

# Decoding Strategies

Week 9: From Probabilities to Text

November 2025

## Learning Goals

1. Discover why predicting one word at a time creates text quality problems
2. Learn how different decoding strategies solve these problems
3. Practice selecting and tuning methods for different use cases

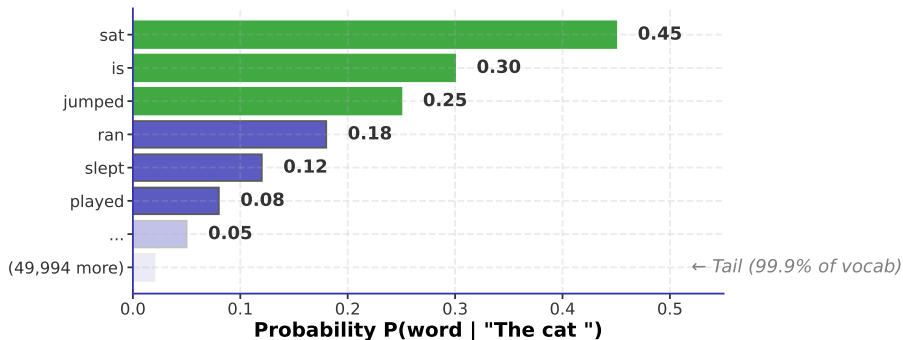
## Plan for Today

1. The challenge: Word predictions → Complete text
2. The toolbox: 6 strategies for text generation
3. Hands-on: Compare methods in practice

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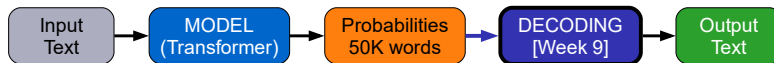
**Notebook available:** `week09_decoding_lab.ipynb`

## The Decoding Challenge: Choose From 50,000 Words



**The Question:** Given these probabilities for “The cat \_\_”, which word should we pick?

At each step, model outputs probability distribution over entire vocabulary - how do we choose?



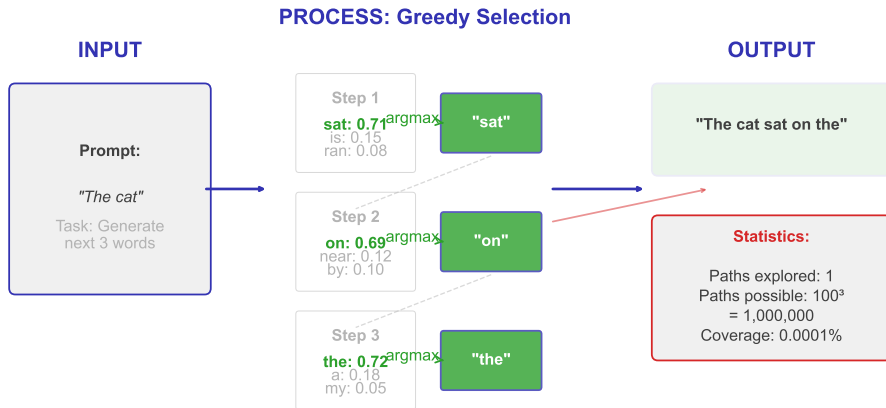
### Our Journey:

1. We trained models (Weeks 3-7: RNN  $\rightarrow$  Transformers  $\rightarrow$  BERT/GPT)
2. They learned to predict:  $P(\text{word}|\text{context})$
3. They output probability distributions over 50,000+ words
4. **Today:** How do we convert these probabilities into actual text?

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Models predict probabilities. Decoding converts probabilities to text.

# Extreme Case 1: Greedy Decoding (Too Narrow)



**Extreme 1: Too narrow - misses 99.999999% of search space**

## What If We Explored More Paths?

**Greedy chose:** “The cat **sat**” ( $P=0.68$ )

**But it ignored these alternatives:**

“The cat <b>walked</b> ”	$P=0.12$	(might lead to better text)
“The cat <b>jumped</b> ”	$P=0.08$	(more interesting)
“The cat <b>slept</b> ”	$P=0.06$	(different story)
“The cat <b>ran</b> ”	$P=0.04$	(action-oriented)

**Question:** What if we kept ALL 100 words at each step?

Think:  $100 \times 100 \times 100 \times 100 \times 100 = ?$

## What If We Explored More Paths?

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**Question:** What if we kept ALL 100 words at each step?

Think:  $100 \times 100 \times 100 \times 100 \times 100 = ?$

**Answer:** 10 billion paths! Let's see what happens...

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From 1 path to ALL paths - what could go wrong?

# Greedy's Fatal Flaw: Missing Better Paths

## Two Paths: Greedy Choice vs. Better Alternative

### GREEDY CHOOSES

*"The cat sat on the"*

Input: *"The cat"*

Step 1: *"sat"*  $P=0.71$  CHOSEN

Step 2: *"on"*  $P=0.15$  CHOSEN

Step 3: *"the"*  $P=0.72$  CHOSEN

Total: 0.077

Result: Generic, predictable

### GREEDY MISSES

*"The cat spotted something"*

Input: *"The cat"*

Step 1: *"spotted"*  $P=0.08$  IGNORED

Step 2: *"something"*  $P=0.85$

Step 3: *"moving"*  $P=0.91$

Total: 0.062

Result: Engaging, creates tension

*Greedy chooses the safer path ( $P=0.076$ ) but misses better narratives ( $P=0.062$ )*

**The Problem:** High first-step probability  $\neq$  Best complete sentence

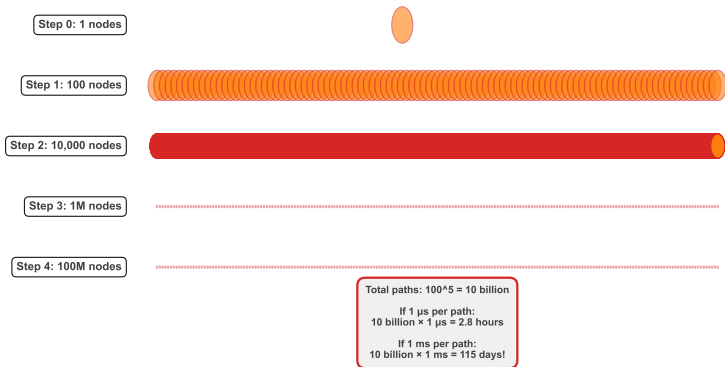
Greedy commits early, missing narratively richer paths despite lower initial probability



