Chapter 6-Relations

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Relations:

A *relation R on a set A* is a subset of $A \times A$,

(R is a set of ordered pairs that are made with elements taken from A that fit a specific criteria/relationship)

xRy

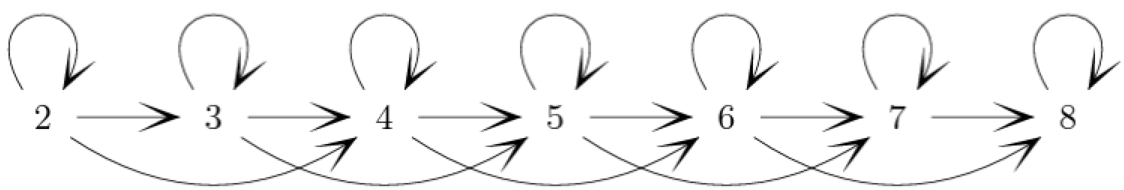
If x is related to y,

$x \not{R} y$

If x is not related to y.

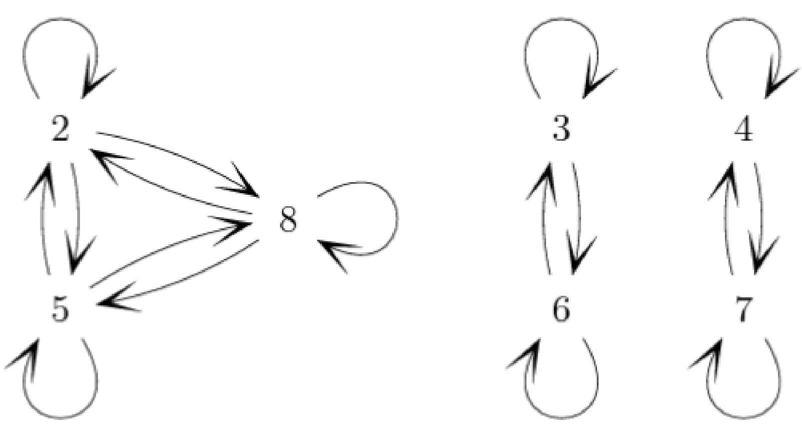
For example, suppose we let $A = \{2, 3, 4, 5, 6, 7, 8\}$. We can define a relation W on A by xWy if and only if $x \leq y \leq x + 2$. Then W contains pairs like $(3, 4)$ and $(4, 6)$, but not the pairs $(6, 4)$ and $(3, 6)$. Under this relation, each element of A is related to itself. So W also contains pairs like $(5, 5)$.

Relation graph:

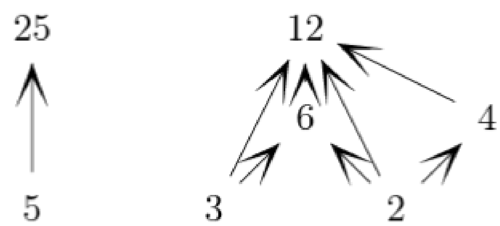


Graphs can also look like:

X ≡ y (mod 3)



xTy if $x|y$ and $x \neq y$.



Relation Properties:

Reflexive: every element is related to itself.

Irreflexive: no element is related to itself.

Neither reflexive nor irreflexive: some elements are related to themselves but some aren't.

(note that the inverse/negative of reflexive is NOT irreflexive!)

not reflexive: there is an $x \in A, x \not R x$

Symmetric: if xRy in R, yRx is also true. Mostly occur in relations that resemble equality, like

xXy iff $|x| = |y|$

(**only** 2-way arrows in relations graph)

symmetric: for all $x, y \in A$, xRy implies yRx

Antisymmetric: if xRy in R, yRx is not true. Mostly occur in relations that put elements into an order, like

xWy if and only if $x \leq y \leq x + 2$.

