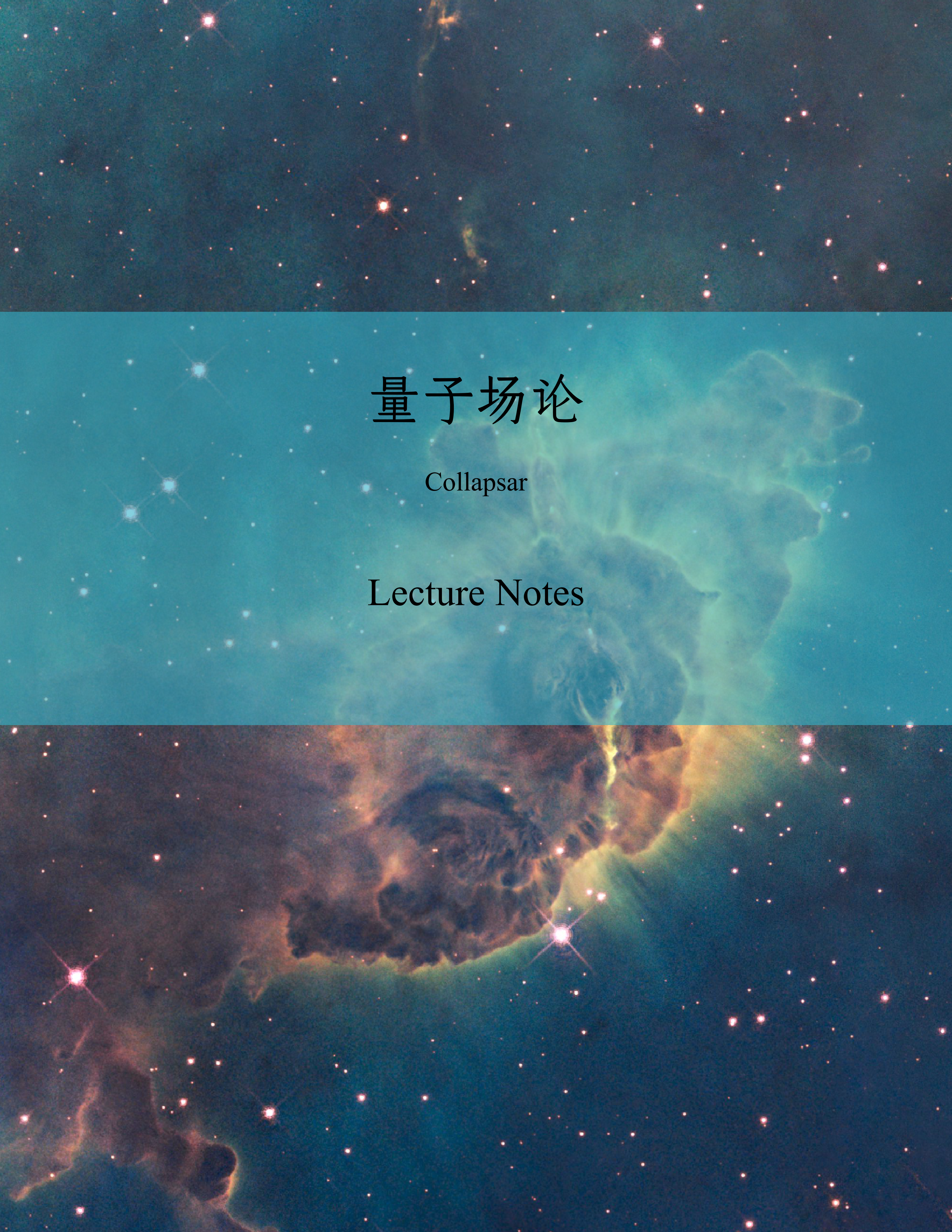

量子场论

Collapsar

Lecture Notes

SUMMER RESEARCH INTERNSHIP, UNIVERSITY OF WESTERN ONTARIO

GITHUB.COM/LAURETHTEX/CLUSTERING

This research was done under the supervision of Dr. Pauline Barmby with the financial support of the MITACS Globalink Research Internship Award within a total of 12 weeks, from June 16th to September 5th of 2014.

First release, August 2014

Contents

1. 预备知识

1.1 基本相互作用力与基本粒子

自然界存在的 4 种基本相互作用为

- 引力相互作用, gravitational interaction
- 电磁相互作用, electromagnetic interaction
- 强相互作用, strong interaction
- 弱相互作用, weak interaction

基本粒子是指尚未发现内部结构的粒子. 组成物质的基本单元是粒子 (particle).3 代基本费米子 (fermion) 由带电轻子 (lepton) + 中微子 (neutrino, 中性轻子) + 下型夸克 + 上型夸克构成, 具体为

- 1st: 电子 (e) + 电子型中微子 (ν_e) + 下夸克 (d) + 上夸克 (u)
- 2nd: μ 子 (μ) + μ 子型中微子 (ν_μ) + 奇夸克 (s) + 粲夸克 (c)
- 3rd: τ 子 (τ) + τ 子型中微子 (ν_τ) + 底夸克 (b) + 顶夸克 (t)

某代某种费米子与它在另一代中相应的费米子具有相同量子数, 但是质量不同.

夸克的种类称为味道 (flavor), 6 种味道的夸克具有不同的质量, 每一味夸克都具有 3 种颜色, 同味异色的夸克具有相同质量, 严格构成颜色三重态, 与描述强相互作用的量子色动力学有关. 多个夸克通过强相互作用构成强子 (hadron), 如

- 介子 (meson) = 正夸克 + 反夸克
- 重子 (baryon) = 三个正夸克/三个反夸克

除了三代中微子以外的基本费米子都有电荷, 参与电磁相互作用, 相应的理论称为量子电动力学. 所有的基本费米子都参与弱相互作用, 与电磁相互作用统一由电弱规范理论描述, 电弱规范理论和量子色动力学构成标准模型 (standard model). 标准模型中费米子的相互作用

由一些基本玻色子传递

- 胶子 (gluon): 传递夸克间强相互作用的规范玻色子
- 光子 (photon): 传递电磁相互作用的规范玻色子
- $W^{\pm}, Z^0$ 玻色子: 传递弱相互作用的规范玻色子
- *Higgs* 玻色子: 与电弱规范对称性的自发破缺以及基本粒子的质量起源有关

1.2 *Klein – Gordon* 方程

Klein-Gordon 方程用以描述单粒子的相对论性运动，它是第一个相对论性的波函数方程，形式为

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \psi(\boldsymbol{x}, t) = (-\hbar^2 c^2 \nabla^2 + m^2 c^4) \psi(\boldsymbol{x}, t). \tag{1.1}$$

Klein-Gordon 方程存在如下问题:

- 负能量困难.Klein-Gordon 方程给出的自由粒子能量为

  $$E = \pm \sqrt{|\boldsymbol{p}|^2 c^2 + m^2 c^4}, \tag{1.2}$$

  其中 $\boldsymbol{p}$ 为粒子动量，m 为粒子静止质量. 能量可以为正也可以为负.
- 负概率困难. 通过 Klein-Gordon 方程构造的符合概率守恒的连续性方程要求粒子在空间中的概率密度为

  $$\rho = \frac{i\hbar}{2mc^2} \left(\psi^* \frac{\partial \psi}{\partial t} - \frac{\partial \psi^*}{\partial t} \psi \right), \tag{1.3}$$

  这样定义的概率密度并不总是正的. 负概率问题的根源在于方程中含有波函数对时间的二阶导数.

1.3 *Dirac* 方程与量子场论

Dirac 方程克服了负概率问题，只包括对时间的一阶导数，且具有 *Lorentz* 协变性，用以描述自旋 1/2 的粒子，一开始是用来描述电子的.

Dirac 方程能够保证概率密度正定和概率守恒，但负能量问题依旧存在. 为此，Dirac 提出：真空 (vacuum) 是所有 $E<0$ 的态都被填满而所有 $E>0$ 的态都为空的状态. 如此，泡利不相容原理会阻止正能量的电子跃迁到负能量的态，因而激发态电子能量总是正的. 如果负能海中缺少一个带有电荷 $-e$ 和负能量的电子，即产生一个空穴 (hole)，则空穴的行为等价于一个带有电荷 e 和相应正能量的“反粒子”(antiparticle)，称为正电子 (positron).

Dirac 方程存在如下问题:

- 并未观测到无穷多个负能电子具有的无穷大电荷密度所引起的电场.
- Dirac 方程一开始作为描述单粒子波函数方程提出来,但 Dirac 的解释包含了无穷多个粒子.
- 整数自旋的玻色子不满足泡利不相容原理,空穴理论无法解释它们的负能量问题.
- Dirac 方程不能解决整数自旋粒子的负概率困难.

使用相对论性波函数方程描述单粒子遇到这么多困难,是因为

- 量子力学:时间与空间不平权. 时间 t 作为一个观测量并没有使用厄米算符描写,而空间 $\boldsymbol{x}$ 则使用位置算符 $\hat{x}$ 描写.
- 狭义相对论:*Lorentz* 协变性将时间和空间完全对等起来.

为此,存在以下两种解决方法:

- 将时间提升为一个厄米算符. 实际操作非常困难.
- 将空间位置降格为一个参数,不再由厄米算符描述. 具体来说是在每个空间点 $\boldsymbol{x}$ 处定义一个算符 $\hat{\phi}(\boldsymbol{x})$,所有这些算符的集合称为量子场. 如此,量子化的对象变成是由依赖于时空坐标的场组成的动力学系统,称为量子场论.

量子场论平等的描述正反粒子,由正反粒子产生和湮灭算符表达的哈密顿量是正定的,不再出现负能量困难. 不再将 ρ 解释为单粒子概率密度,而是解释为单位体积内正反粒子数目之差,不存在负概率困难.

1.4 自然单位制

自然单位制取 $\hbar = c = 1$, 而且

$$1\text{GeV}^{-1} = 6.582 \times 10^{-25}\text{s} = 1.973 \times 10^{-14}\text{cm}, \tag{1.4}$$

自然单位制中,速度没有量纲;长度与时间的量纲相同,是能量量纲的倒数;能量、质量和动量具有相同的量纲. 在量子场论中,通常再取真空介电常数 $\varepsilon_0 = 1$, 这样精细结构常数 (fine-structure constant) α 为

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{e^2}{4\pi} \approx \frac{1}{137}, \tag{1.5}$$

同时可得真空磁导率 $\mu_0 = (\varepsilon_0 c^2)^{-1} = 1$,这样的单位制被称为自然单位制 (natural unit system).

1.5 作用量原理

1.5.1 质点系统的欧拉-拉格朗日方程

Theorem 1.5.1 — 质点系统的欧拉-拉格朗日方程.

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = 0, i = 1, \ldots, n$$

Proof. 对于 n 个自由度的质点系统, 其作用量 S 的定义为拉格朗日量 $L(q_i, \dot{q}_i)$ 的时间积分

$$S = \int_{t_1}^{t_2} \mathrm{d}t L(q_i, \dot{q}_i), \tag{1.6}$$

其中,$q_i, \dot{q}_i \equiv \frac{\mathrm{d}q_i}{\mathrm{d}t}$ 分别为系统的广义坐标和广义速度.

