

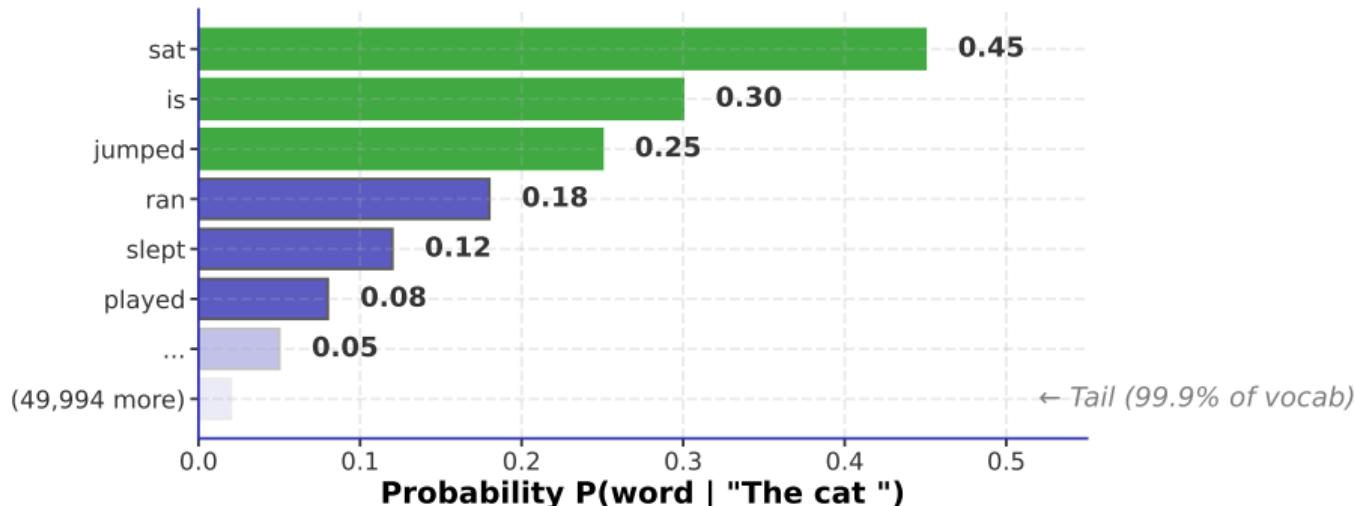
# Decoding Strategies

Week 9: From Probabilities to Text

November 2025

## The Decoding Challenge: Choosing From 50,000 Words

### The Decoding Challenge: Choose From 50,000 Words

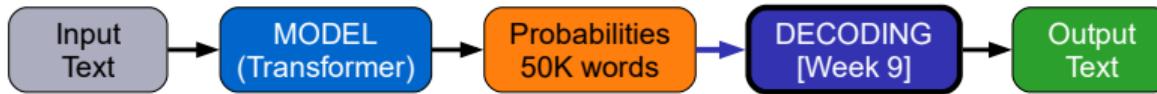


**The Question:** Given these probabilities for "The cat \_", which word should we pick?

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At each step, model outputs probability distribution over entire vocabulary - how do we choose?

## Context: How We Got Here



### Our Journey:

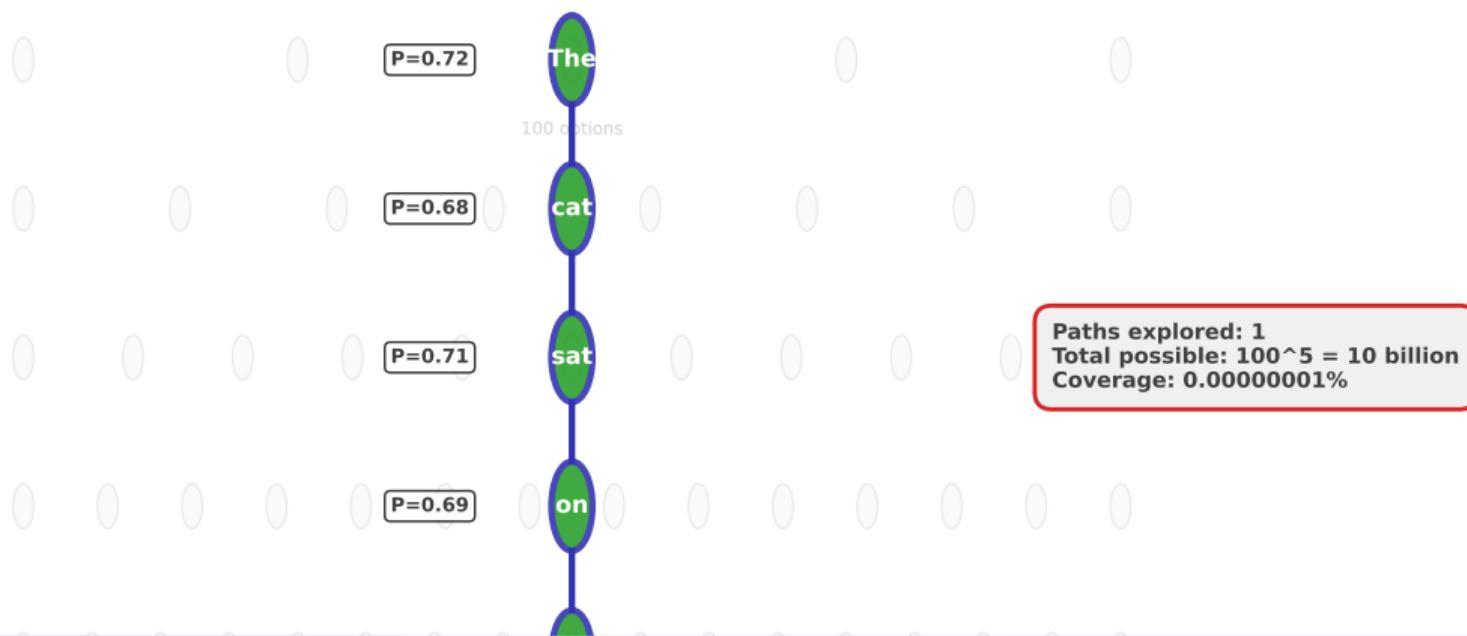
1. We trained models (Weeks 3-7: RNN → Transformers → BERT/GPT)
2. They learned to predict:  $P(\text{word}|\text{context})$
3. They output probability distributions over 50,000+ words
4. **Today:** How do we convert these probabilities into actual text?

---

Models predict probabilities. Decoding converts probabilities to text.

## Extreme Case 1: Greedy Decoding

Vocabulary size = 100, but explores only 1 path per step



## What If We Explored More Paths?

Greedy chose: "The cat **sat**" ( $P=0.68$ )

But it ignored these alternatives:

"The cat <b>walked</b> "	$P=0.12$	(might lead to better text)
"The cat <b>jumped</b> "	$P=0.08$	(more interesting)
"The cat <b>slept</b> "	$P=0.06$	(different story)
"The cat <b>ran</b> "	$P=0.04$	(action-oriented)

**Question:** What if we kept ALL 100 words at each step?

Think:  $100 \times 100 \times 100 \times 100 \times 100 = ?$

## What If We Explored More Paths?

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"The cat <b>ran</b> "	$P=0.04$	(action-oriented)

**Question:** What if we kept ALL 100 words at each step?

Think:  $100 \times 100 \times 100 \times 100 \times 100 = ?$

**Answer:** 10 billion paths! Let's see what happens...

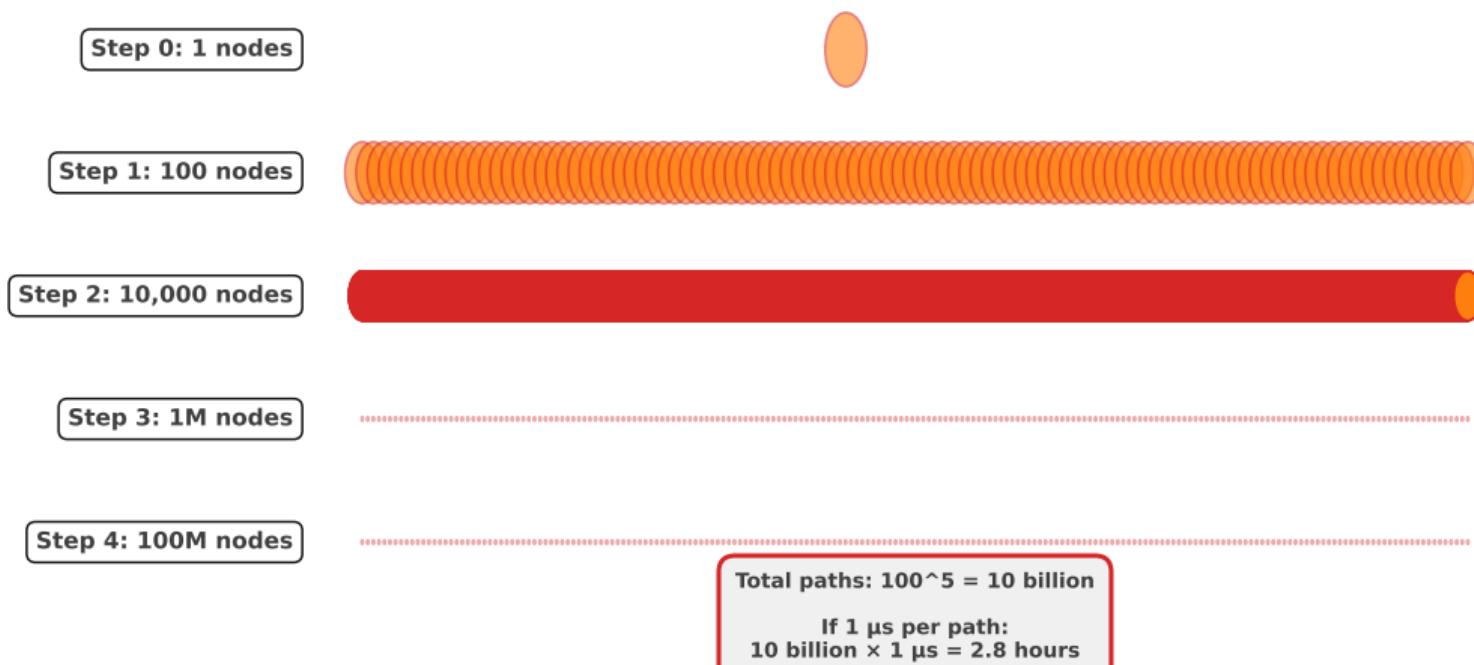
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From 1 path to ALL paths - what could go wrong?

