Chapter 7-Functions and onto

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12:24 AM

Functions:

A Function from A to B is an assignment of exactly one element of B (i.e. the output value/*co-domain*) to each element of A (i.e. the input value/*domain*).
*(Note that every input has to have **1** output for a function to be valid)*

Type Signature:

$f : A \to B$

For example, suppose P is a set of five people:

$$P = \{\text{Margaret}, \text{Tom}, \text{Chen}, \text{LaSonya}, \text{Emma}\}$$

And suppose C is a set of colors:

$$C = \{\text{red}, \text{blue}, \text{green}, \text{purple}, \text{yellow}, \text{orange}\}$$

We can define a function $f : P \to C$ which maps each person to their favorite color. For example, we might have the following input/output pairs:

$$\begin{aligned} f(\text{Margaret}) &= \text{Blue} \\ f(\text{Tom}) &= \text{Red} \\ f(\text{LaSonya}) &= \text{Purple} \\ f(\text{Emma}) &= \text{Red} \\ f(\text{Chen}) &= \text{Blue} \end{aligned}$$

Since the value of each element in P does NOT depend on the previous values, each element has |C| possibilities.
(so if there are x elements in P and y elements in C, then there are x^y possible ways to construct/define the function)

An ***Identity Function*** maps each value to itself in a set.

$\text{id}_A : A \to A$ and $\text{id}_A(x) = x$

For two functions to be equal, they have to

1. Have the same output values for all inputs, and

2. Have the same *type signature (which means that the input and output sets have to be same, regardless of how the inputs map to the outputs)*

Examples of equal functions:

$f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = |x|$.

$f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = \max(x, -x)$.

$f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = x$ if $x \geq 0$ and $f(x) = -x$ if $x \leq 0$.

Image:

The image of a function is a set containing all the values produced by the function.

$f(A) = \{f(x) : x \in A\}$

Onto:

A function is *onto* if the image of the function is its entire co-domain(output set).

$\forall y \in B, \exists x \in A, f(x) = y$

Not Onto:

Negate the definition of Onto:

$\neg \forall y \in B, \exists x \in A, f(x) = y$

(Treat this as

$\neg(\forall y \in B, \exists x \in A, f(x) = y)$

So the definition of not onto is:

$\neg \forall y \in B, \exists x \in A, f(x) = y$

$\equiv \exists y \in B, \neg \exists x \in A, f(x) = y$

$\equiv \exists y \in B, \forall x \in A, \neg(f(x) = y)$

$\equiv \exists y \in B, \forall x \in A, f(x) \neq y$

When using onto, the **order** of the **universal** and the **existential quantifiers** matters!

For example:

For every person p in the Fleck family, there is a toothbrush t such that p brushes their teeth with t.

Is not the same as:

There is a toothbrush t, such that for every person p in the Fleck family, p brushes their teeth with t.
(the quantifiers are flipped, and they mean different things-saying everyone uses a toothbrush vs everyone uses the same toothbrush)
(This reminded me that I need to buy a new toothbrush)

Proving that a Function is Onto (1-D):

Claim 30 *Define the function g from the integers to the integers by the formula $g(x) = x - 8$. g is onto.*

> Proof: We need to show that for every integer y, there is an integer x such that $g(x) = y$.
>
> So, let y be some arbitrary integer. Choose x to be $(y + 8)$. x is an integer, since it's the sum of two integers. But then $g(x) = (y + 8) - 8 = y$, so we've found the required pre-image for y and our proof is done.

The trick is to pick a good pre-image for y (in this case (y+8)).

Proving that a Function is Onto (2-D):

Here's a sample function whose domain is 2D. Let $f : \mathbb{Z}^2 \to \mathbb{Z}$ be defined by $f(x, y) = x + y$. I claim that f is onto.

To prove that f is onto, we need to pick some arbitrary element y in the co-domain. That is to say, y is an integer. Then we need to find a sample value in the domain that maps onto y, i.e. a "preimage" of y. At this point, it helps to fiddle around on our scratch paper, to come up with a suitable preimage. In this case, $(0, y)$ will work nicely. So our proof looks like:

> Proof: Let y be an element of $\mathbb{Z}$. Then $(0, y)$ is an element of $f : \mathbb{Z}^2$ and $f(0, y) = 0 + y = y$. Since this construction will work for any choice of y, we've shown that f is onto.

Trickier than 1-D, but you just need to find a **formula/pre-image** that can cover all possibilities of the **co-domain/output**. (note that not all of the *domain* needs to be covered)

Composing functions:

$f : A \to B$ and $g : B \to C$ are functions

$(g \circ f)(x) = g(f(x))$

(Make sure you know that the ordering matters and the domain can be different from the co-domain-input string and output integer, or input char and output real numbers, etc.)

Avoid using the term "range" as it may refer to either the **image** (the set containing all possible outputs of a specific function), or the **co-domain** (the set containing all elements of a specific type, like reals or integers)