最小作用量原理指出, 作用量的变分极值 ($\delta S = 0$) 对应于系统的经典运动轨迹. 以下假设不作时间坐标的变换, 即时间的变分 $\delta t = 0$.

由于变分运算 δ 与微分 d 或者微商运算可以交换次序, 所以有

$$\delta \dot{q}_i = \delta \frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\delta q_i, \tag{1.7}$$

表明时间导数的变分等于变分的时间导数.

对式1.6左右两边取变分, 考虑到变分运算 δ 和积分运算 $\int$ 也可以交换次序, 所以有

$$\delta S = \delta \int_{t_1}^{t_2} \mathrm{d}t L(q_i, \dot{q}_i) = \int_{t_1}^{t_2} \mathrm{d}t\, \delta L(q_i, \dot{q}_i), \tag{1.8}$$

复合函数 $L(q_i, \dot{q}_i)$ 的变分运算法则与微分运算法则完全相同, 只需要将微分运算的 d 换成 δ, 即

$$\delta L(q_i, \dot{q}_i) = \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \overset{\delta \dot{q}_i = \frac{\mathrm{d}}{\mathrm{d}t}\delta q_i}{=\!=\!=\!=\!=} \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}}{\mathrm{d}t}\delta q_i, \tag{1.9}$$

后一个等号用到了式1.7. 利用分部积分, 式1.9最右边一项可以改写为

$$\frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}}{\mathrm{d}t}\delta q_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) - \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i, \tag{1.10}$$

代入到1.9就有

$$
\begin{aligned}
\delta L(q_i,\dot{q}_i) &= \frac{\partial L}{\partial q_i}\delta q_i + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) - \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i \\
&= \left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right).
\end{aligned}
\tag{1.11}
$$

于是式1.8为

$$
\begin{aligned}
\delta S &= \int_{t_1}^{t_2} \mathrm{d}t\,\delta L(q_i,\dot{q}_i) \\
&= \int_{t_1}^{t_2} \mathrm{d}t\left[\left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)\right] \\
&= \int_{t_1}^{t_2} \left[\mathrm{d}t\left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \mathrm{d}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)\right] \\
&= \int_{t_1}^{t_2} \mathrm{d}t\left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \int_{t_1}^{t_2} \mathrm{d}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) \\
&= \int_{t_1}^{t_2} \mathrm{d}t\left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \left.\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right|_{t_1}^{t_2},
\end{aligned}
\tag{1.12}
$$

假设初始和结束时刻广义坐标的变分为零, 即 $\delta q_i(t_1) = \delta q_i(t_2) = 0$, 于是上式最后一行第二项为零. 又因为变分 $\delta q_i(t)$ 在 $t_1 < t < t_2$ 时任意, 所以 $\delta S = 0$ 将会导致式1.5.1.

. . . . ■

1.5.2 质点系统的哈密顿方程

Theorem 1.5.2 — 质点系统的哈密顿方程.

$$
\begin{cases}
\dot{q}_i = \dfrac{\partial H}{\partial p_i}, \\
\dot{p}_i = -\dfrac{\partial H}{\partial q_i}.
\end{cases}
\quad i = 1,\ldots,n.
$$

质点系统的哈密顿方程相当于用 $2n$ 个一阶方程代替原来的 n 个二阶欧拉-拉格朗日方程. 广义坐标 q_i 与广义动量 p_i 统称为正则变量.

. . . .

Proof. 引入广义动量

$$
p_i \equiv \frac{\partial L}{\partial \dot{q}_i}, i = 1,\cdots,n.
\tag{1.13}
$$

求解上述方程, 将广义速度表示为 q_i, p_i 的函数 $\dot{q}_i(q_i, p_i)$, 通过 *Legendre* 变换定义哈密顿量

$$H(q_i, p_i) \equiv p_i\dot{q}_i - L, \tag{1.14}$$

用 H 取代 L 来表示作用量 S,

$$S = \int_{t_1}^{t_2} \mathrm{d}t L(q_i, \dot{q}_i) = \int_{t_1}^{t_2} \mathrm{d}t \left[p_i\dot{q}_i - H(q_i, p_i)\right], \tag{1.15}$$

对上式左右两边取变分, 注意到

$$\begin{aligned} \delta\left(p_i\dot{q}_i\right) = \delta p_i\dot{q}_i + p_i\delta\dot{q}_i \xlongequal{\delta\dot{q}_i = \frac{\mathrm{d}}{\mathrm{d}t}\delta q_i} \delta p_i\dot{q}_i + p_i\frac{\mathrm{d}}{\mathrm{d}t}\delta q_i \\ = \delta p_i\dot{q}_i + \frac{\mathrm{d}}{\mathrm{d}t}\left(p_i\delta q_i\right) - \frac{\mathrm{d}p_i}{\mathrm{d}t}\delta q_i, \end{aligned} \tag{1.16}$$

以及

$$\delta H(q_i, p_i) = \frac{\partial H}{\partial q_i}\delta q_i + \frac{\partial H}{\partial p_i}\delta p_i, \tag{1.17}$$

于是就有

$$\begin{aligned} \delta S &= \delta\int_{t_1}^{t_2} \mathrm{d}t \left[p_i\dot{q}_i - H(q_i, p_i)\right] = \int_{t_1}^{t_2} \mathrm{d}t\, \delta\left[p_i\dot{q}_i - H(q_i, p_i)\right] \\ &= \int_{t_1}^{t_2} \mathrm{d}t \left[\delta(p_i\dot{q}_i) - \delta H(q_i, p_i)\right] \\ &= \int_{t_1}^{t_2} \mathrm{d}t \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right)\delta p_i + \frac{\mathrm{d}}{\mathrm{d}t}(p_i\delta q_i) - \left(\dot{p}_i + \frac{\partial H}{\partial q_i}\right)\delta q_i\right] \\ &= \int_{t_1}^{t_2} \mathrm{d}t \left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right)\delta p_i + \int_{t_1}^{t_2} \mathrm{d}(p_i\delta q_i) - \int_{t_1}^{t_2} \mathrm{d}t \left(\dot{p}_i + \frac{\partial H}{\partial q_i}\right)\delta q_i \\ &= \int_{t_1}^{t_2} \mathrm{d}t \left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right)\delta p_i + (p_i\delta q_i)\big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathrm{d}t \left(\dot{p}_i + \frac{\partial H}{\partial q_i}\right)\delta q_i, \end{aligned} \tag{1.18}$$

上式最后一行第二项同样由于初始和结束时刻变分为零而消失, 而且 $\delta p_i, \delta q_i$ 任意, 于是作用量的变分为零将会导致式1.5.2.

.... ■

1.5.3 场论中的欧拉-拉格朗日方程

Theorem 1.5.3 — 场论中的欧拉-拉格朗日方程.

$$\frac{\partial \mathscr{L}}{\partial \phi_a} - \partial_\mu \left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)}\right] = 0, a = 1, \cdots, n,$$

Proof. 经典场论中, 场 $\phi(\boldsymbol{x}, t)$ 是系统的广义坐标. 局域场论中, 拉格朗日量表示为

$$L(t) = \int \mathrm{d}^3 x \mathscr{L}(x), \tag{1.19}$$

其中 $\mathscr{L}(x)$ 为拉格朗日量密度, 简称拉氏量, 并且假设它是系统中 n 个场 $\phi_a(\boldsymbol{x}, t), a = 1, \cdots, n$ 及其时空导数 $\partial_\mu \phi_a$ 的函数, 即 $\mathscr{L} = \mathscr{L}[\phi_a(\boldsymbol{x}, t), \partial_\mu \phi_a]$. 如此, 作用量可以表达为

$$S = \int \mathrm{d}t L = \int \mathrm{d}^4 x \mathscr{L}(\phi_a, \partial_\mu \phi_a), \tag{1.20}$$

上图描绘了时空区域 R 上的场 $\phi_a(\boldsymbol{x}, t)$, 其中 $\boldsymbol{x}$ 表示三维空间坐标.$S = \partial R$ 为 R 的边界面. 假设不作时空坐标的变换, 即时空坐标的变分 $\delta x^\mu = 0$, 那么对场的时空导数的变分等于场的变分的时空导数,

$$\delta(\partial_\mu \phi_a) = \partial_\mu(\delta \phi_a). \tag{1.21}$$

于是拉式量的变分为

$$\delta \mathscr{L} = \frac{\partial \mathscr{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \xlongequal{\delta(\partial_\mu \phi_a) = \partial_\mu(\delta \phi_a)} \frac{\partial \mathscr{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \partial_\mu(\delta \phi_a), \tag{1.22}$$

利用分部积分, 将上式第二行第二项改写为

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \partial_\mu(\delta \phi_a) = \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a\right) - \partial_\mu \left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)}\right] \delta \phi_a, \tag{1.23}$$

于是作用量 S 的变分为

$$\begin{aligned}
\delta S &= \int \mathrm{d}^4 x \delta \mathscr{L}(\phi_a, \partial_\mu \phi_a) \\
&= \int \mathrm{d}^4 x \left\{\frac{\partial \mathscr{L}}{\partial \phi_a} \delta \phi_a + \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a\right) - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)}\right) \delta \phi_a\right\} \\
&= \int \mathrm{d}^4 x \left\{\left[\frac{\partial \mathscr{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)}\right)\right] \delta \phi_a + \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a\right)\right\} \\
&= \int \mathrm{d}^4 x \left[\frac{\partial \mathscr{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)}\right)\right] \delta \phi_a + \int \mathrm{d}^4 x \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a\right),
\end{aligned} \tag{1.24}$$

上式最后一行第二项是关于时空坐标的散度, 利用 *Stokes* 公式将其改写为对于积分区域边界面 S 的积分

$$\int \mathrm{d}^4 x \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a \right) = \int_S \mathrm{d}s \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a, \tag{1.25}$$

其中 $\mathrm{d}s$ 是 S 上的面元, 并且假设边界面 S 上面 $\delta\phi = 0$, 则上式为零. 通常讨论时空区域上的场, 相当于假设无穷远时空边界上 $\delta\phi = 0$. 如此,$\delta S = 0$ 将会导致式1.5.3.

.... ■

1.5.4 场论中的哈密顿方程

Theorem 1.5.4 — 场论中的哈密顿方程.

$$\begin{cases} \dot{\phi}_a = \dfrac{\partial H}{\partial \pi_a}, \\ \dot{\pi}_a = \nabla \cdot \dfrac{\partial \mathscr{H}}{\partial(\nabla \phi_a)} - \dfrac{\partial \mathscr{H}}{\partial \phi_a}. \end{cases}$$

....