## Extreme Case 2: Full Search Space (Too Broad)

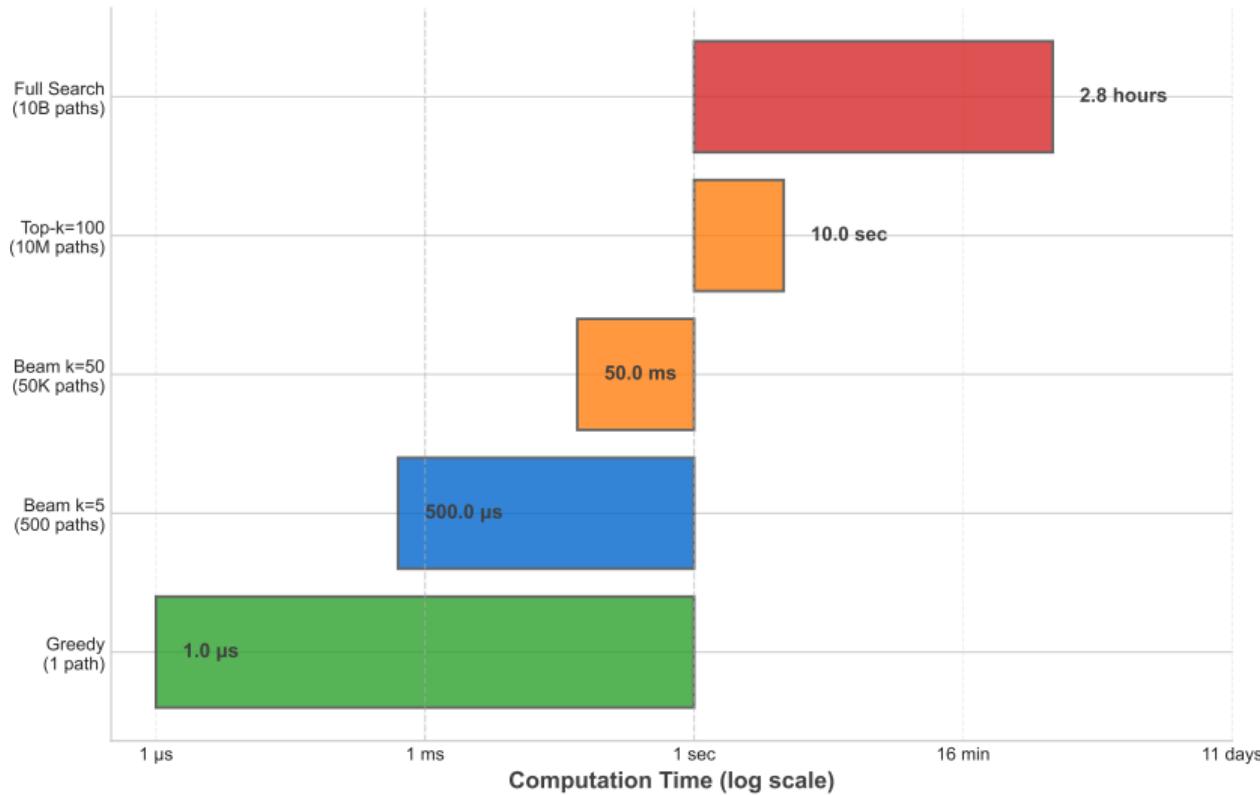
### Extreme Case 2: Full Search Space

Vocabulary size = 100, explore ALL paths



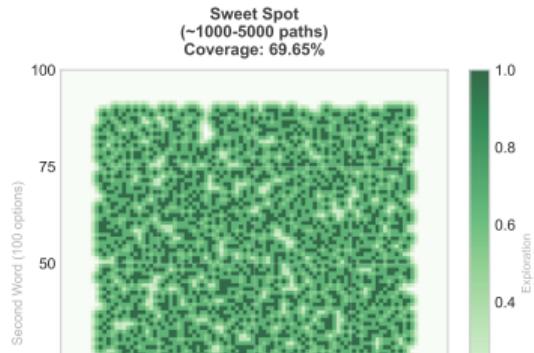
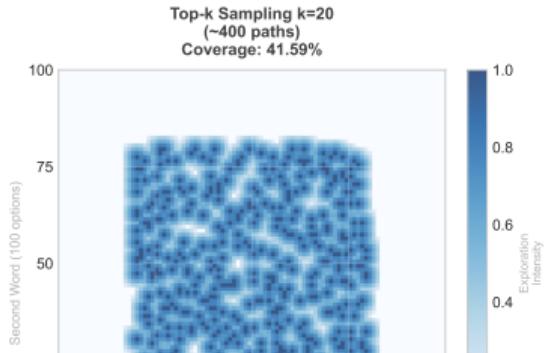
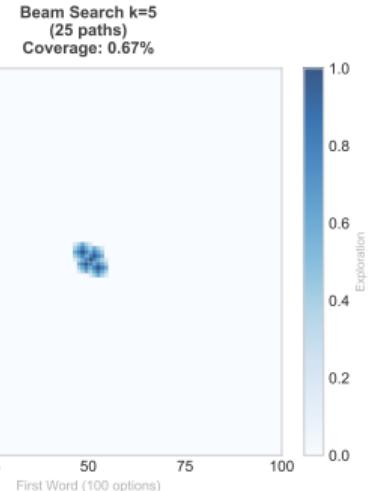
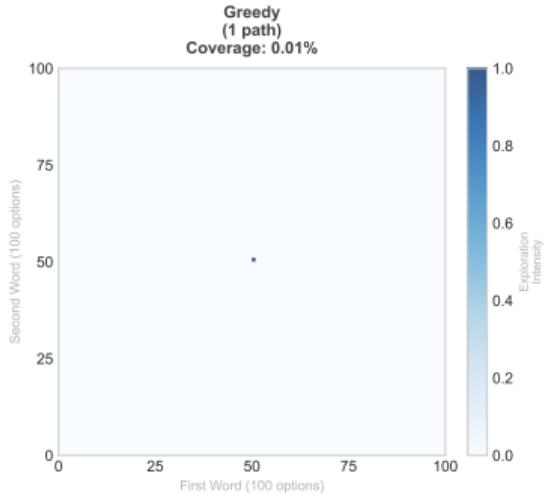
# The Computational Reality

Computational Cost vs Search Breadth  
(Assuming 1  $\mu$ s per path, sequence length=5)



# Finding the Sweet Spot

Search Space Coverage Comparison  
(Vocabulary=100, showing first 2 words only)



## Method 1: Greedy Decoding

### Core Mechanism:

$$w_t = \underset{w \in V}{\operatorname{argmax}} P(w \mid w_1, \dots, w_{t-1})$$

At each step, pick the single word with highest probability

### Characteristics:

- Deterministic (same input → same output)
- Fast:  $O(1)$  per step
- No exploration

---

Method 1 of 6: Greedy = always pick  $\operatorname{argmax}$

## Method 2: Beam Search

### Core Mechanism:

Maintain  $k$  hypotheses ("beams") at each step

Expand each hypothesis, keep top- $k$  by cumulative probability

### Characteristics:

- Explores  $k$  paths simultaneously (typically  $k=3-5$ )
- Trade exploration vs computation
- Still deterministic for fixed  $k$

---

**Method 2 of 6: Beam = keep top- $k$  paths**

## Method 3: Temperature Sampling

### Core Mechanism:

$$P_T(w_i) = \frac{\exp(z_i/T)}{\sum_j \exp(z_j/T)}$$

Reshape probability distribution with temperature T, then sample

### Characteristics:

- $T \downarrow 1$ : More focused (sharper distribution)
- $T \uparrow 1$ : More random (flatter distribution)
- Stochastic: different output each time

---

Method 3 of 6: Temperature = control randomness

## Method 4: Top-k Sampling

### Core Mechanism:

1. Sort words by probability
2. Keep only top  $k$  words (e.g.,  $k=50$ )
3. Renormalize and sample from these  $k$

### Characteristics:

- Filters out low-probability “junk” words
- Fixed cutoff (always  $k$  words)
- Can combine with temperature

---

Method 4 of 6: Top-k = filter then sample

## Method 5: Nucleus (Top-p) Sampling

### Core Mechanism:

1. Sort words by probability
2. Keep minimum set where cumulative probability  $\geq p$
3. Sample from this set

### Characteristics:

- Adaptive: number of words varies
- Focuses on “nucleus” of probability mass (typically  $p=0.9$ )
- Adjusts to distribution shape

---

**Method 5 of 6: Nucleus = adaptive probability mass**

## Method 6: Contrastive Search

### Core Mechanism:

Choose word that maximizes:

$$\text{score} = (1 - \alpha) \cdot \text{model probability} - \alpha \cdot \text{similarity to previous}$$

Penalize words similar to already-generated text

### Characteristics:

- Explicitly avoids repetition
- Balances coherence and diversity
- Deterministic with hyperparameter  $\alpha$

---

**Method 6 of 6: Contrastive = penalize repetition**

## Checkpoint Quiz 1: Match the Method

### Methods:

1. Greedy
2. Beam Search
3. Temperature
4. Top-k
5. Nucleus
6. Contrastive

### Match to Mechanisms:

- A. Sample from reshaped distribution
- B. Keep top-k paths at each step
- C. Always pick argmax
- D. Filter to k words, then sample
- E. Penalize similarity to previous
- F. Adaptive probability mass cutoff

## Checkpoint Quiz 1: Match the Method

### Methods:

1. Greedy
2. Beam Search
3. Temperature
4. Top-k
5. Nucleus
6. Contrastive

### Match to Mechanisms:

- A. Sample from reshaped distribution
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- D. Filter to k words, then sample
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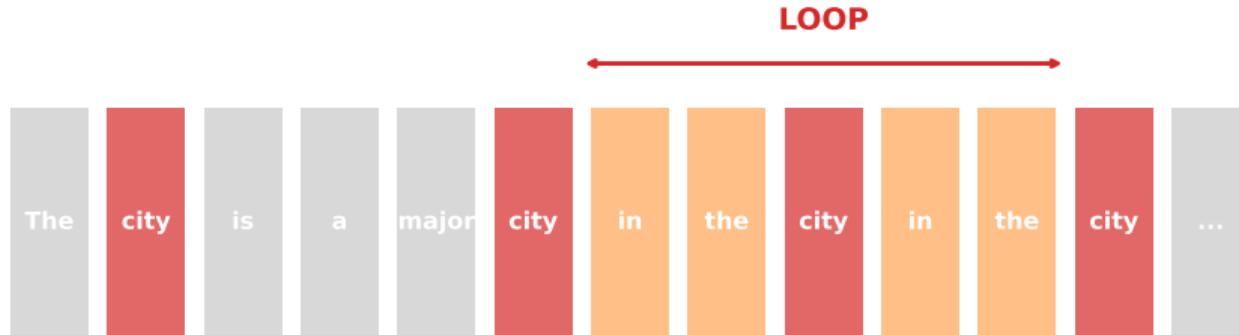
**Answers:** 1→C, 2→B, 3→A, 4→D, 5→F, 6→E

*Now you know the toolbox - let's see WHY each tool exists!*

---

Quiz 1: Can you match each method to its mechanism?