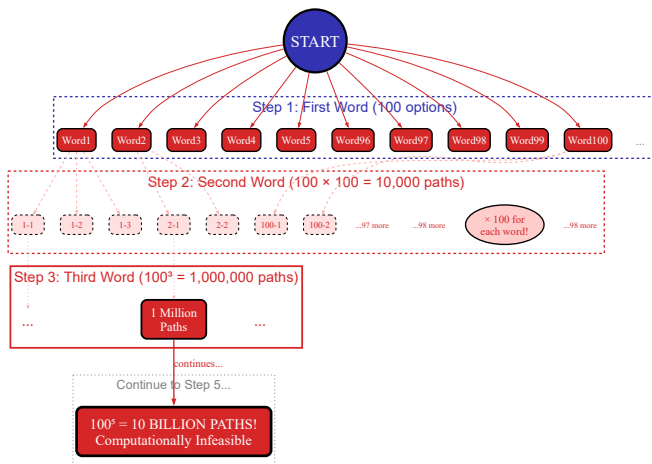
# Extreme Case 2: Full Search Space (Too Broad)

## Extreme Case 2: Full Search Space

*Vocabulary size = 100, explore ALL paths*



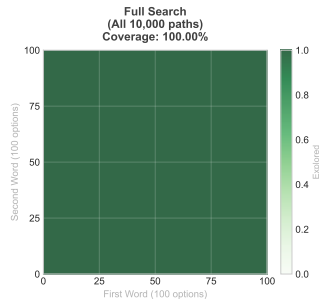
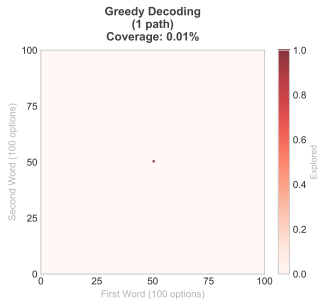
# Full Exploration: From One Path to Billions



**The Problem:** How do we explore more than 1 but less than 10 billion paths?

# The Extremes: Why Neither Works

The Extremes: Coverage Comparison  
(Vocabulary=100, showing first 2 words only)



## Greedy (0.01% coverage):

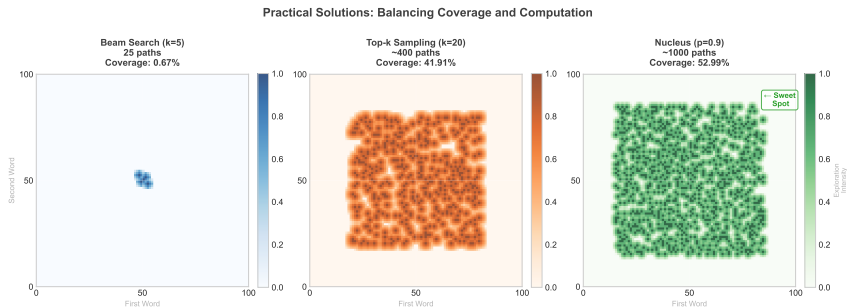
- Too narrow - misses better paths
- Fast but low quality
- Prone to repetition loops

## Full Search (100% coverage):

- Too broad - computationally infeasible
- Perfect in theory, impossible in practice
- Would take days/years to complete

**Key Insight:** We need methods that explore 1-5% of space intelligently

# The Sweet Spot: Balanced Exploration



## The Solution: 1-5% Coverage

- **Not too narrow:** Explores enough paths to find good sequences
- **Not too broad:** Computationally feasible (seconds, not days)
- **Strategic exploration:** Focus on promising regions of search space

**Coming Next:** Learn 6 specific methods that achieve this balance

The sweet spot: Methods that intelligently explore 1-5% of the search space

# Method 1: Greedy Decoding

## Core Mechanism:

$$w_t = \operatorname{argmax}_{w \in V} P(w \mid w_1, \dots, w_{t-1})$$

At each step, pick the single word with highest probability

## Characteristics:

- Deterministic (same input  $\rightarrow$  same output)
- Fast:  $O(1)$  per step
- No exploration

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Method 1 of 6: Greedy = always pick argmax

## Method 2: Beam Search

### Core Mechanism:

Maintain  $k$  hypotheses ("beams") at each step

Expand each hypothesis, keep top- $k$  by cumulative probability

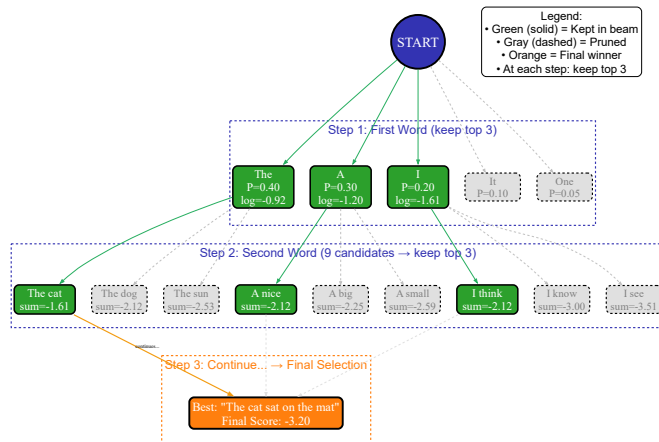
### Characteristics:

- Explores  $k$  paths simultaneously (typically  $k=3-5$ )
- Trade exploration vs computation
- Still deterministic for fixed  $k$

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Method 2 of 6: Beam = keep top- $k$  paths