Proof. 场的共轭动量密度或者正则共轭场定义为

$$\pi_a(\boldsymbol{x}, t) \equiv \frac{\partial \mathscr{L}}{\partial \dot{\phi}_a}. \tag{1.26}$$

通过 *Legendre* 变换定义哈密顿量为

$$H \equiv \int \mathrm{d}^3 x \mathscr{H} = \int \mathrm{d}^3 x \pi_a \dot{\phi}_a - L, \tag{1.27}$$

其中 $\mathscr{H}(\phi_a, \pi_a, \nabla\phi_a) = \pi_a \dot{\phi}_a - \mathscr{L}$ 为哈密顿量密度. 于是拉氏量的变分为

$$\delta \mathscr{L} = \delta(\pi_a \dot{\phi}_a - \mathscr{H}) = \delta \pi_a \dot{\phi}_a + \pi_a \delta \dot{\phi}_a - \delta \mathscr{H}. \tag{1.28}$$

利用分部积分改写上式中的 $\pi_a \delta \dot{\phi}_a$

$$\pi_a \delta \dot{\phi}_a = \pi_a \delta \frac{\mathrm{d}\phi_a}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\pi_a \delta \phi_a) - \frac{\mathrm{d}\pi_a}{\mathrm{d}t} \delta \phi_a. \tag{1.29}$$

而 $\mathscr{H} = \mathscr{H}(\phi_a, \pi_a, \nabla\phi_a)$ 的变分为

$$\delta \mathscr{H} = \frac{\partial \mathscr{H}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathscr{H}}{\partial \pi_a} \delta \pi_a + \frac{\partial \mathscr{H}}{\partial(\nabla \phi_a)} \cdot \delta(\nabla \phi_a), \tag{1.30}$$

考虑到 $\delta(\nabla\phi_a)=\nabla(\delta\phi_a)$ 以及矢量分析公式

$$\nabla\cdot(f\boldsymbol{g})=\nabla f\cdot\boldsymbol{g}+f\nabla\cdot\boldsymbol{g}, \tag{1.31}$$

其中 f 为标量函数, 而 $\boldsymbol{g}$ 为矢量函数. 式1.30最后一项改写为

$$\frac{\partial\mathscr{H}}{\partial(\nabla\phi_a)}\cdot\nabla(\delta\phi_a)=\nabla\cdot\left[\frac{\partial\mathscr{H}}{\partial(\nabla\phi_a)}\delta\phi_a\right]-\delta\phi_a\left[\nabla\cdot\frac{\partial\mathscr{H}}{\partial(\nabla\phi_a)}\right], \tag{1.32}$$

于是作用量的变分可以表示为

$$\begin{aligned}
\delta S=&\int \mathrm{d}^4x\delta\mathscr{L}\overset{1.28}{=\!=}\int \mathrm{d}^4x(\delta\pi_a\dot{\phi}_a+\pi_a\delta\dot{\phi}_a-\delta\mathscr{H})\\
\overset{1.29,1.30}{=\!=\!=}&\int \mathrm{d}^4x\left[\delta\pi_a\dot{\phi}_a+\frac{\mathrm{d}}{\mathrm{d}t}(\pi_a\delta\phi_a)-\frac{\mathrm{d}\pi_a}{\mathrm{d}t}\delta\phi_a\right]\\
&-\int \mathrm{d}^4x\left[\frac{\partial\mathscr{H}}{\partial\phi_a}\delta\phi_a+\frac{\partial\mathscr{H}}{\partial\pi_a}\delta\pi_a+\frac{\partial\mathscr{H}}{\partial(\nabla\phi_a)}\cdot\delta(\nabla\phi_a)\right]\\
\overset{1.32}{=\!=}&\int \mathrm{d}^4x\left[\delta\pi_a\dot{\phi}_a+\frac{\mathrm{d}}{\mathrm{d}t}(\pi_a\delta\phi_a)-\frac{\mathrm{d}\pi_a}{\mathrm{d}t}\delta\phi_a\right]\\
&-\int \mathrm{d}^4x\left\{\frac{\partial\mathscr{H}}{\partial\phi_a}\delta\phi_a+\frac{\partial\mathscr{H}}{\partial\pi_a}\delta\pi_a+\nabla\cdot\left[\frac{\partial\mathscr{H}}{\partial(\nabla\phi_a)}\delta\phi_a\right]-\delta\phi_a\left[\nabla\cdot\frac{\partial\mathscr{H}}{\partial(\nabla\phi_a)}\right]\right\}\\
=&\int \mathrm{d}^4x\left(\dot{\phi}_a-\frac{\partial\mathscr{H}}{\partial\pi_a}\right)\delta\pi_a-\int \mathrm{d}^4x\left(\frac{\partial\mathscr{H}}{\partial\phi_a}+\dot{\pi}_a-\nabla\cdot\frac{\partial\mathscr{H}}{\partial(\nabla\phi_a)}\right)\delta\phi_a\\
&+\int \mathrm{d}^4x\frac{\mathrm{d}}{\mathrm{d}t}(\pi_a\delta\phi_a)-\int \mathrm{d}^4x\nabla\cdot\left[\frac{\partial\mathscr{H}}{\partial(\nabla\phi_a)}\delta\phi_a\right],
\end{aligned} \tag{1.33}$$

上式最后一行的两项由于 $\delta\phi_a=0$ 而消失. 所以作用量变分为零必然导致式1.5.4. 场 ϕ_a 和它的共轭动量密度 π_a 构成系统的正则变量.

.... ■

1.6 Noether 定理、对称性与守恒律

连续变换对应的对称性称为连续对称性.Noether 定理指出,

Theorem 1.6.1 如果系统具有一种连续对称性, 就必然存在一条对应的守恒定律.

它适用于所有物理行为由作用量原理决定的系统.

Noether 守恒流(conserved current) 定义为

$$j^\mu\equiv\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\bar{\delta}\phi_a+\mathscr{L}\delta x^\mu, \tag{1.34}$$

守恒流方程为

$$\partial_\mu j^\mu = 0, \tag{1.35}$$

上式两边对空间区域 $\tilde{R}$ 积分, 应用 Gauss 定理, 有

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\tilde{R}} \mathrm{d}^3x j^0 + \int_{\tilde{S}} \boldsymbol{j}\cdot \mathrm{d}\boldsymbol{\sigma} = 0, \tag{1.36}$$

其中,$\mathrm{d}\boldsymbol{\sigma}$ 是边界面 $\tilde{S}$ 上的定向面元. 引入守恒荷(conserved charge)$Q \equiv \int_{\tilde{R}} \mathrm{d}^3x j^0$, 上式改写为

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\int_{\tilde{S}} \boldsymbol{j}\cdot \mathrm{d}\boldsymbol{\sigma}, \tag{1.37}$$

即区域 $\tilde{R}$ 中的守恒荷减少率 (增加率) 等于从边界面 $\tilde{S}$ 出来 (进入) 的流. j^0 是守恒荷的空间密度.

对于三维空间, 边界面 $\tilde{S}$ 位于无穷远处, 通常假设 ϕ_a 在无穷远处消失, 从而无穷远处 $\boldsymbol{j} \to 0$, 于是全空间的守恒荷

$$Q = \int \mathrm{d}^3x j^0. \tag{21}$$

场论中, 若系统具有某种连续对称性, 则存在相应的守恒流, 满足守恒流方程, 而全空间的守恒荷不随时间变化, 对应着一条守恒定律.

2. 量子标量场

2.1 简谐振子的正则量子化

一维简谐振子的哈密顿量表达式为

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, \tag{2.1}$$

其中,m 为质量,ω 是角频率. 量子力学中将坐标 x 和动量 p 这两个正则变量视为厄米算符, 要求满足正则对易关系

$$[x, p] \equiv xp - px = i. \tag{2.2}$$

构造两个非厄米的无量纲算符:湮灭算符a(annihilation operator) 和产生算符(creation operator)$a^\dagger$ 分别满足

$$\begin{cases} a = \dfrac{1}{\sqrt{2m\omega}}(m\omega x + ip) \\ a^\dagger = \dfrac{1}{\sqrt{2m\omega}}(m\omega x - ip) \end{cases} \tag{2.3}$$

Theorem 2.1.1 — 产生、湮灭算符的对易关系.

$$\begin{cases} [a, a^\dagger] = 1 \\ [a, a] = 0 \\ [a^\dagger, a^\dagger] = 0 \end{cases}$$

由式2.3得到 x, p 为

$$\begin{cases} x = \dfrac{1}{\sqrt{2m\omega}}(a+a^\dagger) \\ p = -i\sqrt{\dfrac{m\omega}{2}}(a-a^\dagger) \end{cases} \tag{2.4}$$

于是哈密顿量写为

$$H = \frac{\omega}{2}(aa^\dagger + a^\dagger a) = \omega\left(N + \frac{1}{2}\right), \tag{2.5}$$

其中 $N = a^\dagger a$ 为粒子数算符, 是一个半正定算符, 即 $\forall |\psi\rangle, N$ 的期望值(expectation value) 非负.

. . . .

Proof.

$$\begin{aligned}
[a, a^\dagger] &= \frac{1}{2m\omega}[m\omega x + ip, m\omega x - ip] \\
&= \frac{1}{2m\omega}([m\omega x, -ip] + [ip, m\omega x]) \\
&= \frac{1}{2m\omega} \cdot m\omega([x, -ip] + [ip, x]) \\
&= \frac{1}{2}(-i \cdot [x, p] + i[p, x]) \\
&= \frac{1}{2}(-i \cdot i + i \cdot (-i)) \\
&= 1
\end{aligned}$$

$$\begin{aligned}
H &= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \\
&= -\frac{m\omega}{2} \cdot \frac{1}{2m}(a-a^\dagger)^2 + \frac{1}{2}m\omega^2 \cdot \frac{1}{2m\omega}(a+a^\dagger)^2 \\
&= -\frac{\omega}{4}(aa + a^\dagger a^\dagger - aa^\dagger - a^\dagger a) + \frac{\omega}{4}(aa + a^\dagger a^\dagger + aa^\dagger + a^\dagger a) \\
&= \frac{\omega}{4}(2aa^\dagger + 2a^\dagger a) = \frac{\omega}{2}(aa^\dagger + a^\dagger a) = \frac{\omega}{2}(a^\dagger a + 1 + a^\dagger a) \\
&= \omega\left(a^\dagger a + \frac{1}{2}\right) = \omega\left(N + \frac{1}{2}\right).
\end{aligned}$$

$$\langle\psi|N|\psi\rangle = \langle\psi|a^\dagger a|\psi\rangle = \langle a\psi|a\psi\rangle \geqslant 0.$$

. . . . ■

设 $|n\rangle$ 是 N 的本征态, 满足归一化条件 $\langle n|n\rangle = 1$ 和本征方程

$$N|n\rangle = n|n\rangle, n \geqslant 0, n \in \mathbb{R}. \tag{2.6}$$

Theorem 2.1.2 — 粒子数算符与产生、湮灭算符的对易关系.

$$\begin{cases} [N,a^\dagger] = a^\dagger \\ [N,a] = -a \end{cases}$$

Proof.

$$n = \langle n|n|n\rangle = \langle n|N|n\rangle \geqslant 0.$$

$$[N,a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger[a,a^\dagger] = a^\dagger.$$
$$[N,a] = [a^\dagger a, a] = [a^\dagger, a]a = -a.$$

