a = -1$ with probability π; and everything described up to here is common knowledge.

   (a) Find the perfect Bayesian Equilibrium when $\pi = 0.4$.

   (b) Find the perfect Bayesian Equilibrium when $\pi = 0.9$.

   (c) What would happen if the game is repeated 100 times?

3. Consider a buyer and a seller. The seller owns an object, whose value for himself is 0. The value of the object for the buyer is $v \in \{1, 2\}$. The seller believes that $v = 2$ with

probability π, while the buyer knows what v is. We have two dates, $t = 0, 1$. At $t = 0$, the buyer offers a price p_0. If the seller accepts, trade occurs at price p_0, when the payoffs of the buyer and the seller are $v - p_0$ and p_0, respectively. If the seller rejects, at $t = 1$, the seller sets another price p_1. If the buyer accepts the price, the trade occurs at price p_1, when the payoffs of the buyer and the seller are $0.9\,(v - p_1)$ and $0.9p_1$, respectively. Otherwise, there will be no trade, when both get 0. Everything described up to here is common knowledge.

(a) Compute the perfect Bayesian Nash equilibrium for $\pi = 0.4$.

(b) Compute the perfect Bayesian Nash equilibrium for $\pi = 0.6$.

4. Gibbons 4.3.a.

5. Gibbons 4.4.

14.12 Game Theory
Fall 2001
Problem Set 5 Solutions

December 14, 2001

1. The possibilities are: Pooling on A, pooling on D, and hybrid equilibria (since there are only two possible actions for player 1, but three possible types of player 1). Let us write the strategies in the form $(t_{11}, t_{21}, t_{31}; t_{12}, t_{22}, t_{32})$ for player 1, where t_ij indicates the action taken by type i at the jth move, and (a) for the strategy of player 2, which gets to move only once and has only one information set. The only equilibrium here is $(A, A, D; d, a, d)$ and (δ). Player 2 moves δ since $u_2(\alpha) = 5/6(3) + 1/6(-5) = 5/3 < 2 = u_2(\delta)$ – note how beliefs are updated. Finally, each type of player 1 does not have an incentive to deviate: top type is getting 5 as opposed to 4 (if deviates and plays D); middle type is getting 0 as opposed to -1 (if deviates and plays D); bottom type is getting 0 as opposed to -1 (if deviates and plays A).

2. In the second stage of the game, it is clear that each type will always play the static best response: $a = 2$ type will fight and $a = -1$ type will accomodate.

   The possibilities for the first stage are:

   (a) Pooling on Fight: If $\pi = 0.4$, then at $t = 2$, given no updating of beliefs (so that $\mu(a = 2|F) = \pi$), the entrant stays out since $EU(Enter) = 2\pi - 1 = -0.2 < 0 = EU(Out)$. Then, in $t = 1$, incumbents pool on fight. The PBE equilibrium is $[(F, F; F, A), (O, O, E)]$, where we write the strategy for player 1 as $(t_{11}, t_{21}; t_{12}, t_{22})$ where t_ij is the action taken by type i in stage j and for player 2 as $(a_1, a_{21}, a_{22}$ where a_1 refers to the first information set in $t = 1$, and the other two refer to the information sets in $t = 2$: a_{21} being the information set after the incumbent fights in the first period, and a_{22} being the information set after the incumbent fights in the first period. Beliefs are not updated on a_{21} and off the equilibrium belief on a_{22} is $\mu(a = -1|accomodate) = 1$.

   (b) Separating equilibrium: If $\pi = 0.9$, then entrant always enters since $EU(Enter) = 2\pi - 1 = 0.8 > 0 = EU(Out)$, a=2 type fights and a=-1 type accomodates. The PBE is $[(F, A; F, A), (E, O, E)]$. Beliefs are updated such that at a_{22} we have $\mu(a = -1|accomodate) = 1$ and at a_{21} we have $\mu(a = 2|fight) = 1$.

If the game is repeated 100 times a similar structure applies.

3. In $t = 1$, buyer of type 1 will accept if and only if:

   $0.9(1 - p_1) \geq 0$, that is $p_1 \leq 1$
   $0.9(2 - p_1) \geq 0$, that is $p_1 \leq 2$

   and seller will offer $p_1 \geq 0$.

   If seller offers $1 \leq p_1 \leq 2$, then type 1 rejects the offer and type 2 accepts it:

   $EU(seller) = (1 - \pi)(0) + p_1\pi = p_1\pi$

   or simply 2π, which is the value that maximizes the seller's utility.

   If seller offers $0 \leq p_1 \leq 1$ then both types accept the offer:

   $EU(seller) = p_1(1 - \pi) + p_1\pi = p_1$

   or simply 1, which is the value that maximizes the seller's utility.

   If $\pi > 1/2$, then seller offers $p_1 = 2$ in $t = 1$ and she accepts offers in $t = 0$ $p_0 \geq 0.9$ since if she waits until $t = 1$ she can get 0.9(1)=0.9 . If $\pi < 1/2$, then seller offers $p_1 = 1$ and she accepts offers in $t = 0$ $p_0 \geq 1.8$ since in $t = 1$ she can get 0.9(2)=1.8.

4.
   - Proposed pooling strategy of Sender: Suppose there is an equilibrium in which the Sender's strategy is (R, R), where (m', m'') means that type t_1 chooses m' and type t_2 chooses m''.
   - Receiver's belief at R: Then the Receiver's information set corresponding corresponding to R is on the equilibrium path, so the Receiver's belief $(p, 1 - p)$ at this information set is determined by Bayes'rule and the Sender's strategy: $p = .5$, the prior distribution (since there is no updating of belief in pooling).
   - Receiver's best response at R: Given this belief, the Receiver's best response following R is to play d since

     $u_2(u|R) = 1(.5) + 0(.5) = .5 < u_2(d|R) = 0(.5) + 2(.5) = 1$
   - Receiver's belief and best response at information set for L: To determine whether both Sender types are willing to choose R, we need to specify how the Receiver would react to L. In particular, we need to pin down the beliefs such that $u_2(u|L) > u_2(d|L)$ in order for the Sender's willingness to choose R. The Receiver's belief at the information set corresponding to L needs to be, in order for u to be the best response, $q \geq 1/3$. Thus, if there is an equilibrium in which the Sender's strategy is (R, R) then the Receiver's strategy must be (d, u) where (a', a'') means that the Receiver plays a' following R and a'' following L.
   - Pooling perfect Bayesian equilibrium: $[(R, R), (d, u), p = .5, q]$ for $q \geq 1/3$.

5. Part a.

- Pooling on R: Beliefs on the equilibrium path: $\mu(t_1|R) = .5$

  Receiver's best response: $u_2(u|R) = .5(2) = 1 > u_2(d|R) = .5(1) = .5$

  Sender's payoff: $u_{t_1}(R) = 2$,$u_{t_2}(R) = 1$.

  Any play by Receiver after observing L sustains (R, R).

  PBE: $[(R, R), (u, u), q > 1/2], [(R, R), (u, d), q < 1/2], [(R, R), (u, \alpha u + (1-\alpha)d), q = 1/2]$

- Pooling on L: Not an equilibrium, since t_2 will always want to play R.

- t_1 plays R, t_2 plays L: Then both of the Receiver's information sets are on the equilibrium path, so both beliefs are determined by Bayes' rule and the Sender's strategy: $\mu(t_1|R) = 1$, $\mu(t_2|L) = 1$.

  Receiver's best response: (u, d)

  t_2, however, will always play R.

- t_1 plays L, t_2 plays R: $\mu(t_1|L) = 1$, $\mu(t_2|R) = 1$.

  Receiver's best response: (u, d)

  Sender's payoff: $u_{t_1}(L) = 1$,$u_{t_2}(R) = 1$.

  PBE: $[(L, R), (u, d)]$

Part b.

- Pooling on R: Beliefs on the equilibrium path: $\mu(t_1|R) = .5$

  Receiver's best response: $u_2(u|R) = .5(2) = 1 > u_2(d|R) = .5(1) = .5$

  Sender's payoff: $u_{t_1}(R) = 0$,$u_{t_2}(R) = 1$.

  However, this is not an equilibrium since t_1 would always play L.

- Pooling on L: Beliefs on the equilibrium path: $\mu(t_1|R) = .5$

  Receiver's best response: $u_2(u|L) = .5(3) = 1.5 > u_2(d|L) = .5(1) + .5(1) = 1$

  Sender's payoff: $u_{t_1}(L) = 3$,$u_{t_2}(L) = 3$.

  Receiver's belief and best response at information set for R: $u_2(u|R) = 2(1-p) > u_2(d|R) = p$ when $p < 2/3$.

  PBE: $[(L, L), (u, u), p < 2/3]$

- t_1 plays R, t_2 plays L: Then both of the Receiver's information sets are on the equilibrium path, so both beliefs are determined by Bayes' rule and the Sender's strategy: $\mu(t_1|R) = 1$, $\mu(t_2|L) = 1$.

  Receiver's best response: (d, u)

  No deviations are profitable.

  PBE: $[(R, L), (d, u)]$

- t_1 plays L, t_2 plays R: $\mu(t_1|L) = 1$, $\mu(t_2|R) = 1$.

  Receiver's best response: (d, u)

  No deviations are profitable.

  PBE: $[(L, R), (d, u)]$

14.12 Game Theory – Midterm I
10/10/2001

Prof. Muhamet Yildiz

Instructions. This is an open book exam; you can use any written material. You have one hour and 20 minutes. Each question is 25 points. Good luck!

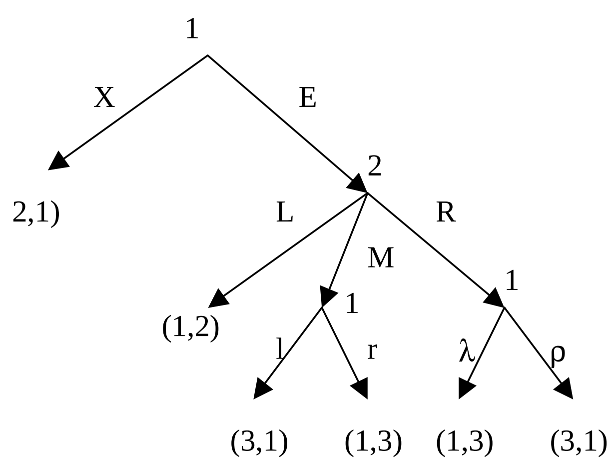
1. Find all the Nash equilibria in the following game:

1\2	L	M	R
T	1,0	0,1	5,0
B	0,2	2,1	1,0

2. Find all the pure strategies that are consistent with the common knowledge of rationality in the following game. (State the rationality/knowledge assumptions corresponding to each operation.)

1\2	L	M	R
T	1,1	0,4	2,2
M	2,4	2,1	1,2
B	1,0	0,1	0,2

3. Consider the following extensive form game.



   (a) Using Backward Induction, compute an equilibrium of this game.
   (b) Find the normal form representation of this game.
   (c) Find all pure strategy Nash equilibria.

4. In this question you are asked to compute the rationalizable strategies in linear Bertrand-duopoly with discrete prices. We consider a world where the prices must be the positive multiples of cents, i.e.,

$$P = \{0.01, 0.02, \ldots, 0.01n, \ldots\}$$

is the set of all feasible prices. For each price $p \in P$, the demand is

$$Q(p) = \max\{1 - p, 0\}.$$

We have two firms $N = \{1, 2\}$, each with zero marginal cost. Simultaneously, each firm i sets a price $p_i \in P$. Observing the prices p_1 and p_2, consumers buy from the firm with the lowest price; when the prices are equal, they divide their demand equally between the firms. Each firm i maximizes its own profit

$$\pi_i (p_1, p_2) = \begin{cases} p_i Q (p_i) & \text{if } p_i < p_j \\ p_i Q (p_i) /2 & \text{if } p_i = p_j \\ 0 & \text{otherwise,} \end{cases}$$

where $j \neq i$.

(a) Show that any price p greater than the monopoly price $p^{mon} = 0.5$ is strictly dominated by some strategy that assigns some probability $\epsilon > 0$ to the price $p^{\min} = 0.01$ and probability $1 - \epsilon$ to the price $p^{mon} = 0.5$.

(b) Iteratively eliminating all the strictly dominated strategies, show that the only rationalizable strategy for a firm is $p^{\min} = 0.01$.

(c) What are the Nash equilibria of this game?

14.12 Game Theory – Midterm I
ANSWERS
10/10/2001

Prof. Muhamet Yildiz

Instructions. This is an open book exam; you can use any written material. You have one hour and 20 minutes. Each question is 25 points. Good luck!

1. Find all the Nash equilibria in the following game:

1\2	L	M	R
T	1,0	0,1	5,0
B	0,2	2,1	1,0

   Answer: By inspection, there is no pure-strategy equilibrium in this game. There is one mixed strategy equilibrium. Since R is strictly dominated, player 2 will assign 0 probability to R. Let p and q be the equilibrium probabilities for strategies T and L, respectively; the probabilities for B and R are $1-p$ and $1-q$, respectively. If 1 plays T, his expected payoff is $q1 + (1-q)\,0 = q$. If he plays B, his expected payoff is $2\,(1-q)$. Since he assigns positive probabilities to both T and B, he must be indifferent between T and B. Hence, $q = 2\,(1-q)$, i.e., $q = 2/3$. Similarly, for player 2, the expected payoffs from playing L and M are $2\,(1-p)$ and 1, respectively. Hence, $2\,(1-p) = 1$, i.e., $p = 1/2$.

2. Find all the pure strategies that are consistent with the common knowledge of rationality in the following game. (State the rationality/knowledge assumptions corresponding to each operation.)

1\2	L	M	R
T	1,1	0,4	2,2
M	2,4	2,1	1,2
B	1,0	0,1	0,2

   Answer:

   (a) 1. For player 1, M strictly dominates B. Since **Player 1 is rational**, he will not play B, and we eliminate this strategy:

1\2	L	M	R
T	1,1	0,4	2,2
M	2,4	2,1	1,2

   2. Since **Player 2 knows that Player 1 is rational**, he will know that 1 will not play B. Given this, the mixed strategy that assigns probability 1/2 to each of the strategies L and M strictly dominates R. Since **Player 2 is rational**, in that case, he will not play R. We eliminate this strategy:

1\2	L	M
T	1,1	0,4
M	2,4	2,1

3. Since **Player 1 knows that Player 2 is rational and that Player 2 knows that Player 1 is rational**, he will know that 2 will not play R. Given this, M strictly dominates T. Since **Player 1 is rational**, he will not play T, either. We are left with

1\2	L	M
M	2,4	2,1

.

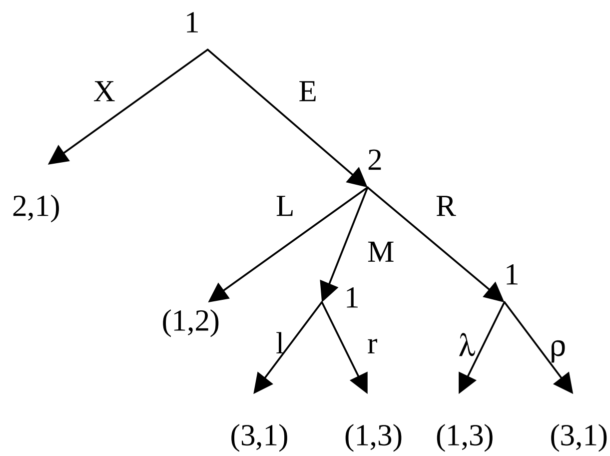
4. Since **Player 2 knows that Player 1 is rational, and that Player 1 knows that Player 2 is rational, and that Player 1 knows that Player 2 knows that Player 1 is rational**, he will know that Player 1 will not play T or B. Given this, L strictly dominates M. Since **Player 2 is rational**, he will not play M, either. He will play L.

1\2	L
M	2,4

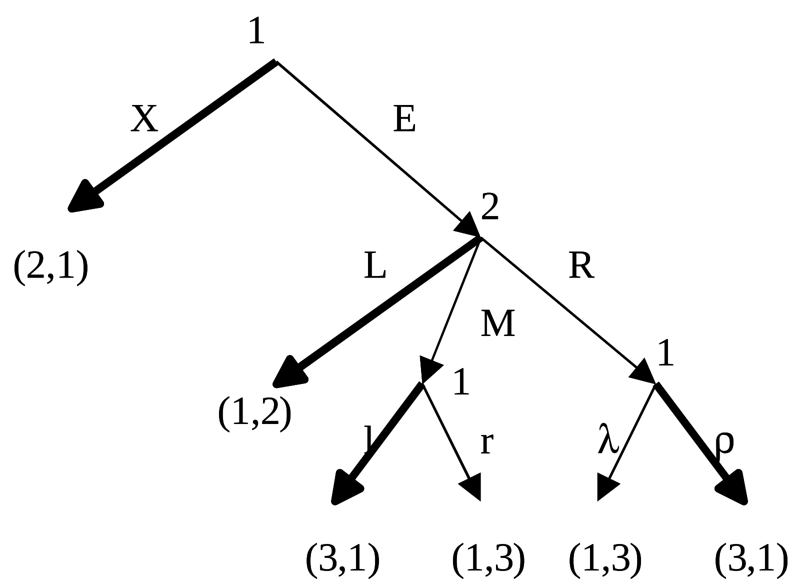
.

Thus, the only strategies that are consistent with the common knowledge of rationality are M for player 1 and L for player 2.

3. Consider the following extensive form game.



(a) Using Backward Induction, compute an equilibrium of this game.



(b) Find the normal form representation of this game.

1\2	L	M	R
$Xl\lambda$	2,1	2,1	2,1
$Xl\rho$	2,1	2,1	2,1
$Xr\lambda$	2,1	2,1	2,1
$Xr\rho$	2,1	2,1	2,1
$El\lambda$	1,2	3,1	1,3
$El\rho$	1,2	3,1	3,1
$Er\lambda$	1,2	1,3	1,3
$Er\rho$	1,2	1,3	3,1

The points will be taken off from the people who did not distinguish the strategies that start with X from each other.

(c) Find all pure strategy Nash equilibria.

1\2	L	M	R
$Xl\lambda$	**2,1**	2,1	2,1
$Xl\rho$	**2,1**	2,1	2,1
$Xr\lambda$	**2,1**	2,1	2,1
$Xr\rho$	**2,1**	2,1	2,1
$El\lambda$	1,2	3,1	1,3
$El\rho$	1,2	3,1	3,1
$Er\lambda$	1,2	1,3	1,3
$Er\rho$	1,2	1,3	3,1

The Nash equilibria are ($Xl\lambda$,L), ($Xl\rho$,L), ($Xr\lambda$,L), ($Xr\rho$,L).

4. In this question you are asked to compute the rationalizable strategies in linear Bertrand-duopoly with discrete prices. We consider a world where the prices must be the positive multiples of cents, i.e.,

$$P = \{0.01, 0.02, \ldots, 0.01n, \ldots\}$$

is the set of all feasible prices. For each price $p \in P$, the demand is

$$Q(p) = \max\{1 - p, 0\}.$$

We have two firms $N = \{1, 2\}$, each with zero marginal cost. Simultaneously, each firm i sets a price $p_i \in P$. Observing the prices p_1 and p_2, consumers buy from the firm with the lowest price; when the prices are equal, they divide their demand equally between the firms. Each firm i maximizes its own profit

$$\pi_i(p_1, p_2) = \begin{cases} p_i Q(p_i) & \text{if } p_i < p_j \\ p_i Q(p_i)/2 & \text{if } p_i = p_j \\ 0 & \text{otherwise,} \end{cases}$$

where $j \neq i$.

(a) Show that any price p greater than the monopoly price $p^{mon} = 0.5$ is strictly dominated by some strategy that assigns some probability $\epsilon > 0$ to the price $p^{\min} = 0.01$ and probability $1 - \epsilon$ to the price $p^{mon} = 0.5$.

Answer: Take any player i and any price $p_i > p^{mon}$. We want to show that the mixed strategy σ^ϵ with $\sigma^\epsilon(p^{mon}) = 1 - \epsilon$ and $\sigma^\epsilon(p^{\min}) = \epsilon$ strictly dominates p_i for some $\epsilon > 0$.

Take any strategy $p_j > p^{mon}$ of the other player j. We have

$$\pi_i(p_i, p_j) \leq p_i Q(p_i) = p_i(1 - p_i) \leq 0.51 \cdot 0.49 = 0.2499,$$

where the first inequality is by definition and the last inequality is due to the fact that $p_i \geq 0.51$. On the other hand,

$$\begin{aligned} \pi_i(\sigma^\epsilon, p_j) &= (1 - \epsilon) p^{mon}(1 - p^{mon}) + \epsilon p^{\min}(1 - p^{\min}) \\ &> (1 - \epsilon) p^{mon}(1 - p^{mon}) \\ &= 0.25(1 - \epsilon). \end{aligned}$$

Thus, $\pi_i(\sigma^\epsilon, p_j) > 0.2499 \geq \pi_i(p_i, p_j)$ whenever $0 < \epsilon \leq 0.0004$. Choose $\epsilon = 0.0004$.