# Beam Search: Step-by-Step Example



Worked example shows why beam search finds better sequences than greedy

## Algorithm:

1. Start: Keep top-k tokens
2. Expand: Generate continuations for each
3. Score: Multiply probabilities
4. Prune: Keep top-k sequences
5. Repeat until END token

## Scoring:

$$\text{score}(y_1 \dots y_t) = \prod_{i=1}^t P(y_i | y_{<i})$$

With length normalization:

$$\text{score} = \frac{1}{t} \sum_{i=1}^t \log P(y_i | y_{<i})$$

## Best For:

- Machine translation
- Summarization
- Question answering
- Tasks with “correct” answer

## Parameters:

Width = 3-5 (translation)

Width = 10 (diverse outputs)

## Tradeoffs:

- + Better quality than greedy
- + Diverse hypotheses
- Still deterministic
- 4-5× slower than greedy

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Beam search is the workhorse for deterministic tasks



## Method 3: Temperature Sampling

### Core Mechanism:

$$P_T(w_i) = \frac{\exp(z_i / T)}{\sum_j \exp(z_j / T)}$$

Reshape probability distribution with temperature  $T$ , then sample

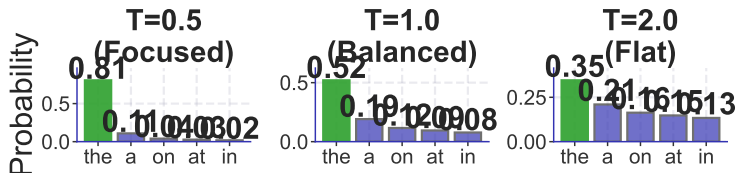
### Characteristics:

- $T < 1$ : More focused (sharper distribution)
- $T > 1$ : More random (flatter distribution)
- Stochastic: different output each time

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Method 3 of 6: Temperature = control randomness

### Temperature Effects on Probability Distribution



**Key Insight:** Temperature reshapes probability distribution

$T < 1$ : more focused.  $T = 1$ : unchanged.  $T > 1$ : more random

# Temperature: Worked Example

**Original probabilities:** "The cat \_"

- sat:  $P = 0.40$
- is:  $P = 0.30$
- jumped:  $P = 0.20$
- walked:  $P = 0.10$

**After temperature  $T = 0.5$**  (more focused):

- sat:  $P = 0.52$  (increased)
- is:  $P = 0.28$
- jumped:  $P = 0.15$
- walked:  $P = 0.05$  (decreased)

**After temperature  $T = 2.0$**  (more random):

- sat:  $P = 0.30$  (decreased)
- is:  $P = 0.28$
- jumped:  $P = 0.24$
- walked:  $P = 0.18$  (increased)

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Lower  $T$  sharpens distribution, higher  $T$  flattens it

## Method 4: Top-k Sampling

### Core Mechanism:

1. Sort words by probability
2. Keep only top k words (e.g.,  $k=50$ )
3. Renormalize and sample from these k

### Characteristics:

- Filters out low-probability “junk” words
- Fixed cutoff (always k words)
- Can combine with temperature

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Method 4 of 6: Top-k = filter then sample

# Top-k Sampling: Filter the Tail

## How it works:

1. Sort all 50,000 words by probability (descending)
2. Keep only the top  $k$  words (e.g.,  $k = 50$ )
3. Discard the remaining 49,950 low-probability words
4. Renormalize probabilities to sum to 1.0
5. Sample from the filtered  $k$  words

## Result:

- Prevents unlikely/nonsensical words
- Fixed vocabulary size (always  $k$  words)
- Combines well with temperature

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Prevents sampling from long tail of unlikely words

### Top-k Example: k=3

Original Probabilities:

cat: 0.45, dog: 0.18, bird: 0.15 = 0.78  
fish: 0.07, mouse: 0.08 = 0.15  
.... 0.04

**Result: Sample from {cat: 58%, dog: 23%, bird: 19%}**

*Prevents sampling from long tail ("mouse" eliminated)*

Concrete numbers show k=50 filtering process

## Method 5: Nucleus (Top-p) Sampling

### Core Mechanism:

1. Sort words by probability
2. Keep minimum set where cumulative probability  $\geq p$
3. Sample from this set

### Characteristics:

- Adaptive: number of words varies
- Focuses on “nucleus” of probability mass (typically  $p=0.9$ )
- Adjusts to distribution shape

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Method 5 of 6: Nucleus = adaptive probability mass

# Nucleus (Top-p) Sampling: Dynamic Cutoff

**Process** ( $p = 0.9$  example):

1. Sort words by probability: sat (0.40), is (0.30), jumped (0.20), ...
2. Add cumulative probabilities: 0.40, 0.70, 0.90, ...
3. Stop when cumulative  $\geq p$  (here: 3 words reach 0.90)
4. Sample from these words only

**Adaptive behavior:**

- Peaked distribution: fewer words needed (e.g., 3 words)
- Flat distribution: more words needed (e.g., 100 words)
- Same  $p$  value, different vocabulary sizes

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Nucleus adapts to distribution shape - not fixed like top-k



## Method 6: Contrastive Search

### Core Mechanism:

Choose word that maximizes:

$$\text{score} = (1 - \alpha) \cdot \text{model probability} - \alpha \cdot \text{similarity to previous}$$

Penalize words similar to already-generated text

### Characteristics:

- Explicitly avoids repetition
- Balances coherence and diversity
- Deterministic with hyperparameter  $\alpha$

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Method 6 of 6: Contrastive = penalize repetition

## The Degeneration Problem: Model Repetition

### Real Output from Greedy Decoding:

*"The city of New York is a major city in the United States. The city is known for its diverse culture and the city has many tourist attractions. The city is also home to the city's financial district..."*

**Problem: "the city" appears 6 times in 4 sentences!**

*Why? Always picking argmax → same patterns repeated*

**Solution: Penalize tokens similar to recent context (Contrastive Search)**

**Discovery Question:** Why do models repeat themselves?

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Greedy and beam search maximize probability - but high probability = repeating recent context

# Contrastive Search: Penalize Repetition

## The scoring function:

$$\text{score}(w) = (1 - \alpha) \cdot P(w \mid \text{context}) - \alpha \cdot \max_{w' \in \text{past}} \text{sim}(w, w')$$

## Two components:

- Model probability: how likely is this word?
- Similarity penalty: how similar to words we already used?
- $\alpha$  balances the two (typically 0.6)

## Effect:

- High-probability words that repeat context get penalized
- Forces model to use semantically different words
- Prevents “the cat sat on the cat on the cat...”

