Analyse appliquée : des lois de la physique à l'analyse fonctionnelle

D'après le cours de Frédéric Lagoutière

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Modeling of continuum mechanics, analysis, numerical analysis…

To consider basic laws of physics (classical physics) and apply them to derive PDE model for continuum mechanics, we want to :

- analyse the PDEs,
- design numerical algorithms to approximate solutions,
- analyse the numerical schemes and program them.

We're going to see in this lecture :

- Point particles, second Newton's law,
- Transport and diffusion models,
- Perfect fluids,
- Viscious (Newtonian) fluids,
- Linear elasticity.

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Chapter 1

Systems of point particles and Newton's second law

1.1 Systems of point particles

Consider a point particle with mass m, position $x(t) \in \mathbb{R}^3$, with $t \in \mathbb{R}$ the time variable.
We denote :

$$\dot{x}(t) = x'(t) = \frac{\partial}{\partial t} x(t) = u(t) \tag{1.1}$$

is the volicity of the particle :

$$p(t) = m \cdot u(t) \tag{1.2}$$

the momentum of the particle at time t.
Assume that the particle is submitted to (external) forces F $(= \sum F_i)$.

1.2 Newton's second law

Newton's second law in an inertial reference frame, one has :

$$\frac{\partial}{\partial t} p(t) = F = mu'(t) = m\dot{u} = m\ddot{x} \tag{1.3}$$

No forces means that velocity is constant. We can state what is the gravity force.
This is an ODE if F is known :

$$\ddot{x} = \frac{F}{m} \tag{1.4}$$

We make use of the Cauchy-Lipschitz theorem to state the existence and uniqueness of solutions to Cauchy problems :

$$(\mathcal{C}) \begin{cases} \ddot{x} = \frac{F}{m}(t, x) \\ x(0) = x_0 \in \mathbb{R}^3 \\ \dot{x}(0) = x_1 \in \mathbb{R}^3 \end{cases} \tag{1.5}$$

Remark :

$$\begin{aligned}
\ddot{x} &= \frac{F}{m}(t,x) \\
\text{and if we notice that } \dot{x} &= u \\
\text{then } \dot{u} &= \frac{F}{m}(t,x) \\
\Leftrightarrow \dot{\xi} &= \Phi(t,\xi) \\
\text{with } \xi(t) &= \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}
\end{aligned}$$

When F is enough smooth, the Cauchy problem $(\mathcal{C})$ admits a unique maximal solution.
Kinetic energy : in our frame, the quantity :

$$E_k(t) = \frac{1}{2}m|\dot{x}|^2(t) \tag{1.6}$$

is called the **kinetic energy** of the particle.
Power : the quantity $F.u(t)$ is called the **power** of F at time t. The power of a force is 0 if this force is not able to move an object.
Work : the quantity :

$$\int_{t_1}^{t_2} F.u(t)dt \tag{1.7}$$

is the **work** of F between times t_1 and t_2.
Theorem :

$$E_k(t_2) - E_k(t_1) = \int_{t_1}^{t_2} F.u(t)dt \tag{1.8}$$

PROOF.

$$\begin{aligned}
m\ddot{x}(t) &= F(t) \\
\Rightarrow \underbrace{m\ddot{x}.\dot{x}}_{\frac{1}{2}m(|\dot{x}|^2)} &= F.\dot{x} \\
\Leftrightarrow \frac{1}{2}m\frac{\partial}{\partial t}|\dot{x}|^2(t) &= F.\dot{x}(t) \\
\Rightarrow E_k(t_2) - E_k(t_1) &= \int_{t_1}^{t_2} F.\dot{x}(t)dt
\end{aligned}$$

Potential force : the force F is said a **potential force** if there exists a **scalar potential** :

$$\left\{ \begin{array}{cccc} V : & x & \rightarrow & V(x) \\ & \mathbb{R}^3 & \rightarrow & \mathbb{R} \end{array} \right. \tag{1.9}$$

such that :

$$F(x) = -\nabla_x V(x). \tag{1.10}$$

F is derivating from a potential force.
Theorem : assume that the particle is submitted to

$$F = -\nabla V. \tag{1.11}$$

Then :

$$E(t) = E_k(t) + V(t) = E_k(t) + V(x(t)) \tag{1.12}$$

does not depend on t.
PROOF.

