Chapter 3-Proofs

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1:34 AM

Proofs:

Simplest way to prove a claim with the below form: Pick a value and plug it in

$$\forall x \in A, P(x)$$

Example:

For every rational number q, $2q$ is rational.

Define "Rational":

A real number r is *rational* if there are integers m and n, $n \neq 0$, such that $r = \frac{m}{n}$.

Proof: Let q be any rational number. From the definition of "rational," we know that $q = \frac{m}{n}$ where m and n are integers and n is not zero. So $2q = 2\frac{m}{n} = \frac{2m}{n}$. Since m is an integer, so is $2m$. So $2q$ is also the ratio of two integers and, therefore, $2q$ is rational.

(first expand the word "rational" into its definition, then at the end go the other way)

Definitions are always intended to work in both directions.

(So use iff instead of if)

Collect up all your given information right at the start of the proof, so you know what you have to work with.

Example 2:

Claim 1 *For any integer k, if k is odd then k^2 is odd.*

Definition 2 *An integer n is odd if there is an integer m such that $n = 2m + 1$.*

Proof of Claim 1: Let k be any integer and suppose that k is odd. We need to show that k^2 is odd.

Since k is odd, there is an integer j such that $k = 2j + 1$. Then we have

$$k^2 = (2j + 1)^2 = 4j^2 + 4j + 1 = 2(2j^2 + 2j) + 1$$

Since j is an integer, $2j^2 + 2j$ is also an integer. Let's call it m. Then $k^2 = 2m + 1$. So, by the definition of odd, k^2 is odd.

(The new variable used at the end of the proof-m-matches the definition, which helps keep the proof organized)

Existential Statement Proofs:

Easiest way is to list specific examples:

Claim 2 *There is an integer k such that $k^2 = 0$.*

Proof: Zero is such an integer. So the statement is true.

(Do NOT go for an overkill and write out a long general statement/argument proof, an example is enough)
(unless you have too much time left on the exam)

Disproving a Universal Claim:

Come up with an existential statement proof (a case in which the claim is false), see above.

Disproving an Existential Statement:

Come up with a universal statement proof (in which the claim is false)

<u>Summary:</u>

	prove	disprove
universal	general argument	specific counter-example
existential	specific example	general argument

It's important to use a fresh variable name each time you expand a definition. (even if it is the same definition)

Q.E.D. at the end of proof means "what we needed to show." (Latin), this lets the reader know that the proof has ended.

Proof by Cases:

When the claim that needs to be proven has an "or" in it, do part of the proof two or more times, once for each of the possibilities in the "or."
Example:

Claim 9 *For all integers j and k, if j is even or k is even, then jk is even.*

We can prove this as follows:

Proof: Let j and k be integers and suppose that j is even or k is even. There are two cases:

Case 1: j is even. Then $j = 2m$, where m is an integer. So the $jk = (2m)k = 2(mk)$. Since m and k are integers, so is mk. So jk must be even.

Case 2: k is even. Then $k = 2n$, where n is an integer. So the $jk = j(2n) = 2(nj)$. Since n and j are integers, so is nj. So jk must be even.

So jk is even in both cases, which is what we needed to show.

Cases can overlap, but all possibilities have to be covered.

Rephrasing Claims:
Turn claim into a convenient form: **a universal if/then statement whose hypothesis contains positive (not negated) facts.**

Turn claim into contrapositive: **by negating the conclusion of the original claim, we gain access to the basic quantities to help us prove the derived quantity that was originally on the "if" side.**

For example, suppose that we want to prove

Claim 15 *For any integers a and b, if $a + b \geq 15$, then $a \geq 8$ or $b \geq 8$.*

This is hard to prove in its original form, because we're trying to use information about a derived quantity to prove something about more basic quantities. If we rephrase as the contrapositive, we get

Claim 16 *For any integers a and b, if it's not the case that $a \geq 8$ or $b \geq 8$, then it's not the case that $a + b \geq 15$.*

And this is equivalent to:

Claim 17 *For any integers a and b, if $a < 8$ and $b < 8$, then $a + b < 15$.*

Notice that when we negated the conclusion of the original statement, we needed to change the "or" into an "and" (DeMorgan's Law).

Proof: We'll prove the contrapositive of this statement. That is, for any integers a and b, if $a < 8$ and $b < 8$, then $a + b < 15$.

So, suppose that a and b are integers such that $a < 8$ and $b < 8$. Since they are integers (not e.g. real numbers), this implies that $a \leq 7$ and $b \leq 7$. Adding these two equations together, we find that $a + b \leq 14$. But this implies that $a + b < 15$. $\square$