Now, pick any $p_j \leq p^{mon}$. Since $p_i > p^{mon}$, we now have $\pi_i(p_i, p_j) = 0$. But

$$\pi_i(\sigma^\epsilon, p_j) = (1 - \epsilon) p^{mon}(1 - p^{mon}) + \epsilon p^{\min}(1 - p^{\min}) \geq \epsilon p^{\min}(1 - p^{\min}) > 0.$$

That is, $\pi_i(\sigma^\epsilon, p_j) > \pi_i(p_i, p_j)$. Therefore, σ^ϵ strictly dominates p_i.

(b) Iteratively eliminating all the strictly dominated strategies, show that the only rationalizable strategy for a firm is $p^{\min} = 0.01$.

Answer: We have already eliminated the strategies that are larger than p^{mon}. At any iteration t assume that, for each player, the set of all remaining strategies are $P^t = \{0.01, 0.02, \ldots, \bar{p}\}$ where $p^{\min} < \bar{p} \leq p^{mon}$. We want to show that $\bar{p}$ is strictly dominated by the mixed strategy $\sigma_{\bar{p}}^{\epsilon}$ with $\sigma_{\bar{p}}^{\epsilon}(\bar{p} - 0.01) = 1 - \epsilon$ and $\sigma_{\bar{p}}^{\epsilon}\left(p^{\min}\right) = \epsilon$, and eliminate the strategy $\bar{p}$. This process will end when $P^s = \{0.01\}$, completing the proof. Now, for player i,

$$\pi_i(\bar{p}, p_j) = \begin{cases} \bar{p}(1-\bar{p})/2 & \text{if } p_j = \bar{p}, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$\begin{aligned} \pi_i\left(\sigma_{\bar{p}}^{\epsilon}, \bar{p}\right) &= (1-\epsilon)(\bar{p} - 0.01)(1 - \bar{p} + 0.01) + \epsilon p^{\min}\left(1 - p^{\min}\right) \\ &> (1-\epsilon)(\bar{p} - 0.01)(1 - \bar{p} + 0.01) \\ &= (1-\epsilon)\left[\bar{p}(1-\bar{p}) - 0.01(1 - 2\bar{p})\right]. \end{aligned}$$

Then, $\pi_i\left(\sigma_{\bar{p}}^{\epsilon}, \bar{p}\right) > \pi_i(\bar{p}, p_j)$ whenever

$$\epsilon \leq 1 - \frac{\bar{p}(1-\bar{p})/2}{\bar{p}(1-\bar{p}) - 0.01(1 - 2\bar{p})}.$$

But $\bar{p} \geq 0.02$, hence $0.01(1 - 2\bar{p}) < \bar{p}(1-\bar{p})/2$, thus the right hand side is greater than 0. Choose

$$\epsilon = 1 - \frac{\bar{p}(1-\bar{p})/2}{\bar{p}(1-\bar{p}) - 0.01(1 - 2\bar{p})} > 0$$

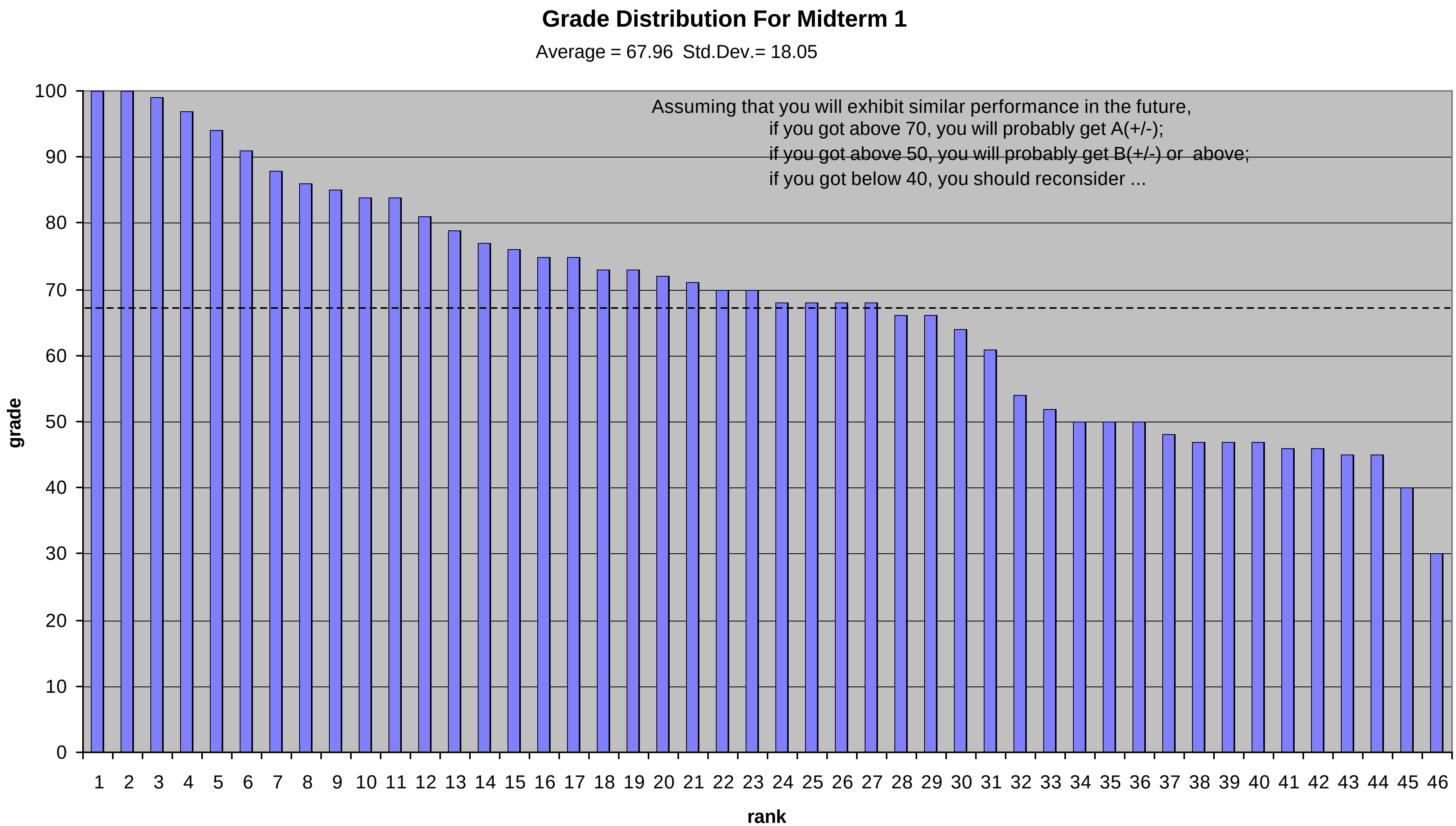
so that $\pi_i\left(\sigma_{\bar{p}}^{\epsilon}, \bar{p}\right) > \pi_i(\bar{p}, p_j)$. Moreover, for any $p_j < \bar{p}$,

$$\begin{aligned} \pi_i\left(\sigma_{\bar{p}}^{\epsilon}, p_j\right) &= (1-\epsilon)(\bar{p} - 0.01)(1 - \bar{p} + 0.01) + \epsilon p^{\min}\left(1 - p^{\min}\right) \\ &\geq \epsilon p^{\min}\left(1 - p^{\min}\right) > 0 = \pi_i(\bar{p}, p_j), \end{aligned}$$

showing that $\sigma_{\bar{p}}^{\epsilon}$ strictly dominates $\bar{p}$, and completing the proof.

(c) What are the Nash equilibria of this game?

Answer: Since any Nash equilibrium is rationalizable, and since the only rationalizable strategy profile is $\left(p^{\min}, p^{\min}\right)$, the only Nash equilibrium is $\left(p^{\min}, p^{\min}\right)$. (Since this is a finite game, there is always a Nash equilibrium — possibly in mixed strategies.)


Grade Distribution For Midterm 1
Average = 67.96 Std.Dev.= 18.05
Assuming that you will exhibit similar performance in the future,
if you got above 70, you will probably get A(+/-);
if you got above 50, you will probably get B(+/-) or above;
if you got below 40, you should reconsider ...
grade
rank
0
10
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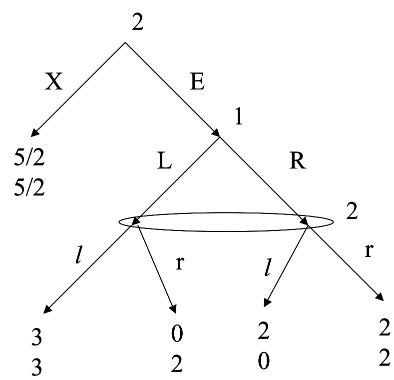
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46

14.12 Game Theory – Midterm II
10/10/2001

Prof. Muhamet Yildiz
Instructions. This is an open book exam; you can use any written material. You have one hour and 20 minutes. Each question is 25 points. Good luck!

1. Consider the following game:



   Compute all the pure-strategy subgame-perfect equilibria. Use a forward induction argument to eliminate one of these equilibria.

2. Below, there are pairs of stage games and strategy profiles. For each pair, check whether the strategy profile is a subgame-perfect equilibrium of the game in which the stage game is repeated infinitely many times. Each agent tries to maximize the discounted sum of his expected payoffs in the stage game, and the discount rate is $\delta = 0.99$.

   (a) **Stage Game:**

1\2	L	M	R
T	2,-1	0,0	-1,2
M	0,0	0,0	0,0
B	-1,2	0,0	2,-1

   Strategy profile: Until some player deviates, player 1 plays T and player 2 alternates between L and R. If anyone devites, then each play M thereafter.

   (b) **Stage Game:**

1\2	A	B
A	2,2	1,3
B	3,1	0,0

   Strategy profile: The play depends on three states. In state S_0, each player plays A; in states S_1 and S_2, each player plays B. The game start at state S_0. In state S_0, if each player plays A or if each player plays B, we stay at S_0, but if a player i plays B while the other is playing A, then we switch to state S_i. At any S_i, if player i plays B, we switch to state S_0; otherwise we state at state S_i.

3. Consider the following first-price, sealed-bid auction where an indivisible good is sold.There are $n \geq 2$ buyers indexed by $i = 1, 2, \ldots, n$. Simultaneously, each buyer i submits a bid $b_i \geq 0$. The agent who submits the highest bid wins. If there are $k > 1$ players submitting the highest bid, then the winner is determined randomly among these players — each has probability $1/k$ of winning. The winner i gets the object and pays his bid b_i, obtaining payoff $v_i - b_i$, while the other buyers get 0, where $v_1, \ldots, v_n$ are independently and identically distributed with probability density function f where

$$f(x) = \begin{cases} 3x^2 & x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Compute the symmetric, linear Bayesian Nash equilibrium.

   (b) What happens as $n \to \infty$?

   [**Hint:** Since $v_1, v_2, \ldots, v_n$ is independently distributed, for any $w_1, w_2, \ldots, w_k$, we have

$$\Pr(v_1 \leq w_1, v_2 \leq w_2, \ldots, v_k \leq w_k) = \Pr(v_1 \leq w_1) \Pr(v_2 \leq w_2) \ldots \Pr(v_k \leq w_k).]$$

4. This question is about a thief and a policeman. The thief has stolen an object. He can either hide the object INSIDE his car on in the TRUNK. The policeman stops the thief. He can either check INSIDE the car or the TRUNK, but not both. (He cannot let the thief go without checking, either.) If the policeman checks the place where the thief hides the object, he catches the thief, in which case the thief gets -1 and the police gets 1; otherwise, he cannot catch the thief, and the thief gets 1, the police gets -1.

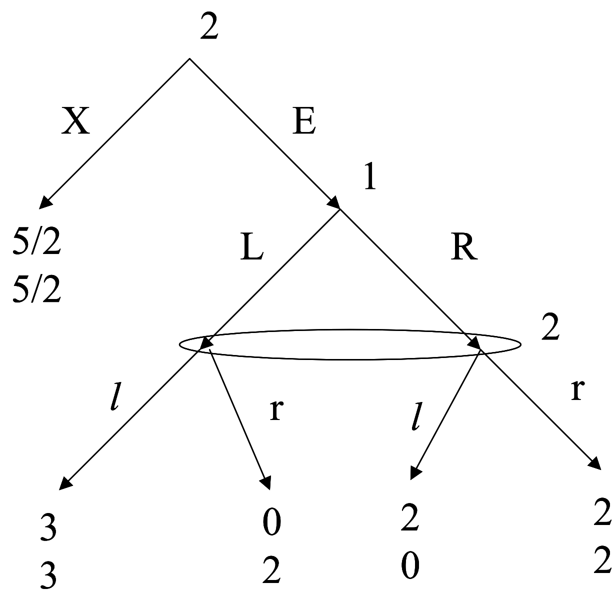
   (a) Compute all the Nash equilibria.

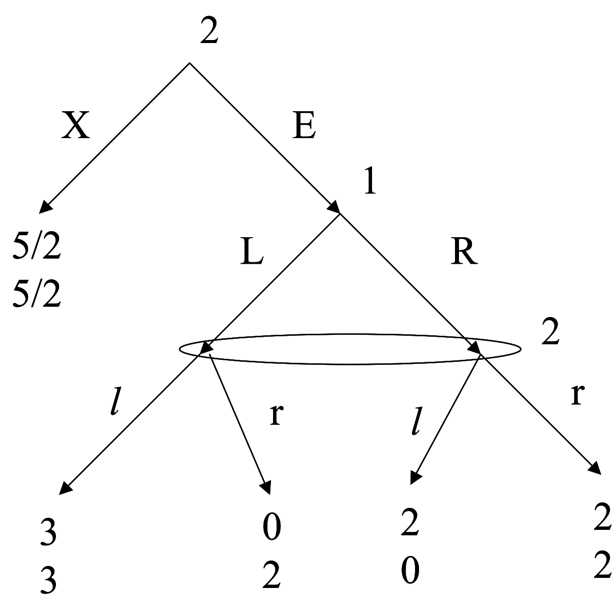
   (b) Now imagine that we have 100 thieves and 100 policemen, indexed by $i = 1, \ldots, 100$, and $j = 1, \ldots, 100$. In addition to their payoffs above, each thief i gets extra payoff b_i form hiding the object in the TRUNK, and each policeman j gets extra payoff d_i from checking the TRUNK. We have

$$\begin{aligned} b_1 &< b_2 < \cdots < b_{50} < 0 < b_{51} < \cdots < b_{100}, \\ d_1 &< d_2 < \cdots < d_{50} < 0 < d_{51} < \cdots < d_{100}. \end{aligned}$$

   Policemen cannot distinguish the thieves from each other, nor can the thieves distinguish the policemen from each other. Each thief has stolen an object, hiding it either in the TRUNK or INSIDE the car. Then, each of them is randomly matched to a policeman. Each matching is equally likely. Again, a policeman can either check INSIDE the car or the TRUNK, but not both. Compute a pure-strategy Bayesian Nash equilibrium of this game.

The game for problem 1.



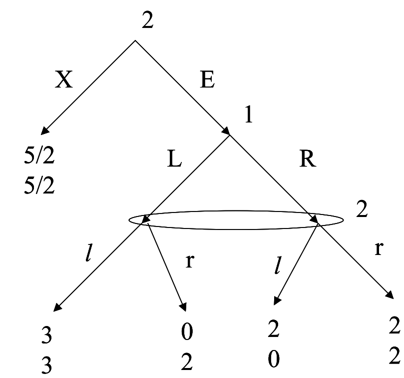


14.12 Game Theory – Midterm II
10/10/2001

Prof. Muhamet Yildiz

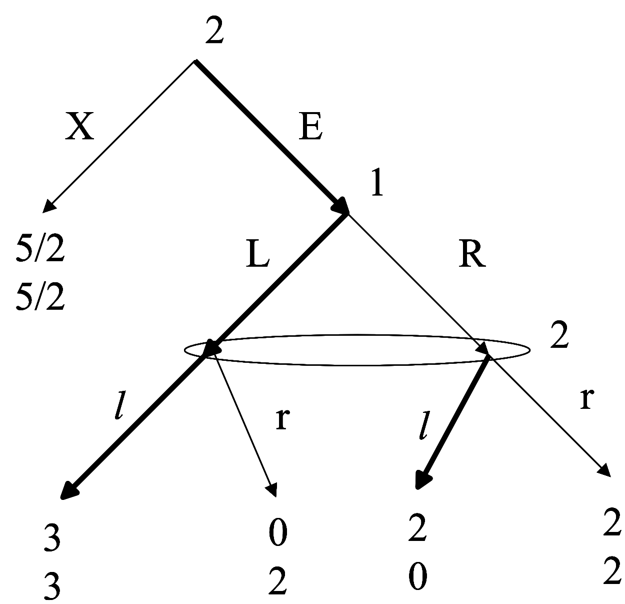
Instructions. This is an open book exam; you can use any written material. You have one hour and 20 minutes. Each question is 25 points. Good luck!

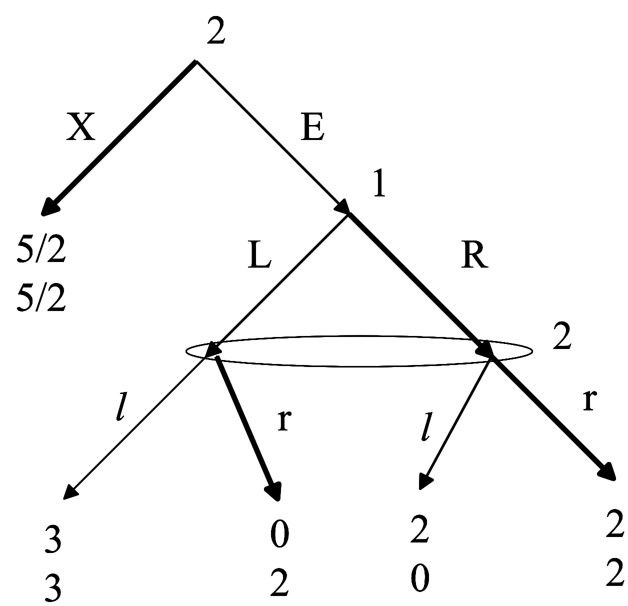
1. Consider the following game:



Compute all the pure-strategy subgame-perfect equilibria. Use a forward induction argument to eliminate one of these equilibria.

Answer: There are two pure strategy Nash equilibria in the proper subgame, yielding subgame-perfect equilibria:





For player 2, Er is strictly dominated by Xr, while El is not dominated. Hence, if player 1 keeps believing that 2 is rational whenever it is possible, then when he sees that 2 played E, he ought to believe that 2 plays strategy El — not the dominated strategy Ex. In that case, 1 would play L, and 2 would play E. Therefore, the equilibrium on the left is eliminated.

2. Below, there are pairs of stage games and strategy profiles. For each pair, check whether the strategy profile is a subgame-perfect equilibrium of the game in which the stage game is repeated infinitely many times. Each agent tries to maximize the discounted sum of his expected payoffs in the stage game, and the discount rate is $\delta = 0.99$.

   (a) **Stage Game:**

1\2	L	M	R
T	2,-1	0,0	-1,2
M	0,0	0,0	0,0
B	-1,2	0,0	2,-1

   Strategy profile: Until some player deviates, player 1 plays T and player 2 alternates between L and R. If anyone deviates, then each play M thereafter.

   Answer: It is subgame perfect. Since (M,M) is a Nash equilibrium of the stage game, we only need to check if any player wants to deviate when player 1 plays T and player 2 alternates between L and R. In this regime, the present value of player 1's payoffs is

$$V_{1L} = \frac{2}{1-\delta} - \frac{\delta}{1-\delta} = \frac{2-\delta}{1-\delta} > 0$$

   when 2 is to play L and

$$V_{1R} = \frac{2\delta}{1-\delta} - \frac{1}{1-\delta} = \frac{2\delta - 1}{1-\delta} = 98$$

   when 2 is to play R. When 2 plays L, 1 cannot gain by deviating. When 2 plays R, the best 1 gets by deviating is

$$2 + 0 < 98$$

   (when he plays B). The only possible profitable deviation for player 2 is to play R when he is supposed to play left. In that contingency, if he follows the strategy he gets $V_{1R} = 98$; if he deviates, he gets $2 + 0 < V_{1R}$.

   (b) **Stage Game:**

1\2	A	B
A	2,2	1,3
B	3,1	0,0

   Strategy profile: The play depends on three states. In state S_0, each player plays A; in states S_1 and S_2, each player plays B. The game start at state S_0. In state S_0, if each player plays A or if each player plays B, we stay at S_0, but if a player i plays B while the other is playing A, then we switch to state S_i. At any S_i, if player i plays B, we switch to state S_0; otherwise we state at state S_i.