$$\begin{aligned}
\frac{\partial}{\partial t}E(t) &= \frac{\partial}{\partial t}E_k(t) + \frac{\partial}{\partial t}(V \circ x(t)) \\
&= m\ddot{x}.\dot{x} + \nabla V(x(t)).\dot{x}(t) \\
&= F.\dot{x} + \nabla V(x(t)).\dot{x}(t) \\
&= 0 \qquad \text{(via potential force : } F = -\nabla V\text{)} \\
\Leftrightarrow \frac{\partial}{\partial t}E(t) &= 0 \\
\Rightarrow E(t) &= c \quad , \qquad c \in \mathbb{R}
\end{aligned}$$

1.3 Systems of particles

n particles with masses $(m_i)_{i=1}^n$ and positions $(x_i(t))_{i=1}^n$ submitted to forces F_i^{ext} (**external forces**) and $F_{j,i}^{\text{int}}$ (**internal forces** exerted by j).

Example : the gravity force acting on a particle near the earth derives of a potential :

$$V = -g\frac{m.M}{|x|}$$

with M the earth mass and the value 0 is the position of the center of the earth.
Then, thanks to (1.11), we have the potential force (of gravity) :

$$F_g = -g\frac{mMx}{|x|^3}.$$

One has :

$$m\ddot{x}_i = F_i^{\text{ext}} + \sum_{j\neq i} F_{j,i}^{\text{int}} \tag{1.13}$$

This can be put under the form :

$$M\ddot{x} = F^{\text{ext}} + F^{\text{int}} \tag{1.14}$$

where :

$$M = diag\left((m_i)_{i=1,\cdots,n}\right) \in \mathcal{M}_{3n}(\mathbb{R}) \tag{1.15}$$

$$F^{\text{ext}} = \left(F_i^{\text{ext}}\right)_{i=1,\cdots,n} \in \mathbb{R}^{3n} \tag{1.16}$$

$$F^{\text{int}} = \left(F_{i,j}^{\text{int}}\right)_{i=1,\cdots,n \quad j=1,\cdots,n-1} \in \mathcal{M}_{3n\times(n-1)}(\mathbb{R}) \tag{1.17}$$

To be 'potential' : the internal forces are said to be **potential** if there exists :

$$V : \mathbb{R} \to \mathbb{R} \tag{1.18}$$

such that for all i and j,

$$\begin{aligned} F^{\text{int}}_{i,j} &= -\nabla_{x_j}\left(x_j \to V(|x_j - x_i|)\right) && (1.19)\\ &= -V'(|x_j - x_i|)\nabla_{x_j}(x_j \to |x_j - x_i|) && (1.20)\\ &= -V'(|x_j - x_i|)e_{ij} && (1.21) \end{aligned}$$

where $e_{i,j}$ is the unit vector in $\mathbb{R}^3$ parallel to $(x_j - x_i)$.
Total potential : let us define the **total potential** V^{int} :

$$V^{\text{int}} = \sum_{i<j} V(|x_j - x_i|) \tag{1.22}$$

We have :

$$F^{\text{int}} = -\nabla_{x_1,\cdots,x_n} V^{\text{int}} \tag{1.23}$$

Compute this equation to recover every force. (The potential is depending on the distance, but i and j !)

Proof. See A.1. □

Example : gravity external forces :

$$V_{i,j} = -g\frac{m_i m_j}{|x_i - x_j|}$$

Kinetic energy : the kinetic energy of the system is :

$$E_k(t) = \sum_i \frac{1}{2} m_i |\dot{x}_i|^2 \tag{1.24}$$

Theorem : Assume the system is subject to external forces F^{ext} and internal forces deriving from V^{int}. Then, one has :

$$\frac{\partial}{\partial t}(E_k + V^{\text{int}}) = F^{\text{ext}}.\dot{x} \tag{1.25}$$

with V^{int} the potential energy.
PROOF.
...
Remark : The mathematical frame used is the Cauchy-Lipschitz theory and in our context :

$$\left\{ \begin{array}{lcl} \dot{x} &=& u \\ M\dot{u} &=& F \end{array} \right. \tag{1.26}$$

so there is :

$$x(t) = x(0) + \int_0^t u(s)ds \tag{1.27}$$

"Théorème des bouts" : let us define the maximal solution $\dot{\xi} = \Phi(t, \xi)$ on the interval $[0, \tau[$.
This solution $(t, \xi(t))$ goes outside from any compact set in the definition domain of Φ as $t \to \tau^-$.
This theorem in our context tells that if $\tau < \infty$ <u>either</u> :

$$(t, \xi(t)) \xrightarrow[t\to\tau^-]{} \Phi|_{\partial\mathcal{U}} \tag{1.28}$$

