Diffraction, Causality & Phase Stretch Transform

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1. Stationary Phase Approximation in Paraxial Far-field Diffraction

Stationary phase approximation for the case of paraxial diffraction results in a mapping of the input signal from the transverse spatial domain variables (x, y) to transverse spatial frequency variables (k_x, k_y).

We start with the general solution to the homogenous electromagnetic wave equation in rectangular coordinates (x, y, z)

$$\mathrm{E(x,y,z)} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{E}(k_x, k_y, 0)\, e^{+jk_z z}\, e^{j(k_x x + k_y y)}\, dk_x dk_y \tag{1}$$

where $\tilde{E}(k_x, k_y, 0)$ is the spatial spectrum of the input field $E(x, y, 0)$ (which may have additional modulation beyond that of the source). The Far-field diffraction kernel along the z-axis is an isotropic phase kernel represented by the expression $e^{jk_z z}$.

For waves propagating close to the optical axis, "Paraxial Approximation", the propagation constant along the z axis, k_z, is equal to

$$k_z = \sqrt{k^2 - (k_x^2 + k_y^2)} \approx k - \frac{(k_x^2 + k_y^2)}{2 \cdot k} \tag{2}$$

by using the paraxial approximation and binomial approximation as derived before. We can then rewrite Eq. (1) as following:

$$\mathrm{E(x,y,z)} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{E}(k_x, k_y, 0)\; e^{j(k_x x + k_y y)}\; e^{jz(k - \frac{k_x^2 + k_y^2}{2 \cdot k})}\, dk_x dk_y \tag{3}$$

where the term e^{jzk} is a constant scalar quantity and is independent of variable x and y. Hence, it can be taken out of the integration which results in:

$$\mathrm{E(x,y,z)} = e^{jzk} \cdot \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{E}(k_x, k_y, 0)\; e^{-j\frac{z}{2k}(k_x^2 + k_y^2 - 2 \cdot \frac{xk}{z} \cdot k_x - 2 \cdot \frac{yk}{z} \cdot k_y)}\, dk_x dk_y \tag{4}$$

We now complete the squares in the exponent by adding and subtracting $\frac{x^2k^2}{z^2}$ and $\frac{y^2k^2}{z^2}$ to the term inside the paranthesis, resulting in,

$$\tilde{E}\mathrm{(x,y,z)} = e^{j\left(zk + \frac{x^2 k}{2z} + \frac{y^2 k}{2z}\right)} \cdot \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{E}(k_x, k_y, 0)\; e^{-j\frac{z}{2k}\left((k_x - \frac{xk}{z})^2 + (k_y - \frac{yk}{z})^2\right)}\, dk_x dk_y \tag{5}$$

A camera or any other photodetector responds to the intensity (power per unit area) which is the magnitude of the E-field squared,

$$\mathrm{I} \propto E.E^* = |\mathrm{E(x,y,z)}|^2 = \left| \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{E}(k_x, k_y, 0)\, e^{-j\frac{z}{2k}\left((k_x - \frac{xk}{z})^2 + (k_y - \frac{yk}{z})^2\right)}\, dk_x dk_y \right|^2$$

Far-field regime

Let's consider the phase term inside the integral. For a given k=$\frac{2\pi}{\lambda}$, if $\frac{z}{\mathrm{k}}$ is very large, then as k_x and k_y are varied to compute the integral, the phase term oscillates between +1 and -1 and the integral is zero, except for particular values of wavevector near which the phase is zero:

$$k_x = \frac{k}{z}x \quad \text{and} \quad k_y = \frac{k}{z}y \tag{6}$$

In other words, only k_x and k_y values near these values contribute to the field at x and y. This is called the Stationary Phase Approximation (SPA) and leads to a direct mapping of spatial frequency to space. In other words if we observe the diffraction pattern in space, we see the Fourier transform of the input. Note that this happens at large values of z so it is valid in the "far field", i.e. far away from the source.