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Explicitly prevents degeneration in long text generation

## Contrastive vs Nucleus: Direct Comparison

### Same Prompt, Different Methods

Prompt: "The future of artificial intelligence is"

"...is pro  
many inc  
significa  
educa

"...is rapidly evolving, bringing  
many unprecedented opportunities across  
sectors ranging from medicine to  
climate science, while raising  
important ethical questions."

+ Diverse  
+ Creative  
+ No repetition

*Contrastive Search explicitly penalizes copying recent context*

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Contrastive adds explicit similarity penalty that Nucleus lacks

# Checkpoint Quiz 1: Match the Method

## Methods:

1. Greedy
2. Beam Search
3. Temperature
4. Top-k
5. Nucleus
6. Contrastive

## Match to Mechanisms:

- A. Sample from reshaped distribution
- B. Keep top-k paths at each step
- C. Always pick argmax
- D. Filter to k words, then sample
- E. Penalize similarity to previous
- F. Adaptive probability mass cutoff

## Checkpoint Quiz 1: Match the Method

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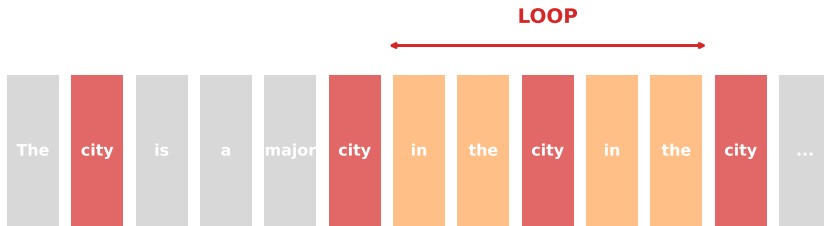
**Answers:** 1→C, 2→B, 3→A, 4→D, 5→F, 6→E

*Now you know the toolbox - let's see WHY each tool exists!*

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Quiz 1: Can you match each method to its mechanism?

### Greedy Decoding Gets Stuck



*Output: "The city is a major city in the city in the city..."*

**Greedy's Problem:** Trapped in loops, can't escape

**Why Beam Helps:** Explores  $k=3-5$  paths, avoids greedy trap

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Problem 1 of 6: Greedy decoding creates loops → Beam search explores alternatives

## High Temperature Creates Nonsense

The	<b>glorp</b>	is	very	<b>blorptastic</b>	
She	likes	to	eat	<b>qwerty</b>	food
I	went	to	the	<b>flurb</b>	yesterday
The	weather	is	<b>zxqp</b>	today	

*Generated words not in vocabulary!*

**Greedy & Beam's Problem:** Same input → same output always

**Why Temperature Helps:** Sampling introduces randomness, enables creativity

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Problem 2 of 6: Deterministic methods lack variation → Temperature adds controlled randomness



## Zero Creativity: Always Same Output

#9:	The weather is nice today.	#10:	The weather is nice today.
#7:	The weather is nice today.	#8:	The weather is nice today.
#5:	The weather is nice today.	#6:	The weather is nice today.
#3:	The weather is nice today.	#4:	The weather is nice today.
#1:	The weather is nice today.	#2:	The weather is nice today.

100x

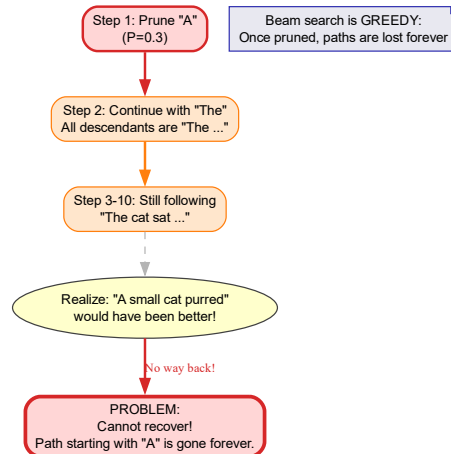
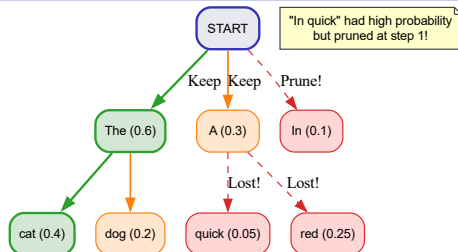
*Asked 100 times → Always: "The weather is nice today."*

**Temperature's Problem:** Pure sampling includes low-quality words

**Why Top-k Helps:** Filter to  $k=50$  best words, then sample

Problem 3 of 6: Can't balance quality & creativity → Top-k filters unlikely words

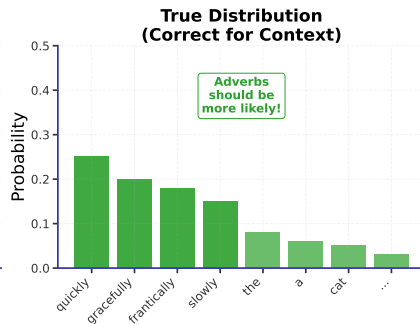
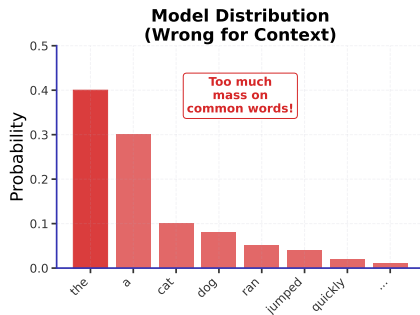
# Beam Search Limitation: Missing Better Paths