   Answer: It is not subgame-perfect. At state S_2, player 2 is to play B, and we will switch to state S_0 no matter what 1 plays. In that case, 1 would gain by deviating and playing A (in state S_2).

3. Consider the following first-price, sealed-bid auction where an indivisible good is sold. There are $n \geq 2$ buyers indexed by $i = 1, 2, \ldots, n$. Simultaneously, each buyer i submits a bid $b_i \geq 0$. The agent who submits the highest bid wins. If there are $k > 1$ players submitting the highest bid, then the winner is determined randomly among these players — each has probability $1/k$ of winning. The winner i gets the object and pays his bid b_i, obtaining payoff $v_i - b_i$, while the other buyers get 0, where $v_1, \ldots, v_n$ are independently and identically distributed with probability density function f where

$$f(x) = \begin{cases} 3x^2 & x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the symmetric, linear Bayesian Nash equilibrium.

Answer: We look for an equilibrium of the form

$$b_i = a + cv_i$$

where $c > 0$. Then, the expected payoff from bidding b_i with type v_i is

$$\begin{aligned} U(b_i; v_i) &= (v_i - b_i) \Pr(b_i > a + cv_j \quad \forall j \neq i) \\ &= (v_i - b_i) \prod_{j \neq i} \Pr(b_i > a + cv_j) \\ &= (v_i - b_i) \prod_{j \neq i} \Pr\left(v_j < \frac{b_i - a}{c}\right) \\ &= (v_i - b_i) \prod_{j \neq i} \left(\frac{b_i - a}{c}\right)^3 \\ &= (v_i - b_i) \left(\frac{b_i - a}{c}\right)^{3(n-1)} \end{aligned}$$

for $b_i \in [a, a + c]$. The first order condition is

$$\frac{\partial U(b_i; v_i)}{\partial b_i} = -\left(\frac{b_i - a}{c}\right)^{3(n-1)} + 3(n-1)\frac{1}{c}(v_i - b_i)\left(\frac{b_i - a}{c}\right)^{3(n-1)-1} = 0;$$

i.e.,

$$-\left(\frac{b_i - a}{c}\right) + 3(n-1)\frac{1}{c}(v_i - b_i) = 0;$$

i.e.,

$$b_i = \frac{a + 3(n-1)v_i}{3(n-1)+1}.$$

Since this is an identity, we must have

$$a = \frac{a}{3(n-1)+1} \Longrightarrow a = 0,$$

and

$$c = \frac{3(n-1)}{3(n-1)+1}.$$

(b) What happens as $n \to \infty$?

Answer: As $n \to \infty$,

$$b_i \to v_i.$$

In the limit, each bidder bids his valuation, and the seller extracts all the gains from trade.

[**Hint:** Since $v_1, v_2, \ldots, v_n$ is independently distributed, for any $w_1, w_2, \ldots, w_k$, we have

$$\Pr(v_1 \leq w_1, v_2 \leq w_2, \ldots, v_k \leq w_k) = \Pr(v_1 \leq w_1) \Pr(v_2 \leq w_2) \ldots \Pr(v_k \leq w_k).]$$

4. This question is about a thief and a policeman. The thief has stolen an object. He can either hide the object INSIDE his car on in the TRUNK. The policeman stops the thief. He can either check INSIDE the car or the TRUNK, but not both. (He cannot let the thief go without checking, either.) If the policeman checks the place where the thief hides the object, he catches the thief, in which case the thief gets -1 and the police gets 1; otherwise, he cannot catch the thief, and the thief gets 1, the police gets -1.

   (a) Compute all the Nash equilibria.

   Answer: This is a matching-pennies game. There is a unique Nash equilibrium, in which Thief hides the object INSIDE or the TRUNK with equal probabilities, and the Policeman checks INSIDE or the TRUNK with equal probabilities.

   (b) Now imagine that we have 100 thieves and 100 policemen, indexed by $i = 1, \ldots, 100$, and $j = 1, \ldots, 100$. In addition to their payoffs above, each thief i gets extra payoff b_i form hiding the object in the TRUNK, and each policeman j gets extra payoff d_j from checking the TRUNK. We have

$$\begin{aligned} b_1 &< b_2 < \cdots < b_{50} < 0 < b_{51} < \cdots < b_{100}, \\ d_1 &< d_2 < \cdots < d_{50} < 0 < d_{51} < \cdots < d_{100}. \end{aligned}$$

   Policemen cannot distinguish the thieves from each other, nor can the thieves distinguish the policemen from each other. Each thief has stolen an object, hiding it either in the TRUNK or INSIDE the car. Then, each of them is randomly matched to a policeman. Each matching is equally likely. Again, a policeman can either check INSIDE the car or the TRUNK, but not both. Compute a pure-strategy Bayesian Nash equilibrium of this game.

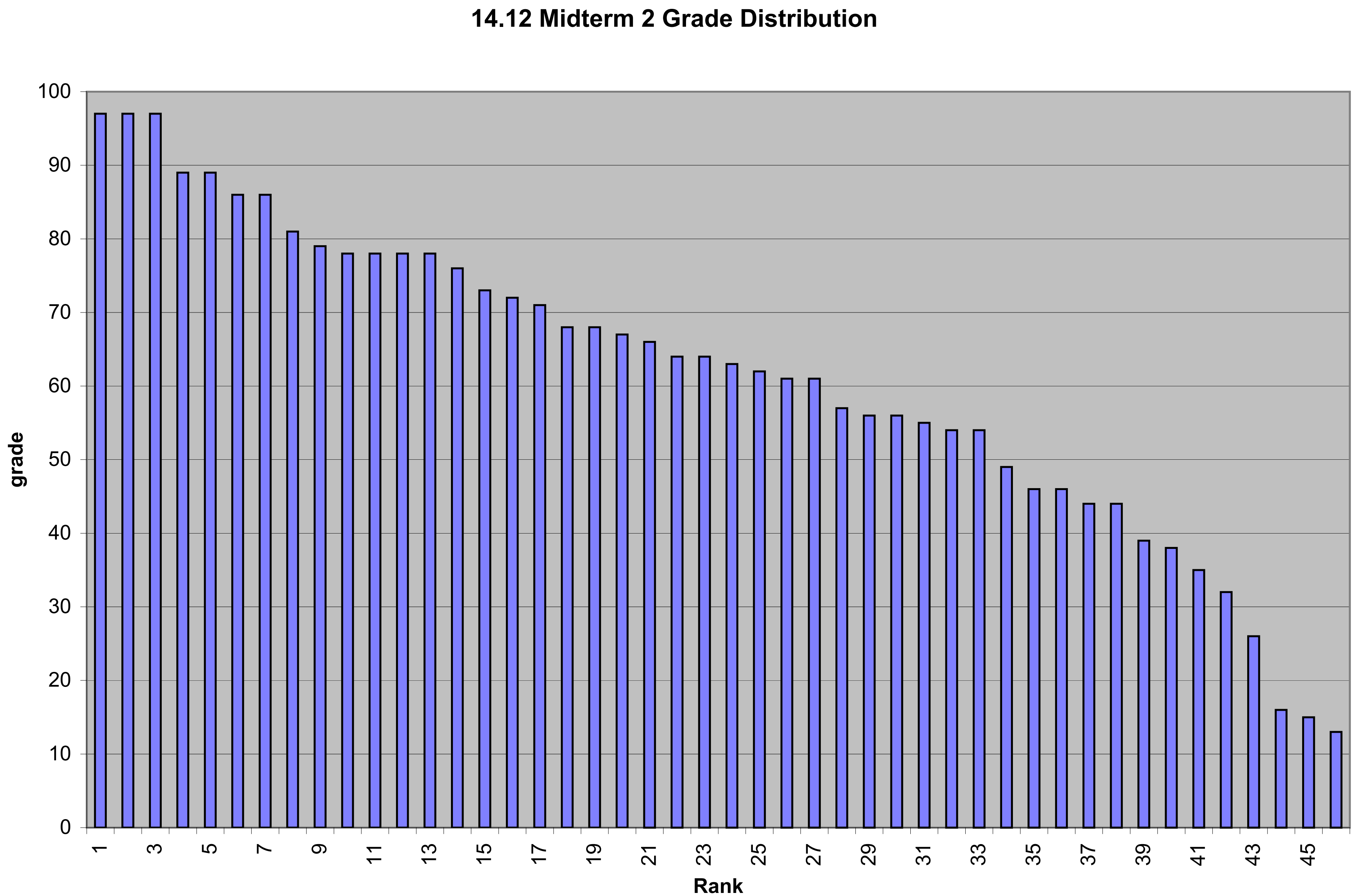
   Answer: A Bayesian Nash equilibrium: A thief i hides the object in

$$\begin{array}{ll} \text{INSIDE} & \text{if } b_i < 0 \\ \text{TRUNK} & \text{if } b_i > 0; \end{array}$$

   a policeman j checks

$$\begin{array}{ll} \text{INSIDE} & \text{if } d_j < 0 \\ \text{TRUNK} & \text{if } d_j > 0. \end{array}$$

   This is a Bayesian Nash equilibrium, because, from the thief's point of view the policeman is equally likely to to check TRUNK or INSIDE the car, hence it is the best response for him to hide in the trunk iff the extra benefit from hiding in the trunk is positive. Similar for the policemen.


14.12 Midterm 2 Grade Distribution
grade
Rank
0
10
20
30
40
50
60
70
80
90
100
1
3
5
7
9
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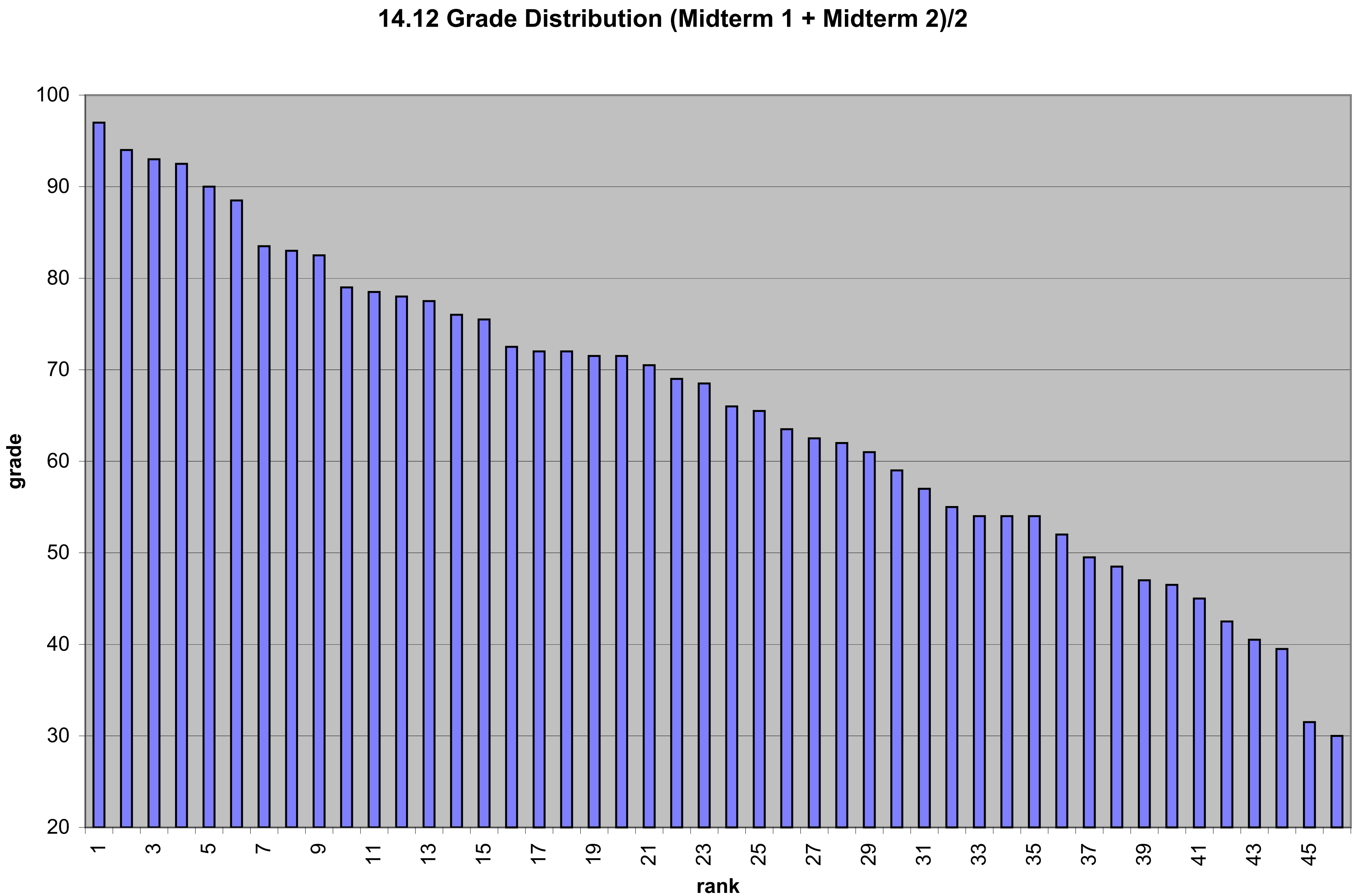
45


14.12 Grade Distribution (Midterm 1 + Midterm 2)/2
grade
rank
100
90
80
70
60
50
40
30
20
1
3
5
7
9
11
13
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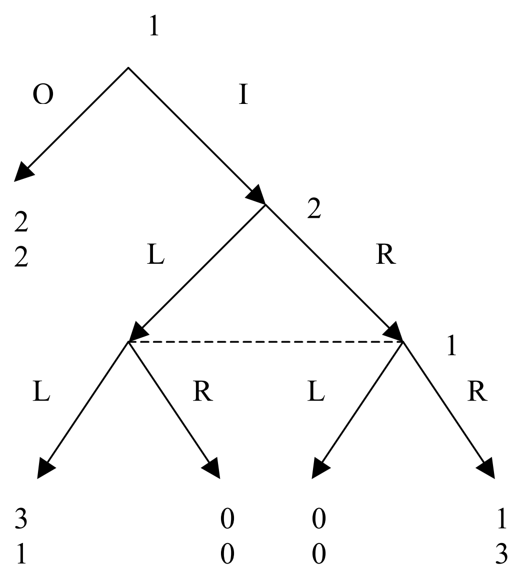
14.12 Game Theory – Final
12/10/2001

Prof. Muhamet Yildiz

Instructions. This is an open book exam; you can use any written material. You have 2 hours 50 minutes. Each question is 20 points. Good luck!

1. Consider the following extensive form game.



   (a) Find the normal form representation of this game.

   (b) Find all rationalizable pure strategies.

   (c) Find all pure strategy Nash equilibria.

   (d) Which strategies are consistent with all of the following assumptions?

      (i) 1 is rational.

      (ii) 2 is sequentially rational.

      (iii) at the node she moves, 2 knows (i).

      (iv) 1 knows (ii) and (iii).

2. This question is about a milkman and a customer. At any day, with the given order,

   - Milkman puts $m \in [0, 1]$ liter of milk and $1 - m$ liter of water in a container and closes the container, incurring cost cm for some $c > 0$;
   - Customer, without knowing m, decides on whether or not to buy the liquid at some price p. If she buys, her payoff is $vm - p$ and the milkman's payoff is $p - cm$. If she does not buy, she gets 0, and the milkman gets $-cm$. If she buys, then she learns m.

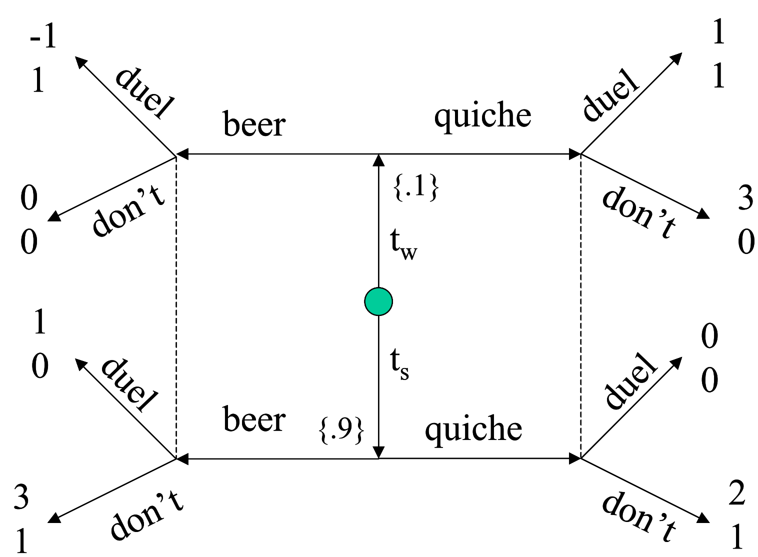
(a) Assume that this is repeated for 100 days, and each player tries to maximize the sum of his or her stage payoffs. Find all subgame-perfect equilibria of this game.

(b) Now assume that this is repeated infinitely many times and each player tries to maximize the discounted sum of his or her stage payoffs, where discount rate is $\delta \in (0, 1)$. What is the range of prices p for which there exists a subgame perfect equilibrium such that, everyday, the milkman chooses $m = 1$, and the customer buys on the path of equilibrium play?

3. For the game in question 3.a, assume that with probability 0.001, milkman strongly believes that there is some entity who knows what the milkman does and will punish him severely on the day 101 for each day the milkman dilutes the milk (by choosing $m < 1$). Call this type irrational. Assume that this is common knowledge. For $v > p > c$, find a perfect Bayesian equilibrium of this game.

   Bonus: [10 points] Discuss what would happen if the irrational type were known to dilute the milk by accident with some small but positive probability.

4. Find a perfect Bayesian equilibrium of the following game.



5. A risk-neutral entrepreneur has a project that requires \$100,000 as an investment, and will yield \$300,000 with probability 1/2, \$0 with probability 1/2. There are two types of entrepreneurs: rich who has a wealth of \$1,000,000, and poor who has \$0. For some reason, the wealthy entrepreneur cannot use his wealth as an investment towards this project. There is also a bank that can lend money with interest rate π. That is, if the entrepreneur borrows \$100,000 to invest, after the project is completed he will pay back $\$100,000\,(1+\pi)$ — if he has that much money. If his wealth is less than this amount at the end of the project, he will pay all he has. The order of the events is as follows:

   - First, bank posts π.

- Then, entrepreneur decides whether to borrow ($100,000) and invest.
- Then, uncertainty is resolved.

(a) Compute the subgame perfect equilibrium for the case when the wealth is common knowledge.

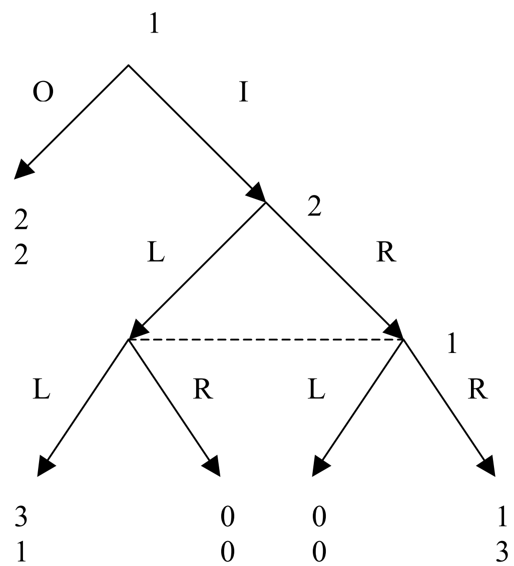
(b) Now assume that the bank does not know the wealth of the entrepreneur. The probability that the entrepreneur is rich is 1/4. Compute the perfect Bayesian equilibrium.

14.12 Game Theory – Final
12/10/2001

Prof. Muhamet Yildiz

Instructions. This is an open book exam; you can use any written material. You have 2 hours 50 minutes. Each question is 20 points. Good luck!

1. Consider the following extensive form game.



(a) Find the normal form representation of this game.

A:

	L	R
OL	2,2	2,2
OR	2,2	2,2
IL	3,1	0,0
IR	0,0	1,3

(b) Find all rationalizable pure strategies.

	L	R
OL	2,2	2,2
OR	2,2	2,2
IL	3,1	0,0

(c) Find all pure strategy Nash equilibria.

	L	R
OL	**2,2**	**2,2**
OR	**2,2**	**2,2**
IL	**3,1**	0,0

(d) Which strategies are consistent with all of the following assumptions?

   (i) 1 is rational.

   (ii) 2 is sequentially rational.

   (iii) at the node she moves, 2 knows (i).

   (iv) 1 knows (ii) and (iii).

   ANSWER: By (i) 1 does not play IR. Hence, by (iii), at the node she moves, 2 knows that 1 does not play IR, hence he knows IL. Then, by (ii), 2 must play L. Therefore, by (i) and (iv), 1 must play IL. The answer is (IL,L).