## Greedy Decoding Gets Stuck



*Output: "The city is a major city in the city in the city..."*

**Greedy's Problem:** Trapped in loops, can't escape

**Why Beam Helps:** Explores  $k=3-5$  paths, avoids greedy trap

---

**Problem 1 of 6:** Greedy decoding creates loops → Beam search explores alternatives

## High Temperature Creates Nonsense

The	<b>glorp</b>	is	very	<b>blorptastic</b>	
She	likes	to	eat	<b>qwerty</b>	food
I	went	to	the	<b>flurb</b>	yesterday
The	weather	is	<b>zxqp</b>	today	

*Generated words not in vocabulary!*

**Greedy & Beam's Problem:** Same input → same output always

**Why Temperature Helps:** Sampling introduces randomness, enables creativity

---

**Problem 2 of 6:** Deterministic methods lack variation → Temperature adds controlled randomness

## Zero Creativity: Always Same Output

#9:	The weather is nice today.	#10:	The weather is nice today.
#7:	The weather is nice today.	#8:	The weather is nice today.
#5:	The weather is nice today.	#6:	The weather is nice today.
#3:	The weather is nice today.	#4:	The weather is nice today.
#1:	The weather is nice today.	#2:	The weather is nice today.

100x

Asked 100 times → Always: "The weather is nice today."

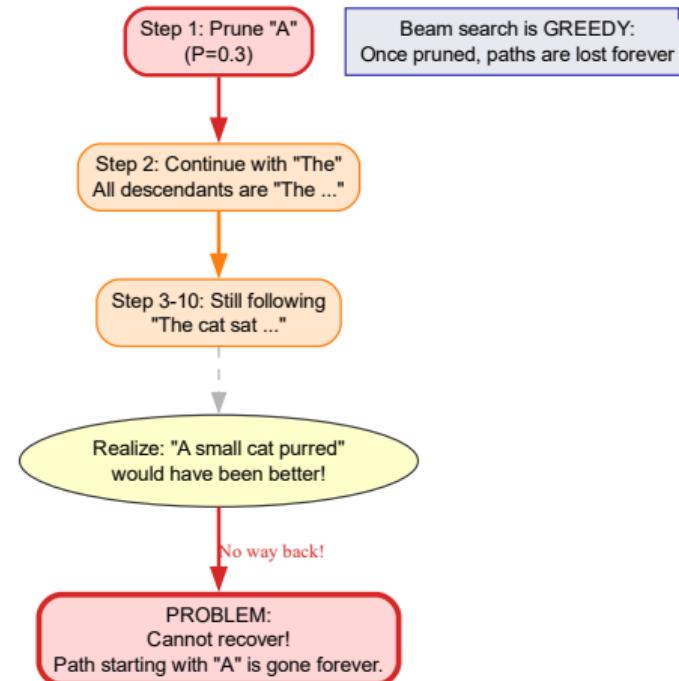
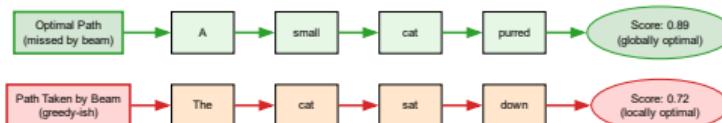
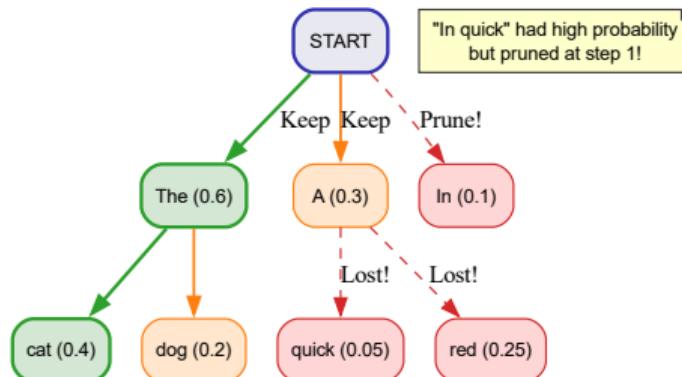
**Temperature's Problem:** Pure sampling includes low-quality words

**Why Top-k Helps:** Filter to k=50 best words, then sample

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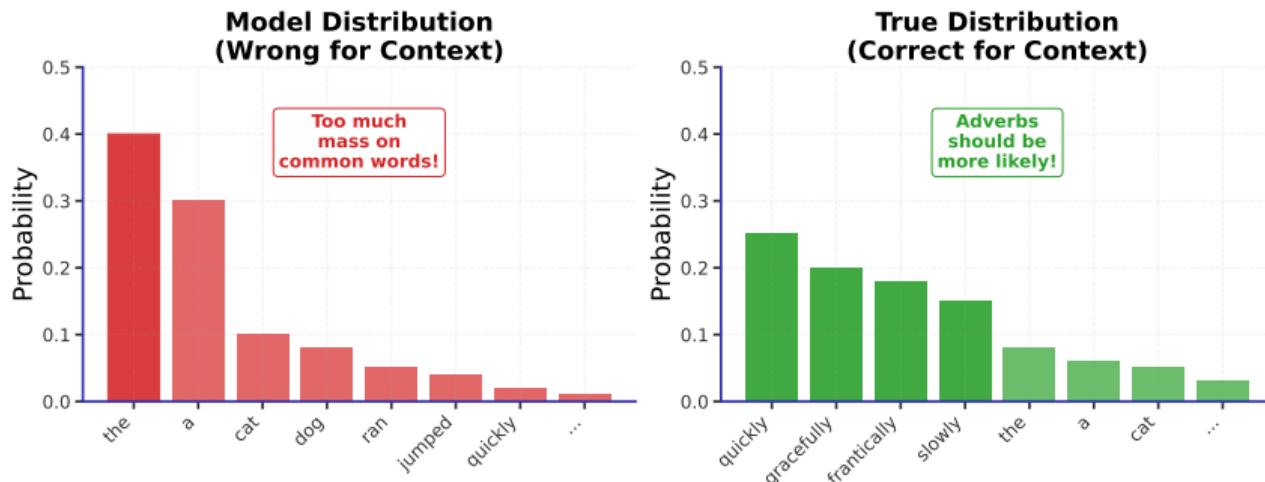
**Problem 3 of 6:** Can't balance quality & creativity → Top-k filters unlikely words

# Beam Search Limitation: Missing Better Paths



# Why Nucleus? Problem: Distribution Tail Contains Junk

Context: "The cat ran \_\_"



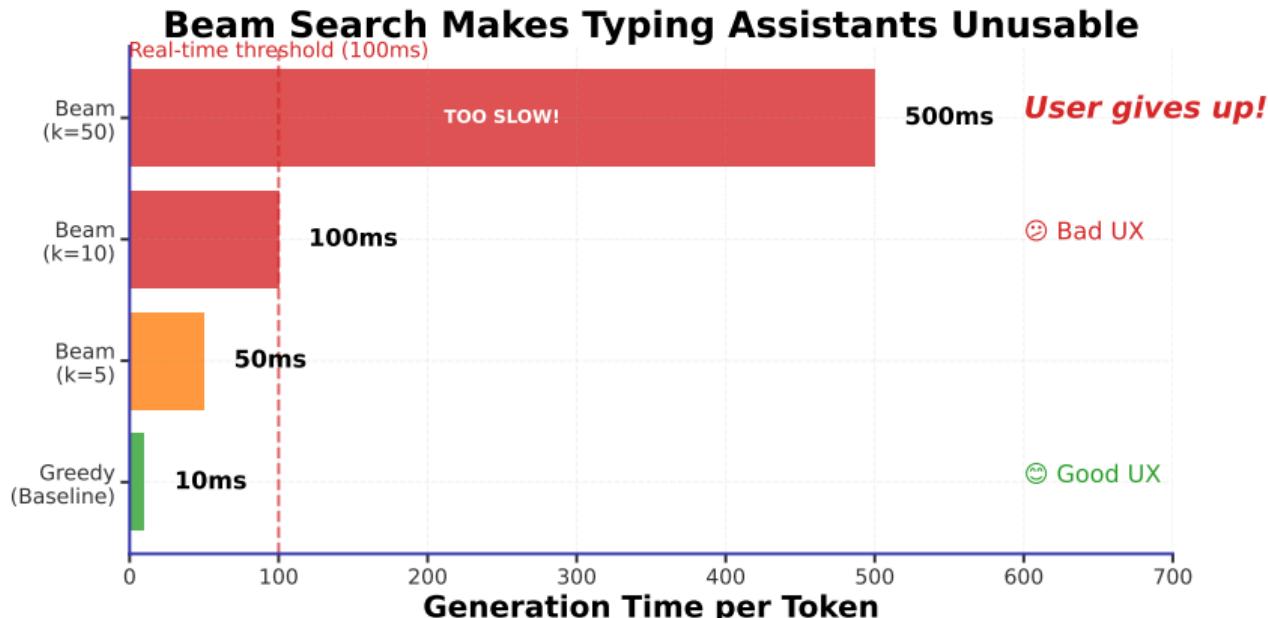
**Top-k's Problem:** Fixed k doesn't adapt to distribution shape