■

Theorem 2.1.3 — 产生、湮灭算符对粒子数算符本征态的作用.

$$\begin{cases} a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \\ a|n\rangle = \sqrt{n}\,|n-1\rangle \end{cases}$$

Proof. 据定理2.1.2有

$$\begin{aligned} Na^\dagger|n\rangle &= (a^\dagger N + a^\dagger)|n\rangle = a^\dagger N|n\rangle + a^\dagger|n\rangle = (n+1)a^\dagger|n\rangle, \\ Na|n\rangle &= (aN - a)|n\rangle = aN|n\rangle - a|n\rangle = (n-1)a|n\rangle. \end{aligned} \tag{2.7}$$

可见,$a^\dagger|n\rangle$ 和 $a|n\rangle$ 分别是粒子数算符 N 属于本征值 $n+1$ 和 $n-1$ 的本征态, 不妨记作

$$\begin{aligned} a^\dagger|n\rangle &= c_1|n+1\rangle, \\ a|n\rangle &= c_2|n-1\rangle. \end{aligned} \tag{2.8}$$

于是有

$$\begin{aligned} n+1 &= \langle n|N+1|n\rangle = \langle n|a^{\dagger}a+1|n\rangle = \langle n|aa^{\dagger}|n\rangle = c_1\cdot c_1^{*}\langle n+1|n+1\rangle = |c_1|^2, \\ n &= \langle n|N|n\rangle = \langle n|a^{\dagger}a|n\rangle = c_2\cdot c_2^{*}\langle n-1|n-1\rangle = |c_2|^2. \end{aligned} \tag{2.9}$$

将 c_1, c_2 都取为实数, 即 $c_1=\sqrt{n+1}, c_2=\sqrt{n}$.

■

Theorem 2.1.4 — 用真空态 $|0\rangle$ 展开任意态 $|n\rangle$.

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^{\dagger})^n|0\rangle.$$

Proof. 从 N 的某个本征态 $|n\rangle$ 出发, 用湮灭算符 a 依次作用, 得到本征值依次减少的一系列本征态

$$\underset{n-1}{\underset{\downarrow}{a|n\rangle}}, \underset{n-2}{\underset{\downarrow}{a^2|n\rangle}}, \underset{n-3}{\underset{\downarrow}{a^3|n\rangle}}, \cdots, \underset{n_0}{\underset{\downarrow}{|n_0\rangle}} \tag{2.10}$$

其中 n_0 是由于 $n\geqslant 0$ 必定存在的一个最小的本征值, 其本征态为 $|n_0\rangle$ 满足 $a|n_0\rangle=0$. 于是就有

$$\begin{aligned} &N|n_0\rangle = a^{\dagger}a|n_0\rangle = 0 = 0|n_0\rangle = n_0|n_0\rangle \\ &\qquad\Rightarrow n_0=0, |n_0\rangle=|0\rangle. \end{aligned} \tag{2.11}$$

从 $|0\rangle$ 出发, 依次用产生算符 $a^{\dagger}$ 作用, 得到本征值依次增加的一系列本征态

$$\underset{1}{\underset{\downarrow}{a^{\dagger}|0\rangle}}, \underset{2}{\underset{\downarrow}{(a^{\dagger})^2|n\rangle}}, \underset{3}{\underset{\downarrow}{(a^{\dagger})^3|n\rangle}}, \cdots \tag{2.12}$$

所以, 本征值 n 的取值是非负整数, 是量子化的.

设 $|n\rangle = c_3(a^\dagger)^n|0\rangle$, 就有

$$
\begin{aligned}
\langle n|n\rangle &= c_3\cdot c_3^*\langle 0|a^n(a^\dagger)^n|0\rangle \\
&= 1\cdot|c_3|^2\langle 1|a^{n-1}(a^\dagger)^{n-1}|1\rangle \\
&= 1\cdot 2\cdot|c_3|^2\langle 2|a^{n-2}(a^\dagger)^{n-2}|2\rangle \\
&= \cdots \\
&= (n-1)!|c_3|^2\langle n-1|aa^\dagger|n-1\rangle \\
&= n!|c_3|^2\langle n|n\rangle\,. \\
&\Rightarrow c_3 = \frac{1}{\sqrt{n!}}.
\end{aligned}
\tag{2.13}
$$

■

Theorem 2.1.5 — 哈密顿算符 H 的能量本征值 E_n.

$$E_n = \omega\left(n+\frac{1}{2}\right).$$

Proof.

$$H|n\rangle = \omega\left(N+\frac{1}{2}\right)|n\rangle = \omega\left(n+\frac{1}{2}\right)|n\rangle = E_n|n\rangle\,. \tag{2.14}$$

■

R $|0\rangle$ 为真空态,$n>0$ 的 $|n\rangle$ 为包含 n 个声子 (phonon) 的激发态, 每一个声子具有一份能量 ω. 如此,

- n 表示声子数目, 粒子数算符 N 描述声子数.
- $a^\dagger$ 的作用是产生一个声子, 增加一份能量.
- a 的作用是湮灭一个声子, 减少一份能量.

2.2 场论中的正则量子化

2.2.1 绘景变换

在 *Schrödinger* 绘景中,

- *Hilbert* 空间上的算符 O^S 不依赖于时间.

- 态矢量 $|\psi(t)\rangle^S$ 代表随时间演化的物理态. 当系统哈密顿量不显含时间时,$|\psi(t)\rangle^S$ 与初始时刻态矢量 $|\psi(0)\rangle^S$ 通过幺正变换 e^{-iHt} 联系起来

$$|\psi(t)\rangle^S = e^{-iHt}|\psi(0)\rangle^S. \tag{2.15}$$

- 态矢量 $|\psi(t)\rangle^S$ 的演化满足 *Schrödinger* 方程

$$i\frac{\partial}{\partial t}|\psi(t)\rangle^S = H|\psi(t)\rangle^S. \tag{2.16}$$

在 *Heisenberg* 绘景中,

- 态矢量 $|\psi\rangle^H$ 不随时间演化, 定义为

$$|\psi\rangle^H = e^{iHt}|\psi(t)\rangle^S = |\psi(0)\rangle^S. \tag{2.17}$$

- 算符 $O^H(t)$ 依赖于时间, 通过一个含时的相似变换与 O^S 联系起来

$$O^H(t) = e^{iHt}O^S e^{-iHt}. \tag{2.18}$$

- $O^H(t)$ 的演化满足 *Heisenberg* 运动方程

$$i\frac{\partial}{\partial t}O^H(t) = [O^H(t), H]. \tag{2.19}$$

Proof. 式2.19是因为

$$\begin{aligned}
i\partial_0 O^H(t) &= i\frac{\partial}{\partial t}\left(e^{iHt}O^S e^{-iHt}\right) \\
&= i\frac{\partial}{\partial t}\left(e^{iHt}\right)O^S e^{-iHt} + ie^{iHt}\frac{\partial}{\partial t}\left(O^S\right)e^{-iHt} + ie^{iHt}O^S\frac{\partial}{\partial t}\left(e^{-iHt}\right) \\
&= i\cdot iHe^{iHt}O^S e^{-iHt} + 0 + ie^{iHt}O^S\cdot(-iH)e^{-iHt} \\
&= -HO^H(t) + O^H(t)H = [O^H(t), H].
\end{aligned} \tag{2.20}$$

其中用到了 H 与 e^{-iHt} 的可对易性.

■

在两种绘景下,

- 哈密顿量 H 的形式不变.
- 力学量在态上的期望值相同.

- 正则对易关系的形式不变, 即有同一时刻 t 成立的等时对易关系

$$[x(t), p(t)] = i. \tag{2.21}$$

Proof. 因 $[H,H]=0$ 有

$$e^{iHt}He^{-iHt} = He^{iHt}e^{-iHt} = H \Rightarrow H^H = H^S = H. \tag{2.22}$$

由式2.15和2.18有

$$^H\langle\psi|\,O^H(t)\,|\psi\rangle^H = {}^H\langle H|\,e^{iHt}O^Se^{-iHt}\,|\psi\rangle^H = {}^S\langle\psi(t)|\,O^S\,|\psi(t)\rangle^S. \tag{2.23}$$

对于正则对易关系, 有

$$\begin{aligned}
[x^H(t), p^H(t)] &= [e^{iHt}x^Se^{-iHt}, e^{iHt}p^Se^{-iHt}] \\
&= e^{iHt}x^Se^{-iHt}\cdot e^{iHt}p^Se^{-iHt} - e^{iHt}p^Se^{-iHt}\cdot e^{iHt}x^Se^{-iHt} \\
&= e^{iHt}x^Sp^Se^{-iHt} - e^{iHt}p^Sx^Se^{-iHt} \\
&= e^{iHt}[x^S, p^S]e^{-iHt} \\
&= e^{iHt}ie^{-iHt} = i.
\end{aligned} \tag{2.24}$$

■

2.2.2 δ 函数相关性质总结

一维 *Dirac* δ 函数定义为

$$\delta(x) = \begin{cases} +\infty, & x=0 \\ 0, & x\neq 0 \end{cases} \tag{2.25}$$

具有如下性质

1. 对在零点附近连续的任意函数 $f(x)$, 恒有

$$\int_{-\infty}^{+\infty}\delta(x)f(x)\mathrm{d}x = f(0) \xlongequal{f(x)=1} \int_{-\infty}^{+\infty}\delta(x)\mathrm{d}x = 1. \tag{2.26}$$

2. δ 函数的任意阶导数都存在

$$\int_{-\infty}^{+\infty} \delta^{(n)}(x) f(x) \mathrm{d}x = (-1)^n f^{(n)}(0) \xlongequal{n=1} \int_{-\infty}^{+\infty} \delta'(x) f(x) \mathrm{d}x = -f'(0). \tag{2.27}$$

3. 对于任意函数 $f(x)$ 有

$$\begin{aligned} f(x)\delta(x-y) &= f(y)\delta(x-y). \\ f(x)\delta(x) &= f(0)\delta(x) \xlongequal{f(x)=x} x\delta(x) = 0. \end{aligned} \tag{2.28}$$

4. δ 函数是偶函数, 求导一次奇偶性就改变一次

$$\begin{aligned} \delta(x) &= \delta(-x). \\ \delta'(-x) &= -\delta'(x). \end{aligned} \tag{2.29}$$

5. δ 函数宗量变换关系

$$\delta(ax) = \frac{1}{|a|}\delta(x). \tag{2.30}$$

6. 定义函数 $f(x)$ 的 Fourier 变换及其逆变换为

$$\begin{cases} \tilde{f}(p) = \displaystyle\int_{-\infty}^{+\infty} \mathrm{d}x\, e^{-ipx} f(x) \\ f(x) = \displaystyle\int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi}\, e^{ipx} \tilde{f}(p) \end{cases} \tag{2.31}$$

则 $2\pi\delta(p)$ 是函数 $f(x)=1$ 的 Fourier 变换, 即

$$2\pi\delta(p) = \int_{-\infty}^{+\infty} \mathrm{d}x\, e^{\pm ipx}. \tag{2.32}$$

7. δ 函数的变上限积分和微分表示

$$\begin{aligned} &\int_{-\infty}^{x} \delta(x)\mathrm{d}x = \eta(x) \\ &\delta(x) = \frac{\mathrm{d}\eta(x)}{\mathrm{d}x} \end{aligned} \tag{2.33}$$

其中 $\eta(x)$ 为 Heaviside 单位阶跃函数, 定义为

$$\eta(x)=\begin{cases}1,x>0\\0,x<0\end{cases} \tag{2.34}$$

8. 若方程 $f(x)=0$ 具有若干分立的单根 x_i, 则有

$$\delta[f(x)]=\sum_i\frac{\delta(x-x_i)}{f'(x_i)}. \tag{2.35}$$

三维形式的 *Dirac* δ 函数满足的性质与一维情况类似, 其定义为

$$\delta^{(3)}(\mathbf{x})=\delta(x^1)\delta(x^2)\delta(x^3). \tag{2.36}$$

2.2.3 量子场论中的正则对易关系

Theorem 2.2.1 — 量子场论中的等时对易关系.

$$\begin{cases}[\phi_a(\mathbf{x},t),\pi_b(\mathbf{y},t)]=i\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})\\ [\phi_a(\mathbf{x},t),\phi_b(\mathbf{y},t)]=0\\ [\pi_a(\mathbf{x},t),\pi_b(\mathbf{y},t)]=0\end{cases}$$

....