Problem 4 of 6: Even beam search prunes early, cannot recover optimal path - 4 perspectives

# Why Nucleus? Problem: Distribution Tail Contains Junk

Context: "The cat ran \_\_\_\_"

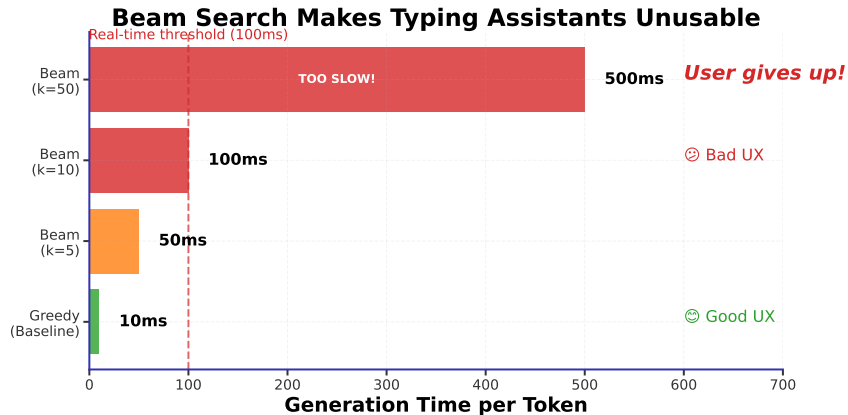


**Top-k's Problem:** Fixed k doesn't adapt to distribution shape

**Why Nucleus Helps:** Adaptive cutoff at  $p=0.9$  probability mass

Problem 5 of 6: Tail contains junk → Nucleus adapts to distribution

## Why Contrastive? Problem: Generic Repetitive Text

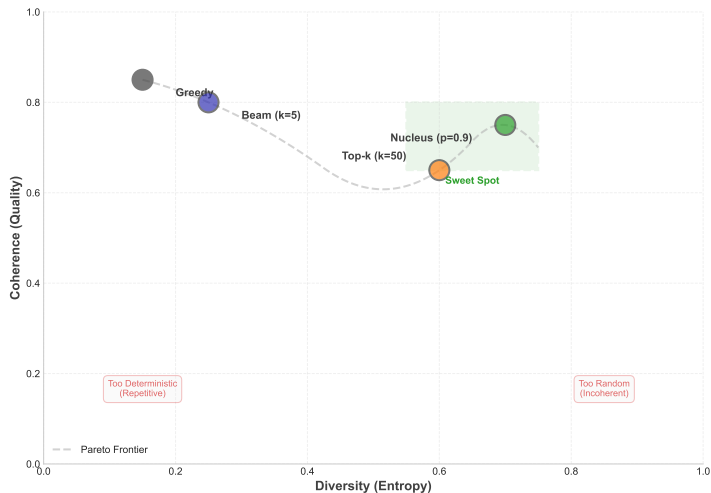


**All Methods' Problem:** Can still produce generic, repetitive text

**Why Contrastive Helps:** Explicitly penalizes similarity to previous tokens

Problem 6 of 6: Generic text persists → Contrastive reduces repetition

# The Quality-Diversity Tradeoff



**Key Insight:** We saw problems and their solutions. Each method balances quality vs diversity.

All 6 problems relate to balancing coherence with creativity

## Checkpoint Quiz 2: Which Method for Which Problem?

### Match Solution to Problem:

1. Beam Search → ?
2. Temperature → ?
3. Top-k → ?
4. Nucleus → ?
5. Contrastive → ?

### Problems to Solve:

- A. Too boring OR too crazy
- B. Repetition loops
- C. No diversity
- D. Fixed k doesn't adapt
- E. Generic repetitive text

## Checkpoint Quiz 2: Which Method for Which Problem?

### Match Solution to Problem:

1. Beam Search → ?
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### Problems to Solve:

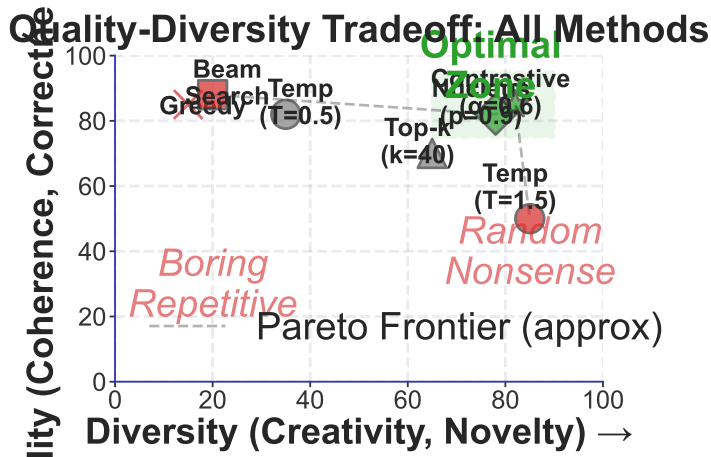
- A. Too boring OR too crazy
- B. Repetition loops
- C. No diversity
- D. Fixed k doesn't adapt
- E. Generic repetitive text

**Answers:** 1→B, 2→C, 3→A, 4→D, 5→E

*Each method targets a specific failure mode!*

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Quiz 2: Understanding the method-problem mapping



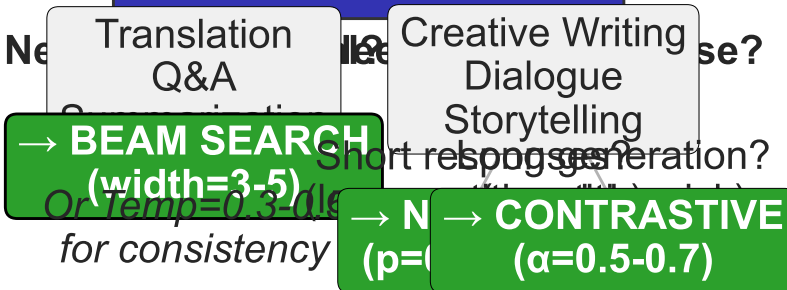
Pareto Frontier: No method dominates all others