Stationary Phase Approximation

Goda, K., Solli, D. R., Tsia, K. K., & Jalali, B. (2009). Theory of amplified dispersive Fourier transformation. *Physical Review A*, *80*(4), 043821.

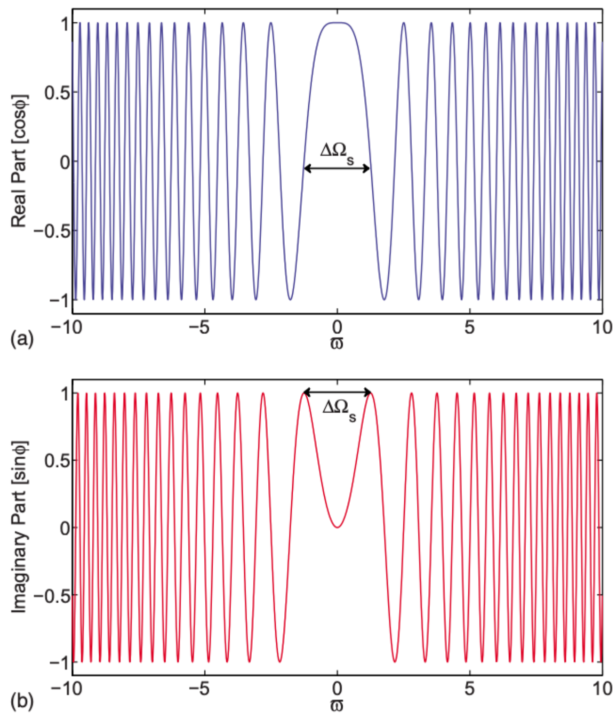


FIG. 2. (Color online) The real (top) and imaginary (bottom) parts of the exponential in the integral in Eq. (17). ϕ is defined to be $\phi=\beta_2 z(\omega-\Omega_S)^2/2=a\varpi^2$, where $a=\beta_2 z\Omega_S^2/2$ and $\varpi=\omega/\Omega_S-1$, which is the normalized frequency. $a=1$ is used for the figure. Both cos ϕ and sin ϕ oscillate rapidly as ϖ varies from the origin. Only the frequencies in the central region of the two plots contribute to the temporal waveform at a particular time instant. Contributions from all the other frequencies oscillate rapidly between positive and negative values and cancel out when integrated over all frequencies.

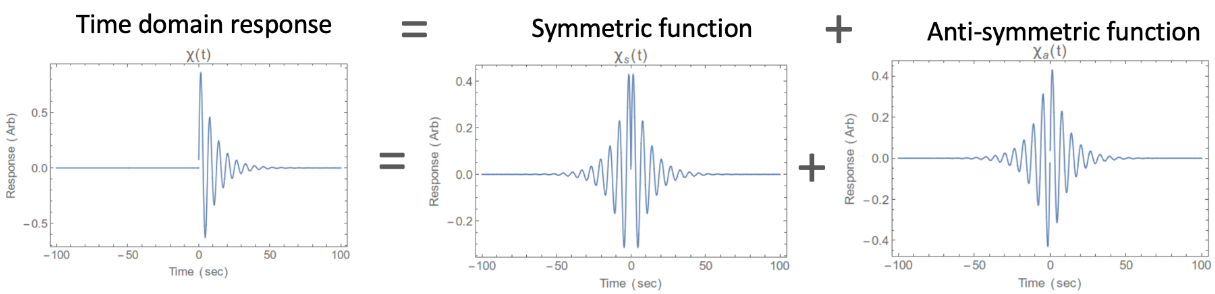
Minimum Phase Signals and Phase Retrieval

A signal is considered minimum phase if it is causal (i.e., it has no output before an input is applied) and stable (i.e., bounded input leads to bounded output)

Phase and Magnitude Relationship: The **phase response is uniquely determined by the magnitude response**. This means that for a given magnitude response, there is only one phase response that will result in a minimum phase system. This property allows the phase response to be inferred from the magnitude response in a process known as **phase retrieval** used for signal reconstruction.