(**only** 1-way arrows in relations graph)

antisymmetric: for all x and y in A with $x \neq y$, if xRy, then $y \not R x$

antisymmetric: for all x and y in A, if xRy and yRx, then $x = y$

(both definitions are equivalent)

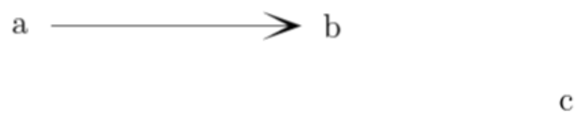
Transitivity:

transitive: for all $a, b, c \in A$, if aRb and bRc, then aRc

(transitivity means that whenever there is an **indirect** path from x to y, then there must also be a **direct** arrow from x to y)

not transitive: there are $a, b, c \in A$, aRb and bRc and $a \not R c$

Note that the transitivity definition is a condition, so again if the "if" part is false, then the statement is true regardless.



(This thing is transitive, even though there's no arrow from a to c)

Types of Relations:

A **partial order** is a relation that is reflexive, antisymmetric, and transitive.

A **linear order** (also called a total order) is a partial order R in which every pair of elements are **comparable**. That is, for any two elements x and y, either xRy or yRx.

A **strict partial order** is a relation that is irreflexive, antisymmetric, and transitive.

(Think of Linear Orders like relating all integers with ≤, every integer can be related to another random integer using ≤)
(Think of Partial Orders like Linear Orders but some pairs of elements are not related. For example, for divides, 5 doesn't divide 7, and 7 also doesn't divide 5)
(Think of Strict Partial Order like a Partial Order except elements are not related to themselves.)

Definition: An **equivalence relation** is a relation that is reflexive, symmetric, and transitive.
If R is some specified relation on a set A, and x is an element in A, the equivalence class of x is all elements y where xRy.
$[x]_R = \{y \in A \mid xRy\}$
(Equivalence Relation is like the general case/definition of relations like Congruence Mod k on the set of integers)

Proving that a Relation is an Equivalence Relation:

Just take the relation and prove all 3 conditions (reflexive, symmetric, and transitive) individually

Example:
Let F be the set of all fractions

$$F = \{\frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\}$$

So:
$\sim$ defined by: $\frac{x}{y} \sim \frac{p}{q}$ if and only if $xq = yp$

Proof: Reflexive: For any x and y, $xy = xy$. So the definition of $\sim$ implies that $\frac{x}{y} \sim \frac{x}{y}$.
Symmetric: if $\frac{x}{y} \sim \frac{p}{q}$ then $xq = yp$, so $yp = xq$, so $py = qx$, which implies that $\frac{p}{q} \sim \frac{x}{y}$.

Transitive: Suppose that $\frac{x}{y} \sim \frac{p}{q}$ and $\frac{p}{q} \sim \frac{s}{t}$. By the definition of $\sim$, $xq = yp$ and $pt = qs$. So $xqt = ypt$ and $pty = qsy$. Since $ypt = pty$, this means that $xqt = qsy$. Cancelling out the q's, we get $xt = sy$. By the definition of $\sim$, this means that $\frac{x}{y} \sim \frac{s}{t}$.

Since $\sim$ is reflexive, symmetric, and transitive, it is an equivalence relation.

Same process when trying to prove other kinds of relations: split the proof into multiple parts and prove each requirement one by one.
(For example: proving antisymmetry:
relation. Consider the set of intervals on the real line $J = \{(a, b) \mid a, b \in \mathbb{R} \text{ and } a < b\}$. Define the containment relation C as follows:

$$(a, b)\ C\ (c, d) \text{ if and only if } a \leq c \text{ and } d \leq b$$

For proving antisymmetry, it's typically easiest to use this form of the definition of antisymmetry: if xRy and yRx, then $x = y$. Notice that C is a relation on intervals, i.e. pairs of numbers, not single numbers. Substituting the definition of C into the definition of antisymmetry, we need to show that

For any intervals (a, b) and (c, d), if $(a, b)\ C\ (c, d)$ and $(c, d)\ C\ (a, b)$, then $(a, b) = (c, d)$.

So, suppose that we have two intervals (a, b) and (c, d) such that $(a, b)\ C\ (c, d)$ and $(c, d)\ C\ (a, b)$. By the definition of C, $(a, b)\ C\ (c, d)$ implies that $a \leq c$ and $d \leq b$. Similarly, $(c, d)\ C\ (a, b)$ implies that $c \leq a$ and $b \leq d$.

Since $a \leq c$ and $c \leq a$, $a = c$. Since $d \leq b$ and $b \leq d$, $b = d$. So $(a, b) = (c, d)$.