Proof. 将空间离散化, 划分为无穷多个小体积元 V_i, 再取 $V_i\to 0$ 的极限得到连续空间的结果. 体积元 V_i 中定义相应的广义坐标

$$\phi_i(t)=\equiv\frac{1}{V_i}\int_{V_i}\mathrm{d}^3x\phi(\mathbf{x},t), \tag{2.37}$$

这是场 $\phi(\mathbf{x},t)$ 在 V_i 中的平均值. 于是 $\partial_\mu\phi$ 和拉格朗日量密度 $\mathscr{L}(\phi,\partial_\mu\phi)$ 在 V_i 中的平均值为

$$\partial_\mu\phi_i\equiv\frac{1}{V_i}\int_{V_i}\mathrm{d}^3x\partial_\mu\phi. \tag{2.38}$$

$$\mathscr{L}_i\equiv\frac{1}{V_i}\int_{V_i}\mathrm{d}^3x\mathscr{L}(\phi,\partial_\mu\phi). \tag{2.39}$$

当 $V_i\to 0$ 时,$\mathscr{L}_i\to\mathscr{L}_i(\phi_i,\partial_\mu\phi_i)$. 于是拉格朗日量可以写为

$$L(t)=\int\mathrm{d}^3x\mathscr{L}=\sum_i\int_{V_i}\mathrm{d}^3x\mathscr{L}=\sum_i V_i\frac{1}{V_i}\int_{V_i}\mathrm{d}^3x\mathscr{L}=\sum_i V_i\mathscr{L}_i(\phi_i,\partial_\mu\phi_i). \tag{2.40}$$

从而广义动量可以表示为

$$\Pi_i(t) = \frac{\partial L}{\partial(\partial_0\phi_i)} = \sum_j V_j \frac{\partial \mathscr{L}_j}{\partial(\partial_0\phi_i)} = \sum_j V_j \delta_{ji} \frac{\partial \mathscr{L}_i}{\partial(\partial_0\phi_i)} = V_i \frac{\partial \mathscr{L}_i}{\partial(\partial_0\phi_i)} := V_i \pi_i(t). \tag{2.41}$$

于是, 等时对易关系为

$$\begin{cases} [\phi_i(t), \Pi_j(t)] = i\delta_{ij} \Longleftrightarrow [\phi_i(t), \pi_j(t)] = i\dfrac{\delta_{ij}}{V_j} \\ [\phi_i(t), \phi_j(t)] = 0 \\ [\Pi_i(t), \Pi_j(t)] = 0 \Longleftrightarrow [\pi_i(t), \pi_j(t)] = 0 \end{cases} \tag{2.42}$$

在 $V_i \to 0$ 的极限下, 有

$$\begin{cases} \dfrac{\delta_{ij}}{V_j} \to \delta^{(3)}(\mathbf{x}-\mathbf{y}) \\ \phi_i(t) \to \phi(\mathbf{x},t) \\ \partial_\mu \phi_i \to \partial_\mu \phi(\mathbf{x},t) \\ \mathscr{L}_i \to \mathscr{L}(\mathbf{x},t) \end{cases} \Longrightarrow \pi_i(t) = \frac{\partial \mathscr{L}_i}{\partial(\partial_0\phi_i)} \to \frac{\partial \mathscr{L}}{\partial(\partial_0\phi)} = \pi(\mathbf{x},t) \tag{2.43}$$

于是就有场论中的等时对易关系

$$\begin{cases} [\phi(\mathbf{x},t), \pi(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\ [\phi(\mathbf{x},t), \phi(\mathbf{y},t)] = 0 \\ [\pi(\mathbf{x},t), \pi(\mathbf{y},t)] = 0 \end{cases} \tag{2.44}$$

进一步推广到包含若干场的系统, 并假设不同场相互独立, 得到定理2.2.1. 此时,$\phi_a(\mathbf{x},t)$ 和 $\pi_a(\mathbf{x},t)$ 都是 Hilbert 空间上的算符.

.... ■

2.3 实标量场的正则量子化

2.3.1 自由实标量场

场 $\phi(x)$ 如果是 Lorentz 标量, 称它是标量场. 固有保时向 Lorentz 变换下, 若时空坐标的变换为 $x' = \Lambda x$, 则标量场 $\phi(x)$ 的变换形式为

$$\phi'(x') = \phi(x). \tag{2.45}$$

实标量场满足自共轭条件, 即

$$\phi^\dagger(x) = \phi(x). \tag{2.46}$$

进一步假设 $\phi(x)$ 是不参与相互作用的自由实标量场, 相应的Lorentz 不变的拉氏量为

$$\mathscr{L} = \underbrace{\frac{1}{2}(\partial^\mu\phi)\partial_\mu\phi}_{\text{动能项}} - \underbrace{\frac{1}{2}m^2\phi^2}_{\text{质量项}}, \tag{2.47}$$

$m > 0$ 为实标量场的质量. 将其代入场的欧拉-拉格朗日方程, 即可解出实标量场 $\phi(x)$ 满足的经典运动方程——Klein-Gordon 方程

$$(\partial^2 + m^2)\phi(x) = 0. \tag{2.48}$$

....

Proof. 首先对 $\mathscr{L}$ 做一点改写

$$\mathscr{L} = \frac{1}{2}(\partial^\mu\phi)\partial_\mu\phi - \frac{1}{2}m^2\phi^2 = -\frac{1}{2}(\partial_\mu\phi)\partial_\mu\phi - \frac{1}{2}m^2\phi^2 = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2. \tag{2.49}$$

从而有

$$\begin{cases} \dfrac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} = -\partial_\mu\phi = \partial^\mu\phi \\ \dfrac{\partial\mathscr{L}}{\partial\phi} = -m^2\phi \end{cases} \tag{2.50}$$

代入到欧拉-拉格朗日方程, 得到 Klein-Gordon 方程.

.... ■

实标量场 $\phi(x)$ 对应的共轭动量密度$\pi(x)$ 为

$$\pi(x) = \frac{\partial\mathscr{L}}{\partial(\partial_0\phi)} = \partial_0\phi(x). \tag{2.51}$$

即 $\pi(x)$ 是场 $\phi(x)$ 的时间导数.

$\phi(x)$ 的自共轭条件导致 $\pi(x)$ 也满足自共轭条件 $\pi^\dagger(x) = \pi(x)$. 于是 $\phi(x)$ 和 $\pi(x)$ 都可以被量子化为 Hilbert 空间的厄米算符, 只需要要求它们满足等时对易关系2.44. 这就是正则量子化.

2.3.2 平面波展开

无界空间单粒子波函数 ψ 的平面波解(plane-wave solution) 为

$$\psi(\mathbf{x},t) = e^{-iEt+i\mathbf{p}\cdot\mathbf{x}}. \tag{2.52}$$

记 $p^\mu = (E,\mathbf{p}), x^\mu = (t,\mathbf{x})$, 于是就有

$$p\cdot x = p_\mu x^\nu = g^\mu{}_\nu p_\mu x^\nu = Et - p_x x - p_y y - p_z z = Et - \mathbf{p}\cdot\mathbf{x}, \tag{2.53}$$

平面波解可以改写为

$$\psi(x) = e^{-ip\cdot x}. \tag{2.54}$$

定义四维动量微分算符

$$\hat{p}^\mu = i\left(\frac{\partial}{\partial t}, -\nabla\right) = i\partial^\mu, \tag{2.55}$$

p^μ 正是 $\hat{p}^\mu = i\partial^\mu$ 的本征值, 平面波解2.54描述的是四维动量为 p^μ 的粒子.

. . . .

Proof. 由于

$$\begin{cases} i\partial^0 e^{-ip\cdot x} = i\partial^0 e^{-iEt+i\mathbf{p}\cdot\mathbf{x}} = Ee^{-ip\cdot x}. \\ i\partial^i e^{-ip\cdot x} = i\partial^i e^{-iEt+i\mathbf{p}\cdot\mathbf{x}} = -i\partial_i e^{-iEt+i\mathbf{p}\cdot\mathbf{x}} = -i\cdot(ip)e^{-iEt+i\mathbf{p}\cdot\mathbf{x}} = pe^{-ip\cdot x}. \end{cases} \tag{2.56}$$

所以

$$i\partial^\mu\psi = i\partial^\mu e^{-ip\cdot x} = p^\mu e^{-ip\cdot x} = p^\mu\psi. \tag{2.57}$$

. . . . ■

Theorem 2.3.1 — 场算符和动量密度算符的一般形式. 满足 Klein-Gordon 方程的场算符 $\phi(\mathbf{x},t)$ 的平面波展开式为

$$\phi(\mathbf{x},t)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x}+a_{\mathbf{p}}^{\dagger}e^{ip\cdot x}\right),$$

其中 $p\cdot x=p^0t-\mathbf{p}\cdot\mathbf{x}$, 而且 $p^0>0$, 满足质壳条件 $p^0=E_p\equiv\sqrt{|\mathbf{p}|^2+m^2}$.

与场算符 $\phi(\mathbf{x},t)$ 共轭的动量密度算符 $\pi(\mathbf{x},t)$ 的平面波展开式为

$$\pi(\mathbf{x},t)=\partial_0\phi(\mathbf{x},t)=\int\frac{d^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x}-a_{\mathbf{p}}^{\dagger}e^{ip\cdot x}\right).$$

....

