Chapter 11-Induction

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<u>Induction:</u>

A proof by induction has the following outline:

Claim: $P(n)$ is true for all positive integers n.

Proof: We'll use induction on n.

Base: We need to show that $P(1)$ is true.

Induction: Suppose that $P(n)$ is true for $n = 1, 2, \ldots, k-1$. We need to show that $P(k)$ is true.

(Basically writing a "recursive" proof)

Manually calculate/show/proof the Base value, then write the induction proof to show that the "next" value (k+1) can also be proven ***given that the "current" value (k) is already proven***

Example:

Claim 38 *For any positive integer n, $\Sigma_{i=1}^{n}\ i = \frac{n(n+1)}{2}$.*

Proof: We will show that $\Sigma_{i=1}^{n}\ i = \frac{n(n+1)}{2}$ for any positive integer n, using induction on n.

Base: We need to show that the formula holds for $n = 1$. $\Sigma_{i=1}^{1}\ i = 1$. And also $\frac{1 \cdot 2}{2} = 1$. So the two are equal for $n = 1$.

Induction: Suppose that $\Sigma_{i=1}^{n}\ i = \frac{n(n+1)}{2}$ for $n = 1, 2, \ldots, k-1$. We need to show that $\Sigma_{i=1}^{k}\ i = \frac{k(k+1)}{2}$.

By the definition of summation notation, $\Sigma_{i=1}^{k}\ i = (\Sigma_{i=1}^{k-1}\ i) + k$

Our inductive hypothesis states that at $n = k-1$, $\Sigma_{i=1}^{k-1}\ i = (\frac{(k-1)k}{2})$.

Combining these two formulas, we get that $\Sigma_{i=1}^{k}\ i = (\frac{(k-1)k}{2}) + k$.

But $(\frac{(k-1)k}{2}) + k = (\frac{(k-1)k}{2}) + \frac{2k}{2} = (\frac{(k-1+2)k}{2}) = \frac{k(k+1)}{2}$.

So, combining these equations, we get that $\Sigma_{i=1}^{k}\ i = \frac{k(k+1)}{2}$ which is what we needed to show.

(only really works for natural numbers or infinite subsets of natural numbers like integers)

Note that the P(k) must be a true of false **statement**, **NOT** a formula that gives a number.
P must also depend on k, and k is known as the induction variable.

Claim 39 *For any natural number n, $n^3 - n$ is divisible by 3.*

Proof: By induction on n.

Base: Let $n = 0$. Then $n^3 - n = 0^3 - 0 = 0$ which is divisible by 3.

Induction: Suppose that $n^3 - n$ is divisible by 3, for $n = 0, 1, \ldots, k$. We need to show that $(k+1)^3 - (k+1)$ is divisible by 3.

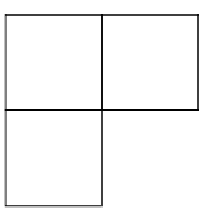
$$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1) = (k^3 - k) + 3(k^2 + k)$$

From the inductive hypothesis, $(k^3 - k)$ is divisible by 3. And $3(k^2 + k)$ is divisible by 3 since $(k^2 + k)$ is an integer. So their sum is divisible by 3. That is $(k+1)^3 - (k+1)$ is divisible by 3.

$\square$

Sometimes there is more than 1 base case written out just for good measure.

The Right Triominoes Example:



Claim 40 *For any positive integer n, a $2^n \times 2^n$ checkerboard with any one square removed can be tiled using right triominoes.*

Proof: by induction on n.

Base: Suppose $n = 1$. Then our $2^n \times 2^n$ checkerboard with one square removed is exactly one right triomino.

Induction: Suppose that the claim is true for $n = 1, \ldots, k$. That is a $2^n \times 2^n$ checkerboard with any one square removed can be tiled using right triominoes as long as $n \leq k$.

Suppose we have a $2^{k+1} \times 2^{k+1}$ checkerboard C with any one square removed. We can divide C into four $2^k \times 2^k$ sub-checkerboards P, Q, R, and S. One of these sub-checkerboards is already missing a square. Suppose without loss of generality that this one is S. Place a single right triomino in the middle of C so it covers one square on each of P, Q, and R.

Now look at the areas remaining to be covered. In each of the sub-checkerboards, exactly one square is missing (S) or already covered (P, Q, and R). So, by our inductive hypothesis, each of these sub-checkerboards minus one square can be tiled with right triominoes. Combining these four tilings with the triomino we put in the middle, we get a tiling for the whole of the larger checkerboard C. This is what we needed to construct.

Try to find ways to relate the P(k+1) part back to the P(k) and the definition of P (P is the equation/statement/algorithm/thing that is in the claim)

The Greedy Graph Coloring Algorithm:

Claim 41 *For any positive integer D, if all nodes in a graph G have degree $\leq D$, then G can be colored with $D + 1$ colors.*

Proof: Let's pick a positive integer D and prove the claim by induction on the number of nodes in G.

Base: Since $D \geq 1$, the graph with just one node can obviously be colored with $D + 1$ colors.

Induction: Suppose that any graph with at most $k - 1$ nodes and maximum node degree $\leq D$ can be colored with $D + 1$ colors.

Let G be a graph with k nodes and maximum node degree $\leq D$. Remove some node v (and its edges) from G to create a smaller graph G'.

G' has $k - 1$ nodes. Also, the maximum node degree of G' is no larger than D, because removing a node can't increase the degree. So, by the inductive hypothesis, G' can be colored with $D + 1$ colors.

Because v has at most D neighbors, its neighbors are only using D of the available colors, leaving a spare color that we can assign to v. The coloring of G' can be extended to a coloring of G with $D + 1$ colors.

So it just goes through all the nodes, at every node checks and see what colors have been assigned to its neighbors. If there is a previously used color not assigned to a neighbor, we re-use that color. Otherwise, we assign a new color to that node and move on.
(Note that it is not the most efficient, since D+1 is the upper bound, and some different coloring method can easily color the same graph with less colors)

A ***Strong Inductive Hypothesis*** is when the inductive step assumed that P(n) is true for all previous steps of n from the base up to k − 1.
When the inductive step requires more than 1 previous value (not enough info from just k-1), you usually need a Strong Inductive Hypothesis and sometimes prove more than 1 base case to ensure that ***all*** *previous steps were true.*

A ***Weak Inductive Hypothesis*** is when the inductive step **only** assumes that P(k-1) is true.

You can use a weak inductive hypothesis when k-1 gives enough info to finish the proof.

The safe way (for dummies like me) is just to use Strong Inductive Hypothesis everywhere.