2. This question is about a milkman and a customer. At any day, with the given order,

   - Milkman puts $m \in [0,1]$ liter of milk and $1-m$ liter of water in a container and closes the container, incurring cost cm for some $c > 0$;
   - Customer, without knowing m, decides on whether or not to buy the liquid at some price p. If she buys, her payoff is $vm - p$ and the milkman's payoff is $p - cm$. If she does not buy, she gets 0, and the milkman gets $-cm$. If she buys, then she learns m.

(a) Assume that this is repeated for 100 days, and each player tries to maximize the sum of his or her stage payoffs. Find all subgame-perfect equilibria of this game.

ANSWER: The stage game has a unique Nash equilibrium, in which $m = 0$ and the customer does not buy. Therefore, this finitely repeated game has a unique subgame-perfect equilibrium, in which the stage equilibrium is repeated.

(b) Now assume that this is repeated infinitely many times and each player tries to maximize the discounted sum of his or her stage payoffs, where discount rate is $\delta \in (0,1)$. What is the range of prices p for which there exists a subgame perfect equilibrium such that, everyday, the milkman chooses $m = 1$, and the customer buys on the path of equilibrium play?

ANSWER: The milkman can guarantee himself 0 by always choosing $m = 0$. Hence, his continuation value at any history must be at least 0. Hence, in the worst equilibrium, if he deviates customer should not buy milk forever, giving the milkman exactly 0 as the continuation value. Hence, the SPE we are looking for is *the milkman always chooses m=1 and the customer buys until anyone deviates, and the milkman chooses m=0 and the customer does not buy thereafter.* If the milkman does not deviate, his continuation value will be

$$V = \frac{p-c}{1-\delta}.$$

The best deviation for him (at any history on the path of equilibrium play) is to choose $m = 0$ (and not being able to sell thereafter). In that case, he will get

$$V_d = p + \delta 0 = p.$$

In order this to be an equilibrium, we must have $V \geq V_d$; i.e.,

$$\frac{p-c}{1-\delta} \geq p,$$

i.e.,

$$p \geq \frac{c}{\delta}.$$

In order that the customer buy on the equilibrium path, we must also have $p \leq v$. Therefore,

$$v \geq p \geq \frac{c}{\delta}.$$

3. For the game in question 2.a, assume that with probability 0.001, milkman strongly believes that there is some entity who knows what the milkman does and will punish him severely on the day 101 for each day the milkman dilutes the milk (by choosing $m < 1$). Call this type irrational. Assume that this is common knowledge. For some $v > p > c$, find a perfect Bayesian equilibrium of this game. [If you find it easier, take the customers at different dates different, but assume that each customer knows whatever the previous customers knew.]

ANSWER: [It is very difficult to give a rigorous answer to this question, so you would get a big partial grade for an informal answer that shows that you understand the reputation from an incomplete-information point of view.] Irrational type always sets $m = 1$. Since he will be detected whenever he sets $m < 1$ and the customer buys, the rational type will set either $m = 1$ or $m = 0$. We are looking for an equilibrium in which early in the relation the rational milkman will always set $m = 1$ and the customer will always buy, but near the end of the relation the rational milkman will mix between $m = 1$ and $m = 0$, and the customer will mix between buy and not buy.

In this equilibrium, if the milkman sets $m < 1$ or the customer does not buy at any t, then the rational milkman sets $m = 0$ at each $s > t$. In that case, if in addition the costumer buys at some dates in the period $\{t+1, t+2, \ldots, s-1\}$ and if the milkman chooses $m = 1$ at each of those days, then the costumer will assign probability 1 to that the milkman is irrational and buy the milk at s; otherwise, he will not buy the milk. On the path of such play, if the milkman sets $m < 1$ or the customer does not buy at any t, then the rational milkman sets $m = 0$ and the costumer does not buy at each $s > t$. In order this to be an equilibrium, the probability μ_t that the milkman is irrational at such history must satisfy

$$\mu_t (100 - t)(v - p) - (1 - \mu_t) p \leq 0,$$

where the first term is the expected benefit from experimenting (if the milkman happens to be irrational) and the second term is the cost (if he is rational). That is,

$$\mu_t \leq \frac{p}{p + (100 - t)(v - p)}.$$

Now we determine what happens if the milkman has always been setting $m = 1$, and the customer has been buying. In the last date, the rational type will set $m = 0$, and the rational type will set $m = 1$; hence, the buyer will buy iff

$$\mu_{100} (v - p) - (1 - \mu_{100}) p \geq 0,$$

i.e.,

$$\mu_{100} \geq \frac{p}{v}.$$

Since we want him to mix, we set

$$\mu_{100} = \frac{p}{v}.$$

We derive μ_t for previous dates using the Bayes' rule and the indifference condition necessary for the customer's mixing. Let's write α_t for the probability that the rational milkman sets $m = 1$ at t, and $a_t = \mu_t + (1 - \mu_t)\,\alpha_t$ for the total probability that $m = 1$ at date t. Since the customer will be indifferent between buying and not buying at $t+1$, his expected payoff at $t+1$ will be 0. Hence, his expected payoff from buying at t is

$$a_t\,(v - p) + (1 - a_t)\,(-p)\,.$$

For indifference, this must be equal to zero, thus

$$a_t = \frac{p}{v}.$$

On the other hand, by Bayes' rule,

$$\mu_{t+1} = \frac{\mu_t}{a_t}.$$

Therefore,

$$\mu_t = a_t \mu_{t+1} = \frac{p}{v}\mu_{t+1}.$$

That is,

$$\begin{aligned} \mu_{100} &= \frac{p}{v} \\ \mu_{99} &= \left(\frac{p}{v}\right)^2 \\ \mu_{98} &= \left(\frac{p}{v}\right)^3 \\ &\vdots \end{aligned}$$

Note that

$$a_t = \frac{p}{v} = \mu_t + (1 - \mu_t)\,\alpha_t \Rightarrow \alpha_t = \frac{\frac{p}{v} - \mu_t}{1 - \mu_t}.$$

Assume that $(p/v)^{100} < 0.001$. Then, we will have a date t^* such that

$$\left(\frac{p}{v}\right)^{101-t^*} < 0.001 < \left(\frac{p}{v}\right)^{100-t^*}.$$

At each date $t > t^*$, we will have $\mu_t = (p/v)^{101-t}$ and the players will mix so that $a_t = \frac{p}{v}$. At each date $t < t^*$, the milkman will set $m = 1$ and the customer will buy. At date t^*, the rational milkman will mix so that

$$\left(\frac{p}{v}\right)^{100-t^*} = \mu_{t^*+1} = \frac{\mu_{t^*}}{a_{t^*}} = \frac{0.001}{a_{t^*}},$$

hence

$$a_{t^*} = \frac{0.001}{\left(\frac{p}{v}\right)^{100-t^*}}.$$

Note that $a_{t^*} > p/v$, hence the customer will certainly buy at t^*.

Let's write β_t for the probability that the customer will buy at day t. In the day 99, if the rational milkman sets $m = 1$, he will get

$$U = \beta_{99}\left(p - c\right) + \beta_{99}\beta_{100}p + \left(1 - \beta_{99}\right)\left(-c\right),$$

where the first term is the profit from selling at day 99, the second term is the profit from day 100 (when he will set $m = 0$), and the last term is the loss if the customer does not buy at day 99. If he sets $m = 0$, he will get $\beta_{99}p$ (from the sale at 99, and will get zero thereafter). Hence, he will set $m = 1$ iff

$$\beta_{99}\left(p - c\right) + \beta_{99}\beta_{100}p + \left(1 - \beta_{99}\right)\left(-c\right) \geq \beta_{99}p$$

i.e.,

$$\beta_{99}\beta_{100} \geq \frac{c}{p}.$$

We are looking for an indifference, hence we set

$$\beta_{99}\beta_{100} = \frac{c}{p}.$$

Similarly, at day 98 the rational milkman will set $m = 1$ iff

$$\beta_{98}\left(p - c\right) + \beta_{98}\beta_{99}p + \left(1 - \beta_{98}\right)\left(-c\right) \geq \beta_{98}p,$$

where the second term is due to the fact that at date 99 he will be indifferent between choosing $m = 0$ and $m = 1$. For indifference, we set

$$\beta_{98}\beta_{99} = \frac{c}{p}.$$

We will continue on like this as long as we need the milkman to mix. That is, we will have

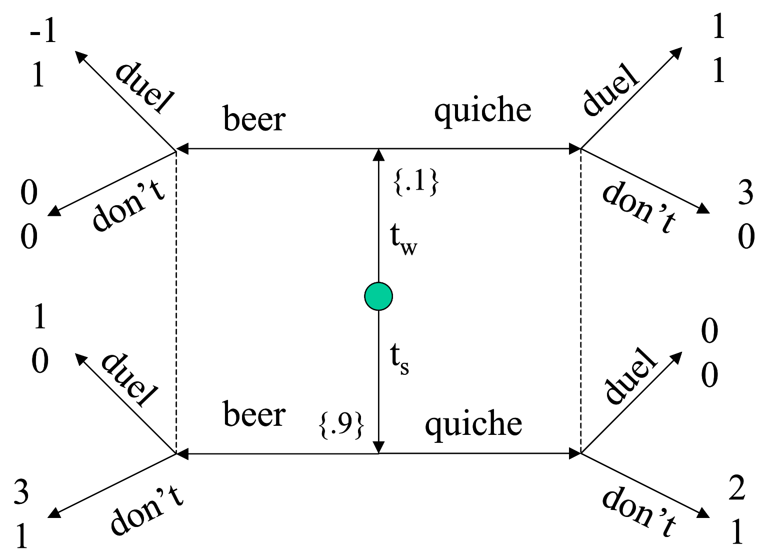
$$\begin{aligned}
\beta_{t^*}\beta_{t^*+1} &= \frac{c}{p}, \\
\beta_{t^*+1}\beta_{t^*+2} &= \frac{c}{p}, \\
&\vdots \\
\beta_{98}\beta_{99} &= \frac{c}{p}, \\
\beta_{99}\beta_{100} &= \frac{c}{p}.
\end{aligned}$$

As we noted before, $\beta_{t^*} = 1$. Hence, $\beta_{t^*+1} = \frac{c}{p}$. Hence, $\beta_{t^*+2} = 1$, ... That is,

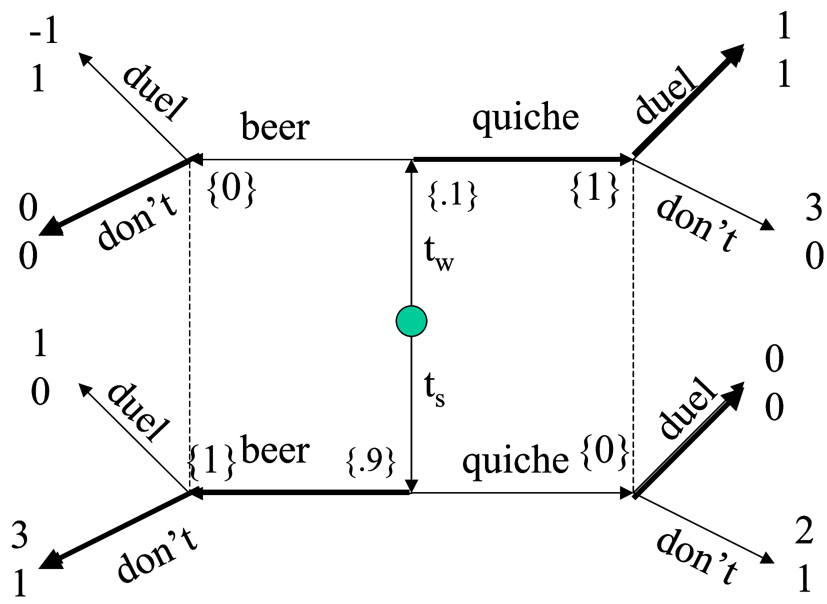
$$\begin{aligned}
\beta_{t^*} &= 1, \\
\beta_{t^*+1} &= \frac{c}{p}, \\
\beta_{t^*+2} &= 1, \\
\beta_{t^*+3} &= \frac{c}{p}, \\
&\vdots
\end{aligned}$$

Bonus: [10 points] Discuss what would happen if the irrational type were known to dilute the milk by accident with some small but positive probability.

4. Find a perfect Bayesian equilibrium of the following game.



ANSWER:



5. A risk-neutral entrepreneur has a project that requires \$100,000 as an investment, and will yield \$300,000 with probability 1/2, \$0 with probability 1/2. There are two types of entrepreneurs: rich who has a wealth of \$1,000,000, and poor who has \$0. For some reason, the wealthy entrepreneur cannot use his wealth as an investment towards this project. There is also a bank that can lend money with interest rate π. That is, if the entrepreneur borrows \$100,000 to invest, after the project is completed he will pay back $\$100,000\,(1+\pi)$ — if he has that much money. If his wealth is less than this amount at the end of the project, he will pay all he has. The order of the events is as follows:

- First, bank posts π.
- Then, entrepreneur decides whether to borrow (\$100,000) and invest.
- Then, uncertainty is resolved.

(a) Compute the subgame perfect equilibrium for the case when the wealth is common knowledge.

ANSWER: The rich entrepreneur is always going to pay back the loan in full amount, hence his expected payoff from investing (as a change from not investing) is

$$(0.5)(300,000) - 100,000\,(1+\pi)\,.$$

Hence, he will invest iff this amount is non-negative, i.e.,

$$\pi \leq 1/2.$$

Thus, the bank will set the interest rate at

$$\pi_R = 1/2.$$

The poor entrepreneur is going to pay back the loan only when the project succeeds. Hence, his expected payoff from investing is

$$(0.5)(300,000 - 100,000\,(1+\pi)).$$

He will invest iff this amount is non-negative, i.e.,

$$\pi \leq 2.$$

Thus, the bank will set the interest rate at

$$\pi_P = 2.$$

(b) Now assume that the bank does not know the wealth of the entrepreneur. The probability that the entrepreneur is rich is 1/4. Compute the perfect Bayesian equilibrium.

ANSWER: As in part (a), the rich type will invest iff $\pi \leq \pi_R = .5$, and the poor type will invest iff $\pi \leq \pi_P = 2$. Now, if $\pi \leq \pi_R$, the bank's payoff is

$$\begin{aligned} U(\pi) &= \frac{1}{4}100,000\,(1+\pi) + \frac{3}{4}\left[\frac{1}{2}100,000\,(1+\pi) + \frac{1}{2}0\right] - 100,000 \\ &= \frac{5}{8}100,000\,(1+\pi) - 100,000 \\ &\leq \frac{5}{8}100,000\,(1+\pi_R) - 100,000 \\ &= \frac{5}{8}100,000\,(1+1/2) - 100,000 = -\frac{1}{16}100,000 < 0. \end{aligned}$$

If $\pi_R < \pi \leq \pi_P$, the bank's payoff is

$$\begin{aligned} U(\pi) &= \frac{3}{4}\left[\frac{1}{2}100,000\,(1+\pi) + \frac{1}{2}0 - 100,000\right] \\ &= \frac{3}{8}100,000\,(\pi - 1), \end{aligned}$$

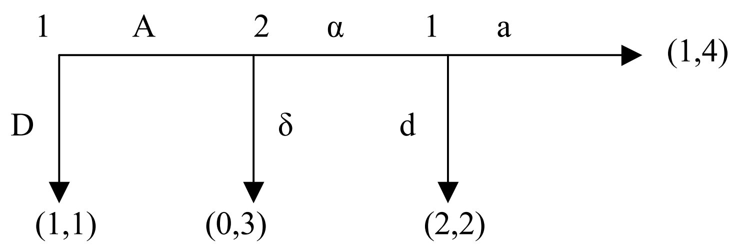
which is maximized at π_P, yielding $\frac{3}{8}100,000$. If $\pi > \pi_P$, $U(\pi) = 0$. Hence, the bank will choose $\pi = \pi_P$.

14.12 Game Theory – Final
12/21/2001

Prof. Muhamet Yildiz

Instructions. This is an open book exam; you can use any written material. You have 2 hours 50 minutes. Each question is 20 points. Good luck!

1. Consider the following extensive form game.



(a) Find the normal form representation of this game.

1\2	α	δ
Aa	1,4	0,3
Ad	2,2	0,3
Da	1,1	1,1
Dd	1,1	1,1

(b) Find all rationalizable pure strategies.

1\2	α	δ
Ad	2,2	0,3
Da	1,1	1,1
Dd	1,1	1,1

(c) Find all pure strategy Nash equilibria.

1\2	α	δ
Ad	2,2	0,3
Da	1,1	**1,1**
Dd	1,1	**1,1**

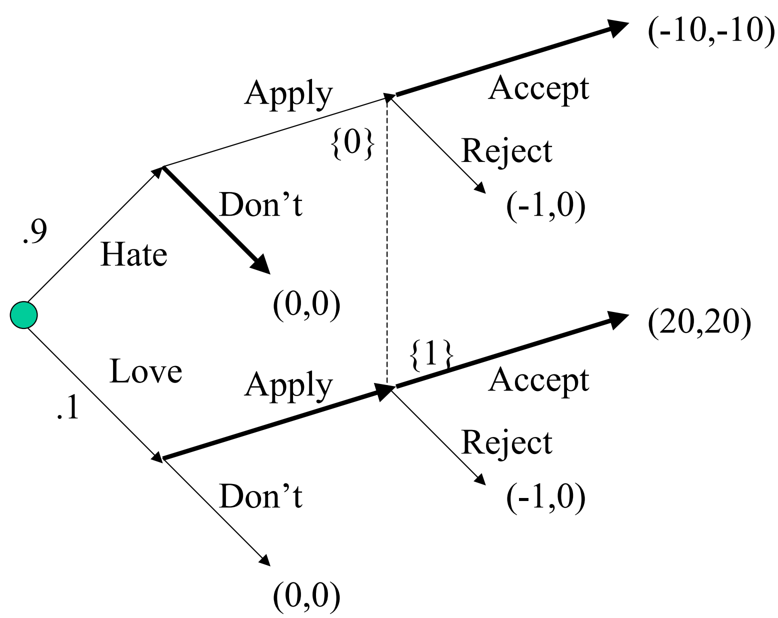
(d) Which strategies are consistent with all of the following assumptions?

(i) 1 is rational.
(ii) 2 is sequentially rational.
(iii) at the node she moves, 2 knows (i).
(iv) 1 knows (ii) and (iii).

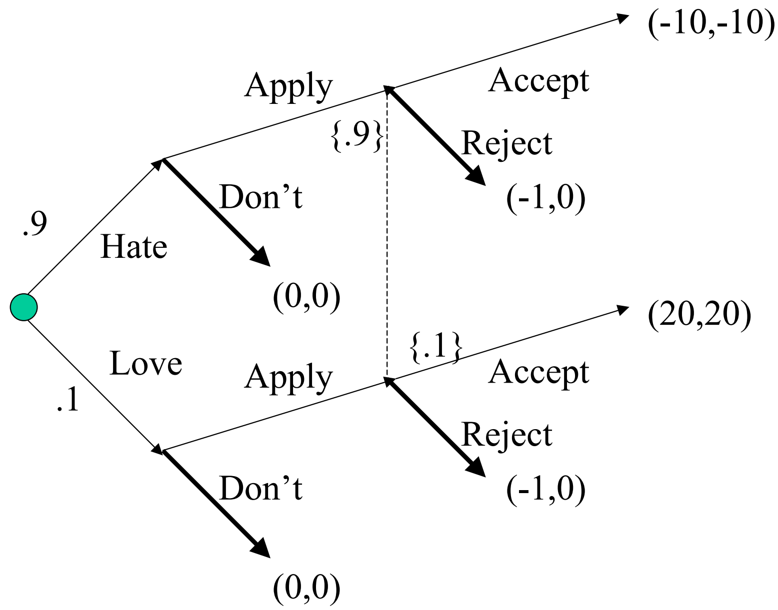
ANSWER: By (i) 1 does not play Aa. Hence, by (iii), at the node she moves, 2 knows that 1 does not play Aa, hence he knows that 1 plays Ad. Then, by (ii), 2 must play δ. Therefore, by (i) and (iv), 1 must play Ad or Aa. The answer is 1 plays A, given chance 2 would play δ.

2. This question is about a game between a possible applicant (henceforth student) to a Ph.D. program in Economics and the Admission Committee. Ex-ante, Admission Committee believes that with probability .9 the student hates economics and with probability .1 he loves economics. After Nature decides whether student loves or hates economics with the above probabilities and reveals it to the student, the student decides whether or not to apply to the Ph.D. program. If the student does not apply, both the student and the committee get 0. If student applies, then the committee is to decide whether to accept or reject the student. If the committee rejects, then committee gets 0, and student gets -1. If the committee accepts the student, the payoffs depend on whether the student loves or hates economics. If the student loves economics, he will be successful and the payoffs will be 20 for each player. If he hates economics, the payoffs for both the committee and the student will be -10. Find a separating equilibrium and a pooling equilibrium of this game.