Choose based on task: deterministic tasks (left), creative tasks (right)



## Choosing the Right Decoding Method

**START: What kind of task?**



### Special: Code Generation

- Greedy or Beam (correctness critical)
  - *Then verify syntax/semantics*

# Task-Specific Decoding Recommendations (2025)

Task	Recommended Decoder	Parameter	Why?
Machine Translation	Beam Search	width=3-5	Deterministic, quality critical
Factual Q&A	Greedy / Low Temp	T=0.1-0.3	Single correct answer needed
Summarization	Beam Search	width=4	Balance coverage + conciseness
Code Generation	Greedy	T=0	Syntax errors costly
Creative Writing	Nucleus / Contrastive	p=0.9, $\alpha=0.6$	Diverse but coherent
Dialogue Systems	Nucleus	p=0.85-0.95	Natural variation needed
Story Generation	Contrastive	$\alpha=0.5-0.7$	Avoid repetition in long text
Long-form Articles	Contrastive	$\alpha=0.6$ , p=0.9	Degeneration prevention

Comprehensive mapping from 8 common tasks to optimal decoding strategies

## Checkpoint Quiz 3: Choose the Right Method

**Given these tasks, which method would you use?**

**1. Medical report summary**

- Needs: Accuracy, no hallucination

**2. Creative story writing**

- Needs: Diversity, creativity

**3. Code generation**

- Needs: Correctness, explore options

**4. Customer service chat**

- Needs: Natural, varied responses

**5. Legal document**

- Needs: Precise, formal

**6. Long blog post**

- Needs: Coherent, no repetition

## Checkpoint Quiz 3: Choose the Right Method

Given these tasks, which method would you use?

1. **Medical report summary**

- Needs: Accuracy, no hallucination

2. **Creative story writing**

- Needs: Diversity, creativity

3. **Code generation**

- Needs: Correctness, explore options

4. **Customer service chat**

- Needs: Natural, varied responses

5. **Legal document**

- Needs: Precise, formal

6. **Long blog post**

- Needs: Coherent, no repetition

**Answers:**

1. Greedy/Low temp ( $T=0.1-0.3$ )   2. Nucleus ( $p=0.95$ ,  $T=1.0$ )   3. Beam Search ( $k=3-5$ )  
4. Nucleus ( $p=0.9$ ,  $T=0.7$ )   5. Greedy ( $T=0$ )   6. Contrastive ( $\alpha=0.6$ )

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Quiz 3: Real-world task selection is crucial for quality

# Key Takeaways

1. **6 Problems** → **6 Solutions**: Each method solves specific failure mode
2. **Deterministic** (Greedy, Beam): High quality, no diversity - factual tasks
3. **Stochastic** (Temperature, Top-k, Nucleus): Diverse but variable quality
4. **Balanced** (Contrastive): Explicit degeneration prevention
5. **Task matters**: Translation → Beam — Dialogue → Nucleus — Stories → Contrastive
6. **Tradeoffs**: Speed vs Quality, Diversity vs Coherence

**Modern Standard**: Nucleus (top-p=0.9) + Temperature (T=0.7) for most applications

**Next**: Lab - Implement all 6 methods, measure quality-diversity tradeoffs

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Decoding strategy matters as much as model architecture



## What We Learned:

- Models give us probability distributions (Week 3-7)
- Converting to text has 6 fundamental challenges
- Each decoding method addresses specific problems
- No universal best - choose based on task requirements
- Production systems use hybrid methods (Nucleus + Temperature)

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Complete pipeline from model training to text generation

# Technical Appendix

25 slides: Complete mathematical treatment

**A1-A5: Beam Search Mathematics**

**A6-A10: Sampling Mathematics**

**A11-A14: Contrastive Search & Degeneration**

**A15-A19: Advanced Topics & Production**

**A20-A25: The 6 Problems - Technical Analysis (NEW)**

# A1: Beam Search Formulation

**Objective:** Find sequence  $y^* = \operatorname{argmax} P(y|x)$

**Decomposition:**

$$P(y|x) = \prod_{t=1}^T P(y_t|y_{<t}, x)$$

**Log-probability** (more stable):

$$\log P(y|x) = \sum_{t=1}^T \log P(y_t|y_{<t}, x)$$

**Beam Search Approximation:**

Instead of exploring all  $V^T$  sequences, maintain top-k hypotheses at each step

**Complexity:**

Time:  $O(k \cdot V \cdot T)$  where  $k$  = beam width,  $V$  = vocabulary,  $T$  = length

Space:  $O(k \cdot T)$  to store hypotheses

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Beam search is tractable approximation to exact search



## A2: Length Normalization

**Problem:** Longer sequences have lower probabilities (more terms multiplied)

$$P(y_1, y_2, y_3, y_4) = \underbrace{0.5}_{y_1} \times \underbrace{0.5}_{y_2} \times \underbrace{0.5}_{y_3} \times \underbrace{0.5}_{y_4} = 0.0625$$

$$P(y_1, y_2) = 0.5 \times 0.5 = 0.25 > 0.0625$$

Bias toward shorter sequences!