Signal Reconstruction: In communication applications, we like to reconstruct a signal from its magnitude spectrum. If the original signal is known to be minimum phase, this reconstruction is possible.

Impulse Response: The impulse response of a minimum phase system has the property of having the least possible delay. This is particularly useful in **realtime signal recovery** where minimizing delay is important.



- For t<0, the symmetric and anti-symmetric parts will cancel out perfectly
- The symmetric part of corresponds to the real part of spectrum.
- The anti-symmetrized part of corresponds to the imaginary part of spectrum.
- Spectrum is complex with magnitude and phase that are related by the Kramer-Kronig relation

Kramers-Kronig relationship between the real and imaginary components of the spectrum of a function describing a causal process

The Kramers-Kronig relationship **arises from causality**, which states that the response of a system at a given time can only depend on the stimulus at the present or past times, not on future inputs.

Connects Real and Imaginary Parts: The Kramers-Kronig relations express the real part of a complex function as an integral over the imaginary part and vice versa.

Essentially, if you know the real part of a function over all frequencies, you can calculate the imaginary part, and similarly, if you know the imaginary part, you can determine the real part.

Complex Analytic Signals: The relations apply to complex functions that are analytic in the upper half of the complex frequency plane. This typically involves functions representing the frequency response of a system, such as the **complex refractive index or the complex permittivity of a material**.

Applications in Optics: In optics, these relations are used to **connect the absorption and dispersion properties of a medium**. For instance, they can relate the absorption coefficient of a material to its refractive index.

Physical Interpretation: The real part of a response function often represents a dispersive process (altering the phase of a wave), while the imaginary part usually corresponds to an absorptive process (dissipating energy). The Kramers-Kronig relations mathematically link these two physical phenomena.

Limitations: It **assumes linearity and time-invariance**. They may not be applicable in non-linear or time-variant scenarios.

Linearity Assumption: The system's response to a sum of inputs is equal to the sum of the responses to each input individually (superposition principle).

Limitations: The Kramers-Kronig relations specifically describe the relationship between the phase and amplitude of a linear system's response to a sinusoidal input. In nonlinear systems, the phase-amplitude relationship can be much more complex and cannot be captured by these linear relations.

Complexity in Nonlinear Systems: The behavior of nonlinear systems can be much more complex and less predictable compared to linear systems. The interactions of different signal components in a nonlinear medium can lead to effects like modulation, mixing, or creation of new frequency components (harmonics), which are not contained the linear framework of the Kramers-Kronig relations.

2. Differentiation using Diffraction & Coherent Detection (Phase Stretch Transform)

https://en.wikipedia.org/wiki/Phase_stretch_transform

Let's rewrite the signal after diffractive propagation in terms of the input multiplied by a transfer function which contains the phase induced by propagation

$$\tilde{E}_o(k_x, k_y, z) = \tilde{E}(k_x, k_y, 0)\,\tilde{H}(k_x, k_y) \qquad (7)$$

such that diffraction phase kernel

$$\tilde{H}(k_x, k_y) = e^{jk_z z} \qquad (8)$$

where $k_x = 2\pi/\Delta\, x$ and $k_y = 2\pi/\Delta\, y$. According to the **paraxial approximation**, the spatial features $\Delta\, x$ and $\Delta\, y$ are large compared to the wavelength ($\Delta\, x$ and $\Delta\, y >> \lambda/n$) and therefore, k_x and k_y are very small compared to k. Hence,

$$k_z = k - (k_x^2 + k_y^2)/(2 \cdot k) \qquad (9)$$

$$\tilde{H}(k_x, k_y) = e^{jk_z z} = e^{j\,[k-(k_x^2+k_y^2)/(2\cdot k)]\,z} = e^{j\phi_o - j[\phi(k_x^2)+\phi(k_y^2)]}$$