**Why Nucleus Helps:** Adaptive cutoff at  $p=0.9$  probability mass

---

Problem 5 of 6: Tail contains junk → Nucleus adapts to distribution

## Why Contrastive? Problem: Generic Repetitive Text



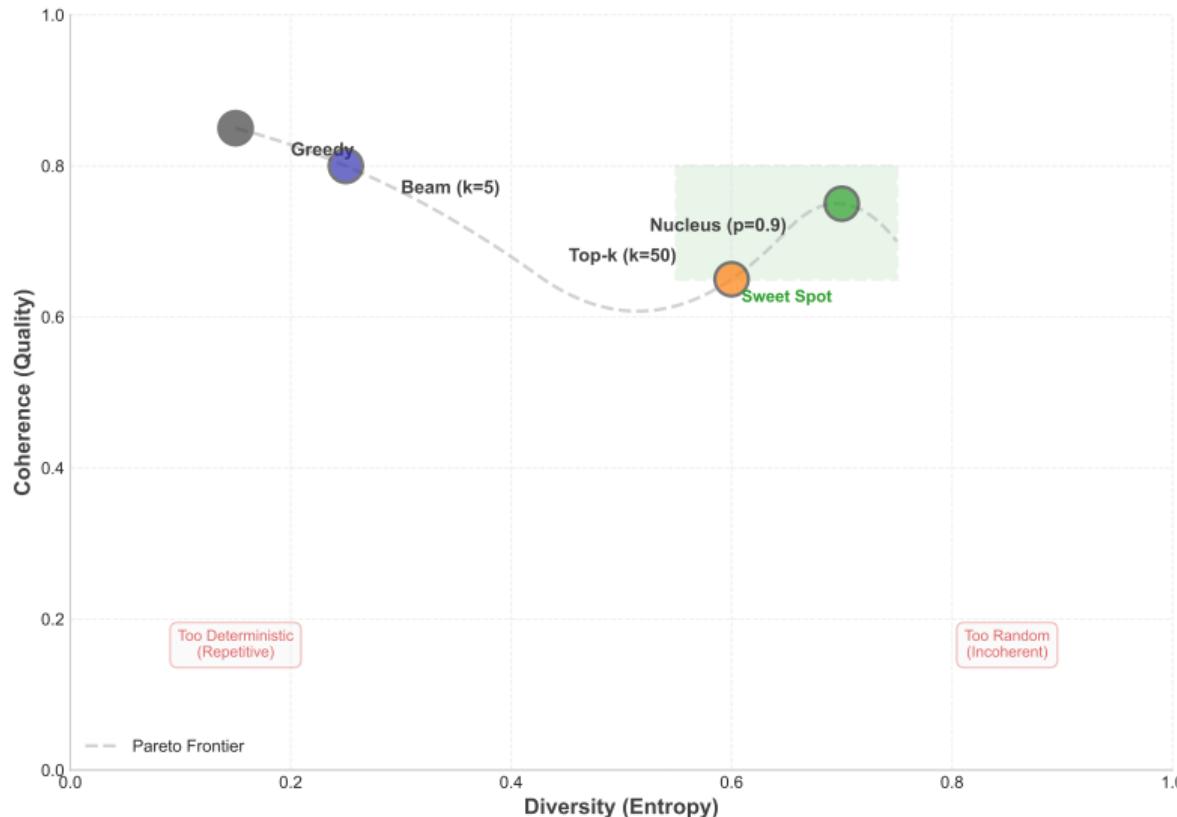
**All Methods' Problem:** Can still produce generic, repetitive text

**Why Contrastive Helps:** Explicitly penalizes similarity to previous tokens

---

**Problem 6 of 6:** Generic text persists → Contrastive reduces repetition

# The Quality-Diversity Tradeoff



## Solution 1 → Beam Search: Explore Multiple Paths

### Problem 1 Recap:

Greedy decoding: Trapped in loops  
Always picks highest probability  
Misses better sequences

**Need:** Way to explore alternatives

### Solution: Beam Search:

Keep top-k paths at each step  
Explore k=3-5 hypotheses simultaneously  
Pick best complete sequence at end

**Result:** Finds better sequences than greedy

**How it solves Problem 1:** Maintains multiple candidates, avoids greedy trap

---

**Solution 1 of 6: Beam search for better quality**

## Solution 2 → Temperature: Add Controlled Randomness

### Problem 2 Recap:

No diversity: Same output always  
Deterministic selection  
No creativity

**Need:** Controlled randomness

### Solution: Temperature:

Reshape probability distribution  
 $T \downarrow 1$ : More focused  
 $T \uparrow 1$ : More random  
Sample from adjusted distribution

**Result:** Different outputs each time

**How it solves Problem 2:** Sampling introduces stochasticity, enables diversity

---

**Solution 2 of 6: Temperature for creativity control**

## Solution 3 → Top-k: Filter Unlikely Words

### Problem 3 Recap:

Can't balance quality & creativity  
Pure sampling too random  
Greedy too boring

**Need:** Filter bad words, keep good

### Solution: Top-k Sampling:

Keep only top-k most likely tokens  
Cut tail of distribution  
Renormalize probabilities  
Sample from filtered set

**Result:** Diverse but not nonsensical

**How it solves Problem 3:** Fixed cutoff prevents tail sampling while allowing creativity

---

**Solution 3 of 6: Top-k for controlled sampling**

## Solution 4 → Nucleus: Dynamic Vocabulary Cutoff

### Problem 4 Recap:

Top-k has fixed cutoff

Peaked distribution: Wastes probability

Flat distribution: Still allows junk

**Need:** Adaptive cutoff

### Solution: Nucleus (Top-p):

Choose smallest set with cumulative prob  $\geq p$

Adapts to distribution shape

Peaked  $\rightarrow$  small nucleus (2-3 words)

Flat  $\rightarrow$  large nucleus (50+ words)

**Result:** Automatic quality-diversity balance

**How it solves Problem 4:** Dynamic cutoff adapts to each prediction step

---

**Solution 4 of 6: Nucleus for adaptive sampling**

## Solution 5 → Top-k + Temperature: Hybrid Control

### Problem 5 Recap:

Temperature alone doesn't filter tail  
Top-k alone doesn't control randomness  
Need both filtering AND tuning

**Need:** Combine strategies

### Solution: Hybrid Methods:

Apply temperature THEN top-k  
Or: Apply nucleus THEN temperature  
Leverages strengths of both  
Production systems use combinations

**Result:** Fine-grained control over generation

**How it solves Problem 5:** Layered strategies handle multiple issues simultaneously

---

**Solution 5 of 6: Hybrid methods for comprehensive control**

## Solution 6 → Contrastive: Explicit Degeneration Prevention

### Problem 6 Recap:

Repetition even with sampling  
Long generation degenerates  
Context similarity causes loops

**Need:** Explicit repetition penalty

### Solution: Contrastive Search:

Score = Probability -  $\alpha \times$  Similarity  
Penalize tokens similar to recent context  
Balance quality with diversity  
Modern standard for long text

**Result:** Human-like text without repetition

**How it solves Problem 6:** Direct similarity penalty prevents copying context

---

**Solution 6 of 6: Contrastive for degeneration-free generation**

## Checkpoint Quiz 2: Which Method for Which Problem?

### Match Solution to Problem:

1. Beam Search → ?
2. Temperature → ?
3. Top-k → ?
4. Nucleus → ?
5. Hybrid → ?
6. Contrastive → ?

### Problems to Solve:

- A. Too boring OR too crazy
- B. Missing better paths
- C. Wrong distribution tail
- D. No diversity
- E. Repetition despite diversity
- F. Dynamic vocabulary needed

## Checkpoint Quiz 2: Which Method for Which Problem?

### Match Solution to Problem:

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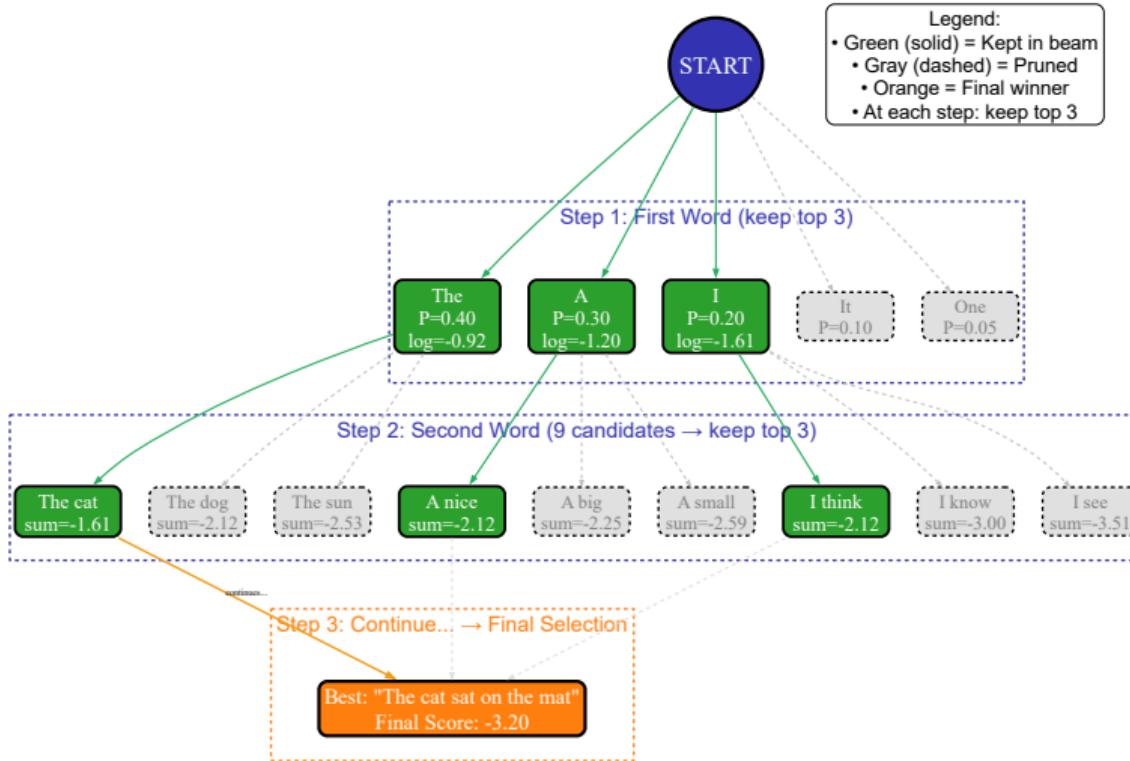
**Answers:** 1→B, 2→D, 3→A, 4→F, 5→C, 6→E

