Chapter 8-Functions and one-to-one

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One-to-One:

A function is one-to-one if it never assigns two input values to the same output value.

$\forall x, y \in A, x \neq y \rightarrow f(x) \neq f(y)$

Or the contrapositive (more useful definition):

$\forall x, y \in A, f(x) = f(y) \rightarrow x = y$

Bijection/Bijective:

A function that is both one-to-one and onto.

f maps from A to B, then f^{-1} maps from B to A

An *onto* function from A to B requires that A have at least as many elements as B;
A *one-to-one* function from A to B requires that B have at least as many values as A;
So, if there is a *bijection* between A and B, then the two sets must contain the **same number of elements**.

Pigeonhole principle: Suppose you have n objects and assign k labels to these objects. If $n > k$, then two objects must get the same label.

(sometimes can be used as a clever trick in a proof that seems unrelated)
(or just include it in a random proof for fun if you want)

Permutations:

An arrangement of n objects in order.

(For example, there is set A and set B with the same size n, there are many ways to construct a one-to-one function using A and B. After choosing the output for each input, there will be 1 less output to choose from for the following inputs, so the total number of ways is n!)

$$P_k^n = \frac{n!}{(n-k)!}$$

(where n is the total number of objects available to choose from, and k is the number of objects we need to choose)

Proving a Function is One-to-One:

Claim 33 *Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 3x + 7$. f is one-to-one.*

Let's prove this using our definition of one-to-one.

Proof: We need to show that for every integers x and y, $f(x) = f(y) \to x = y$.

So, let x and y be integers and suppose that $f(x) = f(y)$. We need to show that $x = y$.

We know that $f(x) = f(y)$. So, substituting in our formula for f, $3x + 7 = 3y + 7$. So $3x = 3y$ and therefore $x = y$, by high school algebra. This is what we needed to show.

One-to-One Composition:

Claim 34 *For any sets A, B, and C and for any functions $f : A \to B$ and $g : B \to C$, if f and g are one-to-one, then $g \circ f$ is also one-to-one.*

Proof: Let A, B, and C be sets. Let $f : A \to B$ and $g : B \to C$ be functions. Suppose that f and g are one-to-one.

We need to show that $g \circ f$ is one-to-one. So, choose x and y in A and suppose that $(g \circ f)(x) = (g \circ f)(y)$

Using the definition of function composition, we can rewrite this as $g(f(x)) = g(f(y))$. Combining this with the fact that g is one-to-one, we find that $f(x) = f(y)$. But, since f is one-to-one, this implies that $x = y$, which is what we needed to show.