Proof. 无界空间中, 假设实标量场 $\phi(x)$ 满足的 Klein-Gordon 方程具有 $\varphi(x)=e^{-ik\cdot x}$ 这样的平面波解, 于是就有

$$\partial^2\varphi=\partial^\mu\partial_\mu e^{-ik\cdot x}=\partial^\mu(-i)k_\mu e^{-ik\cdot x}=(-i)^2k^\mu k_\mu e^{-ik\cdot x}=-k^2\varphi. \tag{2.58}$$

代入到 Klein-Gordon 方程中就有

$$0=(\partial^2+m^2)\varphi=(m^2-k^2)\varphi=(m^2-k_0^2+|\mathbf{k}|^2)\varphi \tag{2.59}$$

于是

$$k_0^2=m^2+|\mathbf{k}|^2:=E_{\mathbf{k}}^2, \tag{2.60}$$

并且假设 $E_{\mathbf{k}}>0$. 如此, 对于固定的 $\mathbf{k}$, 将会有两个线性独立的平面波解

$$\begin{cases} k_0=E_{\mathbf{k}}\to\varphi_{\mathbf{k}}^{(+)}(x)=\exp\left[-i(k^0x^0-\mathbf{k}\cdot\mathbf{x})\right]=\exp[-i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})]. \\ k_0=-E_{\mathbf{k}}\to\varphi_{\mathbf{k}}^{(-)}(x)=\exp\left[-i(k^0x^0-\mathbf{k}\cdot\mathbf{x})\right]=\exp[i(E_{\mathbf{k}}t+\mathbf{k}\cdot\mathbf{x})]. \end{cases} \tag{2.61}$$

满足 Klein-Gordon 方程的场算符 $\phi(\mathbf{x},t)$ 的一般解可以写为如下形式

$$\phi(\mathbf{x},t)=\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\left[a_{\mathbf{k}}\varphi_{\mathbf{k}}^{(+)}(x)+\tilde{a}_{\mathbf{k}}\varphi_{\mathbf{k}}^{(-)}(x)\right] \tag{2.62}$$

$$=\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\left[a_{\mathbf{k}}e^{-i(E_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}+\tilde{a}_{\mathbf{k}}e^{i(E_{\mathbf{k}}t+\mathbf{k}\cdot\mathbf{x})}\right], \tag{2.63}$$

其中 $a_{\mathbf{k}}$ 和 $\tilde{a}_{\mathbf{k}}$ 是两个只依赖 $\mathbf{k}$ 的算符, $\frac{1}{\sqrt{2E_{\mathbf{k}}}}$ 是归一化因子. 这实际上是把 $\phi(\mathbf{x},t)$ 展开成三维

动量空间中无穷多个动量模式的叠加.

对式2.63左右两边取厄米共轭, 有

$$\begin{aligned}\phi^\dagger(\mathbf{x},t) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k}^\dagger e^{i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})}+\tilde{a}_\mathbf{k}^\dagger e^{-i(E_\mathbf{k}t+\mathbf{k}\cdot\mathbf{x})}\right]\\ &\xlongequal{\mathbf{k}\to-\mathbf{k}} \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{-\mathbf{k}}}}\left[a_{-\mathbf{k}}^\dagger e^{i(E_{-\mathbf{k}\mathbf{k}}t+\mathbf{k}\cdot\mathbf{x})}+\tilde{a}_{-\mathbf{k}}^\dagger e^{-i(E_{-\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}\right]\\ &\xlongequal{E_{-\mathbf{k}}=E_\mathbf{k}} \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_{-\mathbf{k}}^\dagger e^{i(E_\mathbf{k}t+\mathbf{k}\cdot\mathbf{x})}+\tilde{a}_{-\mathbf{k}}^\dagger e^{-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})}\right].\end{aligned} \tag{2.64}$$

根据实标量场 $\phi(x)=\phi^\dagger(x)$ 的条件, 对比式2.63和式2.64 有

$$\begin{cases} a_{-\mathbf{k}}^\dagger = \tilde{a}_\mathbf{k}. \\ \tilde{a}_{-\mathbf{k}}^\dagger = a_\mathbf{k}. \end{cases} \tag{2.65}$$

利用上式改写 $\phi(\mathbf{x},t)$

$$\begin{aligned}\phi(\mathbf{x},t) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k} e^{-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})}+\tilde{a}_\mathbf{k} e^{i(E_\mathbf{k}t+\mathbf{k}\cdot\mathbf{x})}\right]\\ &\xlongequal{\tilde{a}_\mathbf{k}=a_{-\mathbf{k}}^\dagger} \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k} e^{-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})}+a_{-\mathbf{k}}^\dagger e^{i(E_\mathbf{k}t+\mathbf{k}\cdot\mathbf{x})}\right]\\ &\xlongequal[E_\mathbf{k}=E_{-\mathbf{k}}]{\text{第二项}\mathbf{k}\to-\mathbf{k}} \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left[a_\mathbf{k} e^{-i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})}+a_\mathbf{k}^\dagger e^{i(E_\mathbf{k}t-\mathbf{k}\cdot\mathbf{x})}\right]\\ &\xlongequal{\mathbf{k}\to\mathbf{p}} \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left[a_\mathbf{p} e^{-i(E_\mathbf{p}t-\mathbf{p}\cdot\mathbf{x})}+a_\mathbf{p}^\dagger e^{i(E_\mathbf{p}t-\mathbf{p}\cdot\mathbf{x})}\right]\\ &\xlongequal{p\cdot x=E_\mathbf{p}t-\mathbf{p}\cdot\mathbf{x}} \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left[a_\mathbf{p} e^{-ip\cdot x}+a_\mathbf{p}^\dagger e^{ip\cdot x}\right].\end{aligned} \tag{2.66}$$

共轭动量密度算符为场算符的时间导数, 即

$$\begin{aligned}\pi(\mathbf{x},t)=\partial_0\phi(\mathbf{x},t) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left[(-iE_\mathbf{p})a_\mathbf{p}e^{-ip\cdot x}+(iE_\mathbf{p})a_\mathbf{p}^\dagger e^{ip\cdot x}\right]\\ &= \int \frac{d^3p}{(2\pi)^3}\frac{iE_\mathbf{p}}{\sqrt{2E_\mathbf{p}}}\left[-a_\mathbf{p}e^{-ip\cdot x}+a_\mathbf{p}^\dagger e^{ip\cdot x}\right]\\ &\xlongequal{E_\mathbf{p}=p_0} \int \frac{d^3p}{(2\pi)^3}\frac{ip_0}{\sqrt{2E_\mathbf{p}}}\left[-a_\mathbf{p}e^{-ip\cdot x}+a_\mathbf{p}^\dagger e^{ip\cdot x}\right].\end{aligned} \tag{2.67}$$

.... ■

2.3.3 产生、湮灭算符的对易关系

Theorem 2.3.2 — 产生、湮灭算符的对易关系.

$$\begin{cases} [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}). \\ [a_{\mathbf{p}}, a_{\mathbf{q}}] = 0. \\ [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0. \end{cases}$$

....

Proof. 记 $q\cdot x = q^0 t - \mathbf{q}\cdot\mathbf{x}$, 在三维空间中对 $\varphi(\mathbf{x},t)$ 作 Fourier 变换,

$$\begin{aligned}
&\int \mathrm{d}^3 x e^{iq\cdot x}\phi(x) = \int \mathrm{d}^3 x e^{iq\cdot x}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\int \mathrm{d}x^3\left[a_{\mathbf{p}}e^{i(q-p)\cdot x} + a_{\mathbf{p}}^\dagger e^{i(p+q)\cdot x}\right] \\
&\xlongequal[(q+p)\cdot x=(q^0+p^0)t-(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}]{(q-p)\cdot x=(q^0-p^0)t-(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\int \mathrm{d}x^3\left[a_{\mathbf{p}}e^{i(q^0-p^0)t}e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\right] \\
&\xlongequal{2.32} \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left[a_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})e^{i(q^0-p^0)t} + a_{\mathbf{p}}^\dagger(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q})e^{i(q^0+p^0)t}\right] \\
&\xlongequal[2.26]{\mathbf{p}=\pm\mathbf{q}\Rightarrow p^0=q^0} \frac{1}{\sqrt{2E_{\mathbf{q}}}}a_{\mathbf{q}} + \frac{1}{\sqrt{2E_{-\mathbf{q}}}}a_{-\mathbf{q}}^\dagger e^{2iq^0 t} \xlongequal{E_{-\mathbf{q}}=E_{\mathbf{q}}} \frac{1}{\sqrt{2E_{\mathbf{q}}}}\left(a_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger e^{2iq^0 t}\right),
\end{aligned} \tag{2.68}$$

其中,$p^0 = \sqrt{m^2+|\mathbf{p}|^2}, q^0 = \sqrt{m^2+|\mathbf{q}|^2}$.

$\pi(\mathbf{x},t)$ 的 Fourier 变换为

$$\begin{aligned}
&\int \mathrm{d}^3 x e^{iq\cdot x}\pi(\mathbf{x},t) = \int \mathrm{d}^3 x e^{iq\cdot x}\int \frac{d^3 p}{(2\pi)^3}\frac{-iE_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x} - a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right) \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{-iE_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}}}\int \mathrm{d}x^3\left[a_{\mathbf{p}}e^{i(q-p)\cdot x} - a_{\mathbf{p}}^\dagger e^{i(p+q)\cdot x}\right] \\
&\xlongequal[(q+p)\cdot x=(q^0+p^0)t-(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}]{(q-p)\cdot x=(q^0-p^0)t-(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{-iE_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}}}\int \mathrm{d}x^3\left[a_{\mathbf{p}}e^{i(q^0-p^0)t}e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{i(p^0+q^0)t}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\right] \\
&\xlongequal{2.32} \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{-iE_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}}}\left[a_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})e^{i(q^0-p^0)t} - a_{\mathbf{p}}^\dagger(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q})e^{i(q^0+p^0)t}\right] \\
&\xlongequal[2.26]{\mathbf{p}=\pm\mathbf{q}\Rightarrow p^0=q^0} \frac{-iE_{\mathbf{q}}}{\sqrt{2E_{\mathbf{q}}}}a_{\mathbf{q}} - \frac{-iE_{-\mathbf{q}}}{\sqrt{2E_{-\mathbf{q}}}}a_{-\mathbf{q}}^\dagger e^{2iq^0 t} \xlongequal{E_{-\mathbf{q}}=E_{\mathbf{q}}} \frac{-iE_{\mathbf{q}}}{\sqrt{2E_{\mathbf{q}}}}\left(a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger e^{2iq^0 t}\right).
\end{aligned} \tag{2.69}$$

于是

$$
\begin{aligned}
&\int \mathrm{d}^3 x e^{iq\cdot x}\left[\pi(x) - iE_{\mathbf{q}}\phi(x)\right] = \int \mathrm{d}^3 x e^{iq\cdot x}\pi(x) - iE_{\mathbf{q}}\int \mathrm{d}^3 x e^{iq\cdot x}\phi(x) \\
&= \frac{-iE_{\mathbf{q}}}{\sqrt{2E_{\mathbf{q}}}}\left[a_{\mathbf{q}} - a^\dagger_{-\mathbf{q}}e^{2iq^0 t} + a_{\mathbf{q}} + a^\dagger_{-\mathbf{q}}e^{2iq^0 t}\right] = \frac{-iE_{\mathbf{q}}}{\sqrt{2E_{\mathbf{q}}}}2a_{\mathbf{q}} = -i\sqrt{2E_{\mathbf{q}}}a_{\mathbf{q}}.
\end{aligned} \tag{2.70}
$$