The term $\phi_o = k.z$ is a constant scalar quantity. A constant term in the phase kernel implies a finite delay in transmission and can be safely ignored. The diffractive phase $\phi(k_x, k_y)$ then becomes a quadratic function of spatial frequency k_x and k_y given by the sum of phase terms

$$\tilde{H}(k_x, k_y) = e^{+j\phi(k_x,k_y)} = e^{-j(k_x^2+k_y^2)z/(2\cdot k)} \qquad (10)$$

High pass phase filter

To obtain the field in the spatial domain, we perform and inverse Fourier transform (IFT)

$$E_o(x, y, z) = \text{IFT}\{\,\tilde{E}(k_x, k_y, 0)\,\tilde{H}(k_x, k_y)\,\} \qquad (11)$$

The output field is a complex function. Let's write it as

$$E_o(x, y, z) = |E_o(x, y, z)|\; e^{j\psi(x,y)}$$

where $|E_o(x, y)|$ is the magnitude and $j\psi(x, y)$ is the phase in spatial domain (as opposed to the phase in frequency domain $\phi(k_x, k_y)$.

$$\psi(x, y) = \tan^{-1}\frac{Im\{E_o(x,y)\}}{\text{Re}\{E_o(x,y)\}} \qquad (8)$$

We insert the diffractive phase kernel $\tilde{H}(k_x, k_y) = e^{+j\phi(k_x,k_y)}$ in Eq.(11):

$$E_o(x,y) = \mathrm{IFT}\{\, \tilde{E}(k_x,k_y,0)\ \ e^{j\phi(k_x,k_y)}\,\} \tag{9}$$

which can be expanded as:

$$E_o(x,y) = \mathrm{IFT}\left\{\, \tilde{E}\big(k_x,k_y,0\big)[\cos\,[\phi\big(k_x,k_y\big)] + j sin[\phi\big(k_x,k_y\big)]]\right\} \tag{10}$$

Small Phase Approximation ("Nominal Near Field")

By assuming that the diffractive phase is very small ("nominal near field" condition), we can write the diffraction phase kernel as $\widetilde{H}\big(k_x,k_y\big) = e^{j\phi(k_x,k_y)}$:

$$E_o(x,y) = \mathrm{IFT}\left\{\, \tilde{E}\big(k_x,k_y,0\big)\ \times \big(1 + j\ \phi(k_x,k_y)\big)\right\} \tag{11}$$

which is also equal to:

$$E_o(x,y) = \mathrm{IFT}\left\{\, \tilde{E}\big(k_x,k_y,0\big) \times \Big(1 - j(k_x^2 + k_y^2)\mathrm{z}/(2\cdot k)\Big)\right\} \tag{12}$$

Using the differentiation property of Fourier transform,

$$E_o(x,y,z) = E(x,y,0)\ \ + \frac{\boldsymbol{j.z}}{(\boldsymbol{2\cdot k})}\ \left(\frac{\boldsymbol{d^2E(x,y,0)}}{\boldsymbol{dx^2}} + \frac{\boldsymbol{d^2E(x,y,0)}}{\boldsymbol{dy^2}}\right) \tag{13}$$

Fourier Differentiation property:

$$\mathcal{F}[\frac{d^n}{dt^n}x(t)] = (j\omega)^n X(j\omega)$$

So if we look at the imaginary part of the field in space, it is proportional to the Laplacian of the input field, where the **Laplacian operator** is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Phase Detection (coherent detection)

$$\angle E_o = \arctan\frac{Im(E_o)}{Re(E_o)} = \arctan\left[\frac{z}{(2\cdot k)}\ \frac{\left(\frac{\boldsymbol{d^2E(x,y,0)}}{\boldsymbol{dx^2}} + \frac{\boldsymbol{d^2E(x,y,0)}}{\boldsymbol{dy^2}}\right)}{\boldsymbol{E(x,y,0)}}\right]$$

Phase Stretch Transform (PST)

Therefore, the process of diffractive propagation through a thin medium followed by phase detection is equivalent to:

1. Taking the second derivative of the input, then
2. Dividing by the input, then

3. Squashing it with an arctan function

See PST examples in:
https://en.wikipedia.org/wiki/Phase_stretch_transform