由变分法导出经典力学&场论中的运动方程

> $Collapsar, Aug\ 1, 2024$

经典力学

质点系统的欧拉-拉格朗日方程

对于n个自由度的质点系统,其作用量S的定义为拉格朗日量$L(q_i,\dot{q}_i)$的时间积分

$$S=\int_{t_1}^{t_2}\mathrm{d}tL(q_i,\dot{q}_i), \tag{1}$$

其中,$q_i,\dot{q}_i\equiv\dfrac{\mathrm{d}q_i}{\mathrm{d}t}$分别为系统的广义坐标和广义速度.

最小作用量原理指出,作用量的变分极值($\delta S=0$)} 对应于系统的经典运动轨迹. 以下假设不作时间坐标的变换,即时间的变分$\delta t=0$.

由于变分运算δ与微分d或者微商运算可以交换次序,所以有

$$\delta\dot{q}_i=\delta\frac{\mathrm{d}q_i}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\delta q_i, \tag{2}$$

表明时间导数的变分等于变分的时间导数.

对式(1)左右两边取变分,考虑到变分运算δ和积分运算$\int$也可以交换次序,所以有

$$\delta S=\delta\int_{t_1}^{t_2}\mathrm{d}tL(q_i,\dot{q}_i)=\int_{t_1}^{t_2}\mathrm{d}t\delta L(q_i,\dot{q}_i), \tag{3}$$

复合函数$L(q_i,\dot{q}_i)$的变分运算法则与微分运算法则完全相同,只需要将微分运算的d换成δ,即

$$\delta L(q_i,\dot{q}_i)=\frac{\partial L}{\partial q_i}\delta q_i+\frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \xlongequal{\delta\dot{q}_i=\frac{\mathrm{d}}{\mathrm{d}t}\delta q_i} \frac{\partial L}{\partial q_i}\delta q_i+\frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}}{\mathrm{d}t}\delta q_i, \tag{4}$$

后一个等号用到了式(2). 利用分部积分,式(4)最右边一项可以改写为

$$\frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}}{\mathrm{d}t}\delta q_i=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)-\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i, \tag{5}$$

代入到(4)就有

$$\begin{aligned}\delta L(q_i,\dot{q}_i)&=\frac{\partial L}{\partial q_i}\delta q_i+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)-\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i\\&=\left(\frac{\partial L}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right).\end{aligned} \tag{6}$$

于是式(3)为

$$\begin{aligned}\delta S&=\int_{t_1}^{t_2}\mathrm{d}t\delta L(q_i,\dot{q}_i)\\&=\int_{t_1}^{t_2}\mathrm{d}t\left[\left(\frac{\partial L}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)\right]\\&=\int_{t_1}^{t_2}\left[\mathrm{d}t\left(\frac{\partial L}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i+\mathrm{d}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)\right]\\&=\int_{t_1}^{t_2}\mathrm{d}t\left(\frac{\partial L}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i+\int_{t_1}^{t_2}\mathrm{d}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)\\&=\int_{t_1}^{t_2}\mathrm{d}t\left(\frac{\partial L}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i+\frac{\partial L}{\partial \dot{q}_i}\delta q_i\bigg|_{t_1}^{t_2},\end{aligned} \tag{7}$$

假设初始和结束时刻广义坐标的变分为零,即$\delta q_i(t_1)=\delta q_i(t_2)=0$,于是上式最后一行第二项为零.又因为变分$\delta q_i(t)$在$t_1<t<t_2$时任意,所以$\delta S=0$将会导致

$$\boxed{\frac{\partial L}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}=0, i=1,\cdots,n \qquad (8)}$$

这就是描述质点系统经典运动的欧拉-拉格朗日方程.