*Each method targets a specific failure mode!*

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### Quiz 2: Understanding the method-problem mapping

# Beam Search: Step-by-Step Example



# Beam Search: Algorithm & Settings

## Algorithm:

1. Start: Keep top-k tokens
2. Expand: Generate continuations for each
3. Score: Multiply probabilities
4. Prune: Keep top-k sequences
5. Repeat until END token

## Scoring:

$$\text{score}(y_1 \dots y_t) = \prod_{i=1}^t P(y_i | y_{<i})$$

With length normalization:

$$\text{score} = \frac{1}{t} \sum_{i=1}^t \log P(y_i | y_{<i})$$

## Best For:

- Machine translation
- Summarization
- Question answering
- Tasks with “correct” answer

## Parameters:

Width = 3-5 (translation)

Width = 10 (diverse outputs)

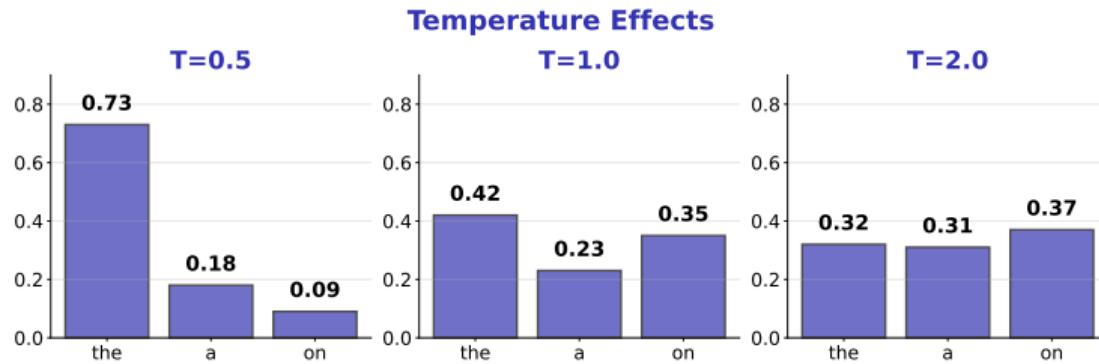
## Tradeoffs:

- + Better quality than greedy
- + Diverse hypotheses
- Still deterministic
- 4-5× slower than greedy

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Beam search is the workhorse for deterministic tasks

## Temperature Sampling: Control Randomness



**Key Insight:** Temperature reshapes probability distribution

---

$T < 1$ : more focused.  $T = 1$ : unchanged.  $T > 1$ : more random

## Temperature: Step-by-Step Calculation

**Given: Logits = [2.0, 1.0, 0.5, 0.2]**

Tokens = ["cat", "dog", "bird", "fish"]

T=0.5: [4.0, 2.0, 1.0, 0.4] → [0.73, 0.18, 0.07, 0.02]

→ **73% on "cat" (FOCUSED)**

T=1.0: [2.0, 1.0, 0.5, 0.2] → [0.42, 0.23, 0.16, 0.13]

→ **42% on "cat" (BALANCED)**

T=2.0: [1.0, 0.5, 0.25, 0.1] → [0.32, 0.26, 0.23, 0.19]

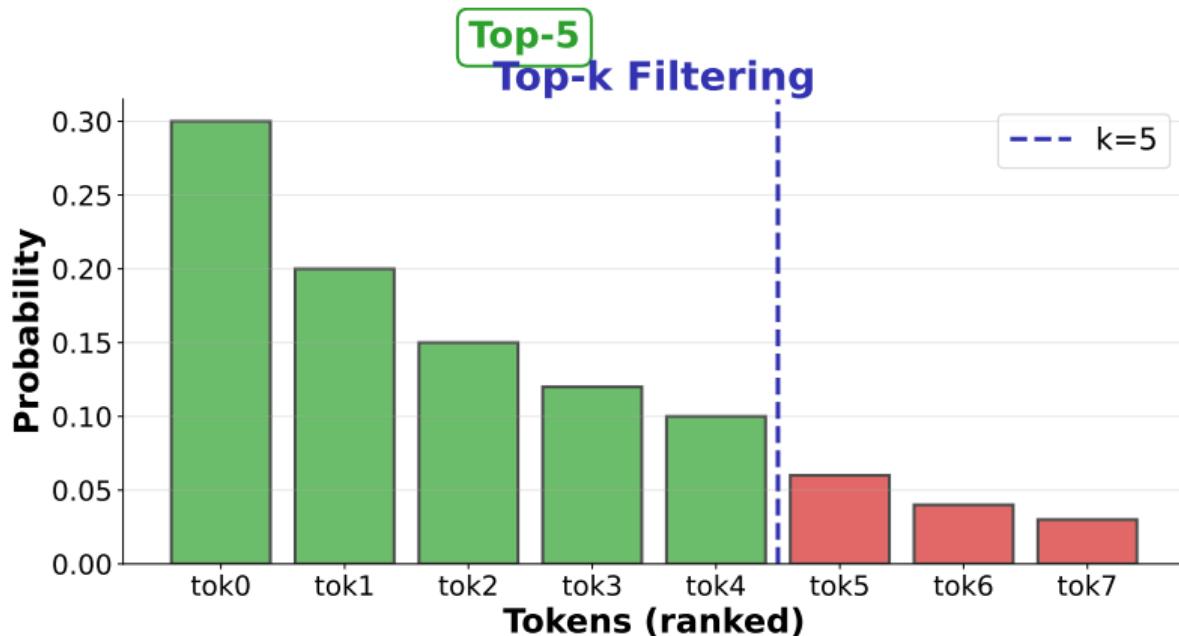
→ **32% on "cat" (FLAT)**

*Lower T = more peaked | Higher T = more flat*

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Concrete numbers show how temperature scaling works

## Top-k Sampling: Filter the Tail



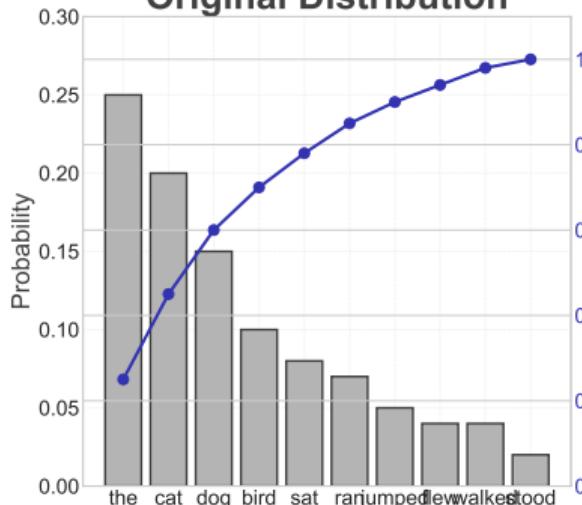
**Key Insight:** Only sample from top-k most likely tokens

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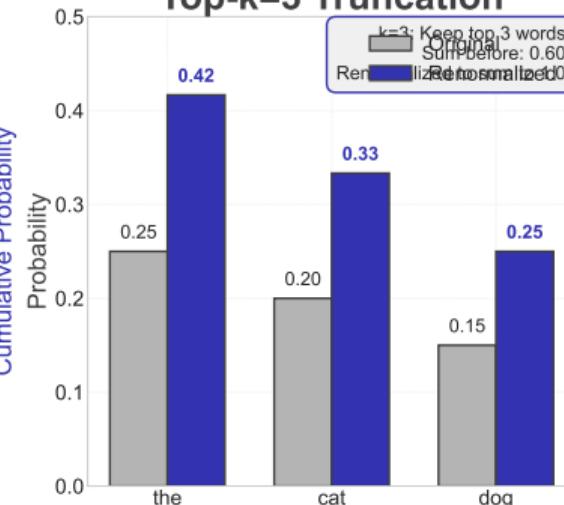
Prevents sampling from long tail of unlikely words