(where $\Phi|_{\partial\mathcal{U}}$ is boundary of the definition domain of Φ which will be called $\mathcal{U}$), or :

$$|u(t)| \xrightarrow[t\to\tau^-]{} +\infty \tag{1.29}$$

Application : Consider 2 particles subject to gravity (internal) forces.
Thus :

$$V = V_{1,2}^{\text{int}} = -g\frac{m_1 m_2}{|x_1 - x_2|}. \tag{1.30}$$

The definition domain of Φ is

$$\mathcal{U} = \left\{(x_1, x_2, u_1, u_2) \in \mathbb{R}^{12} / x_1 \neq x_2\right\}. \tag{1.31}$$

For any Cauchy data : $x_1^0, x_2^0, u_1^0, u_2^0$, there exists a unique maximal solution, on time interval $[0, \tau[$.
Indeed, Φ is localy Lipschitz-continuous on $\mathcal{U}$.
Assume that the solution is not global : $\tau < +\infty$.
Then :

- either $|\xi(t)| \to +_\infty$,
- or $|x_1 - x_2| \to 0$,

thus :

- either $|u(t)| \to +_\infty$,
- or $|x_1 - x_2| \to 0$,

If $|u| = \left|\binom{u_1}{u_2}\right| \to +\infty$:

$$E_k(t) = \frac{1}{2}m_1|u_1|^2(t) + \frac{1}{2}m_2|u_2|^2(t) \xrightarrow[t\to\tau]{} +\infty \tag{1.32}$$

According to the last theorem,

$$\begin{aligned} \frac{\partial}{\partial t}\left(E_k + V^{\text{int}}\right) &= 0, &(1.33)\\ \Rightarrow \underbrace{E_k(t)}_{\xrightarrow[t\to\tau]{}+\infty} + V^{\text{int}}(t) &= E_k(0) + V^{\text{int}}(0) &(1.34)\end{aligned}$$

Thus :

$$\begin{aligned} V^{\text{int}}(t) &\xrightarrow[t\to\tau]{} -\infty, &(1.35)\\ \text{but} \quad V^{\text{int}}(t) &= -g\frac{m_1 m_2}{|x_1 - x_2|} &(1.36)\\ \text{then} \quad |x_1 - x_2| &\to 0 &(1.37)\end{aligned}$$

Chapter 2

Convection and diffusion phenomena

2.1 Convection (or transport)

We consider a fluid with a given solution $u(t,x)$, with $t \in \mathbb{R}, x \in \mathbb{R}^d, d \in \mathbb{N}$.
Trajectory : a **trajectory** is a curve $t \to X(t,x)$ solution to :

$$(\mathcal{C}) \begin{cases} \frac{\partial}{\partial t} X(t,x) & = & u(t, X(t,x)) \\ X(0,x) & = & x \end{cases} \tag{2.1}$$

If u is continuous, and globally Lipschitz-continuous to x, the Cauchy-Lipschitz theory tells us that $(\mathcal{C})$ has a **unique global solution** $\forall x \in \mathbb{R}^d$.
Actually, we can consider a solution $t \to X(t,t_0,x)$ to :

$$(\mathcal{C}) \begin{cases} \frac{\partial}{\partial t} X(t,t_0,x) & = & u(t, X(t,t_0,x)) \\ X(t_0,t_0,x) & = & x \end{cases} \tag{2.2}$$

Streamline : a **streamline** is a curve $s \to Y(s,t,x)$ solution to :

$$\begin{cases} \frac{\partial}{\partial t} Y(s,t,x) & = & u(t, Y(s,t,x)) \\ Y(0,t,x) & = & x \end{cases} \tag{2.3}$$

When u does not depend on t, trajectories $\equiv$ streamlines.

Eulerian time derivative : let us consider a quantity $\Phi(t,x)$ attached to the fluid or the fluid particles. Its **Eulerian time derivative** of Φ is $\frac{\partial}{\partial t}\Phi$.
Lagrangian time derivative : let us consider a quantity $\Phi(t,x)$ attached to the fluid or the fluid particles. Its **Lagrangian time derivative** is the derivative with regard to t of $t \to \Phi(t, X(t,x))$ (with $X(t,x)$ the trajectory) which is :

$$\begin{aligned} \frac{d}{dt}\left(t \to \Phi(t, X(t,x))\right) &= \frac{\partial}{\partial t}\Phi(t,X(t,x)) + \nabla_x \Phi(t, X(t,x)).\frac{\partial}{\partial t} X(t,x) && (2.4) \\ &= \partial_t \Phi(t, X(t,x)) + u(t, X(t,x)).\nabla_x \Phi(t, X(t,x)) && (2.5) \\ &= D_t \Phi && (2.6) \\ &= \frac{D}{Dt}\Phi && (2.7) \end{aligned}$$

