Chapter 5-Sets

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12:39 AM

Sets:

An *unordered* collection of objects. (unordered means as long as the same values are present, it is considered the same set)

Ways to define a set:

1. describe its contents in mathematical English, e.g. “the integers between 3 and 7, inclusive.”
2. list all its members, e.g. {3,4,5,6,7}
3. use so-called set builder notation

   e.g. $\{x \in \mathbb{Z} \mid 3 \leq x \leq 7\}$

   (note that the **|** can be swapped with **:** when **|** needs to be used in the right side of the notation)

In **Tuples** duplicates and ordering matters, but not in sets.

*Note that **(**1,2,2,3**)** is a tuple, and **{**1,2,2,3**}** is a set.*

A tuple cannot contain less than 2 objects, and a set can be empty (with the symbol Ø)

Cardinality:

A finite set X's Cardinality is |X|, which is the number of **different** objects in X.

$A \subseteq B$

A is a **Subset** of B is every element of A is also in B.

$\subseteq$ is like $\leq$

The 2 sets might be the same.

$A \subset B$

A is the **Proper Subset** of B, they may NOT be the same.

$B \supseteq A$ means the same as $A \subseteq B$

Vacuously True:

Weird statements that are true because the condition is always false.

Operations:

Intersection of two sets:

$(A \cap B)$

gives a set containing all objects that are in both A and B.

If the resulting set of the intersection operation is am empty set, then the 2 sets are said to be **Disjoint**. (no element in common)

Union of two sets:

$(A \cup B)$

gives a set containing objects that are in either A or B.

(Adds two sets together)

Difference of two sets:

$(A - B)$

gives a set containing objects that are in A but not in B.

Complement of two sets:

$(\overline{A})$

gives a set containing all objects that are NOT in A.

Note that a *Universal Set* needs to be defined when using the complement operation.

So, if our universe U is all integers, and A contains all the multiples of 3, then $\overline{A}$ is all the integers whose remainder mod 3 is either 1 or 2.

(Universal Set can be whatever set we define it to be)

Cartesian product of two sets:

$(A \times B)$

gives a set containing ordered pairs (x, y) where x is in A and y is in B.

For example, if $A = \{a, b\}$ and $B = \{1, 2\}$, then

$$A \times B = \{(a,1), (a,2), (b,1), (b,2)\}$$
$$B \times A = \{(1,a), (2,a), (1,b), (2,b)\}$$

$A{\times}B{\times}C == \{(a,1,p), (a,1,q), (a,2,p), (a,2,q), (b,1,p), (b,1,q), (b,2,p), (b,2,q)\}$

*(D*istributively *Multiply)*

DeMorgan's Law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\cup$ is like $\vee$

$\cap$ is like $\wedge$

$\overline{A}$ is like $\neg P$

$\emptyset$ (the empty set) is like F

U (the universal set) is like T

<u>Calculating the Number of Elements in the Union of Sets:</u>

Inclusion-Exclusion Principle: $|A \cup B| = |A| + |B| - |A \cap B|$
(sum the size of each set and subtract all overlapping elements)

And then there's this monstrosity:

$$\begin{aligned}
|A \cup B \cup C| &= |A| + |B \cup C| - |A \cap (B \cup C)| \\
&= |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)| \\
&= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|) \\
&= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|
\end{aligned}$$

(distributive property in step 3 and removed the pointless parenthesis and the extra A in the last step)

The Product Rule:

if you have p choices for one part of a task, then q choices for a second part, and your options for the second part don't depend on what you chose for the first part, then you have pq options for the whole task.

*(so if there are p elements in A, q elements in B, then the size of (A X B) is p*q)*

We can often combine the Inclusion-Exclusion Principle and the Product Rule to find the size of complex sets.

General Proof Form to Show A is a Subset of B:

Proof: Let sets A and B be defined as above. Let x be an element of A.

[missing details]

So x is an element of B.

Since x was arbitrarily chosen, we've shown that any element of A is also an element of B. So A is a subset of B.

(and remember that you can start from the top, but also work backwards from the bottom at some point after starting from the top to help fill out most of the missing details using given/obvious information)

A general tip of proofs is that the proof should use all the information in the hypothesis of the claim. If that's not the case, either the proof has a bug, or the claim could be revised to make it more "*interesting*."

Example:

Claim 28 *For any sets A, B, and C, if $A \times B \subseteq A \times C$ and $A \neq \emptyset$, then $B \subseteq C$.*

Proof draft: Suppose that A, B, and C are sets and suppose that $A \times B \subseteq A \times C$ and $A \neq \emptyset$. We need to show that $B \subseteq C$.

So let's choose some $x \in B$. ...

(We need to relate A X B and A X C using individual elements)

So let's choose some $x \in B$. Since $A \neq \emptyset$, we can choose an element t from A. Then $(t, x) \in A \times B$ by the definition of Cartesian product.

Since $(t, x) \in A \times B$ and $A \times B \subseteq A \times C$, we must have that $(t, x) \in A \times C$ (by the definition of subset). But then (again by the definition of Cartesian product) $x \in C$.

So we've shown that if $x \in B$, then $x \in C$. So $B \subseteq C$, which is what we needed to show.

Another Example:

Claim 29 *For any sets A and B, if $(A - B) \cup (B - A) = A \cup B$ then $A \cap B = \emptyset$.*

Notice that the conclusion $A \cap B = \emptyset$ claims that something does not exist (i.e. an object that's in both A and B). So this is a good place to apply proof by contrapositive.

(also the conclusion is a basic statement, while the given is complex, and often it is easier to use something basic to proof something complex, so we take the contrapositive)

Proof: Let's prove the contrapositive. That is, we'll prove that if $A \cap B \neq \emptyset$, then $(A - B) \cup (B - A) \neq A \cup B$.

So, let A and B be sets and suppose that $A \cap B \neq \emptyset$. Since $A \cap B \neq \emptyset$, we can choose an element from $A \cap B$. Let's call it x.

Since x is in $A \cap B$, x is in both A and B. So x is in $A \cup B$.

However, since x is in B, x is not in $A - B$. Similarly, since x is in A, x is not in $B - A$. So x is not a member of $(A - B) \cup (B - A)$. This means that $(A - B) \cup (B - A)$ and $A \cup B$ cannot be equal, because x is in the second set but not in the first. $\square$.