   ANSWER: A separating equilibrium:



   A pooling equilibrium:



3. Consider a bargaining problem where two risk-neutral players are trying to divide a dollar they own, which they cannot use until they reach an agreement. The players do not discount the future, but at the end of each rejection of an offer the bargaining breaks down with probability $1-\delta \in (0,1)$ and each player gets 0.

   (a) Consider the following bargaining procedure. Player 1 makes an offer $(x, 1-x)$, where x is player 1's share. Then, player 2 decides whether or not to accept the offer. If she accepts, they implement the offer, yielding division $(x, 1-x)$. If she rejects the offer, then with probability $1-\delta$, the bargaining breaks down an each gets 0; with probability δ, player 1 makes another offer, which will be accepted or rejected by player 2 as above. (If player 2 rejects the offer, bargaining will break down with probability $1-\delta$ again.) If the offer is rejected and the bargaining did not break down, now player 2 makes a counter offer, and player 1 accepts or rejects this counter offer as above. If the offer is rejected, this time the game will end, and each will get 0. Find the subgame-perfect Nash equilibrium of this game. Compute the expected payoff of each player at the beginning of the game in this equilibrium.

   ANSWER: On the last day, 1 accepts any offer, so 2 offers (0,1). Hence, on the previous day, 2 accepts an offer iff she gets at least δ. Hence, 1 offers $(1-\delta, \delta)$ — accepted. Thus, in the first day, 2 accepts an offer iff she gets at least δ^2. Hence, 1 offers $\left(1-\delta^2, \delta^2\right)$ — accepted. The expected payoffs are $\left(1-\delta^2, \delta^2\right)$.

   (b) Compute the subgame-perfect equilibrium of the game in which the procedure in part (a) is repeated 2 times. (The probability of bargaining breakdown after each rejection is $1-\delta$, except for the end of the game.)

   ANSWER: The last period as above. Let's look at the first period. On the last day of the first period, 1 accepts an offer iff he gets at least $\delta\left(1-\delta^2\right)$, so 2 offers

$\left(\delta\left(1-\delta^2\right), 1-\delta\left(1-\delta^2\right)\right)$. Hence, on the previous day, 2 accepts an offer iff she gets at least $\delta\left(1-\delta\left(1-\delta^2\right)\right)$. Hence, 1 offers

$$\left(1-\delta\left(1-\delta\left(1-\delta^2\right)\right), \delta\left(1-\delta\left(1-\delta^2\right)\right)\right) = \left(1-\delta+\delta^2\left(1-\delta^2\right), \delta\left(1-\delta\left(1-\delta^2\right)\right)\right)$$

— accepted. Thus, in the first day, 2 accepts an offer iff she gets at least $\delta^2\left(1-\delta\left(1-\delta^2\right)\right)$. Hence, 1 offers

$$\left(1-\delta^2\left(1-\delta\left(1-\delta^2\right)\right), \delta\left(1-\delta\left(1-\delta^2\right)\right)\right) = \left(1-\delta^2+\delta^3\left(1-\delta^2\right), \delta^2\left(1-\delta\left(1-\delta^2\right)\right)\right)$$

— accepted.

(c) Find the subgame-perfect equilibrium of the game in which this procedure is repeated until they reach an agreement. Note that player 1 makes two offers, then 2 makes one offer, then 1 makes two offers, and so on. You need to show that the proposed strategy profile is in fact a subgame-perfect equilibrium. (The probability of bargaining breakdown after each rejection is $1-\delta$.)

[Hint: One way is to compute the SPE for the game in which the procedure is repeated n times and let $n \to \infty$. A somewhat easier way is to consider an alternating offer bargaining procedure with some effective discount rates — different for a different player.]

ANSWER: If you compare the calculations above with the calculations with the alternating offer case with asymmetric discount rates, you should realize that the first offer player 1 makes and the offer player 2 makes are identical to the offers players 1 and 2 make, respectively, if the discount rates were $\delta_1 = \delta$ and $\delta_2 = \delta^2$. Intuitively, in his second offer player 1 makes player 2 indifferent between accepting 1's second offer and making an offer next day, and in his first offer he makes her indifferent between accepting the offer and waiting for the second offer. Therefore, 2 is indifferent between accepting 1's first offer and waiting two days two make an offer, as in the the alternating offer case when her discount rate is δ^2. Now conjecture that the subgame-perfect equilibrium would be as in the alternating offer game with above discount rates. That is,

- in his first offer, player 1 offers

$$\begin{aligned}\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, 1-\frac{1-\delta_2}{1-\delta_1\delta_2}\right) &\equiv \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2\left(1-\delta_1\right)}{1-\delta_1\delta_2}\right) \equiv \left(\frac{1-\delta^2}{1-\delta^3}, \frac{\delta^2\left(1-\delta\right)}{1-\delta^3}\right) \\ &\equiv \left(\frac{1+\delta}{1+\delta+\delta^2}, \frac{\delta^2}{1+\delta+\delta^2}\right);\end{aligned}$$

- in his secon offer, he will offer

$$\left(1-\frac{\delta\left(1-\delta_1\right)}{1-\delta_1\delta_2}, \frac{\delta\left(1-\delta_1\right)}{1-\delta_1\delta_2}\right) \equiv \left(\frac{1+\delta^2}{1+\delta+\delta^2}, \frac{\delta}{1+\delta+\delta^2}\right);$$

- player 2 will offer

$$\left(1-\frac{1-\delta_1}{1-\delta_1\delta_2}, \frac{1-\delta_1}{1-\delta_1\delta_2}\right) \equiv \left(\frac{\delta_1\left(1-\delta_2\right)}{1-\delta_1\delta_2}, \frac{1-\delta_1}{1-\delta_1\delta_2}\right) \equiv \left(\frac{\delta+\delta^2}{1+\delta+\delta^2}, \frac{1}{1+\delta+\delta^2}\right).$$

Player 1's first offer and player 2's offer are by formula for alternating offer, player 1's second offer is calculated by backward induction using the player 2's offer in the next period. Using single deviation property, you need to check that this is an equilibrium.

4. We have an employer and a worker, who will work as a salesman. The worker may be a good salesman or a bad one. In expectation, if he is a good salesman, he will make \$200,000 worth of sales, and if he is bad, he will make only \$100,000. The employer gets 10% of the sales as profit. The employer offers a wage w. Then, the worker accepts or rejects the offer. If he accepts, he will be hired at wage w. If he rejects the offer, he will not be hired. In that case, the employer will get 0, the worker will get his outside option, which will pay \$15,000 if he is good, \$8,000 if he is bad. Assume that all players are risk-neutral.

   (a) Assume that the worker's type is common knowledge, and compute the subgame-perfect equilibrium.

   ANSWER: A worker will accept a wage iff it is at least as high as his outside option, and the employer will offer the outside option — as he still makes profit. That is, 15,000 for the good worker 8,000 for the bad.

   (b) Assume that the worker knows his type, but the employer does not. Employer believes that the worker is good with probability 1/4. Find the perfect Bayesian Nash equilibrium.

   ANSWER: Again a worker will accepts an offer iff his wage at least as high as his outside option. Hence if $w \geq 15,000$ the offer will be accepted by both types, yielding

$$U(w) = (1/4)(.1)\,200,000 + (3/4)(.1)\,100,000 - w = 12,500 - w < 0$$

   as the profit for the employer. If $8,000 \leq w < 15,000$, then only the bad worker will accept the offer, yielding

$$U(w) = (3/4)\left[(.1)\,100,000 - w\right] = (3/4)\left[10,000 - w\right]$$

   as profit. If $w < 0$, no worker will accept the offer, and the employer will get 0. In that case, the employer will offer $w = 8,000$, hiring the bad worker at his outside option.

   (c) Under the information structure in part (b), now consider the case that the employer offers a share s in the sales rather than the fixed wage w. Compute the perfect Bayesian Nash equilibrium.

   ANSWER: Again a worker will accept the share s iff his income is at least as high as his outside option. That is, a bad worker will accept s iff

$$100,000s \geq 8,000$$

i.e.,

$$s \geq s_B = \frac{8,000}{100,000} = 8\%.$$

A good worker will accept s iff

$$s \geq s_G = \frac{15,000}{200,000} = 7.5\%.$$

In that case, if $s < s_G$ no one will accept the offer, and the employer will get 0; if $s_G \leq s < s_B$, the good worker will accept the offer and the employer will get

$$(1/4)\,(10\% - s)\,200,000 = 50,000\,(10\% - s)\,,$$

and if $s \geq s_B$, each type will accept the offer and the employer will get

$$(10\% - s)\,[(1/4)\,200,000 + (3/4)\,100,000] = 125,000\,(10\% - s)\,.$$

Since $125,000\,(10\% - s_B) = 2\%125,000 = 2,500$ is larger than $50,000\,(10\% - s_G) = 2.5\%50,000 = 1,250$, he will offer $s = s_B$, hiring both types.

5. As in question 4, We have an employer and a worker, who will work as a salesman. Now the market might be good or bad. In expectation, if the market is good, the worker will make \$200,000 worth of sales, and if the market is bad, he will make only \$100,000 worth of sales. The employer gets 10% of the sales as profit. The employer offers a wage w. Then, the worker accepts or rejects the offer. If he accepts, he will be hired at wage w. If he rejects the offer, he will not be hired. In that case, the employer will get 0, the worker will get his outside option, which will pay \$12,000. Assume that all players are risk-neutral.

   (a) Assume that whether the market is good or bad is common knowledge, and compute the subgame-perfect equilibrium.

   ANSWER: A worker will accept a wage iff it is at least as high as his outside option 12,000. If the market is good, the employer will offer the outside option $w = 12,000$, and make $20,000 - 12,000 = 8,000$ profit. If the market is bad, the return 10,000 is lower than the worker's outside option, and the worker will not be hired.

   (b) Assume that the employer knows whether the market is good or bad, but the worker does not. The worker believes that the market is good with probability 1/4. Find the perfect Bayesian Nash equilibrium.

   ANSWER: As in part (a). [We will have a separating equilibrium.]

   (c) Under the information structure in part (b), now consider the case that the employer offers a share s in the sales rather than the fixed wage w. Compute a perfect Bayesian Nash equilibrium.

   ANSWER: Note that, since the return is 10% independent of whether the market is good or bad, the employer will make positive profit iff $s < 10\%$. Hence, except for $s = 10\%$, we must have a pooling equilibrium. Hence, at any s, the worker's income is

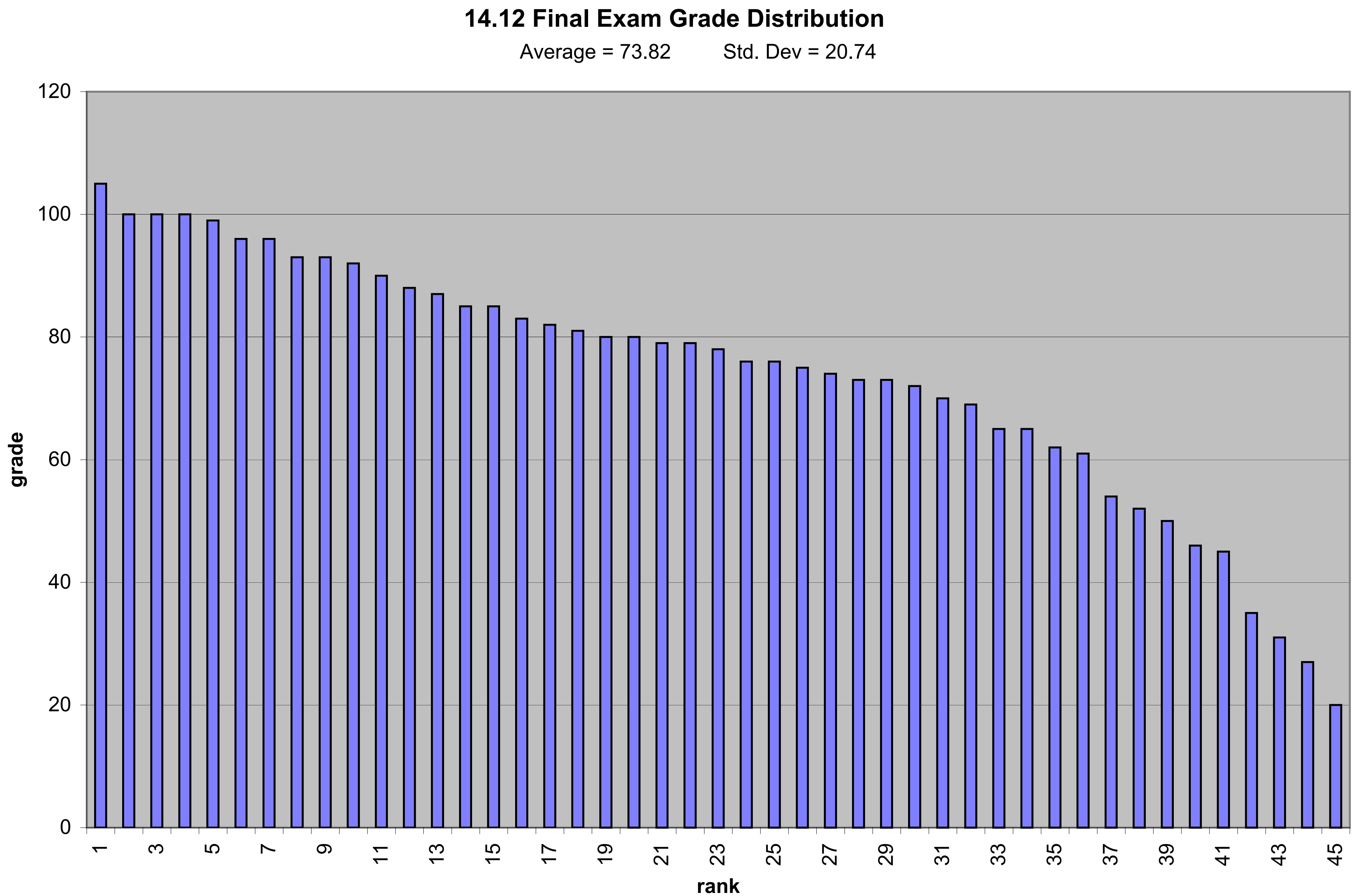
$$[(1/4)\,200,000 + (3/4)\,100,000]\,s = 125,000s.$$

This will be at least as high as his outside option iff

$$s \geq s^* = \frac{12,000}{125,000} = 9.6\% < 10\%.$$

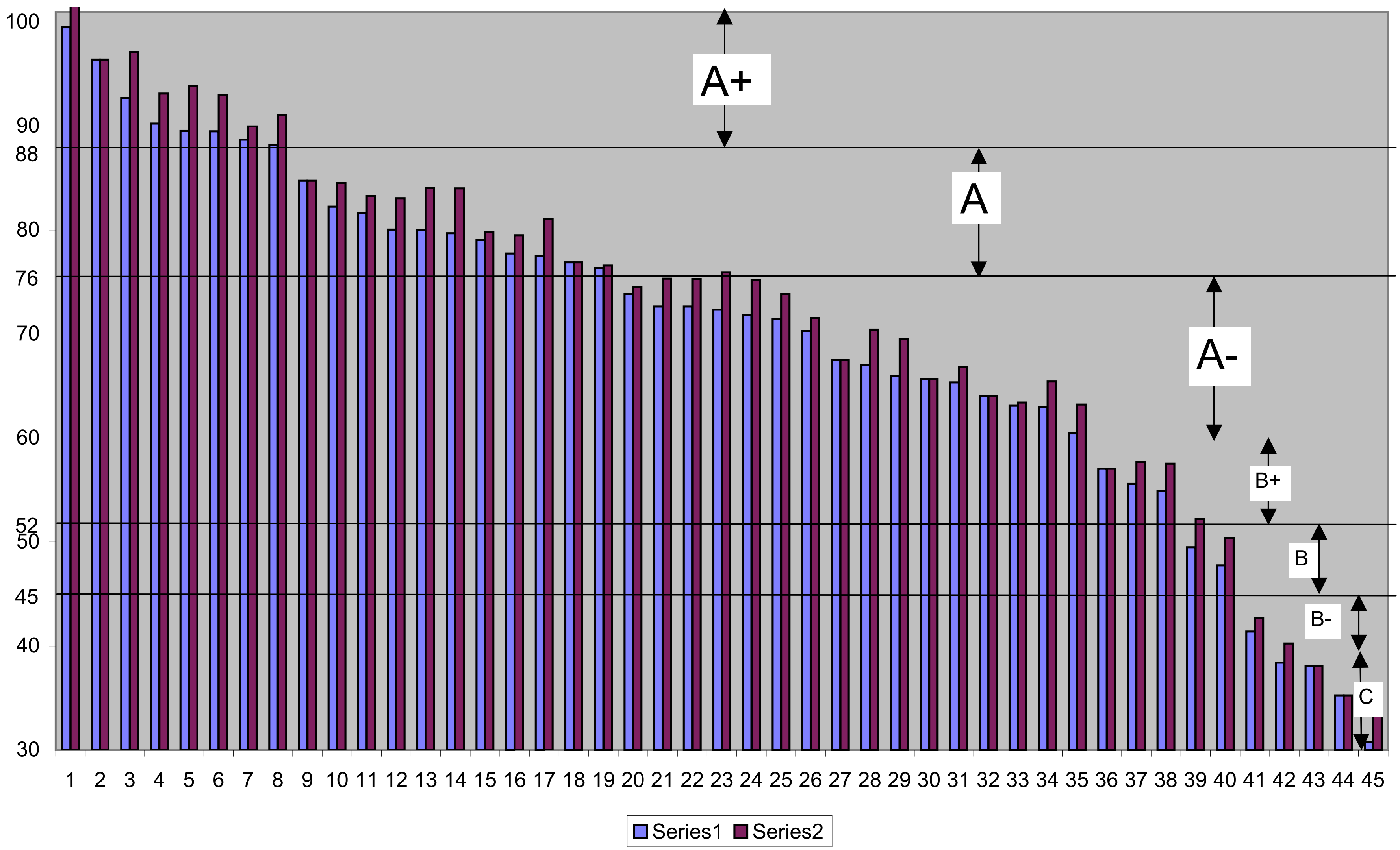
Hence an equilibrium: the worker will accept an offer s iff $s \geq s^*$, and the employer will offer s^*. The worker's beliefs at any offer s is that the market is good with probability 1/4. [Note that this is an inefficient equilibrium. When the market is bad, the gains from trade is less than the outside option.]

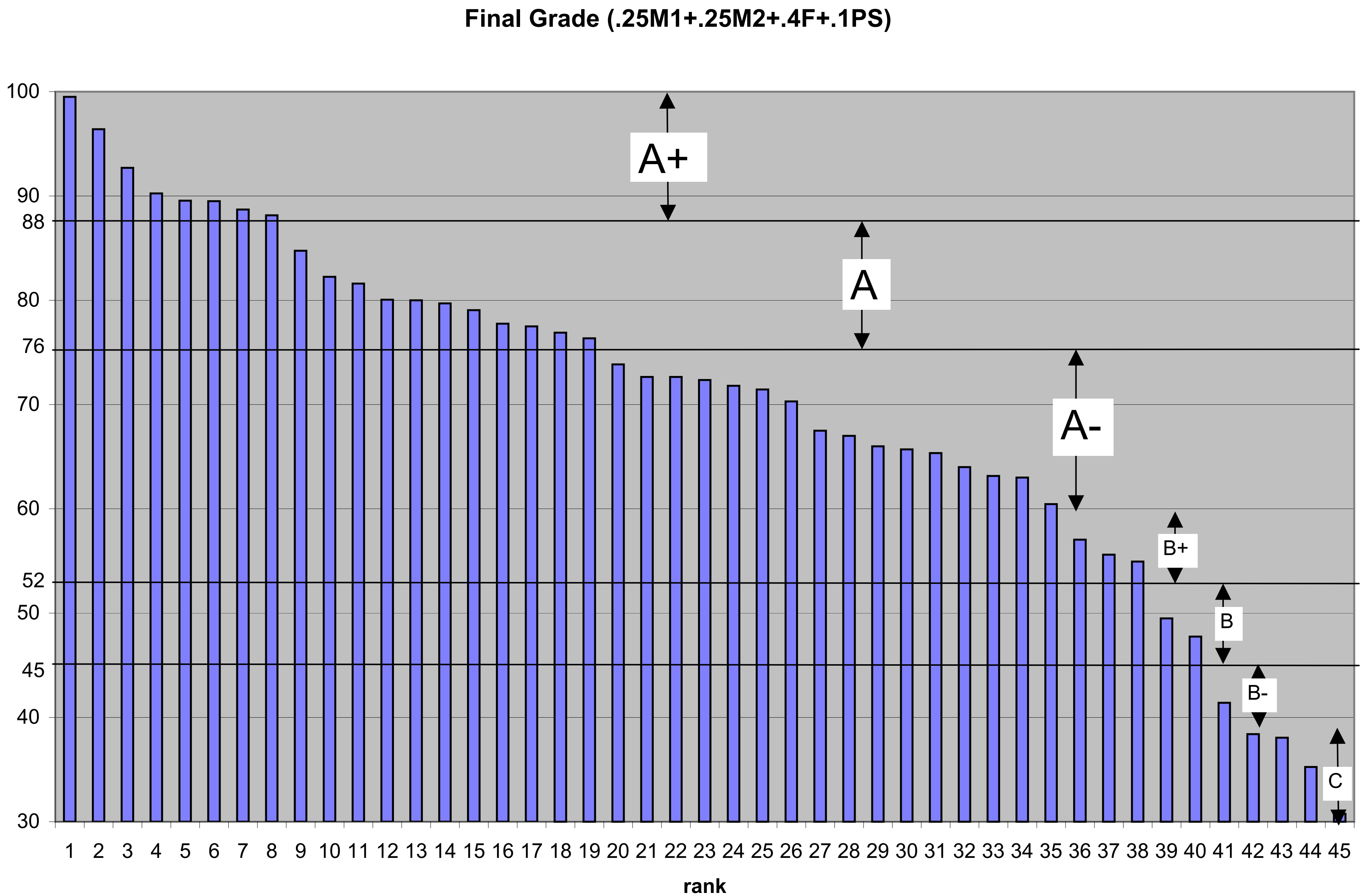
There are other inefficient equilibria where there is no trade (i.e., worker is never hired). In any such equilibrium, worker take any high offer as a sign that the market is bad, and does not accept an offer s unless $s \geq 12,000/100,000 = 12\%$, and the employer offers less than 12%. When the market is good, in any such pure strategy equilibrium, he must in fact be offering less than s^*. (why?) For instance, employer offers $s = 0$ independent of the market, and the worker accept s iff $s > 12\%$.