**Solution:** Length normalization

$$\text{score}(y) = \frac{1}{|y|^\alpha} \log P(y)$$

where  $\alpha \in [0.5, 1.0]$  (typically 0.6-0.7)

**Effect:**

Without: Beam search heavily biases toward short outputs

With: Fair comparison across different lengths

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Length normalization is essential for beam search quality

## A3: Beam Search Variants

### **Diverse Beam Search:**

Partition beams into groups  
Penalize within-group similarity  
Result: More diverse hypotheses

### **Constrained Beam Search:**

Force certain tokens to appear  
Useful for: Keywords, entities  
Applications: Controllable generation

### **Stochastic Beam Search:**

Sample beams instead of argmax  
Combines beam + sampling  
More diverse than standard beam

### **Block n-gram Beam:**

Penalize n-gram repetition  
Prevents “the city is a city” loops  
Common in summarization

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Many beam search variants exist for specific requirements

## A4: Beam Search Stopping Criteria

**When to stop expanding beams?**

**Method 1:** Fixed length

Stop at  $T_{\max}$  tokens (simple but rigid)

**Method 2:** END token

Stop when beam generates special token (most common)

**Method 3:** Score threshold

Stop when best score cannot improve enough

$$\frac{\text{best\_incomplete}}{\text{best\_complete}} < \text{threshold}$$

**Method 4:** Timeout

Computational budget exceeded (production systems)

---

**Choice of stopping criterion affects output length distribution**

## A5: Beam Search Limitations

### Fundamental Issues:

1. **Exposure bias:** Trained with teacher forcing, tested with own outputs
2. **Label bias:** Cannot compare sequences of different prefixes fairly
3. **Repetition:** Still can loop (“the city is a major city”)
4. **Bland outputs:** Maximizes probability, not interestingness
5. **Search errors:** May miss better sequences outside beam

### When Beam Search Fails:

Open-ended generation (dialogue, stories)

Long-form text (repetition accumulates)

Creative tasks (probability  $\neq$  quality)

→ Need sampling-based methods

---

Beam search optimizes wrong objective for creative tasks

## A6: Sampling as Inference

**Goal:** Sample  $y \sim P(y|x)$  instead of  $\operatorname{argmax} P(y|x)$

**Ancestral Sampling:**

For  $t = 1$  to  $T$ :

    Compute  $P(y_t|y_{<t}, x)$

    Sample  $y_t \sim P(\cdot|y_{<t}, x)$

**Properties:**

Stochastic: Different output each time

Explores full distribution (in expectation)

Can generate low-probability sequences

**Variants:**

Temperature: Reshape distribution before sampling

Top-k: Truncate distribution before sampling

Nucleus: Dynamic truncation before sampling

---

**Sampling enables diversity but loses quality guarantees**

### Softmax with Temperature:

$$p_i(T) = \frac{\exp(z_i/T)}{\sum_{j=1}^V \exp(z_j/T)}$$

### Limiting Cases:

$$T \rightarrow 0: p_i \rightarrow \begin{cases} 1 & \text{if } i = \operatorname{argmax} z \\ 0 & \text{otherwise} \end{cases} \quad (\text{greedy})$$

$$T \rightarrow \infty: p_i \rightarrow 1/V \quad (\text{uniform})$$

### Entropy Analysis:

Entropy  $H(p) = -\sum p_i \log p_i$  measures randomness

$H$  increases monotonically with  $T$

Low  $T$  ( $<0.5$ ):  $H \approx 0$  (deterministic)

High  $T$  ( $>2.0$ ):  $H \approx \log V$  (maximum entropy)

---

Temperature provides continuous control over distribution entropy

**Formal Definition:**

Let  $\sigma$  = permutation sorting probabilities descending

$$V_k = \{w_{\sigma(1)}, w_{\sigma(2)}, \dots, w_{\sigma(k)}\}$$

Truncated distribution:

$$p'(w) = \begin{cases} \frac{p(w)}{\sum_{w' \in V_k} p(w')} & \text{if } w \in V_k \\ 0 & \text{otherwise} \end{cases}$$

**Information Loss:**

Original entropy:  $H(p) = -\sum_{i=1}^V p_i \log p_i$

After top-k:  $H(p') = -\sum_{i=1}^k p'_i \log p'_i < H(p)$

Loss  $\approx \sum_{i=k+1}^V p_i \log(1/p_i)$  (tail information)

---

Top-k sacrifices tail probability mass for sampling quality

## A9: Nucleus (Top-p) Mathematics

### Formal Definition:

$$V_p = \min \left\{ V' \subseteq V : \sum_{w \in V'} p(w) \geq p \right\}$$

Smallest set with cumulative mass  $\geq p$

### Dynamic Vocabulary Size:

$$|V_p| = \min \left\{ k : \sum_{i=1}^k p_{\sigma(i)} \geq p \right\}$$

Adapts to distribution shape:

Peaked: Small  $|V_p|$  (2-5 tokens)

Flat: Large  $|V_p|$  (50+ tokens)

### Why Nucleus > Top-k:

Top-k: Fixed  $k$  regardless of  $p(w)$  distribution

Nucleus: Adapts  $k$  to achieve consistent probability mass

---

Nucleus automatically adjusts vocabulary to distribution characteristics



## A10: Sampling Quality Metrics

### Quality Metrics:

**Perplexity:**  $\exp(-\frac{1}{T} \sum \log p(y_t))$   
Lower = better

### BLEU (translation):

N-gram overlap with reference  
0-100 scale

### Human evaluation:

Fluency (1-5)  
Relevance (1-5)

### Diversity Metrics:

**Distinct-n:**  $\frac{\text{unique n-grams}}{\text{total n-grams}}$   
Higher = more diverse

### Self-BLEU:

BLEU of output vs other outputs  
Lower = more diverse

### Repetition Rate:

$\frac{\text{repeated n-grams}}{\text{total n-grams}}$   
Lower = less repetitive

---

Need both quality AND diversity metrics to evaluate decoding

# A11: The Degeneration Problem (Formal)

**Definition:** Model-generated text with unnatural repetitions

**Why It Happens:**

1. Model trained on natural text (low repetition)
2. But generation maximizes  $P(y_t|y_{<t})$
3. Recent context  $y_{<t}$  influences  $P$
4. Creates positive feedback: high prob word  $\rightarrow$  context  $\rightarrow$  same high prob word

**Quantifying Degeneration:**

Repetition rate in greedy: 15-30% (depending on domain)