质点系统的哈密顿正则方程

引入广义动量

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}, i = 1, \cdots, n. \tag{9}$$

求解上述方程,将广义速度表示为q_i, p_i的函数$\dot{q}_i(q_i, p_i)$,通过$Legendre$变换定义哈密顿量

$$H(q_i, p_i) \equiv p_i \dot{q}_i - L, \tag{10}$$

用H取代L来表示作用量S,

$$S = \int_{t_1}^{t_2} \mathrm{d}t L(q_i, \dot{q}_i) = \int_{t_1}^{t_2} \mathrm{d}t \left[p_i \dot{q}_i - H(q_i, p_i)\right], \tag{11}$$

对上式左右两边取变分,注意到

$$\begin{aligned}\delta\left(p_i \dot{q}_i\right) &= \delta p_i \dot{q}_i + p_i \delta \dot{q}_i \xlongequal{\delta \dot{q}_i = \frac{\mathrm{d}}{\mathrm{d}t}\delta q_i} \delta p_i \dot{q}_i + p_i \frac{\mathrm{d}}{\mathrm{d}t} \delta q_i \\ &= \delta p_i \dot{q}_i + \frac{\mathrm{d}}{\mathrm{d}t}(p_i \delta q_i) - \frac{\mathrm{d}p_i}{\mathrm{d}t} \delta q_i,\end{aligned} \tag{12}$$

以及

$$\delta H(q_i, p_i) = \frac{\partial H}{\partial q_i} \delta q_i + \frac{\partial H}{\partial p_i} \delta p_i, \tag{13}$$

于是就有

$$\begin{aligned}\delta S &= \delta \int_{t_1}^{t_2} \mathrm{d}t \left[p_i \dot{q}_i - H(q_i, p_i)\right] = \int_{t_1}^{t_2} \mathrm{d}t \delta \left[p_i \dot{q}_i - H(q_i, p_i)\right] \\ &= \int_{t_1}^{t_2} \mathrm{d}t \left[\delta(p_i \dot{q}_i) - \delta H(q_i, p_i)\right] \\ &= \int_{t_1}^{t_2} \mathrm{d}t \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right) \delta p_i + \frac{\mathrm{d}}{\mathrm{d}t}(p_i \delta q_i) - \left(\dot{p}_i + \frac{\partial H}{\partial q_i}\right) \delta q_i\right] \\ &= \int_{t_1}^{t_2} \mathrm{d}t \left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right) \delta p_i + \int_{t_1}^{t_2} \mathrm{d}(p_i \delta q_i) - \int_{t_1}^{t_2} \mathrm{d}t \left(\dot{p}_i + \frac{\partial H}{\partial q_i}\right) \delta q_i \\ &= \int_{t_1}^{t_2} \mathrm{d}t \left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right) \delta p_i + (p_i \delta q_i)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathrm{d}t \left(\dot{p}_i + \frac{\partial H}{\partial q_i}\right) \delta q_i,\end{aligned} \tag{14}$$

上式最后一行第二项同样由于初始和结束时刻变分为零而消失,而且$\delta p_i, \delta q_i$任意,于是作用量的变分为零

将会导致

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i}, \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i}. \end{cases} \quad i = 1, \cdots, n. \tag{15}$$

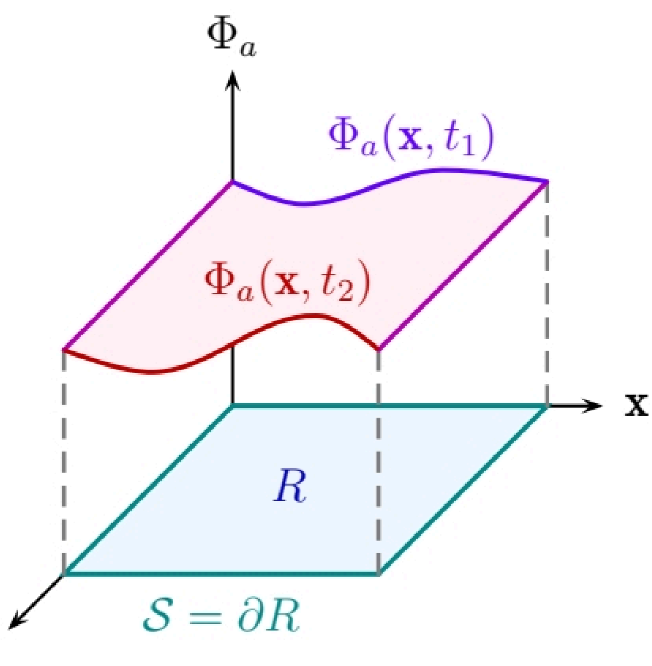
这就是哈密顿正则运动方程,相当于用$2n$个一阶方程代替原来的n个二阶欧拉-拉格朗日方程.广义坐标q_i与广义动量p_i统称为正则变量.