14.12 Final Exam Grade Distribution
Average = 73.82
Std. Dev = 20.74
grade
rank

Final Grade, Final Grade + Quiz





Final Grade (.25M1+.25M2+.4F+.1PS)
100
90
88
80
76
70
60
52
50
45
40
30
A+
A
A-
B+
B
B-
C
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45
rank

14.12
Spring 1995 **Professor Daron Acemoglu**

Midterm I

1. Consider the following game tree:

[Put any extensive-form game here. m.yildiz.]

(i) write the corresponding normal form
(ii) find all the Nash Equilibria
(iii) are all of these reasonable? why or why not?
(iv) find the Subgame Perfect Equilibrium

2. Two firms compete for a number of workers. The firms simultaneously set wages. There is a proportion λ of informed workers that will go only to the highest wage firm. The remaining $(1-\lambda)$ are uninformed and go with equal probability to either firm. All workers are of equal value to the firms y.

(i) write down profits as a function of wages
(ii) find the equilibrium wages (w_1, w_2) when $\lambda = 0$
(iii) find the equilibrium wages (w_1, w_2) when $\lambda = 1$
(iv) show that there is no pure strategy equilibrium when $0 < \lambda < 1$

3. Consider the following game: Each player says an integer between 0 and 100. Let n_1 denote the number said by player 1 and n_2 that by player 2. The player with n_i closer to $\frac{n_1+n_2}{3}$ gets 10, the other gets 0. If they say the same number they share the prize.

(i) which, if any, are strictly dominated strategies, for each player?
(ii) what outcomes survive after iterated elimination of strictly dominated strategies?
(iii) what are the rationalizable outcomes of this game?
(iv) say a NE of this game
(v) can you find a NE for a similar n-player game?

14.12 **Prof. Daron Acemoglu**
Spring 1995

Answers to Midterm Exam

1. (i) normal form:

	ll	lr	rl	rr
L	$(5,5)$	$(5,5)$	$(3,3)$	$(3,3)$
R	$(10,0)$	$(-10,-10)$	$(10,0)$	$(-10,-10)$

(ii) NE are (R, ll) with payoffs $(10,0)$; (R, rl) with payoffs $(10,0)$; and (L, lr) with payoffs $(5,5)$

(iii) (L, lr) is not reasonable because player 2 cannot credibly convince player 1 that he would play r in his lower node, since l is clearly better (0 is better than -10). Similarly (R, rl) is not convincing because player 2 clearly prefers l to r in his upper node.

(iv) SPE is (R, ll). There is a subgame starting at each of player 2 's nodes. Each one of these subgames has a unique NE which is l on the upper one and l on the lower one. This allows us to exclude r in both cases as a possible SPE outcome. Cutting off the "bad" branches we have

L	$(5,5)$
R	$(10,0)$

now the choice for player 1 is clear - he prefers R.

2. (i) Both firms have to choose wages above w_R and:

$$\Pi_1 = (y - w_1)\left((1-\lambda)\frac{N}{2} + \lambda N\right) \quad if \quad w_1 > w_2$$

$$\Pi_1 = (y - w_1)\left(\frac{N}{2}\right) \quad if \quad w_1 = w_2$$

$$\Pi_1 = (y - w_1)\left((1-\lambda)\frac{N}{2}\right) \quad if \quad w_1 < w_2$$

and similarly for firm 2.

(ii) when $\lambda = 0$

$$\Pi_1 = (y - w_1)\left(\frac{N}{2}\right) \quad if \quad w_1 > w_2$$

$\Pi_1 = (y - w_1)\left(\frac{N}{2}\right) \quad if \quad w_1 = w_2$

$\Pi_1 = (y - w_1)\left(\frac{N}{2}\right) \quad if \quad w_1 < w_2$

That is, the profit does not change if a firm lowers the wage, so each firm will offer the lowest possible wage which is w_R.

(iii) when $\lambda = 1$

$\Pi_1 = (y - w_1)(N) \quad if \quad w_1 > w_2$

$\Pi_1 = (y - w_1)\left(\frac{N}{2}\right) \quad if \quad w_1 = w_2$

$\Pi_1 = (y - w_1)(0) \quad if \quad w_1 < w_2$

The firm with the lowest wage gets zero profit. No firm will be at equilibrium offering lower wage than the other. This means that wages offered are equal. But at any wage each firm wants to raise its wage just enough to be above the other's. So the only equilibrium is when neither wants to increase wage any more - that is - $w_1 = w_2 = y$.

(iv) when $0 < \lambda < 1$, rewriting profits

$\Pi_1 = (y - w_1)\left(\frac{N}{2} + \lambda\frac{N}{2}\right) \quad if \quad w_1 > w_2$

$\Pi_1 = (y - w_1)\left(\frac{N}{2}\right) \quad if \quad w_1 = w_2$

$\Pi_1 = (y - w_1)\left(\frac{N}{2} - \lambda\frac{N}{2}\right) \quad if \quad w_1 < w_2$

As long as wages are below y, each firm prefers to increase wages rather than offering the same wage. If wages are equal to y (profits are zero), then firms prefer to lower wage all the way down to w_R. But both firms at this wage is also not an equilibrium, because each is willing to increase it a little.

3.(i) 100 is strictly dominated by 0, for both players. We can delete 100 for both. We cannot delete any more strategies. Note that, for example, 99, is not strictly dominated by 0 - if the other player says 100, 99 and 0 give exactly the same payoff. (99 is weakly dominated but not strictly dominated). Not strictly dominated are [0,99].

	0	1	2	. . .	98	99	100
0	$5,5$	$10,0$	$10,0$	. . .	$10,0$	$10,0$	$10,0$
1	$0,10$	$5,5$	$10,0$	. . .	$10,0$	$10,0$	$10,0$
2	$0,10$	$0,10$	$5,5$	. . .	$10,0$	$10,0$	$10,0$
.							
.							
.							
98	$0,10$	$0,10$	$0,10$		$5,5$	$10,0$	$10,0$
99	$0,10$	$0,10$	$0,10$		$0,10$	$5,5$	$10,0$
100	$0,10$	$0,10$	$0,10$		$0,10$	$0,10$	$5,5$

(ii) Now, once we have deleted 100 for both players, 99 is strictly dominated by 0. That's the only strategy we can eliminate in this iteration.

	0	1	2	. . .	98	99
0	$5,5$	$10,0$	$10,0$	. . .	$10,0$	$10,0$
1	$0,10$	$5,5$	$10,0$	. . .	$10,0$	$10,0$
2	$0,10$	$0,10$	$5,5$	. . .	$10,0$	$10,0$
.						
.						
.						
98	$0,10$	$0,10$	$0,10$		$5,5$	$10,0$
99	$0,10$	$0,10$	$0,10$		$0,10$	$5,5$

In each iteration we can eliminate the highest number. We are left with only [0] after 100 iterations.

(iii) The rationalizable solutions in a 2-player game are those that remain after strict dominance elimination - in this case it's only (0,0).

(iv) NE is (0,0) with payoffs (5,5).

(v) In an n-player game 0 by all players is a NE.

14.12 Economic Applications of Game Theory
Midterm Examination—October 16th, 1997

You have one and a half hours. Answer all three questions. The number of points for each part is given. They add up to 100.

1) Consider an individual with von Neumann-Morgenstern utility function $u(c) = \frac{c^{1+\alpha}}{1+\alpha}$. The individual has some wealth equal to W at the beginning of the period and does not consume until the end. He can either store his wealth, consumes it for certain at the end of the period, or he can invest part of it in the stock market. The individual knows that the stock market pays a gross rate of return $R > 1$ (that is every \$ invested becomes \$$R$) with probability p or collapses and pays nothing with probability $1 - p$.

1. Find the share of wealth x that the individual puts in the stock market. [9 points]

2. What happens to x as W increases? Why is this? [5 points]

3. What happens to x as α increases? Why is this? [5 points]

4. Now suppose that there is a financial analyst who can give advice. He either reports that the market will boom, in which case the return is R with probability $q > p$, or he reports that the market will crash in which case the return is R with probability $1 - q$. Explain why a rational investor should expect the analyst to report that the market will boom with probability p. [5 points]

5. Find the optimal investment strategy of the individual after listening to the advice of the analyst [Hint: you have to find two numbers x_g and x_b the respective proportions of wealth invested after a good and a bad report]. Explain carefully why the individual would pay money to listen to the analyst's advice. [7 points]

2) There are two risk-neutral workers and two jobs. One job offers wage w_1 and the other wage $w_2 < w_1$. Worker A first decides which job to apply to. Then Worker B observes worker A's choice and makes his decision. If there is one applicant for a job, he is employed and receives the associated wage. If both workers apply to the same job, then the firm randomly chooses between the two applicants, and the other is unemployed at wage 0.

1. Draw the extensive form of this game. [5 points]

2. Write down the normal form and find all the pure strategy equilibria. [If you find it useful, you can distinguish between two cases depending on whether $w_1 > 2 \times w_2$]. [10 points]

3. Now find the Subgame Perfect Equilibria. Explain carefully why some of the Nash Equilibria are not subgame perfect. [10 points]

4. Now assume that Worker B does not observe A's choice. Write down the extensive and the normal forms of the game, and find all Nash Equilibria (including mixed strategy equilibria). [9 points]

3) There are two coffee shops each with constant marginal cost of a cup of coffee equal to c. These two stores compete by setting prices. There are N consumers where N is a large number. Each consumer purchases one cup of coffee as long as the price is less than or equal to R. A fraction $1 - \lambda$ of these consumers see both prices and purchase at the lowest price (as long as $\leq R$). The remaining fraction λ randomly walk into one of the two stores and buy a cup as long as the price is less than or equal to R. If the price is greater than R, they do not buy any coffee.

1. Write down the profits of the two firms as functions of the two prices. [5 points]

2. Find the Nash Equilibria when $\lambda = 0$ and then when $\lambda = 1$. [8 points]

3. Show that there exists no pure strategy Nash Equilibria. [20 points]

4. Show that there exists no mixed strategy Nash Equilibria where one of the firms randomizes between a finite number of strategies. [Difficult. Only for bonus points. Do not waste time on this]

Midterm Exam 14.12

Answer all three questions. You have one and half hours. Each question is worth a 33 percent of the grade. Good luck.

1) Consider an individual with von Neumann-Morgenstern utility function $u(c) = \frac{1-\exp\{-\theta c\}}{\theta}$ with $\theta > 0$. The individual has some wealth equal to W at the beginning of the period and does not consume until the end. He can either store his wealth, consumes it for certain at the end of the period, or he can invest part of it in the stock market. The individual knows that the stock market pays a gross rate of return $R > 1$ (that is every \$ invested becomes \$$R$) with probability p or collapses and pays nothing with probability $1 - p$.

1. Find the amount I that the individual puts in the stock market.
2. What happens to I as W increases? What happens to the share of wealth invested? Why is this?
3. What happens to I as θ increases? Why is this?
4. Now suppose that there is a financial analyst who can give advice. He either reports that the market will boom, in which case the return is R with probability $q > p$, or he reports that the market will crash in which case the return is R with probability $1-q$. Find the optimal investment strategy of the individual after listening to the advice of the analyst [Hint: you have to find two numbers I_g and I_b the respective amounts of wealth invested after a good and a bad report]. Explain carefully why the individual would pay money to listen to the analyst's advice.
5. Discuss informally why the individual may not want to listen to the advice of the analyst if the analyst's pay is a fraction of the investment in the risky asset.

2) There are M coffee shops, each with cost of a cup of coffee equal to $c(x) = cx^2$ for x cups of coffee. These stores compete by setting prices. There are N consumers where N is a large number. Each consumer purchases one cup of coffee as long as the price is less than or equal to R. A fraction $1 - \lambda$ of these consumers see all prices and purchase at the lowest price (as long as $\leq R$). The remaining fraction λ randomly walk into one of the stores and buy a cup as long as the price is less than or equal to R. If the price is greater than R, they do not buy any coffee.

1. Write down the profits of each firm as a function of the prices of others.
2. Find the Nash Equilibria when $\lambda = 0$ and then when $\lambda = 1$.
3. Show that there exists no Nash Equilibria with all firms charging the same price when $\lambda \in (0, 1)$.

3) Consider the following two-period political game with two players, the working class and the elite. In the first-period, the elite decide whether to redistribute or not, and then the working class decides whether to carry out a revolution. Redistribution and no revolution gives a utility 10 to the working class and 15 to the elite. If there is no redistribution and no revolution, the working class gets 0 and the elite get 25. And if there is a revolution (irrespective of redistribution), the working class get 15 and the elite get 0. If there is a revolution, that is the end of the game, and the payoffs are final. If there is no revolution, the game proceeds to the second period, where both parties get additional payoffs. But first, nature determines whether the working class has the opportunity to carry out a revolution. The probability that this opportunity exists for the working class is q. Observing whether the working class has the opportunity to carry out a revolution, the elite again decide whether to redistribute. Once again, without redistribution, they get an additional 25 and the working class get 0. With redistribution, the working class get an additional 10 and the elite 15. Also, again a revolution gives 15 to the working class and nothing to the elite (irrespective of whether there is redistribution in the second period or not). There is no discounting between the two periods.

1. Draw the game tree.
2. Find the unique subgame perfect equilibrium of this game.
3. Explain why a high value of q, probability of revolution opportunity in the second period, prevents a revolution.

14.12 Game Theory – Midterm I
10/19/2000

Prof. Muhamet Yildiz

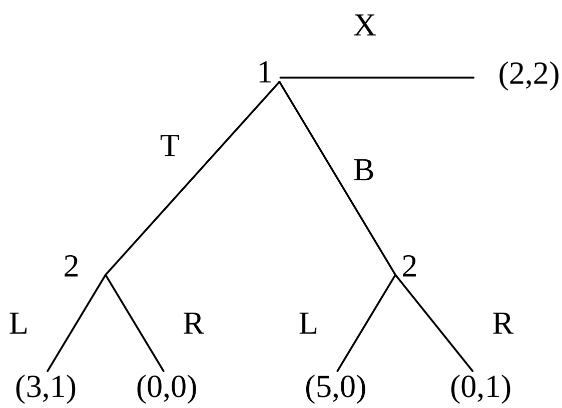
Instructions. This is an open book exam; you can use any written material. You have one hour and 20 minutes. Each question is 33 points. Good luck!

1. Consider the following game.

1\2	L	M	R
T	3,2	4,0	1,1
M	2,0	3,3	0,0
B	1,1	0,2	2,3

a. Iteratively eliminate all the strictly dominated strategies.
b. State the rationality/knowledge assumptions corresponding to each elimination.
c. What are the rationalizable strategies?
d. Find all the Nash equilibria. (Don't forget the mixed-strategy equilibrium!)

2. Consider the following extensive form game.



a. Find the normal form representation of this game.
b. Find all pure strategy Nash equilibria.
c. Which of these equilibria are subgame perfect?

3. Consider two agents $\{1,2\}$ owning one dollar which they can use only after they divide it. Each player's utility of getting x dollar at t is $\delta^t x$ for $\delta \in (0,1)$. Given any $n > 0$, consider the following n-period symmetric, random bargaining model. Given any date $t \in \{0,1,\ldots,n-1\}$, we toss a fair coin; if it comes Head (which comes with probability 1/2), we select player 1; if it comes Tail, we select player 2. The selected player makes an offer $(x,y) \in [0,1]^2$ such that $x+y \leq 1$. Knowing what has been offered, the other player accepts or rejects the offer. If the offer (x,y) is accepted, the game ends, yielding payoff vector $(\delta^t x, \delta^t y)$. If the offer is rejected, we proceed to the next date, when the same procedure is repeated, except for $t = n-1$, after which the game ends, yielding (0,0). The coin tosses at different dates are stochastically independent. And everything described up to here is common knowledge.

a. Compute the subgame perfect equilibrium for $n = 1$. What is the value of playing this game for a player? (That is, compute the expected utility of each player before the coin-toss, given that they will play the subgame-perfect equilibrium.)
b. Compute the subgame perfect equilibrium for $n = 2$. Compute the expected utility of each

player before the first coin-toss, given that they will play the subgame-perfect equilibrium.

c. What is the subgame perfect equilibrium for $n \geq 3$.

14.12 Game Theory – Midterm I Solutions
10/19/2000

Prof. Muhamet Yildiz

1. Consider the following game.

1\2	L	M	R
T	3,2	4,0	1,1
M	2,0	3,3	0,0
B	1,1	0,2	2,3

(a) Iteratively eliminate all the strictly dominated strategies. [10]

Answer: For Player 1, T strictly dominates M, hence we eliminate M of Player 1.

1\2	L	M	R
T	3,2	4,0	1,1
B	1,1	0,2	2,3

Now, R strictly dominates M for Player 2, hence we eliminate M of Player 2, and obtain

1\2	L	R
T	3,2	1,1
B	1,1	2,3

(b) State the rationality/knowledge assumptions corresponding to each elimination. [10]

Answer: For the first elimination, we assume that Player 1 is rational (hence he would not play a strictly dominated strategy). For the second elimination, we assume that Player 2 is rational and that he knows that player 1 is rational. [Many of you lost point for this part; those who stated general rationality assumptions did not get any credit, in general.]

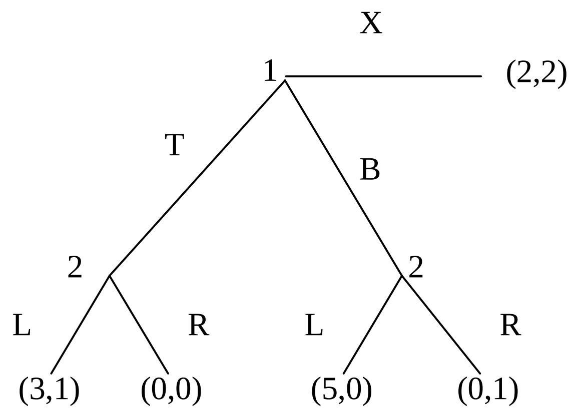
(c) What are the rationalizable strategies? [3]

Answer: {T,B} for Player 1, and {L,R} for player 2.

(d) Find all the Nash equilibria. (Don't forget the mixed-strategy equilibrium!) [10]

Answer: Pure strategy Nash Equilibria are (T,L) and (B,R). [5]. Mixed strategy Nash equilibrium is $[\sigma_1(T) = 2/3; \sigma_1(B) = 1/3; \sigma_2(L) = 1/3; \sigma_2(R) = 2/3]$. [5]

2. Consider the following extensive form game.



(a) Find the normal form representation of this game. [11]

Answer: Player 1 has three strategies: {X,T,B}. Player 2 has four strategies:

- LL = L if Player 1 chooses T; L if Player 1 chooses B.
- LR = L if Player 1 chooses T; R if Player 1 chooses B.
- RL = R if Player 1 chooses T; L if Player 1 chooses B.
- RR = R if Player 1 chooses T; R if Player 1 chooses B.