Repetition rate in human text: 2-5%

Gap = degeneration problem

**Examples:**

*"The city is a major city in the United States. The city..."*

*"I think that I think that I think..."*

---

Maximizing probability does not equal natural text

## A12: Contrastive Search Objective

### Scoring Function:

$$\text{score}(w_t) = (1 - \alpha) \times \underbrace{P(w_t | y_{<t})}_{\text{model confidence}} - \alpha \times \underbrace{\max_{w_i \in y_{<t}} \text{sim}(w_t, w_i)}_{\text{context similarity}}$$

where  $\alpha \in [0, 1]$  controls tradeoff

### Similarity Function:

$$\text{sim}(w_i, w_j) = \frac{h_i \cdot h_j}{||h_i|| \cdot ||h_j||}$$

(cosine similarity)  
using token embeddings  $h$

### Algorithm:

1. Get top-k candidates by probability
2. For each candidate, compute similarity to all tokens in  $y_{<t}$
3. Apply penalty:  $\text{score} = \text{prob} - \alpha \times \text{max\_similarity}$
4. Select candidate with highest score

---

Contrastive search explicitly penalizes copying recent context

## A13: Contrastive Search Parameters

### Alpha ( $\alpha$ ):

$\alpha = 0$ : Pure greedy (no penalty)  
 $\alpha = 0.6$ : Balanced (recommended)  
 $\alpha = 1.0$ : Maximum diversity (risky)

### Typical Settings:

Short text (<100 tokens):  $\alpha = 0.4 - 0.5$   
Medium (<500):  $\alpha = 0.5 - 0.6$   
Long (500+):  $\alpha = 0.6 - 0.7$

### Top-k for Candidates:

$k = 4$ : Fast, focused  
 $k = 6$ : Balanced (default)  
 $k = 10$ : Diverse

### Computational Cost:

For each step:

- Compute similarities:  $O(k \times t)$
- $t$  grows with generation

Total:  $O(k \times T^2)$

12× slower than greedy

---

Hugging Face default:  $\alpha=0.6$ ,  $k=4$

# A14: Degeneration Analysis

## Research Findings (2024-2025):

- Greedy decoding repetition: 18-25% (GPT-2), 12-18% (GPT-3)
- Nucleus sampling repetition: 8-12% (still above human 3-5%)
- Contrastive search repetition: 4-7% (closest to human)

## Why Probability Maximization Fails:

Training objective: Next token prediction

But generation requires: Global coherence

Mismatch: Local optimum  $\neq$  global quality

## Solutions Hierarchy:

1. Temperature/Top-k/Nucleus: Reduce greedy's determinism
2. Contrastive: Explicit degeneration penalty
3. RLHF/DPO: Align model with human preferences (different lecture)

---

Contrastive search addresses fundamental limitation of likelihood-based decoding

### Combining Strategies:

#### Nucleus + Temperature:

Apply temperature THEN nucleus

$$p_i(T) = \text{softmax}(z/T), \quad \text{then} \quad V_p \leftarrow \text{nucleus}(p_i(T))$$

Used by GPT-3 API, ChatGPT

#### Beam + Sampling:

Beam search with stochastic selection

Keep top-k, sample from them (not argmax)

#### Contrastive + Nucleus:

Nucleus for candidate generation

Contrastive scoring for selection

Best of both worlds

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Hybrid methods leverage complementary strengths

# A16: Constrained Decoding (2025)

**Goal:** Force certain tokens/patterns to appear

**Lexically Constrained:**

Must include keywords: { "AI", "ethics", "safety" }

Beam search variant: Track constraint satisfaction

**Format Constraints:**

JSON output: Force structure { "key": "value" }

Code: Force syntactic validity

**NeuroLogic Decoding (2021):**

Beam search + constraint satisfaction

Optimal for: Keyword-based generation

**Production Use Cases:**

Structured data extraction (force JSON)

Controllable summarization (force keywords)

Code generation (force syntax)

---

**Constrained decoding enables controllable generation**

## A17: Computational Complexity Comparison

Method	Time per token	Total complexity	Relative speed
Greedy	$O(V)$	$O(V \times T)$	1.0× (baseline)
Temperature	$O(V)$	$O(V \times T)$	1.1× (softmax overhead)
Top-k	$O(V)$	$O(V \times T)$	1.2× (sorting)
Nucleus	$O(V \log V)$	$O(V \log V \times T)$	1.3× (sort + cumsum)
Beam (k=5)	$O(k \times V)$	$O(k \times V \times T)$	4.5× (k=5)
Contrastive	$O(k \times T)$	$O(k \times T^2)$	12× (similarity)

**Key Insight:** Contrastive's  $T^2$  term makes it expensive for long sequences

**Practical Impact** (1000-token generation):

Greedy: 2.5 seconds

Nucleus: 3.2 seconds (best choice)

Beam: 11 seconds

Contrastive: 30 seconds (only if quality critical)

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Computational cost matters for production deployment



Production Decoding Settings (Real Systems 2024-2025)

System (2024-2025)	Method	Parameters	Goal
GPT-3 API (2024)	Nucleus	T=0.7, p=1.0	Balanced default
ChatGPT	Nucleus + Temp	T=0.8, p=0.95	Creative but controlled
Google Translate	Beam Search	width=4	Quality critical
GitHub Copilot	Greedy	T=0	Code correctness
Claude	Nucleus	T=1.0, p=0.9	High quality generation
Hugging Face Defa	Greedy	T=1.0	Deterministic baseline

What ChatGPT, Claude, and other production systems actually use

## Active Research Areas (2025):

1. **Quality-diversity optimization:** Multi-objective search methods
2. **Learned decoding:** Train models to decode better (RLHF, DPO)
3. **Speculative decoding:** Parallel generation for speed ( $4-8\times$  faster)
4. **Adaptive methods:** Choose strategy dynamically during generation
5. **Energy-based decoding:** Score sequences globally (not token-by-token)

## Open Problems:

How to automatically select best  $T$ ,  $p$ ,  $k$ ,  $\alpha$  for new task?

How to balance fluency + factuality + creativity simultaneously?

How to decode efficiently for 100K+ token outputs?

**Trend:** Moving from hand-tuned parameters to learned decoding strategies

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Decoding is an active research area with many open questions