亦即

$$
a_{\mathbf{q}} = \frac{i}{\sqrt{2E_{\mathbf{q}}}}\int \mathrm{d}^3 x e^{iq\cdot x}\left[\pi(x) - iE_{\mathbf{q}}\phi(x)\right]. \tag{2.71}
$$

于是 $a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}$ 的表达式为

$$
\begin{aligned}
a_{\mathbf{p}} &= \frac{i}{\sqrt{2E_{\mathbf{p}}}}\int \mathrm{d}^3 x e^{ip\cdot x}\left[\pi(x) - iE_{\mathbf{p}}\phi(x)\right]. \\
a^\dagger_{\mathbf{p}} &= \frac{-i}{\sqrt{2E_{\mathbf{p}}}}\int \mathrm{d}^3 x e^{-ip\cdot x}\left[\pi^\dagger(x) + iE_{\mathbf{p}}\phi^\dagger(x)\right] \\
&\xlongequal[\phi^\dagger(x)=\phi(x)]{\pi^\dagger(x)=\pi(x)} \frac{-i}{\sqrt{2E_{\mathbf{p}}}}\int \mathrm{d}^3 x e^{-ip\cdot x}\left[\pi(x) + iE_{\mathbf{p}}\phi(x)\right].
\end{aligned} \tag{2.72}
$$

从而

$$
\begin{aligned}
[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] &= \left[\frac{i}{\sqrt{2E_{\mathbf{p}}}}\int \mathrm{d}^3 x e^{ip\cdot x}[\pi(x) - iE_{\mathbf{p}}\phi(x)], \frac{-i}{\sqrt{2E_{\mathbf{q}}}}\int \mathrm{d}^3 y e^{-iq\cdot y}[\pi(y) + iE_{\mathbf{q}}\phi(y)]\right] \\
&= \frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\int \mathrm{d}^3 x \mathrm{d}^3 y \left[e^{i(E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x})}[\pi(x) - iE_{\mathbf{p}}\phi(x)], e^{-i(E_{\mathbf{q}}t - \mathbf{q}\cdot\mathbf{y})}[\pi(y) + iE_{\mathbf{q}}\phi(y)]\right] \\
&= \frac{e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t}}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\int \mathrm{d}^3 x \mathrm{d}^3 y e^{-i(\mathbf{p}\cdot\mathbf{x} - \mathbf{q}\cdot\mathbf{y})}\left[\pi(x) - iE_{\mathbf{p}}\phi(x), \pi(y) + iE_{\mathbf{q}}\phi(y)\right].
\end{aligned} \tag{2.73}
$$

又因为

$$
\begin{aligned}
[\pi(x) - iE_{\mathbf{p}}\phi(x), \pi(y) + iE_{\mathbf{q}}\phi(y)] &= [\pi(x), iE_{\mathbf{q}}\phi(y)] + [-iE_{\mathbf{p}}\phi(x), \pi(y)] \\
&= iE_{\mathbf{q}}(-i)\delta^{(3)}(\mathbf{y} - \mathbf{x}) - iE_{\mathbf{p}}i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \\
&= (E_{\mathbf{p}} + E_{\mathbf{q}})\delta^{(3)}(\mathbf{x} - \mathbf{y}).
\end{aligned} \tag{2.74}
$$

所以

$$
\begin{aligned}
[a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}] &= \frac{e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t}}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \int \mathrm{d}^3x\mathrm{d}^3y e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} \left[\pi(x) - iE_{\mathbf{p}}\phi(x), \pi(y) + iE_{\mathbf{q}}\phi(y)\right] \\
&= \frac{e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t}}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \int \mathrm{d}^3x\mathrm{d}^3y e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} (E_{\mathbf{p}} + E_{\mathbf{q}})\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\
&= \frac{e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t}}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \int \mathrm{d}^3x e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} (E_{\mathbf{p}} + E_{\mathbf{q}}) \\
&= \frac{e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t}}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} (2\pi)^3\delta(\mathbf{p}-\mathbf{q})(E_{\mathbf{p}} + E_{\mathbf{q}}) \\
&= \frac{e^{i(E_{\mathbf{p}}-E_{\mathbf{p}})t}}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{p}}}} (2\pi)^3\delta(\mathbf{p}-\mathbf{q})(E_{\mathbf{p}} + E_{\mathbf{p}}) = (2\pi)^3\delta(\mathbf{p}-\mathbf{q}).
\end{aligned} \tag{2.75}
$$

类似的,

$$
\begin{aligned}
[a_{\mathbf{p}}, a_{\mathbf{q}}] &= \left[\frac{i}{\sqrt{2E_{\mathbf{p}}}} \int \mathrm{d}^3x e^{ip\cdot x}[\pi(x) - iE_{\mathbf{p}}\phi(x)], \frac{i}{\sqrt{2E_{\mathbf{q}}}} \int \mathrm{d}^3y e^{iq\cdot y}[\pi(y) - iE_{\mathbf{q}}\phi(y)]\right] \\
&= \frac{-1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \int \mathrm{d}^3x\mathrm{d}^3y \left[e^{i(E_{\mathbf{p}}t-\mathbf{p}\cdot\mathbf{x})}[\pi(x) - iE_{\mathbf{p}}\phi(x)], e^{i(E_{\mathbf{q}}t-\mathbf{q}\cdot\mathbf{y})}[\pi(y) - iE_{\mathbf{q}}\phi(y)]\right] \\
&= -\frac{e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \int \mathrm{d}^3x\mathrm{d}^3y e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \left[\pi(x) - iE_{\mathbf{p}}\phi(x), \pi(y) - iE_{\mathbf{q}}\phi(y)\right].
\end{aligned} \tag{2.76}
$$

又因为

$$
\begin{aligned}
[\pi(x) - iE_{\mathbf{p}}\phi(x), \pi(y) - iE_{\mathbf{q}}\phi(y)] &= [\pi(x), -iE_{\mathbf{q}}\phi(y)] + [-iE_{\mathbf{p}}\phi(x), \pi(y)] \\
&= -iE_{\mathbf{q}}(-i)\delta^{(3)}(\mathbf{y}-\mathbf{x}) - iE_{\mathbf{p}}i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\
&= (E_{\mathbf{p}} - E_{\mathbf{q}})\delta^{(3)}(\mathbf{x}-\mathbf{y}).
\end{aligned} \tag{2.77}
$$

所以

$$
\begin{aligned}
[a_{\mathbf{p}}, a_{\mathbf{q}}] &= -\frac{e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \int \mathrm{d}^3x \mathrm{d}^3y e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} [\pi(x) - iE_{\mathbf{p}}\phi(x), \pi(y) - iE_{\mathbf{q}}\phi(y)] \\
&= -\frac{e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \int \mathrm{d}^3x \mathrm{d}^3y e^{-i(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} (E_{\mathbf{p}} - E_{\mathbf{q}}) \delta^{(3)}(\mathbf{x}-\mathbf{y}) \\
&= -\frac{e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} \int \mathrm{d}^3x e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} (E_{\mathbf{p}} - E_{\mathbf{q}}) \\
&= -\frac{e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} (E_{\mathbf{p}} - E_{\mathbf{q}}) (2\pi)^3 \delta^{(3)}(\mathbf{p}+\mathbf{q}) \\
&= \frac{e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t}}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} (E_{\mathbf{q}} - E_{\mathbf{p}}) (2\pi)^3 \delta^{(3)}(\mathbf{p}+\mathbf{q}) \xlongequal{\mathbf{p}=-\mathbf{q}\Rightarrow E_{\mathbf{p}}=E_{\mathbf{q}}} 0.
\end{aligned} \tag{2.78}
$$

因此

$$
[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger - a_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger = (a_{\mathbf{q}} a_{\mathbf{p}} - a_{\mathbf{p}} a_{\mathbf{q}})^\dagger = [a_{\mathbf{q}}, a_{\mathbf{p}}]^\dagger = 0. \tag{2.79}
$$

.... ■

2.3.4 哈密顿量和总动量

Theorem 2.3.3 — 场算符与四维动量算符的对易关系. 场算符 $\phi(x)$ 与四维动量算符 P^μ 的对易子相当于将四维动量微分算符 $i\partial^\mu$ 作用在实标量场 $\phi(x)$ 上, 即

$$
[\phi(x), P^\mu] = i\partial^\mu \phi(x).
$$

....

Proof. 自由实标量场的哈密顿量密度为

$$
\begin{aligned}
\mathscr{H} &= \pi\partial_0\phi - \mathscr{L} = (\partial_0\phi)^2 - \frac{1}{2}(\partial^\mu\phi)\partial_\mu\phi + \frac{1}{2}m^2\phi^2 \\
&= (\partial_0\phi)^2 - \left[\frac{1}{2}(\partial^0\phi)\partial_0\phi + \frac{1}{2}(\partial^1\phi)\partial_1\phi + \frac{1}{2}(\partial^2\phi)\partial_2\phi + \frac{1}{2}(\partial^3\phi)\partial_3\phi\right] + \frac{1}{2}m^2\phi^2 \\
&= (\partial_0\phi)^2 - \left[\frac{1}{2}(\partial_0\phi)\partial_0\phi - \frac{1}{2}(\partial_1\phi)\partial_1\phi - \frac{1}{2}(\partial_2\phi)\partial_2\phi - \frac{1}{2}(\partial_3\phi)\partial_3\phi\right] + \frac{1}{2}m^2\phi^2 \\
&= \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \\
&\xlongequal{\partial_0\phi=\pi} \frac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right].
\end{aligned} \tag{2.80}
$$

从而哈密顿量可以表示为

$$H=\int \mathrm{d}^3x\mathscr{H}=\frac{1}{2}\int \mathrm{d}^3x\left[\pi^2+(\nabla\phi)^2+m^2\phi^2\right]. \tag{2.81}$$

实标量场的总动量算符定义为

$$P^i=\int \mathrm{d}^3x\pi\partial^i\phi. \tag{2.82}$$

于是就有

$$\begin{aligned}
[\phi(x),H] &= \frac{1}{2}\int \mathrm{d}^3y[\phi(\mathbf{x},t),\pi^2(\mathbf{y},t)] \\
&= \frac{1}{2}\int \mathrm{d}^3y\{\pi(\mathbf{y},t)[\phi(\mathbf{x},t),\pi(\mathbf{y},t)]+[\phi(\mathbf{x},t),\pi(\mathbf{y},t)]\pi(\mathbf{y},t)\} \\
&= i\int \mathrm{d}^3y\pi(\mathbf{y},t)\delta^{(3)}(\mathbf{x}-\mathbf{y}) = \mathrm{i}\pi(\mathbf{x},t)=i\partial^0\phi(x).
\end{aligned} \tag{2.83}$$

$$\begin{aligned}
[\phi(x),P^i] &= \int \mathrm{d}^3y\left[\phi(\mathbf{x},t),\pi(\mathbf{y},t)\frac{\partial}{\partial y_i}\phi(\mathbf{y},t)\right]=\int \mathrm{d}^3y\,[\phi(\mathbf{x},t),\pi(\mathbf{y},t)]\,\frac{\partial}{\partial y_i}\phi(\mathbf{y},t) \\
&= i\int \mathrm{d}^3y\delta^{(3)}(\mathbf{x}-\mathbf{y})\frac{\partial}{\partial y_i}\phi(\mathbf{y},t)=i\frac{\partial}{\partial x_i}\phi(\mathbf{x},t)=i\partial^i\phi(x).
\end{aligned} \tag{2.84}$$

定义四维动量算符$P^\mu=(H,\mathbf{P})$, 于是上面两个式子就可以合写成定理2.3.3的形式.

