Chapter 2-Logic

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3:20 PM

Logik:

$\wedge$
is shorthand for "and" (equivalent to the AND Operator)

A ***Proposition*** is a statement which is true or false (but never both!). (It can't be a question. It also can't contain variables, and a claim needs to be stated in the proposition) *Not as useful in Proofing.*

$\vee$
Is shorthand for "or" (equivalent to the OR Operator)

$\neg p$
Is shorthand for "Not p" (equivalent to the NOT Operator)

$p \rightarrow q$
Is shorthand for "If p then q"

$p \leftrightarrow q$
Is shorthand for "if p then q, and the converse is also true"

$\neg q \ \rightarrow \ \neg p$
Is shorthand for the contrapositive of if p then q, and **is** equivalent to if p then q.

Ordering in Mathematical Shorthand: apply the "not" operators first, then the "and" and "or". Then you take the results and do the implication operations. (similar to high school algebra rules)

Logical Equivalence:
A frequently useful fact is that $p \rightarrow q$ is logically equivalent to $\neg p \vee q$.

De Morgan's Law:

not (A and B) $\longrightarrow$ not A or not B

not (A or B) $\longrightarrow$ not A and not B

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not (A and B) $\neq$ not A and not B

T and F are special constant propositions with no variables that are, respectively, always true and always false.

$p \wedge \neg p \equiv F$

The equal operator **=** can only be applied to **objects** such as numbers. When comparing **logical expressions** that return true/false values, you must use **≡**.

Distributive Rules:

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(you can distribute either operator over the other)

The Conditional to OR/AND Rule:

$p \rightarrow q \equiv \neg p \vee q$

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

(Recall this is true because in p -> q, when p is false, the statement is always true, and when p is true, the statement is true only when q is true)

A ***<u>Predicate</u>*** is a statement that becomes true or false if you substitute in values for its variables.

Quantifiers:

There are only 3, and the first 2 are the most common:

$\forall$

For All (universal quantifier),

$\exists$

There Exists (existential quantifier),

$\exists!$

There exists a unique … (unique existence, used when there is only 1 object that matches some requirements)

Examples:

$\forall x \in \mathbb{R}, x^2 + 3 \geq 0$

$\exists y \in \mathbb{R}, y = \sqrt{2}$

$\exists! x \in \mathbb{R}, x^2 = 0$

"Such that" is sometimes abbreviated "s.t."
(Mathematicians are lazy)

Contrapositives also exists for statements with quantifiers.
For example:

$\forall x, \text{ if } p(x), \text{ then } q(x)$

And its contrapositive:

$\forall x, \text{ if } \neg q(x), \text{ then } \neg p(x)$

The quantifier stays the same: only transform the if/then part.

Representing 2D points:
Either refer to 2 single variables and pair them up, like so:

$\forall x, y \in \mathbb{Z}$

And refer to them as a pair later on, or
Treat the pair as a single variable, like so:

$\exists (x, y) \in \mathbb{R}^2, x^2 + y^2 = 1$

Or be even more abstract:

$\exists p \in \mathbb{R}^2, p \text{ is on the unit circle}$

(and then later define what "p" and "the unit circle" means)

<u>Predicate Logic "De Morgan's Laws":</u>

$\neg(\forall x, P(x)) \equiv \exists x, \neg P(x)$

"for all x in A, P(x)" is false when (is equivalent) to "there is some value x in A such that P(x) is false"

$\neg(\exists x, P(x)) \equiv \forall x, \neg P(x)$

"there exists x in A such that P(x) is true" is false when (is equivalent to) "for all x in A, P(x) is false"

(P(x) is a statement like $x > 0$ or whatever, and A can be some set like real numbers or whatever)

Example:

$$\forall x, P(x) \rightarrow (Q(x) \wedge R(x))$$

Its **negation** is:

$$\begin{aligned} \neg(\forall x, P(x) \rightarrow (Q(x) \wedge R(x))) &\equiv \exists x, \neg(P(x) \rightarrow (Q(x) \wedge R(x))) \\ &\equiv \exists x, P(x) \wedge \neg(Q(x) \wedge R(x))) \\ &\equiv \exists x, P(x) \wedge (\neg Q(x) \vee \neg R(x)) \end{aligned}$$

Bounding and Scope:

A quantifier **binds** to a variable, but if a variable hasn't been bound by a quantifier, (or otherwise given a value or a set of replacement values), it is called a **Free Variable.**

Free variables don't have a defined truth value, so they **cannot** be a step in a proof.

The "bound" variable in a quantification is only defined for a limited time, called the **Scope** of the binding.

Qualifiers define the scope of the variable. For example, the **i** in

$\sum_{i=0}^{n} \frac{1}{i}$

is only defined while you are still inside the summation.