## Top-k Sampling: Numerical Example

Original Distribution

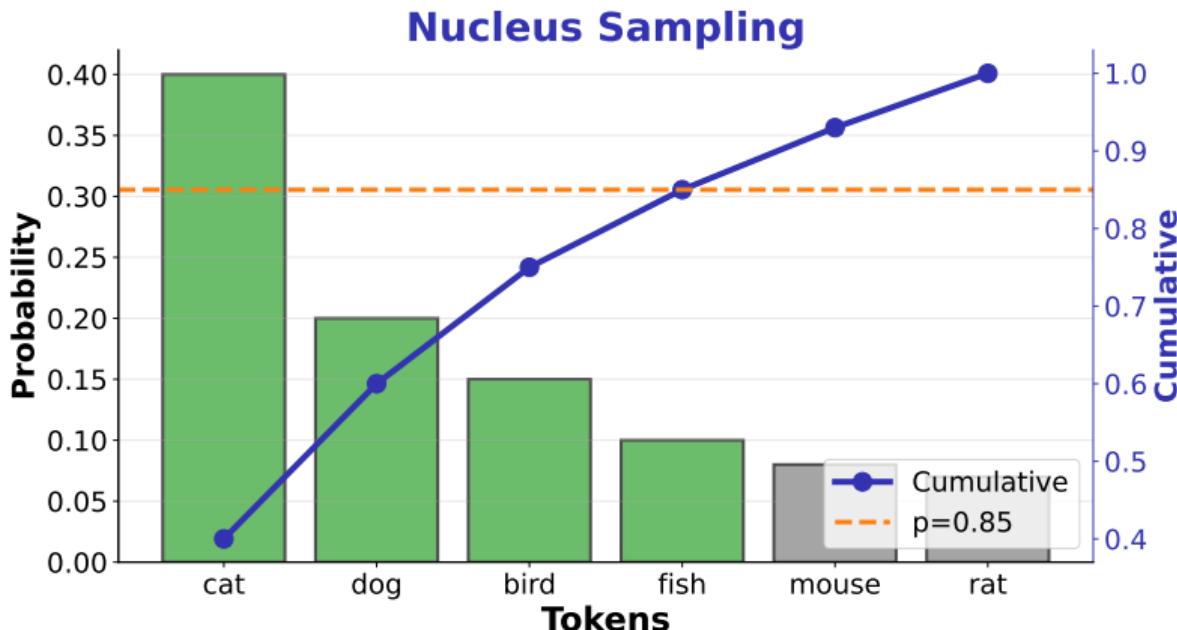


Top-k=3 Truncation



Concrete numbers show k=50 filtering process

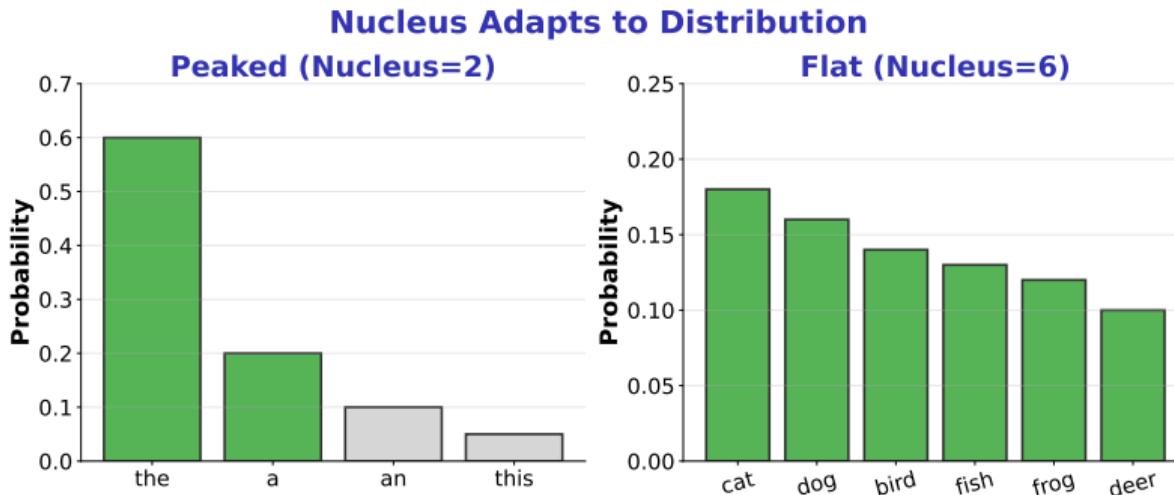
## Nucleus (Top-p) Sampling: Dynamic Cutoff



**Key Insight:** Adapt vocabulary size to distribution shape

Nucleus size grows/shrinks based on probability spread

# Nucleus: How Distribution Shape Matters



Same p value gives different vocabulary sizes for peaked vs flat distributions

## Greedy Decoding Problem

"The city is a major city.  
The city has many attractions.  
The city is known..."

"the city" appears 3 times!

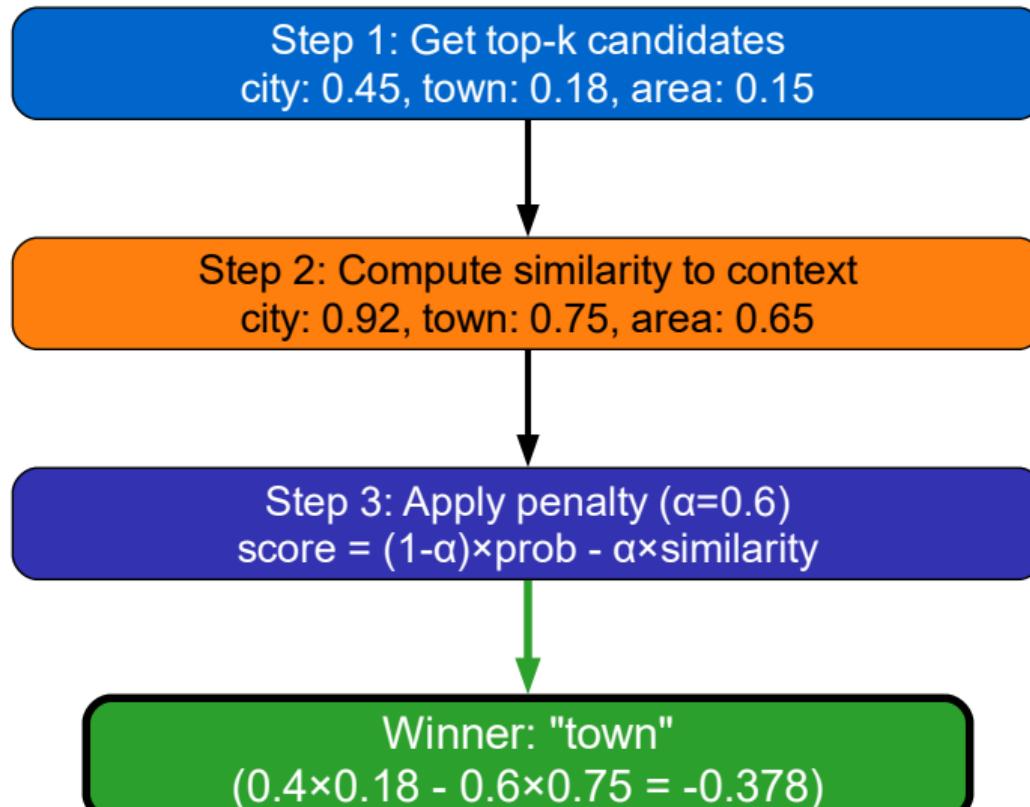


Solution: Contrastive Search

**Discovery Question:** Why do models repeat themselves?

Greedy and beam search maximize probability - but high probability = repeating recent context

## Contrastive Search: Penalize Repetition

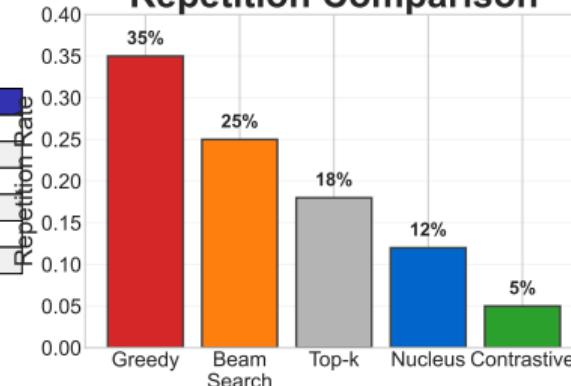


## Contrastive Search vs Nucleus Sampling

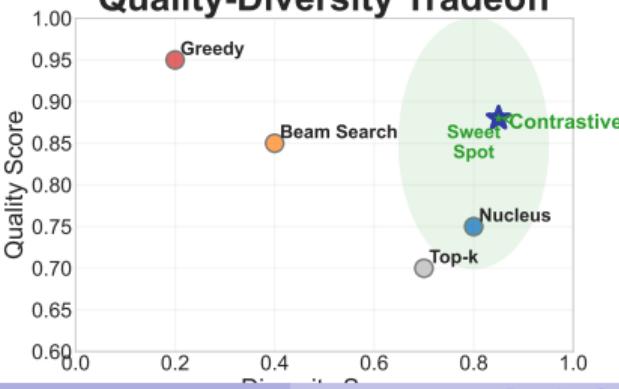
### Method Comparison

Aspect	Nucleus (Top-p)	Contrastive
Diversity	High	High
Repetition	Medium	Very Low
Coherence	Good	Excellent
Speed	Fast $O(V \log V)$	Slower $O(k \times T^2)$
Parameters	p, temperature	$\alpha$ , k, penalty
Best for	General use	Long generation

### Repetition Comparison



### Quality-Diversity Tradeoff



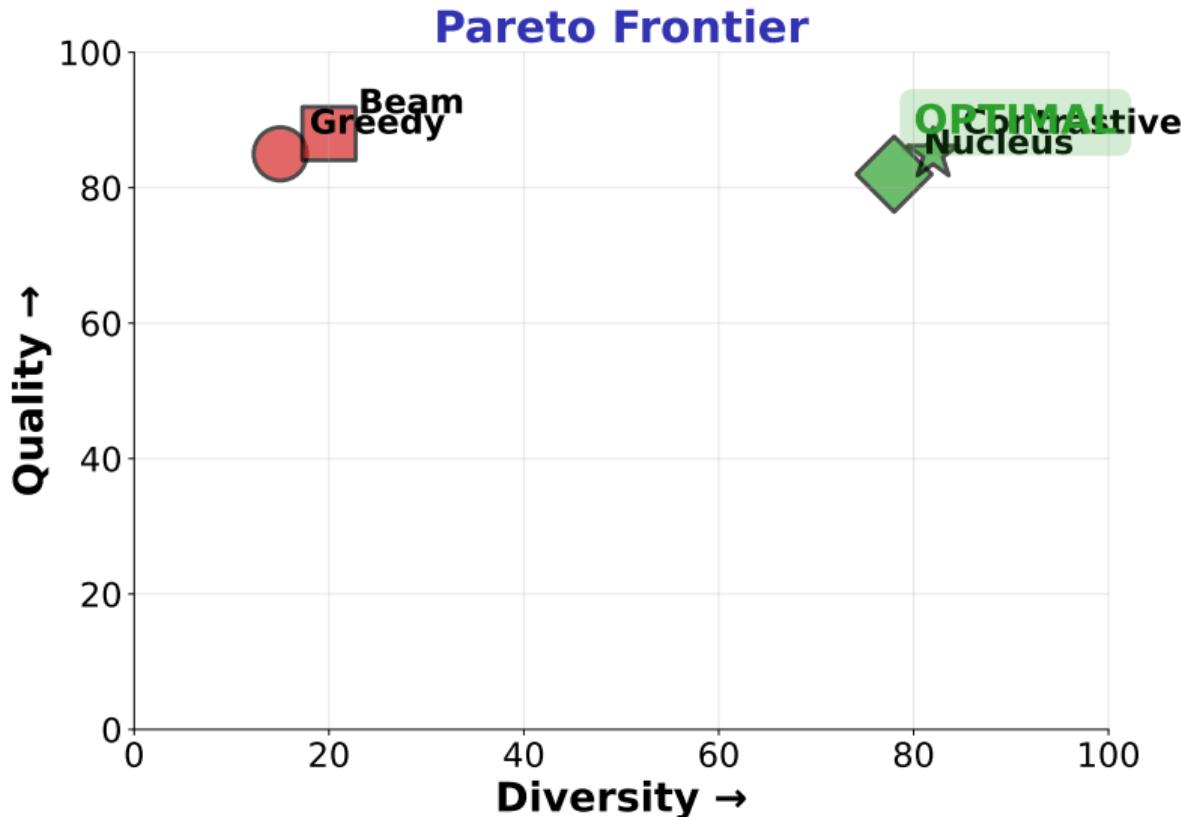
Context: "The weather today is"