场论

场的欧拉-拉格朗日方程

经典场论中,场$\phi(\boldsymbol{x}, t)$是系统的广义坐标.局域场论中,拉格朗日量表示为$L(t) = \int \mathrm{d}^3 x \mathcal{L}(x)$,其中,$\mathcal{L}(x)$为拉格朗日量密度,简称拉氏量,并且假设它是系统中$n$个场$\phi_a(\boldsymbol{x}, t), a = 1, \cdots, n$及其时空导数$\partial_\mu \phi_a$的函数,即$\mathcal{L} = \mathcal{L}[\phi_a(\boldsymbol{x}, t), \partial_\mu \phi_a]$.如此,作用量可以表达为

$$S = \int \mathrm{d}t L = \int \mathrm{d}^4 x \mathcal{L}(\phi_a, \partial_\mu \phi_a), \tag{16}$$



时空区域R上的场$\phi_a(\boldsymbol{x}, t)$

上图描绘了时空区域R上的场$\phi_a(\boldsymbol{x},t)$,其中$\boldsymbol{x}$表示三维空间坐标.$S=\partial R$为R的边界面.假设不作时空坐标的变换,即时空坐标的变分$\delta x^\mu=0$,那么对场的时空导数的变分等于场的变分的时空导数,

$$\delta(\partial_\mu\phi_a)=\partial_\mu(\delta\phi_a). \tag{17}$$

于是拉式量的变分为

$$\begin{aligned}\delta\mathcal{L}&=\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a)\\&\xlongequal{\delta(\partial_\mu\phi_a)=\partial_\mu(\delta\phi_a)}\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu(\delta\phi_a),\end{aligned} \tag{18}$$

利用分部积分,将上式第二行第二项改写为

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu(\delta\phi_a)=\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right)-\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]\delta\phi_a, \tag{19}$$

于是作用量S的变分为

$$\begin{aligned}\delta S&=\int\mathrm{d}^4x\delta\mathcal{L}(\phi_a,\partial_\mu\phi_a)\\&=\int\mathrm{d}^4x\left\{\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a+\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right)-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\delta\phi_a\right\}\\&=\int\mathrm{d}^4x\left\{\left[\frac{\partial\mathcal{L}}{\partial\phi_a}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right]\delta\phi_a+\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right)\right\}\\&=\int\mathrm{d}^4x\left[\frac{\partial\mathcal{L}}{\partial\phi_a}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right]\delta\phi_a+\int\mathrm{d}^4x\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right),\end{aligned} \tag{20}$$

上式最后一行第二项是关于时空坐标的散度,利用$Stokes$公式将其改写为对于积分区域边界面S的积分

$$\int\mathrm{d}^4x\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right)=\int_S\mathrm{d}s\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a, \tag{21}$$

其中$\mathrm{d}s$是S上的面元,并且假设边界面S上面$\delta\phi=0$,则上式为零. 通常讨论时空区域上的场,相当于假设无穷远时空边界上$\delta\phi=0$.如此,$\delta S=0$将会导致

$$\boxed{\frac{\partial\mathcal{L}}{\partial\phi_a}-\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]=0,a=1,\cdots,n,} \tag{22}$$

这就是场的欧拉-拉格朗日方程.

场的哈密顿正则方程

场的共轭动量密度或者正则共轭场定义为

$$\pi_a(\boldsymbol{x},t) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}. \tag{23}$$

通过$Legendre$变换定义哈密顿量为

$$H \equiv \int \mathrm{d}^3 x \mathcal{H} = \int \mathrm{d}^3 x \pi_a \dot{\phi}_a - L, \tag{24}$$

其中$\mathcal{H}(\phi_a, \pi_a, \nabla\phi_a) = \pi_a \dot{\phi}_a - \mathcal{L}$为哈密顿量密度. 于是拉氏量的变分为

$$\delta \mathcal{L} = \delta(\pi_a \dot{\phi}_a - \mathcal{H}) = \delta\pi_a \dot{\phi}_a + \pi_a \delta\dot{\phi}_a - \delta\mathcal{H}. \tag{25}$$

利用分部积分改写上式中的$\pi_a \delta\dot{\phi}_a$

$$\pi_a \delta\dot{\phi}_a = \pi_a \delta \frac{\mathrm{d}\phi_a}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\pi_a \delta\phi_a) - \frac{\mathrm{d}\pi_a}{\mathrm{d}t}\delta\phi_a. \tag{26}$$