■

Theorem 2.3.4 — 哈密顿量. 实标量场的哈密顿量可以表示为

$$H=\int\frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}a_{\mathbf{p}}^\dagger a_{\mathbf{p}}+\tilde{V}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{E_{\mathbf{p}}}{2},$$

其中 $\tilde{V}=(2\pi)^3\delta^{(3)}(\mathbf{0})$ 为进行积分的空间体积.

Proof. 由于

$$\begin{aligned}
\phi(\mathbf{x},t) &= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x}+a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right). \\
\pi(\mathbf{x},t) &= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{-ip_0}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x}-a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right).
\end{aligned} \tag{2.85}$$

所以有

$$\begin{cases} \pi^2 = -\int \frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6}\frac{p_0q_0}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x}-a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\left(a_{\mathbf{q}}e^{-iq\cdot x}-a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right). \\ \nabla\phi = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(-ipa_{\mathbf{p}}e^{-ip\cdot x}+ipa_{\mathbf{p}}^\dagger e^{ip\cdot x}\right). \\ \Rightarrow (\nabla\phi)^2 = -\int \frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6}\frac{pq}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\left(a_{\mathbf{p}}^\dagger e^{ip\cdot x}-a_{\mathbf{p}}e^{-ip\cdot x}\right)\left(a_{\mathbf{q}}^\dagger e^{iq\cdot x}-a_{\mathbf{q}}e^{-iq\cdot x}\right). \\ m^2\phi^2 = m^2\int \frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\left(a_{\mathbf{p}}^\dagger e^{ip\cdot x}+a_{\mathbf{p}}e^{-ip\cdot x}\right)\left(a_{\mathbf{q}}^\dagger e^{iq\cdot x}+a_{\mathbf{q}}e^{-iq\cdot x}\right). \end{cases} \tag{2.86}$$

代入式2.81, 则哈密顿量可以表示为

$$\begin{aligned} H &= \frac{1}{2}\int \frac{\mathrm{d}^3x\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6}\frac{(-p_0q_0-pq)}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x}-a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\left(a_{\mathbf{q}}e^{-iq\cdot x}-a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\ &+\frac{m^2}{2}\int \frac{\mathrm{d}^3x\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\left(a_{\mathbf{p}}^\dagger e^{ip\cdot x}+a_{\mathbf{p}}e^{-ip\cdot x}\right)\left(a_{\mathbf{q}}^\dagger e^{iq\cdot x}+a_{\mathbf{q}}e^{-iq\cdot x}\right) \\ &= \frac{1}{2}\frac{1}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\int \mathrm{d}^3x\mathrm{d}^3p\mathrm{d}^3q[(-p_0q_0-pq)\left(a_{\mathbf{p}}e^{-ip\cdot x}-a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right)\left(a_{\mathbf{q}}e^{-iq\cdot x}-a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right) \\ &+m^2\left(a_{\mathbf{p}}^\dagger e^{ip\cdot x}+a_{\mathbf{p}}e^{-ip\cdot x}\right)\left(a_{\mathbf{q}}^\dagger e^{iq\cdot x}+a_{\mathbf{q}}e^{-iq\cdot x}\right)] \\ &= \frac{1}{2}\frac{1}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\int \mathrm{d}^3x\mathrm{d}^3p\mathrm{d}^3q\left\{(-p_0q_0-pq+m^2)\left[a_{\mathbf{p}}a_{\mathbf{q}}e^{-i(p+q)\cdot x}+a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p+q)\cdot x}\right]\right\} \\ &+\frac{1}{2}\frac{1}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\int \mathrm{d}^3x\mathrm{d}^3p\mathrm{d}^3q\left\{(p_0q_0+pq+m^2)\left[a_{\mathbf{p}}^\dagger a_{\mathbf{q}}e^{i(p-q)\cdot x}+a_{\mathbf{p}}a_{\mathbf{q}}^\dagger e^{i(q-p)\cdot x}\right]\right\} \\ &= \frac{1}{2}\frac{1}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\int \mathrm{d}^3x\mathrm{d}^3p\mathrm{d}^3q\left\{(-p_0q_0-pq+m^2)\left[a_{\mathbf{p}}a_{\mathbf{q}}e^{-i(p_0+q_0)t}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}+a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0+q_0)t}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\right]\right\} \\ &+\frac{1}{2}\frac{1}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\int \mathrm{d}^3x\mathrm{d}^3p\mathrm{d}^3q\left\{(p_0q_0+pq+m^2)\left[a_{\mathbf{p}}^\dagger a_{\mathbf{q}}e^{i(p_0-q_0)t}e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}+a_{\mathbf{p}}a_{\mathbf{q}}^\dagger e^{i(q_0-p_0)t}e^{i(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}}\right]\right\} \\ &= \frac{1}{2}\frac{1}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\int \mathrm{d}^3p\mathrm{d}^3q(-p_0q_0-pq+m^2)(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q})\left(a_{\mathbf{p}}a_{\mathbf{q}}e^{-i(p_0+q_0)t}+a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p_0+q_0)t}\right) \\ &+\frac{1}{2}\frac{1}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\int \mathrm{d}^3p\mathrm{d}^3q(p_0q_0+pq+m^2)(2\pi)^3\delta^{(3)}(\mathbf{q}-\mathbf{p})\left(a_{\mathbf{p}}\dagger a_{\mathbf{q}}e^{i(p_0-q_0)t}+a_{\mathbf{p}}a_{\mathbf{q}}^\dagger e^{-i(p_0-q_0)t}\right) \\ &= \frac{1}{2}\frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3p}{2E_{\mathbf{p}}}(-E_{\mathbf{p}}^2+|\mathbf{p}|^2+m^2)(a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}e^{-2ip_0t}+a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger e^{2ip_0t}) \\ &+\frac{1}{2}\frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3p}{2E_{\mathbf{p}}}(E_{\mathbf{p}}^2+|\mathbf{p}|^2+m^2)(a_{\mathbf{p}}^\dagger a_{\mathbf{p}}+a_{\mathbf{p}}a_{\mathbf{p}}^\dagger). \end{aligned} \tag{2.87}$$

利用质壳条件

$$m^2+|\mathbf{p}|^2=E_{\mathbf{p}}^2, \tag{2.88}$$

和产生、湮灭算符的对易关系

$$[a_{\mathbf{p}},a_{\mathbf{p}}^\dagger]=a_{\mathbf{p}}a_{\mathbf{p}}^\dagger-a_{\mathbf{p}}^\dagger a_{\mathbf{p}}=(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p})=(2\pi)^3\delta^{(3)}(\mathbf{0}), \tag{2.89}$$

上式可以进一步化简为

$$\begin{aligned}
H&=\frac{1}{2}\frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3p}{2E_{\mathbf{p}}}(E_{\mathbf{p}}^2+E_{\mathbf{p}}^2)(a_{\mathbf{p}}^\dagger a_{\mathbf{p}}+a_{\mathbf{p}}a_{\mathbf{p}}^\dagger)=\frac{1}{2}\frac{1}{(2\pi)^3}\int\mathrm{d}^3pE_{\mathbf{p}}(a_{\mathbf{p}}^\dagger a_{\mathbf{p}}+a_{\mathbf{p}}a_{\mathbf{p}}^\dagger)\\
&=\frac{1}{2}\frac{1}{(2\pi)^3}\int\mathrm{d}^3pE_{\mathbf{p}}(2a_{\mathbf{p}}^\dagger a_{\mathbf{p}}+(2\pi)^3\delta^{(3)}(\mathbf{0}))\\
&=\int\frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}a_{\mathbf{p}}^\dagger a_{\mathbf{p}}+(2\pi)^3\delta^{(3)}(\mathbf{0})\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{E_{\mathbf{p}}}{2}\\
&=\int\frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}a_{\mathbf{p}}^\dagger a_{\mathbf{p}}+\int\mathrm{d}^3x\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{E_{\mathbf{p}}}{2}\\
&=\int\frac{\mathrm{d}^3p}{(2\pi)^3}E_{\mathbf{p}}a_{\mathbf{p}}^\dagger a_{\mathbf{p}}+\underbrace{\tilde{V}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{E_{\mathbf{p}}}{2}}_{\text{真空能}}.
\end{aligned} \tag{2.90}$$

第二项是一个无穷大的 c 数 (经典的数, 非算符), 是真空的零点能, 是所有动量模式在全空间贡献的零点能之和. 如果不讨论引力现象, 这一项就不重要.

■

Theorem 2.3.5 — 哈密顿量与产生、湮灭算符的对易关系.

$$\begin{cases}[H,a_{\mathbf{p}}^\dagger]=E_{\mathbf{p}}a_{\mathbf{p}}^\dagger.\\ [H,a_{\mathbf{p}}]=-E_{\mathbf{p}}a_{\mathbf{p}}.\end{cases}$$

Proof.

$$[H,a_{\mathbf{p}}^\dagger]=\int\frac{\mathrm{d}^3q}{(2\pi)^3}E_{\mathbf{q}}[a_{\mathbf{q}}^\dagger a_{\mathbf{q}},a_{\mathbf{p}}^\dagger]=\int\frac{\mathrm{d}^3q}{(2\pi)^3}E_{\mathbf{q}}a_{\mathbf{q}}^\dagger[a_{\mathbf{q}},a_{\mathbf{p}}^\dagger]=\int\mathrm{d}^3qE_{\mathbf{q}}a_{\mathbf{q}}^\dagger\delta^{(3)}(\mathbf{q}-\mathbf{p})=E_{\mathbf{p}}a_{\mathbf{p}}^\dagger. \tag{2.91}$$

$$[H, a_{\mathbf{p}}] = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} E_{\mathbf{q}} [a^\dagger_{\mathbf{q}} a_{\mathbf{q}}, a_{\mathbf{p}}] = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} E_{\mathbf{q}} [a^\dagger_{\mathbf{q}}, a_{\mathbf{p}}] a_{\mathbf{q}} = -\int \mathrm{d}^3 q E_{\mathbf{q}} a_{\mathbf{q}} \delta^{(3)}(\mathbf{q} - \mathbf{p}) = -E_{\mathbf{p}} a_{\mathbf{p}}. \tag{2.92}$$

■

Theorem 2.3.6 — 总动量. 实标量场的总动量可以表示为

$$\mathbf{P} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathbf{p} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}.$$

2.4 复标量场的正则量子化