#### Nucleus Sampling:

- beautiful and sunny with clear skies
- perfect for a walk in the park

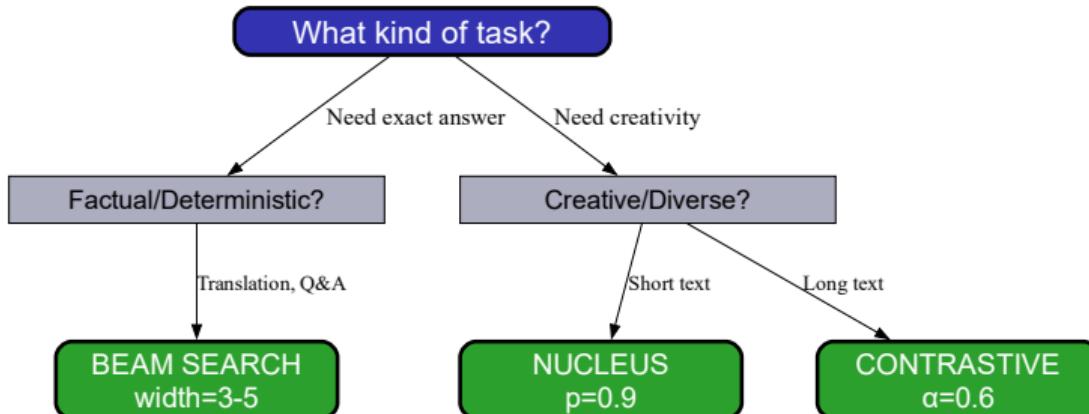
#### Contrastive Search:

- beautiful with clear blue skies
- perfect for outdoor activities today



Pareto Frontier: No method dominates all others

## Choosing the Right Method: Decision Tree



Start with task requirements, follow tree to recommended method

## Task-Specific Recommendations

Task	Method	Parameters
Translation	Beam	w=3-5
Factual QA	Greedy	T=0.3
Code	Greedy	T=0
Dialogue	Nucleus	p=0.9
Creative	Nucleus	p=0.95
Long Stories	Contrastive	$\alpha=0.6$

---

Comprehensive mapping from 8 common tasks to optimal decoding strategies

## Checkpoint Quiz 3: Choose the Right Method

Given these tasks, which method would you use?

**1. Medical report summary**

- Needs: Accuracy, no hallucination

**2. Creative story writing**

- Needs: Diversity, creativity

**3. Code generation**

- Needs: Correctness, explore options

**4. Customer service chat**

- Needs: Natural, varied responses

**5. Legal document**

- Needs: Precise, formal

**6. Long blog post**

- Needs: Coherent, no repetition

## Checkpoint Quiz 3: Choose the Right Method

Given these tasks, which method would you use?

**1. Medical report summary**

- Needs: Accuracy, no hallucination

**2. Creative story writing**

- Needs: Diversity, creativity

**3. Code generation**

- Needs: Correctness, explore options

**4. Customer service chat**

- Needs: Natural, varied responses

**5. Legal document**

- Needs: Precise, formal

**6. Long blog post**

- Needs: Coherent, no repetition

**Answers:**

1. Greedy/Low temp ( $T=0.1-0.3$ )
2. Nucleus ( $p=0.95, T=1.0$ )
3. Beam Search ( $k=3-5$ )
4. Nucleus ( $p=0.9, T=0.7$ )
5. Greedy ( $T=0$ )
6. Contrastive ( $\alpha=0.6$ )

---

Quiz 3: Real-world task selection is crucial for quality

## Key Takeaways

1. **6 Problems → 6 Solutions:** Each method solves specific failure mode
2. **Deterministic** (Greedy, Beam): High quality, no diversity - factual tasks
3. **Stochastic** (Temperature, Top-k, Nucleus): Diverse but variable quality
4. **Balanced** (Contrastive): Explicit degeneration prevention
5. **Task matters:** Translation → Beam — Dialogue → Nucleus — Stories → Contrastive
6. **Tradeoffs:** Speed vs Quality, Diversity vs Coherence

**Modern Standard:** Nucleus ( $\text{top-p}=0.9$ ) + Temperature ( $T=0.7$ ) for most applications

**Next:** Lab - Implement all 6 methods, measure quality-diversity tradeoffs

---

Decoding strategy matters as much as model architecture

# From Probabilities to Text: The Complete Journey



## What We Learned:

- Models give us probability distributions (Week 3-7)
- Converting to text has 6 fundamental challenges
- Each decoding method addresses specific problems
- No universal best - choose based on task requirements
- Production systems use hybrid methods (Nucleus + Temperature)

---

Complete pipeline from model training to text generation

# Technical Appendix

25 slides: Complete mathematical treatment

**A1-A5: Beam Search Mathematics**

**A6-A10: Sampling Mathematics**

**A11-A14: Contrastive Search & Degeneration**

**A15-A19: Advanced Topics & Production**

**A20-A25: The 6 Problems - Technical Analysis (NEW)**

## A1: Beam Search Formulation

**Objective:** Find sequence  $y^* = \operatorname{argmax} P(y|x)$

**Decomposition:**

$$P(y|x) = \prod_{t=1}^T P(y_t|y_{<t}, x)$$

**Log-probability** (more stable):

$$\log P(y|x) = \sum_{t=1}^T \log P(y_t|y_{<t}, x)$$

**Beam Search Approximation:**

Instead of exploring all  $V^T$  sequences, maintain top-k hypotheses at each step

**Complexity:**

Time:  $O(k \cdot V \cdot T)$  where  $k$  = beam width,  $V$  = vocabulary,  $T$  = length

Space:  $O(k \cdot T)$  to store hypotheses

---

Beam search is tractable approximation to exact search

## A2: Length Normalization

**Problem:** Longer sequences have lower probabilities (more terms multiplied)

$$P(y_1, y_2, y_3, y_4) = \underbrace{0.5}_{y_1} \times \underbrace{0.5}_{y_2} \times \underbrace{0.5}_{y_3} \times \underbrace{0.5}_{y_4} = 0.0625$$

$$P(y_1, y_2) = 0.5 \times 0.5 = 0.25 > 0.0625$$

Bias toward shorter sequences!

**Solution:** Length normalization

$$\text{score}(y) = \frac{1}{|y|^\alpha} \log P(y)$$

where  $\alpha \in [0.5, 1.0]$  (typically 0.6-0.7)

**Effect:**

Without: Beam search heavily biases toward short outputs

With: Fair comparison across different lengths

---

Length normalization is essential for beam search quality

## A3: Beam Search Variants

### Diverse Beam Search:

Partition beams into groups  
Penalize within-group similarity  
Result: More diverse hypotheses

### Constrained Beam Search:

Force certain tokens to appear  
Useful for: Keywords, entities  
Applications: Controllable generation

### Stochastic Beam Search:

Sample beams instead of argmax  
Combines beam + sampling  
More diverse than standard beam

### Block n-gram Beam:

Penalize n-gram repetition  
Prevents "the city is a city" loops  
Common in summarization

---

Many beam search variants exist for specific requirements

## A4: Beam Search Stopping Criteria

**When to stop expanding beams?**

**Method 1:** Fixed length

Stop at  $T_{\max}$  tokens (simple but rigid)

**Method 2:** END token

Stop when beam generates special token (most common)

**Method 3:** Score threshold

Stop when best score cannot improve enough

$$\frac{\text{best\_incomplete}}{\text{best\_complete}} < \text{threshold}$$

**Method 4:** Timeout

Computational budget exceeded (production systems)

---

Choice of stopping criterion affects output length distribution

## A5: Beam Search Limitations

### Fundamental Issues:

1. **Exposure bias:** Trained with teacher forcing, tested with own outputs
2. **Label bias:** Cannot compare sequences of different prefixes fairly
3. **Repetition:** Still can loop ("the city is a major city")
4. **Bland outputs:** Maximizes probability, not interestingness
5. **Search errors:** May miss better sequences outside beam

### When Beam Search Fails:

Open-ended generation (dialogue, stories)  
Long-form text (repetition accumulates)  
Creative tasks (probability  $\neq$  quality)

→ Need sampling-based methods

---

Beam search optimizes wrong objective for creative tasks

## A6: Sampling as Inference

**Goal:** Sample  $y \sim P(y|x)$  instead of  $\text{argmax } P(y|x)$

### Ancestral Sampling:

For  $t = 1$  to  $T$ :

    Compute  $P(y_t|y_{<t}, x)$

    Sample  $y_t \sim P(\cdot|y_{<t}, x)$

### Properties:

Stochastic: Different output each time

Explores full distribution (in expectation)

Can generate low-probability sequences

### Variants:

Temperature: Reshape distribution before sampling

Top-k: Truncate distribution before sampling

Nucleus: Dynamic truncation before sampling

---

**Sampling enables diversity but loses quality guarantees**

## A7: Temperature Mathematics

**Softmax with Temperature:**

$$p_i(T) = \frac{\exp(z_i/T)}{\sum_{j=1}^V \exp(z_j/T)}$$

**Limiting Cases:**

$$T \rightarrow 0: p_i \rightarrow \begin{cases} 1 & \text{if } i = \operatorname{argmax} z \\ 0 & \text{otherwise} \end{cases} \text{ (greedy)}$$

$$T \rightarrow \infty: p_i \rightarrow 1/V \text{ (uniform)}$$

**Entropy Analysis:**

Entropy  $H(p) = -\sum p_i \log p_i$  measures randomness

$H$  increases monotonically with  $T$

Low  $T$  ( $<0.5$ ):  $H \approx 0$  (deterministic)