而$\mathcal{H} = \mathcal{H}(\phi_a, \pi_a, \nabla\phi_a)$的变分为

$$\delta\mathcal{H} = \frac{\partial \mathcal{H}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{H}}{\partial \pi_a}\delta\pi_a + \frac{\partial \mathcal{H}}{\partial(\nabla\phi_a)} \cdot \delta(\nabla\phi_a), \tag{27}$$

考虑到$\delta(\nabla\phi_a) = \nabla(\delta\phi_a)$以及矢量分析公式

$$\nabla \cdot (f\boldsymbol{g}) = \nabla f \cdot \boldsymbol{g} + f \nabla \cdot \boldsymbol{g}, \tag{28}$$

其中f为标量函数,而$\boldsymbol{g}$为矢量函数.式(27)最后一项改写为

$$\frac{\partial \mathcal{H}}{\partial(\nabla\phi_a)} \cdot \nabla(\delta\phi_a) = \nabla \cdot \left[\frac{\partial \mathcal{H}}{\partial(\nabla\phi_a)}\delta\phi_a\right] - \delta\phi_a \left[\nabla \cdot \frac{\partial \mathcal{H}}{\partial(\nabla\phi_a)}\right], \tag{29}$$

于是作用量的变分可以表示为

$$
\begin{aligned}
\delta S &= \int \mathrm{d}^4 x \delta \mathcal{L} \\
&\xlongequal{(25)} \int \mathrm{d}^4 x (\delta\pi_a \dot{\phi}_a + \pi_a \delta\dot{\phi}_a - \delta\mathcal{H}) \\
&\xlongequal{(26),(27)} \int \mathrm{d}^4 x \left[\delta\pi_a \dot{\phi}_a + \frac{\mathrm{d}}{\mathrm{d}t}(\pi_a \delta\phi_a) - \frac{\mathrm{d}\pi_a}{\mathrm{d}t}\delta\phi_a\right] \\
&\quad - \int \mathrm{d}^4 x \left[\frac{\partial\mathcal{H}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{H}}{\partial\pi_a}\delta\pi_a + \frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\cdot\delta(\nabla\phi_a)\right] \\
&\xlongequal{(29)} \int \mathrm{d}^4 x \left[\delta\pi_a \dot{\phi}_a + \frac{\mathrm{d}}{\mathrm{d}t}(\pi_a \delta\phi_a) - \frac{\mathrm{d}\pi_a}{\mathrm{d}t}\delta\phi_a\right] \\
&\quad - \int \mathrm{d}^4 x \left\{\frac{\partial\mathcal{H}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{H}}{\partial\pi_a}\delta\pi_a + \nabla\cdot\left[\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\delta\phi_a\right] - \delta\phi_a\left[\nabla\cdot\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\right]\right\} \\
&= \int \mathrm{d}^4 x \left(\dot{\phi}_a - \frac{\partial\mathcal{H}}{\partial\pi_a}\right)\delta\pi_a - \int \mathrm{d}^4 x \left(\frac{\partial\mathcal{H}}{\partial\phi_a} + \dot{\pi}_a - \nabla\cdot\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\right)\delta\phi_a \\
&\quad + \int \mathrm{d}^4 x \frac{\mathrm{d}}{\mathrm{d}t}(\pi_a \delta\phi_a) - \int \mathrm{d}^4 x \nabla\cdot\left[\frac{\partial\mathcal{H}}{\partial(\nabla\phi_a)}\delta\phi_a\right],
\end{aligned}
\tag{30}
$$

上式最后一行的两项由于$\delta\phi_a = 0$而消失.所以作用量变分为零必然导致

$$
\boxed{
\begin{cases}
\dot{\phi}_a = \dfrac{\partial H}{\partial \pi_a}, \\
\dot{\pi}_a = \nabla\cdot\dfrac{\partial\mathcal{H}}{\partial(\nabla\phi_a)} - \dfrac{\partial\mathcal{H}}{\partial\phi_a}.
\end{cases}
}
\tag{31}
$$

这就是场的哈密顿正则运动方程.场ϕ_a和它的共轭动量密度π_a构成系统的正则变量.