Therefore, the normal form is the following.

1\2	LL	LR	RL	RR
X	2,2	2,2	2,2	2,2
T	3,1	3,1	0,0	0,0
B	5,1	0,1	5,1	0,1

[Those who assigned only two strategies to player 2 received no credit. For this is the mistake that those who do not know any Game Theory can make. Many of you were confused; they assigned eight strategies to player 2, assigning 2 trivial moves after player 1 plays X. This is clearly wrong, for player 2 does not have any move after player 1 plays X. They received eight points.]

(b) Find all pure strategy Nash equilibria. [11]

Answer: The pure strategy Nash equilibria are (T, LR), and (X,RR).

(c) Which of these equilibria are subgame perfect? [11]

Answer: (T,LR) is subgame perfect. (X,RR) is not; for 2 would not play R if 1 played T.

3. Consider two agents $\{1, 2\}$ owning one dollar which they can use only after they divide it. Each player's utility of getting x dollar at t is $\delta^t x$ for $\delta \in (0, 1)$. Given any $n > 0$, consider the following n-period symmetric, random bargaining model. Given any date $t \in \{0, 1, \ldots, n-1\}$, we toss a fair coin; if it comes Head (which comes with probability 1/2), we select player 1; if it comes Tail, we select player 2. The selected player makes an offer $(x, y) \in [0, 1]^2$ such that $x + y \leq 1$. Knowing what has been offered, the

other player accepts or rejects the offer. If the offer (x, y) is accepted, the game ends, yielding payoff vector $(\delta^t x, \delta^t y)$. If the offer is rejected, we proceed to the next date, when the same procedure is repeated, except for $t = n-1$, after which the game ends, yielding (0,0). The coin tosses at different dates are stochastically independent. And everything described up to here is common knowledge.

(a) Compute the subgame perfect equilibrium for $n = 1$. What is the value of playing this game for a player? (That is, compute the expected utility of each player before the coin-toss, given that they will play the subgame-perfect equilibrium.) [10]

Answer: If a player rejects an offer, he will get 0, hence he will accept any offer that gives him at least 0. (He is indifferent between accepting and rejecting an offer that gives him exactly 0; but rejecting such an offer is inconsistent with an equilibrium.) Hence, the selected player offers 0 to his opponent, taking entire dollar for himself; and his offer will be accepted. Therefore, in any subgame perfect equilibrium, the outcome is (1,0) if it comes Head, and (0, 1) if it comes Tail. The expected payoffs are

$$V = \frac{1}{2}(1,0) + \frac{1}{2}(0,1) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

(b) Compute the subgame perfect equilibrium for $n = 2$. Compute the expected utility of each player before the <u>first</u> coin-toss, given that they will play the subgame-perfect equilibrium. [10]

Answer: In equilibrium, on the last day, they will act as in part (a). Hence, on the first day, if a player rejects the offer, the expected payoff of each player will be $\delta \cdot 1/2 = \delta/2$. Thus, he will accept an offer if an only if it gives him at least $\delta/2$. Therefore, the selected player offers $\delta/2$ to his opponent, keeping $1 - \delta/2$ for himself, which is more than $\delta/2$, his expected payoff if his offer is rejected. Therefore, in any subgame perfect equilibrium, the outcome is $(1 - \delta/2, \delta/2)$ if it comes Head, and $(\delta/2, 1 - \delta/2)$ if it comes Tail. The expected payoff of each player before the first coin toss is

$$\frac{1}{2}(1 - \delta/2) + \frac{1}{2}(\delta/2) = \frac{1}{2}.$$

(c) What is the subgame perfect equilibrium for $n \geq 3$. [13]

Answer: Part (b) suggests that, if expected payoff of each player at the beginning of date $t+1$ is $\delta^{t+1}/2$, the expected payoff of each player at the beginning of t will be $\delta^t/2$. [Note that in terms of dollars these numbers correspond to $\delta/2$ and $1/2$, respectively.] Therefore, the equilibrium is follows: At any date $t < n-1$, the selected player offers $\delta/2$ to his opponent, keeping $1 - \delta/2$ for himself; and his opponent accepts an offer iff he gets at least $\delta/2$; and at date $n-1$, a player accepts any offer, hence the selected player offers 0 to his opponent, keeping 1 for himself. [You should be able to prove this using mathematical induction and the argument in part (b).]

Economic Applications of Game Theory: 14.12

Second Midterm

Please Answer all three questions. You have 1.30 hours.

1. Consider the infinite horizion bargaining game. John and Beth are trying to split a pie of size 1. They both get linear utility from the share of the cake they obtain (x_J for John and x_B for Beth). Both players discount every period of bargaining with discount factor δ (thus the whole cake next period is equivalent to a share δ of the cake this period). Also, each player can stop bargaining at any point and get his or her outside option d_J and d_B where $d_J + d_B < 1$ (the outside options are also discounted, thus receiving outside option d next period is worth δd this period).

   At the beginning of every period Nature decides which player will make the offer in that period. Both players have a probability $\frac{1}{2}$ of being selected every period (draws independent over time).

   (a) Sketch the game tree.

   (b) Find the subgame perfect equilibrium assuming that $d_J = d_B = 0$.

   (c) How does the equilibrium change when $d_J > 0$ and $d_B > 0$.

2. Consider the following Prisoner's Dilemma type game:

	Cooperate	Cheat
Cooperate	10,10	-1,11
Cheat	11,-1	0,0

Suppose that this game is repeated over time, and we are trying to maintain (Cooperate, Cooperate) as a subgame perfect equilibrium. Also assume that both players have discount factor equal to $\frac{1}{2}$.

(a) Find the trigger strategies that will support (Cooperate, Cooperate) as an equilibrium.

(b) Next, suppose that 'non-forgiving' strategies are not allowed. Instead consider the following Trigger strategy: "If you cheat I will cheat for the next T periods, and then I will cooperate again until you cheat one more time". Write the payoff to cheating in the first period, and then starting to cooperate in period T+1. Show that if T is greater than a cut-off level T^*, then cooperating is preferred to cheating.

(c) Explain in words why it is OK to look at the strategy of cheating now and then cooperating from T+1 onwards rather than cheating all the time?

3. Consider the following incomplete information Bertrand game. Each player can be high or low type with probability $\frac{1}{2}$. These draws are independent. A high type has marginal cost 1 and a low type has marginal cost 0. There

is one unit of demand in the market for all prices less than $R(>1)$. Thus, the profit of a firm who charges price $p < R$ while the other firm charges a price $p' > p$ is equal to $p - c$ where c is the relevant marginal cost.

(a) Show that there exists no pure strategy Bayesian Nash Equilibrium.

(b) **(Difficult- so do not get stuck on this part at the expense of the rest) Let us find the mixed strategy equilibrium. First show that the high types will set $p = 1$. Suppose that player 1 plays the mixed strategy $f(p)$ when he is of low type, where $f(p)$ is the probability density with which he chooses price p. Let $F(p)$ be the cdf. Explain why in this case the profit of the player 2 who chooses price q when he is low type is given as $\pi(q) = \frac{1}{2}q + \frac{1}{2}q\,[1 - F(q)]$ (where this is the ex ante profit before knowing the type of the other player). Then note that for a player of type low to play a mixed strategy, he needs to be indifferent between different prices. Using this information, solve for $F(p)$ such that player 2 is indifferent between different $q's$. Using the information that $F(1) = 1$ characterize $F(p)$ completely. Then explain in words why both players playing $F(p)$ when they are low type and $p = 1$ when they are high type is a Bayesian Nash Equilibrium.**

Economic Applications of Game Theory. Second Midterm Exam. November 20th.

You have one and a half hours. Answer all three questions. [Note: the questions do not have equal weight].

1) [Total: 30 points] Consider the following alternating offer bargaining game between two players A and B. A has a discount factor δ_A and B has δ_B. Initially, they have a cake of size equal to 2. In the first period, A makes an offer. B can either accept or reject. If he rejects this offer, then B gets to make an offer in the second period. A can accept or reject. If A rejects, however, in the third period, the cake shrinks to size 1 (this shrinkage is additional to the usual discounting between periods). From then on, the cake remains of size 1 (but there is still discounting) and the two players alternate in making offers until an offer is accepted.

Draw the game tree and find the unique subgame perfect equilibrium [Hint: first solve the subgame which starts in period 3 using standard techniques, and then use backward induction to find the equilibrium behavior in the first two periods.]

2) [Total: 40 points] Consider the following stage game between players A and B:

$A \setminus B$	l	m	r
L	$(-10, 4)$	$(10, 0)$	$(-1, -1)$
M	$(0, 10)$	$(-1, -1)$	$(-1, 1)$
R	$(4, -10)$	$(-1, -1)$	$(2, 2)$

1. Find the Nash Equilibria of this game. [5 points]

2. Consider a supergame G^T which is obtained by repeating this stage game T times. Find the subgame perfect equilibria of G^T. [5 points]

3. Now consider $G^\infty(\delta)$ which is obtained by repeating this stage game an infinite number of times with discount factor δ for both players. Find an equilibrium which is preferred to playing the Nash Equilibrium of the stage game. [30 points]

3)[Total: 30 points] Consider the following incomplete information Bertrand game. There is one customer who will buy one unit of the good as long as the price, P, is less than R. Firm 1 has cost equal to c. Firm 2 has cost c_L with probability θ and cost c_H with probability $1-\theta$, where $c_L < c < c_H < R$. The cost level of firm 2 is not observed by firm 1, but firm 2 knows its own cost level. Both firms simultaneously choose their prices. The firm with the lower price supplies the customer. If they have equal price, one of the firms is randomly chosen to supply the customer. Both firms are risk-neutral.

1. Write the payoff functions of the two firms. [5 points]

2. Define a Bayesian Nash Equilibrium for this game. [5 points]

3. Show that there exists no pure strategy Bayesian Nash Equilibrium.[20 points]

Economic Applications of Game Theory: 14.12

Mock Midterm

1. Consider the infinite horizion bargaining game. John and Beth are trying to split a pie of size 1. They both get linear utility from the share of the cake they obtain (x_J for John and x_B for Beth). Both players discount every period of bargaining with discount factor δ (thus the whole cake next period is equivalent to a share δ of the cake this period). Also, each player can stop bargaining at any point and get his or her outside option d_J and d_B where $d_J + d_B < 1$ (the outside options are also discounted, thus receiving outside option d next period is worth δd this period).

   At the beginning of every period Nature decides which player will make the offer in that period. Both players have a probability $\frac{1}{2}$ of being selected every period (draws independent over time).

   (a) Sketch the game tree.

   (b) Find the subgame perfect equilibrium assuming that $d_J = d_B = 0$.

   (c) How does the equilibrium change when $d_J > 0$ and $d_B > 0$.

   (d) Compare this to the Nash Solution.

2. Consider the following Prisoner's Dilemma type game:

	Cooperate	Cheat
Cooperate	10,10	-1,11
Cheat	11,-1	0,0

Suppose that this game is repeated over time, and we are trying to maintain (Cooperate, Cooperate) as a subgame perfect equilibrium. Also assume that both players have discount factor equal to $\frac{1}{2}$.

(a) Find the trigger strategies that will support (Cooperate, Cooperate) as an equilibrium.

(b) Next, suppose that 'non-forgiving' strategies are not allowed. Instead consider the following Trigger strategy: "If you cheat I will cheat for the next T periods, and then I will cooperate again until you cheat one more time". Write the payoff to cheating in the first period, and then starting to cooperate in period T+1. Show that if T is greater than a cut-off level T^*, then cooperating is preferred to cheating.

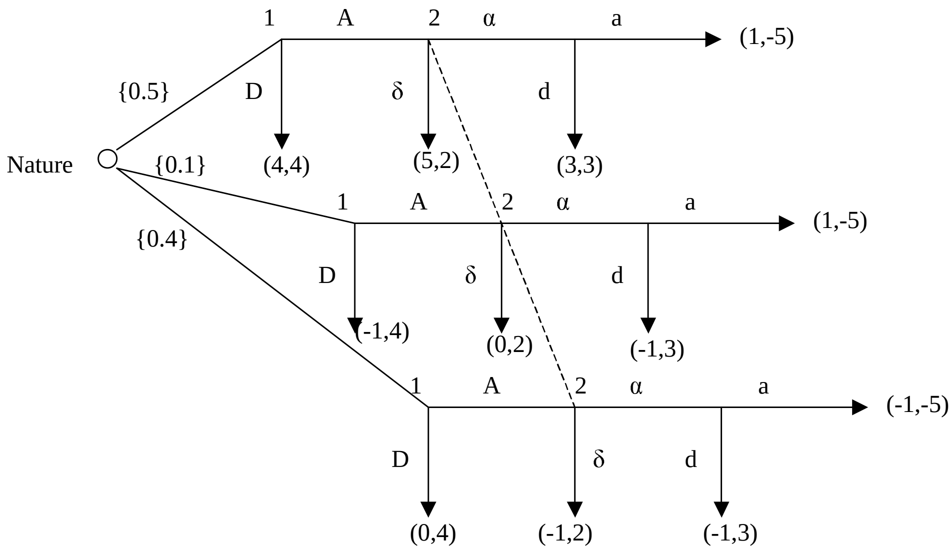
(c) Explain in words why it is OK to look at the strategy of cheating now and then cooperating from T+1 onwards rather than cheating all the time?

14.12 Game Theory – Midterm II
11/16/2000

Prof. Muhamet Yildiz

Instructions. This is an open book exam; you can use any written material. You have one hour and 20 minutes. Each question is 33 points. Good luck!

1. Compute the perfect Bayesian equilibrium of the following game.



2. Consider the following private-value auction of a single object, whose value for the seller is 0. there are two buyers, say 1 and 2. The value of the object for each buyer $i \in \{1,2\}$ is v_i so that, if i buys the object paying the price p, his payoff is $v_i - p$; if he doesn't buy the object, his payoff is 0. We assume that v_1 and v_2 are independently and identically distributed with the probability density function $f : [0,\infty) \to [0,\infty)$, where

$$f(x) = \begin{cases} x & \text{if } x \in [0,2] \\ 0 & \text{otherwise.} \end{cases}$$

   (a) We use sealed-bid first-price auction, where each buyer i simultaneously bids b_i, and the one who bids the highest bid buys the object paying his own bid. Compute the symmetric Bayesian Nash equilibrium in linear strategies, where $b_i = a + cv_i$. Compute the expected utility of a buyer for whom the value of the object is v.

   (b) What is the Bayesian Nash equilibrium for the sealed-bid second-price auction, where the winner pays only the other buyer's bid. What is the expected utility of a buyer for whom the value of the object is v?

14.12 Game Theory – Midterm II
11/17/2000

Important Note: Final Exam is on December 7th.

Prof. Muhamet Yildiz

Instructions. This is an open book exam; you can use any written material. You have one hour and 20 minutes. Each question is 33 points. Good luck!

1. Consider the infinitely repeated game with observable actions where $\delta = 0.99$ and the following Prisoners' Dilemma game is repeated.

	C	D
C	5,5	-6,8
D	8,-6	0,0

Check if any of the following strategy profiles is a subgame-perfect Nash equilibrium.

(a) Each player's strategy is: Play always C.

(b) Each player's strategy is: Play C in the first date; and at the second date and thereafter, play whatever the other player played in the previous date.

(c) There are four modes; Cooperation Mode, Punishment Mode for player 1, Punishment Mode for player 2, and Non-cooperation Mode. In the first date they are in the Cooperation mode. In this mode, each player plays C. In the punishment mode for i, i plays C, while the other player plays D. In non-cooperation mode each player plays D. Once they are in non-cooperation mode they stay in non-cooperation mode forever. In any other mode, if both player stick to their strategy, they go to cooperation mode in the next date; if any player i deviates unilaterally (while the other player sticks with his strategy), they go to the punishment mode for i in the next date; and if both player deviate, in the next date they go to the non-cooperation mode.

2. Consider the following private-value auction of a single object, whose value for the seller is 0. there are two buyers, say 1 and 2. The value of the object for each buyer $i \in \{1, 2\}$ is v_i so that, if i buys the object paying the price p, his payoff is $v_i - p$; if he doesn't buy the object, his payoff is 0. We assume that v_1 and v_2 are independently and identically distributed uniformly on $[\underline{v}, 1]$ where $0 \leq \underline{v} < 1$.

(a) We use sealed-bid first-price auction, where each buyer i simultaneously bids b_i, and the one who bids the highest bid buys the object paying his own bid. Compute the symmetric Bayesian Nash equilibrium in linear strategies, where $b_i = a + cv_i$. Compute the expected utility of a buyer for whom the value of the object is v.

(b) Now assume that v_1 and v_2 are independently and identically distributed uniformly on $[0, 1]$. Now, in order to enter the auction, a player must pay an entry fee $\phi \in (0, 1)$. First, each buyer simultaneously decides whether to enter the auction. Then, we run the sealed-bid auction as in part (a); which players entered is now common knowledge. If only one player enters the auction any bid $b \geq 0$ is accepted. Compute the symmetric perfect Bayesian Nash equilibrium where the buyers use the linear strategies in the auction if both buyer enter the auction. Anticipating this equilibrium, which entry fee the seller must choose? [**Hint**: In the entry stage, there is a cutoff level such that a buyer enters the auction iff his valuation is at least as high as the cutoff level.]

3. Consider the entry deterrence game, where an Entrant decides whether to enter the market; if he enters the Incumbent decides whether to Fight or Accomodate. We consider a game where Incumbent's payoff from the Fight is private information, the entry deterrence game is repeated twice and the discount rate is $\delta = 0.9$. The payoff vectors for the stage game are (0,2) if the Entrant does not enter, $(-1, a)$ if he enters and the Incumbent Fights; and (1,1) if he enters and the Incumbent accomodates, where the first entry in each paranthesis is the payoff for the entrant. Here, a can be either -1 or 2, and is privately known by the Incumbent. Entrants believes that $a = -1$ with probability π; and everything described up to here is common knowledge.

   (a) Find the perfect Bayesian Equilibrium when $\pi = 0.4$.

   (b) Find the perfect Bayesian Equilibrium when $\pi = 0.9$.

Some previous exam questions

1. Consider the infinitely repeated game with observable actions where $\delta = 0.99$ and the following Prisoners' Dilemma game is repeated.

	C	D
C	5,5	-6,8
D	8,-6	0,0

Check if any of the following strategy profiles is a subgame-perfect Nash equilibrium.

(a) Each player's strategy is: Play always C.

(b) Each player's strategy is: Play C in the first date; and at the second date and thereafter, play whatever the other player played in the previous date.

(c) There are four modes; Cooperation Mode, Punishment Mode for player 1, Punishment Mode for player 2, and Non-cooperation Mode. In the first date they are in the Cooperation mode. In this mode, each player plays C. In the punishment mode for i, i plays C, while the other player plays D. In non-cooperation mode each player plays D. Once they are in non-cooperation mode they stay in non-cooperation mode forever. In any other mode, if both player stick to their strategy, they go to cooperation mode in the next date; if any player i deviates unilaterally (while the other player sticks with his strategy), they go to the punishment mode for i in the next date; and if both player deviate, in the next date they go to the non-cooperation mode.

2. Consider the following private-value auction of a single object, whose value for the seller is 0. there are two buyers, say 1 and 2. The value of the object for each buyer $i \in \{1, 2\}$ is v_i so that, if i buys the object paying the price p, his payoff is $v_i - p$; if he doesn't buy the object, his payoff is 0. We assume that v_1 and v_2 are independently and identically distributed uniformly on $[\underline{v}, 1]$ where $0 \leq \underline{v} < 1$.

(a) We use sealed-bid first-price auction, where each buyer i simultaneously bids b_i, and the one who bids the highest bid buys the object paying his own bid. Compute the symmetric Bayesian Nash equilibrium in linear strategies, where $b_i = a + cv_i$. Compute the expected utility of a buyer for whom the value of the object is v.

(b) Now assume that v_1 and v_2 are independently and identically distributed uniformly on $[0, 1]$. Now, in order to enter the auction, a player must pay an entry fee $\phi \in (0, 1)$. First, each buyer simultaneously decides whether to enter the auction. Then, we run the sealed-bid auction as in part (a); which players entered is now common knowledge. If only one player enters the auction any bid $b \geq 0$ is accepted. Compute the symmetric perfect Bayesian Nash equilibrium where the buyers use the linear strategies in the auction if both buyer enter the auction. Anticipating this equilibrium, which entry fee the seller must choose? [**Hint**: In the entry stage, there is a cutoff level such that a buyer enters the auction iff his valuation is at least as high as the cutoff level.]