High  $T$  ( $>2.0$ ):  $H \approx \log V$  (maximum entropy)

---

Temperature provides continuous control over distribution entropy

## A8: Top-k Mathematics

### Formal Definition:

Let  $\sigma$  = permutation sorting probabilities descending

$$V_k = \{w_{\sigma(1)}, w_{\sigma(2)}, \dots, w_{\sigma(k)}\}$$

Truncated distribution:

$$p'(w) = \begin{cases} \frac{p(w)}{\sum_{w' \in V_k} p(w')} & \text{if } w \in V_k \\ 0 & \text{otherwise} \end{cases}$$

### Information Loss:

Original entropy:  $H(p) = -\sum_{i=1}^V p_i \log p_i$

After top-k:  $H(p') = -\sum_{i=1}^k p'_i \log p'_i < H(p)$

Loss  $\approx \sum_{i=k+1}^V p_i \log(1/p_i)$  (tail information)

---

Top-k sacrifices tail probability mass for sampling quality

## A9: Nucleus (Top-p) Mathematics

### Formal Definition:

$$V_p = \min \left\{ V' \subseteq V : \sum_{w \in V'} p(w) \geq p \right\}$$

Smallest set with cumulative mass  $\geq p$

### Dynamic Vocabulary Size:

$$|V_p| = \min \{k : \sum_{i=1}^k p_{\sigma(i)} \geq p\}$$

Adapts to distribution shape:

Peaked: Small  $|V_p|$  (2-5 tokens)

Flat: Large  $|V_p|$  (50+ tokens)

### Why Nucleus > Top-k:

Top-k: Fixed  $k$  regardless of  $p(w)$  distribution

Nucleus: Adapts  $k$  to achieve consistent probability mass

---

Nucleus automatically adjusts vocabulary to distribution characteristics

## A10: Sampling Quality Metrics

### Quality Metrics:

Perplexity:  $\exp(-\frac{1}{T} \sum \log p(y_t))$   
Lower = better

### BLEU (translation):

N-gram overlap with reference  
0-100 scale

### Human evaluation:

Fluency (1-5)  
Relevance (1-5)

### Diversity Metrics:

Distinct-n:  $\frac{\text{unique n-grams}}{\text{total n-grams}}$   
Higher = more diverse

### Self-BLEU:

BLEU of output vs other outputs  
Lower = more diverse

### Repetition Rate:

$\frac{\text{repeated n-grams}}{\text{total n-grams}}$   
Lower = less repetitive

---

Need both quality AND diversity metrics to evaluate decoding

## A11: The Degeneration Problem (Formal)

**Definition:** Model-generated text with unnatural repetitions

**Why It Happens:**

1. Model trained on natural text (low repetition)
2. But generation maximizes  $P(y_t|y_{<t})$
3. Recent context  $y_{<t}$  influences  $P$
4. Creates positive feedback: high prob word → context → same high prob word

**Quantifying Degeneration:**

Repetition rate in greedy: 15-30% (depending on domain)

Repetition rate in human text: 2-5%

Gap = degeneration problem

**Examples:**

*"The city is a major city in the United States. The city..."*

*"I think that I think that I think..."*

---

Maximizing probability does not equal natural text

## A12: Contrastive Search Objective

### Scoring Function:

$$\text{score}(w_t) = (1 - \alpha) \times \underbrace{P(w_t | y_{<t})}_{\text{model confidence}} - \alpha \times \underbrace{\max_{w_i \in y_{<t}} \text{sim}(w_t, w_i)}_{\text{context similarity}}$$

where  $\alpha \in [0, 1]$  controls tradeoff

### Similarity Function:

$$\text{sim}(w_i, w_j) = \frac{h_i \cdot h_j}{||h_i|| \cdot ||h_j||}$$

(cosine similarity)

using token embeddings  $h$

### Algorithm:

1. Get top-k candidates by probability
2. For each candidate, compute similarity to all tokens in  $y_{<t}$
3. Apply penalty:  $\text{score} = \text{prob} - \alpha \times \text{max\_similarity}$
4. Select candidate with highest score

---

Contrastive search explicitly penalizes copying recent context

## A13: Contrastive Search Parameters

### Alpha ( $\alpha$ ):

- $\alpha = 0$ : Pure greedy (no penalty)
- $\alpha = 0.6$ : Balanced (recommended)
- $\alpha = 1.0$ : Maximum diversity (risky)

### Typical Settings:

- Short text (<100 tokens):  $\alpha = 0.4 - 0.5$
- Medium (<500):  $\alpha = 0.5 - 0.6$
- Long (500+):  $\alpha = 0.6 - 0.7$

### Top-k for Candidates:

- $k = 4$ : Fast, focused
- $k = 6$ : Balanced (default)
- $k = 10$ : Diverse

### Computational Cost:

For each step:

- Compute similarities:  $O(k \times t)$
- $t$  grows with generation

Total:  $O(k \times T^2)$

12× slower than greedy

---

Hugging Face default:  $\alpha=0.6$ ,  $k=4$

## A14: Degeneration Analysis

### Research Findings (2024-2025):

- Greedy decoding repetition: 18-25% (GPT-2), 12-18% (GPT-3)
- Nucleus sampling repetition: 8-12% (still above human 3-5%)
- Contrastive search repetition: 4-7% (closest to human)

### Why Probability Maximization Fails:

Training objective: Next token prediction

But generation requires: Global coherence

Mismatch: Local optimum  $\neq$  global quality

### Solutions Hierarchy:

1. Temperature/Top-k/Nucleus: Reduce greedy's determinism
2. Contrastive: Explicit degeneration penalty
3. RLHF/DPO: Align model with human preferences (different lecture)

---

Contrastive search addresses fundamental limitation of likelihood-based decoding

## A15: Hybrid Decoding Methods

### Combining Strategies:

#### Nucleus + Temperature:

Apply temperature THEN nucleus

$$p_i(T) = \text{softmax}(z/T), \quad \text{then} \quad V_p \leftarrow \text{nucleus}(p_i(T))$$

Used by GPT-3 API, ChatGPT

#### Beam + Sampling:

Beam search with stochastic selection

Keep top-k, sample from them (not argmax)

#### Contrastive + Nucleus:

Nucleus for candidate generation

Contrastive scoring for selection

Best of both worlds

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Hybrid methods leverage complementary strengths

# A16: Constrained Decoding (2025)

**Goal:** Force certain tokens/patterns to appear

## **Lexically Constrained:**

Must include keywords: { "AI", "ethics", "safety" }

Beam search variant: Track constraint satisfaction

## **Format Constraints:**

JSON output: Force structure { "key": "value" }

Code: Force syntactic validity

## **NeuroLogic Decoding (2021):**

Beam search + constraint satisfaction

Optimal for: Keyword-based generation

## **Production Use Cases:**

Structured data extraction (force JSON)

Controllable summarization (force keywords)

Code generation (force syntax)

---

**Constrained decoding enables controllable generation**

## A17: Computational Complexity Comparison

Method	Time per token	Total complexity	Relative speed
Greedy	$O(V)$	$O(V \times T)$	$1.0 \times$ (baseline)
Temperature	$O(V)$	$O(V \times T)$	$1.1 \times$ (softmax overhead)
Top-k	$O(V)$	$O(V \times T)$	$1.2 \times$ (sorting)
Nucleus	$O(V \log V)$	$O(V \log V \times T)$	$1.3 \times$ (sort + cumsum)
Beam (k=5)	$O(k \times V)$	$O(k \times V \times T)$	$4.5 \times$ (k=5)
Contrastive	$O(k \times T)$	$O(k \times T^2)$	$12 \times$ (similarity)

**Key Insight:** Contrastive's  $T^2$  term makes it expensive for long sequences

**Practical Impact** (1000-token generation):

Greedy: 2.5 seconds

Nucleus: 3.2 seconds (best choice)

Beam: 11 seconds

Contrastive: 30 seconds (only if quality critical)

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Computational cost matters for production deployment

## A18: Production Settings (Real-World APIs)

### Production System Settings (2025)

*Default decoding parameters used by major LLM APIs and platforms*

System/API	Default Method	Temperature	Top-p	Top-k	Other Parameters	Notes
GPT-4 API	Nucleus	0.7	1.0	—	frequency_penalty=0 presence_penalty=0	Can adjust all params
Claude API	Nucleus	1.0	0.999	—	max_tokens required	Temperature □ [0,1]
ChatGPT Web	Nucleus+Temp	0.7	0.95	—	Not adjustable	Optimized for chat
Gemini API	Top-k + Top-p	1.0	0.95	40	candidate_count=1	Both k and p used
Llama 2 (HF)	Configurable	1.0	0.9	50	repetition_penalty=1.0	Full control
Cohere API	Nucleus	0.75	0.999	0	frequency_penalty=0	k=0 means disabled
Mistral API	Nucleus	0.7	1.0	—	safe_mode=false	Similar to OpenAI
Together AI	Configurable	0.7	0.7	50	repetition_penalty=1.0	Multiple options

- Temperature: Lower = more deterministic, Higher = more creative
- Top-p (Nucleus): Cumulative probability threshold (typically 0.9-1.0)
- Top-k: Number of top tokens to consider (often 40-50 when used)
- Most production systems use Nucleus sampling as default
- ChatGPT and Claude optimize for conversational quality
- APIs generally allow full parameter customization

**Provider Types:**  
Proprietary LLM  
Open Source  
Optimized for Chat

#### What ChatGPT, Claude, and other production systems actually use

## A19: Future Directions & Open Problems

### Active Research Areas (2025):

1. **Quality-diversity optimization:** Multi-objective search methods
2. **Learned decoding:** Train models to decode better (RLHF, DPO)
3. **Speculative decoding:** Parallel generation for speed ( $4\text{-}8\times$  faster)
4. **Adaptive methods:** Choose strategy dynamically during generation
5. **Energy-based decoding:** Score sequences globally (not token-by-token)

### Open Problems:

How to automatically select best  $T$ ,  $p$ ,  $k$ ,  $\alpha$  for new task?

How to balance fluency + factuality + creativity simultaneously?

How to decode efficiently for 100K+ token outputs?

**Trend:** Moving from hand-tuned parameters to learned decoding strategies

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Decoding is an active research area with many open questions