3. Consider the following "repeated" Stackelberg duopoly, where a long-run firm plays against many short-run firms, each of which is in the market only for one date, while the long-run firm remains in the market throughout the game. At each date t, first, the short run firm sets its quantity x_t; then, knowing x_t, the long-run firm sets its quantity y_t; and each sells his good at price $p_t = 1 - (x_t + y_t)$. The marginal costs are all 0. The short-run firm maximizes its profit, which incurs at t. The long-run firm maximizes the present value of its profit stream where the discount rate is $\delta = 0.99$. At the beginning of each date, the actions taken previously are all common knowledge.

(a) What is the subgame perfect equilibrium if there are only finitely many dates, i.e., $t \in \{0, 1, \ldots, T\}$.

(b) Now consider the infinitely repeated game. Find a subgame perfect equilibrium, where $x_t = 1/4$ and $y_t = 1/2$ at each t on the path of equilibrium play, namely in the contingencies that happen with positive probability given the strategies.

(c) Can you find a subgame perfect equilibrium, where $x_t = y_t = 1/4$ for each t on the path of equilibrium play?

4. Consider the following private-value auction of a single object, whose value for the seller is 0. There are n buyers. The value of the object for each buyer $i \in \{1, 2, \ldots, n\}$ is v_i so that, if i buys the object paying the price p, his payoff is $v_i - p$; if he doesn't buy the object, his payoff is 0. We assume that $(v_1, \ldots, v_n)$ are independently and identically distributed uniformly on $[0, 1]$.

   (a) We use sealed-bid first-price auction, where each buyer i simultaneously bids b_i, and the one who bids the highest bid buys the object paying his own bid. Compute the symmetric Bayesian Nash equilibrium in linear strategies, where $b_i = a + cv_i$.

   (b) Compute the expected utility of a buyer for whom the value of the object is v.

   (c) What happens as $n \to \infty$? Can seller have a better mechanism (for himself) in the limit ($n = \infty$)?

5. Consider the following private-value auction of a single object, whose value for the seller is 0. there are two buyers, say 1 and 2. The value of the object for each buyer $i \in \{1, 2\}$ is v_i so that, if i buys the object paying the price p, his payoff is $v_i - p$; if he doesn't buy the object, his payoff is 0. We assume that v_1 and v_2 are independently and identically distributed with the probability density function $f : [0, \infty) \to [0, \infty)$, where

$$f(x) = \begin{cases} x & \text{if } x \in [0, 2] \\ 0 & \text{otherwise.} \end{cases}$$

   (a) We use sealed-bid first-price auction, where each buyer i simultaneously bids b_i, and the one who bids the highest bid buys the object paying his own bid. Compute the symmetric Bayesian Nash equilibrium in linear strategies, where $b_i = a + cv_i$. Compute the expected utility of a buyer for whom the value of the object is v.

   (b) What is the Bayesian Nash equilibrium for the sealed-bid second-price auction, where the winner pays only the other buyer's bid. What is the expected utility of a buyer for whom the value of the object is v?

6. [Total: 40 points] Consider the following stage game between players A and B:

$A \setminus B$	l	m	r
L	$(-10, 4)$	$(10, 0)$	$(-1, -1)$
M	$(0, 10)$	$(-1, -1)$	$(-1, 1)$
R	$(4, -10)$	$(-1, -1)$	$(2, 2)$

   (a) Find the Nash Equilibria of this game. [5 points]

   (b) Consider a supergame G^T which is obtained by repeating this stage game T times. Find the subgame perfect equilibria of G^T. [5 points]

   (c) Now consider $G^\infty(\delta)$ which is obtained by repeating this stage game an infinite number of times with discount factor δ for both players. Find an equilibrium which is preferred to playing the Nash Equilibrium of the stage game. [30 points]

7. [Total: 30 points] Consider the following incomplete information Bertrand game. There is one customer who will buy one unit of the good as long as the price, P, is less than R. Firm 1 has cost equal to c. Firm 2 has cost c_L with probability θ and cost c_H with probability $1 - \theta$, where $c_L < c < c_H < R$. The cost level of firm 2 is not observed by firm 1, but firm 2 knows its own cost level. Both firms simultaneously choose their prices. The firm with the lower price supplies the customer. If they have equal price, one of the firms is randomly chosen to supply the customer. Both firms are risk-neutral.

   (a) Write the payoff functions of the two firms. [5 points]

   (b) Define a Bayesian Nash Equilibrium for this game. [5 points]

   (c) Show that there exists no pure strategy Bayesian Nash Equilibrium.[20 points]

14.12 Solutions for Question 7 (or 8)

Kenichi Amaya[1]

November 9, 2001

(a)

Let's denote firm 2's by c_2, i.e., $c_2 \in \{c_L, c_H\}$. Let p_1 and p_2 be each firm's price (action). Then, each firm's payoff functions can be written as

$$\pi_1(p_1, p_2) = \begin{cases} p_1 - c & if p_1 < p_2 \\ \frac{p_1 - c}{2} & if p_1 = p_2 \\ 0 & if p_1 > p_2 \end{cases},$$

$$\pi_2(p_1, p_2; c_2) = \begin{cases} p_2 - c_2 & if p_2 < p_1 \\ \frac{p_2 - c_2}{2} & if p_2 = p_1 \\ 0 & if p_2 > p_1 \end{cases}$$

(b)

A pure strategy profile $\{p_1^*, (p_{2L}^*, p_{2H}^*)\}$ is a Bayesian Nash equilibrium if

$$p_1^* = \text{argmax}_{p_1} \theta \pi_1(p_1, p_{2L}^*) + (1-\theta)\pi_1(p_1, p_{2H}^*),$$
$$p_{2L}^* = \text{argmax}_{p_2} \pi_2(p_1^*, p_2; c_L),$$
$$p_{2H}^* = \text{argmax}_{p_2} \pi_2(p_1^*, p_2; c_H),$$

(c)

If there exists a pure strategy Bayesia Nash equilibrium, it must be true that

$$p_1^* = \min\{p_{2L}^*, p_{2H}^*\}.$$

To see this, if $p_1^* < \min\{p_{2L}^*, p_{2H}^*\}$, firm 1 is always selling to the whole market. If it increases the price slightly, it still sells the whole market. So it can increase the payoff. If $p_{2k}^* < p_1^*$, where k is either L or H, firm 2 is selling to the whole market when its type is k. If it increases the price slightly, it still sells the whole market. So it can increase the payoff.

Next, if there exists a pure strategy Bayesia Nash equilibrium, it must be true that

$$p_1^* = \min\{p_{2L}^*, p_{2H}^*\} = c.$$

To see this, if $p_1^* < c$, firm 1 is selling positive amount and losing money. Therefore it's better to charge a price higher than R and not to sell to get zero profit. On the other hand, if $p_1^* < c$, firm 1 can increase it's profit by slightly reducing its price.

Finally, $p_1^* = \min\{p_{2L}^*, p_{2H}^*\} = c$ can not be an equilibrium because firm 2 of type L can increase its profit by choosing a price slightly smaller than c.

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14.12 Game Theory – Final

12/7/2000

Prof. Muhamet Yildiz

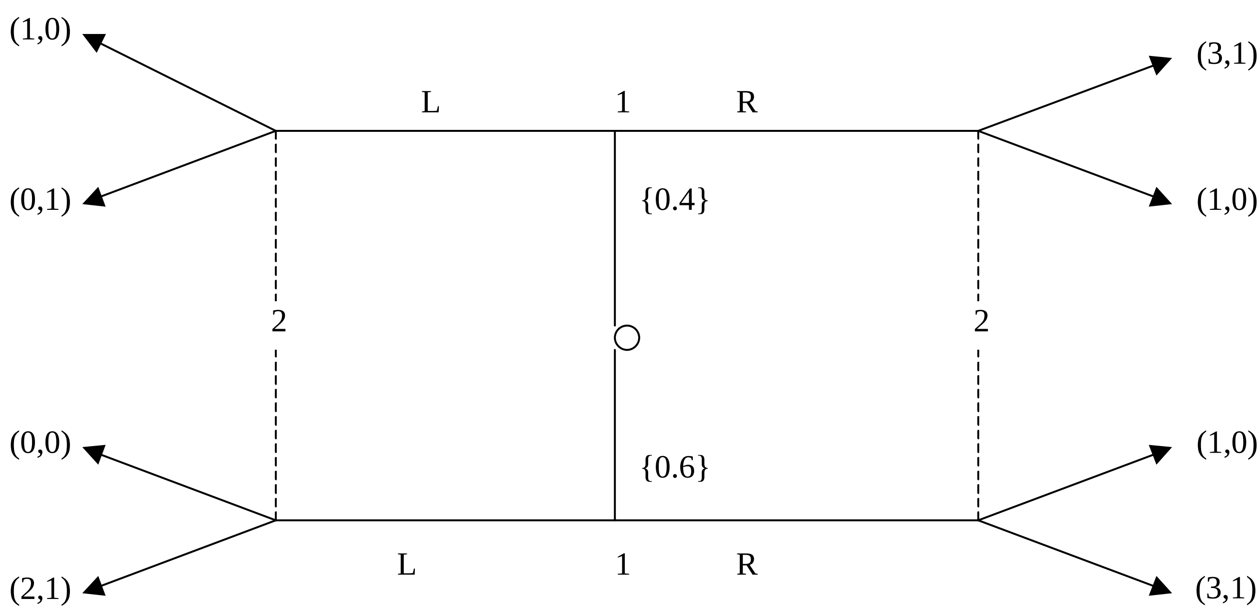
Instructions. This is an open book exam; you can use any written material. You have one hour and 20 minutes. Please answer only three of the following four questions. Each question is 33 points. Good luck!

1. In Spence's Job-market signalling game, assume that there are three types of workers, $t = 1, 2, 3$. Each worker of type t has payoff function $w - e/t$, where w is the wage and e is the education level. The payoff for the firm that hires a worker of type t with wage w is $t - w$. The firms compete for the workers.

   (a) Find a separating equilibrium. (Don't forget the beliefs and wages.)

   (b) Assuming each type is equally likely, find an equilibrium in which the types 2 and 3 are pooling while type 1 is separating.

2. Consider the following game.



   (a) Find a separating equilibrium.

   (b) Find a pooling equilibrium.

   (c) Find an equilibrium in which a type of player one plays a (completely) mixed strategy.

3. Consider a legal case where a plaintiff files a suit against a defendant. It is common knowledge that, when they go to court, the defendant will have to pay \$1000,000 to the plaintiff, and \$100,000 to the court. The court date is set 10 days from now. Before the court date plaintiff and the defendant can settle privately, in which case they do not have the court. Until the case is settled (whether in the court or privately) for

each day, the plaintiff and the defendant pay \$2000 and \$1000, respectively, to their legal team. To avoid all these costs plaintiff and the defendant are negotiating in the following way. In the first day demands an amount of money for the settlement. If the defendant accepts, then he pays the amount and they settle. If he rejects, then he offers a new amount. If the plaintiff accepts the offer, they settle for that amount; otherwise the next day the plaintiff demands a new amount; and they make offers alternatively in this fashion until the court day. Players are risk neutral and do not discount the future. Find the subgame-perfect equilibrium.

4. Consider a worker and a firm. Worker can be of two types, High or Low. The worker knows his type, while the firm believes that each type is equally likely. Regardless of his type, a worker is worth 10 for the firm. The worker's reservation wage (the minimum wage that he is willing to accept) depends on his type. If he is of high type his reservation wage is 5 and if he is of low type his reservation wage is 0. First the worker demands a wage w_0; if the firm accepts it, then he is hired with wage w_0, when the payoffs of the firm and the worker are $10 - w_0$ and w_0, respectively. If the firm rejects it, in the next day, the firm offers a new wage w_1. If the worker accept the offer, he is hired with that wage, when the payoffs of the firm and the worker are again $10 - w_1$ and w_1, respectively. If the worker rejects the offer, the game ends, when the worker gets his reservation wage and the firm gets 0. Find a perfect Bayesian equilibrium of this game.

14.12 Game Theory
Fall 2000
Final Exam Solutions

December 14, 2001

1. (a) Given perfect competition among firms, wages will be $w_1 = 1$,$w_2 = 2$ and $w_3 = 3$ for types $t = 1, 2, 3$ respectively. The incentive compatibility constraints give us the conditions that the education levels must fulfill in equilibrium: $e_2^* \geq 1 + e_1^*$ and $e_3^* \geq 2 + e_1^*$ for $t = 1$, $e_2^* \leq 2 + e_1^*$ and $e_3^* \geq 2 + e_2^*$ for $t = 2$, and $e_3^* \leq 3 + e_2^*$ and $e_3^* \leq 6 + e_1^*$ for $t = 3$.

   If we let, for example, $e_1^* = 0$, then $e_2^* = 1$ and $e_3^* = 3$. Beliefs are $\mu(t = 1|e') = 1$, where e' is in $[0, \inf)$, $\mu(t = 2|e = 1) = 1$ and $\mu(t = 3|e = 3) = 1$.

   (b) Let $w_1 = 1$ as before, since type 1 is separating. Given that types 2 and 3 are pooling, their wage is the expected marginal productivity or $1/2(2) + 1/2(3) = 5/2$. Letting $e_1^* = 0$, the compatibility constraints provide the conditions that the education levels must fulfill in equilibrium: $e^* \geq 3/2$ and $e^* \leq 3$. For instance, we choose $e^* = 2$. Beliefs are $\mu(t = 1|e \neq e^*) = 1$, $\mu(t = 2|e^*) = 1/2$ and $\mu(t = 3|e^*) = 1/2$.

2. (a) Let the separating equilibrium be one where top plays R, bottom plays L. The beliefs are $\mu(top|R) = 1$ and $\mu(bottom|R) = 1$. Player 2, the receiver, plays up in the information set on the right-hand side, and down, on the information set on the left-hand side. Now it is easy to check that player 1's types do not want to deviate: if top type plays L instead, she gets 0 as opposed to 3, while bottom type, if she plays R instead, gets 1 as opposed to 2.

   (b) Pooling on R: player 2 plays down when sees R since $EU(up|R) = 0.4(1) + 0.6(0) = 0.4 < EU(down|R) = 0.4(0) + 0.6(1) = 0.6$. Any beliefs about types given L actually support this equilibrium.

   (c) Mixed strategies equilibrium: Bottom mixes $\alpha R + (1 - \alpha)L$. Player 2 mixes $1/2up{+}1/2down$ when sees R. By applying Bayes' rule and forcing it to be equal to 1/2, because this is the value that would make the bottom type want to mix, we have $\mu(top|R) = 0.4(1)/(0.4{+}0.6(\alpha)) = 1/2$. In particular, for player 2, $EU(up|R) = P(top|R)$ and $EU(down|R) = 1 - P(top|R)$. Setting these two equal, because we need player 2 to mix after observing R in order for bottom type to want to mix, we obtain that $P(top|R) = 1/2$. From the Bayes' rule equation we solve for α above, such that bottom type's strategy is $2/3R + 1/3L$. Top type always plays R. Player 2 plays $1/2up + 1/2down$ when sees R and plays down when sees L. Beliefs are $\mu(bottom|L) = 1$ and $\mu(top|R) = 1/2$.

3. Given no discounting, and the fact that the plaintiff gets to make the first offer, the game ends in the first day, with the plaintiff making an offer w and the defendant accepting the offer. In particular, taking into account the legal fees both parties must pay until court day, and the amount the defendant must pay then, which is certain and common knowledge, the plaintiff will offer $1,110K$ and the defendant will accept all offers less than or equal to this amount, and reject all others.

4. One pooling equilibrium of this game would be for the firm to reject all offers $w_0 > 5$ and accept all others, for both types of workers to offer $w_0 = 5$, and to offer $w_1 = 5$ in the following period if reached. Beliefs: $\mu(type = 0|w \neq 5) = 1$. Workers do not want to deviate because that would lead to a rejection and a wage of 5 in the following period, which is what they can get now (so they are indifferent).

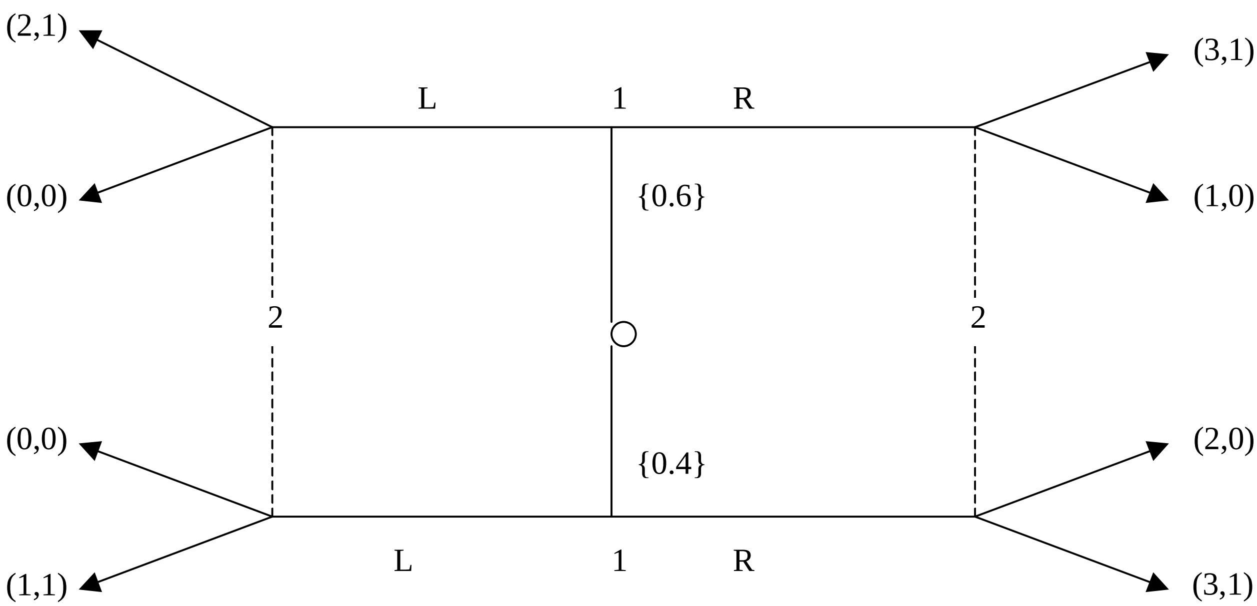


Figure 1:

14.12 Game Theory – Final (Make Up)
12/15/2000

Prof. Muhamet Yildiz

Instructions. This is an open book exam; you can use any written material. You have one hour and 20 minutes. Please answer only three of the following four questions. Each question is 33 points. Good luck!

1. Two players (say A and B) own a company, each of them owning a half of the Company. They want to dissolve the partnership in the following way. Player A sets a price p. Then, player B decides whether to buy A's share or to sell his own share to A,in each case at price p. The value of the Company for players A and B are v_A and v_B, respectively.

   (a) Assume that the values v_A and v_B are commonly known. What would be the price in the subgame-perfect equilibrium?

   (b) Assume that the value of the Company for each player is his own private information, and that these values are independently drawn from a uniform distribution on [0,1]. Compute the perfect Bayesian equilibrium.

2. Consider the following game.

   (a) Find a separating equilibrium.

   (b) Find a pooling equilibrium.

   (c) Find an equilibrium in which a type of player 1 plays a (completely) mixed strategy.

3. Find the subgame-perfect equilibrium of the following 2-person game. First, player 1 picks an integer x_0 with $1 \leq x_0 \leq 10$. Then, player 2 picks an integer y_1 with $x_0 + 1 \leq y_1 \leq x_0 + 10$. Then, player 1 picks an integer x_2 with $y_1 + 1 \leq x_2 \leq y_1 + 10$. In this fashion, they pick integers, alternatively. At each time, the player moves picks an integer, by adding an integer between 1 and 10 to the number picked by the other player last time. Whoever picks 100 wins the game and gets 100; the other loses the game and gets zero.

4. Consider a used-car market where buyers need to use their cars in two periods. There are two types of cars, peaches and lemons. A peach will not break down and the buyer will use it two periods. A lemon can break down at the end of the first period with probability 1/2, in which case buyer will not be able to use in the second period. The value of a car for a buyer is \$2,000 if he uses it only in the first period, and \$4,000 if he uses it in both periods. The value of a car for the seller is \$2,600 if it is a lemon, and \$3,600 if it is a peach. For each car the seller knows whether it is a lemon or peach but the buyer does not know. Both buyers and the sellers are risk neutral, and there are equal number of buyers and sellers (and equal number of peaches and lemons).

   (a) Find the equilibrium price in this market.

   (b) Now, introduce a risk-neutral dealer in the market. He can tell whether a car is a lemon or a peach with certainty. He sells the cars with a warranty, replacing the car with another one if it breaks down, incurring a cost of \$1,800. Now every seller either sells in the market or goes to the dealer. After checking the car, dealer offers him a price. If the seller accepts, the dealer buys the car at that price, and sells it in the market with the warranty. What are the equilibrium prices in the market? [The price of a car sold by the owner, by the dealer, and the amount the dealer offers to each seller.]