Material volume : a material volume $\omega(t)$ is a volume (depending on t) that follows the fluid in its movement :

$$\omega(t) = \{X(t,x)/x \in \omega(0)\} \tag{2.8}$$

Property : consider a quantity $\rho(t,x)$ that is constant in time in any material volume :

$$\int_{\omega(t)} \rho(t,x)dx = \int_{\omega(0)} \rho(0,x)dx \tag{2.9}$$

for any material volume ω.
ρ can be the volumic mass density of the fluid like :

$$\int_{\omega(t)} \rho(t,x)dx = m(\omega(t)) \tag{2.10}$$

Then, $\rho(t,x)$ satisfies

$$\partial_t \rho(t,x) + \nabla.(\rho u)(t,x) = 0, \tag{2.11}$$

where $t \in \mathbb{R}, x \in \mathbb{R}^d$ (that means $\nabla.V = \sum_{i=1}^d \partial_{x_i} V_i$).
In the case of smooth solutions, this rewrites :

$$\partial_t \rho + u\nabla_x \rho + \rho \nabla.u = 0 \tag{2.12}$$

PROOF.

$$\begin{aligned} \int_{\omega(t)} \rho(t,x)dx &= \int_{\omega(0)} \rho(0,x), \forall \omega(0) \\ \text{thus} \quad \frac{d}{dt}\int_{\omega(t)} \rho(t,x)dx &= 0 \end{aligned}$$

Normal (geometry) : in geometry, a normal is an object such as a line or vector that is perpendicular to a given object. In our case, $n(t,x)$ is a unit vector on the boundary that points out of ω.

Reynolds transport theorem :

$$\frac{d}{dt}\int_{\Omega(t)} \mathbf{f}\, dV = \int_{\Omega(t)} \frac{\partial \mathbf{f}}{\partial t}\, dV + \int_{\partial\Omega(t)} \left(\mathbf{v}^b \cdot \mathbf{n}\right) \mathbf{f}\, dA \tag{2.13}$$

in which $n(x,t)$ is the outward-pointing unit normal vector, x is a point in the region and is the variable of integration, dV and dA are volume and surface elements at x.
Then, thanks to Reynolds transport theorem :

$$\frac{d}{dt}\int_{\omega(t)} \rho(t,x)dx = \int_{\omega(t)} \partial_t \rho(t,x)dx + \int_{\partial\omega(t)} \rho(t,x)u(t,x).\underbrace{n(t,x)}_{\textbf{normal}}\, d\sigma \tag{2.14}$$

1. <u>If $d=1$</u>, via la formule de Green :

$$\begin{aligned} \frac{d}{dt}\int_{a(t)}^{b(t)} \rho(t,x)dx &= \int_{a(t)}^{b(t)} \partial_t \rho(t,x)dx - a'(t)\rho(t,a(t)) + b'(t)\rho(t,b(t)) && (2.15) \\ &= \int_{a(t)}^{b(t)} \partial_t \rho(t,x)dx - u(t,a(t))\rho(t,a(t)) + u(t,b(t))\rho(t,b(t)) && (2.16) \\ &= \int_{a(t)}^{b(t)} \partial_t \rho(t,x)dx + [\rho(t,x)u(t,x)]_{a(t)}^{b(t)} && (2.17) \end{aligned}$$

2. If d = 2 :

$$\int_{\omega(t+\epsilon)} \rho(t+\epsilon,x)dx - \int_{\omega(t)} \rho(t,x)dx \quad \text{(en bricolant, on obtient :)} \tag{2.18}$$

$$= \int_{\omega(t)} \rho(t+\epsilon,x) - \rho(t,x)dx + \int_{\omega(t+\epsilon)} \rho(t+\epsilon,x)dx - \int_{\omega(t)} \rho(t+\epsilon,x)dx \tag{2.19}$$

$$= \epsilon \int_{\omega(t)} \partial_t \rho + \circ(\epsilon)dx + \epsilon \int_{\partial\omega(t)} \rho(t,x)u(t,x).\underbrace{n(t,x)}_{\textbf{normal}}\, d\sigma \tag{2.20}$$

Thus :

$$\int_{\omega} \partial_t \rho + \int_{\partial\omega} \rho u.n = 0 \tag{2.21}$$

Thanks to the Ostrogradski formula (based on Integration By Parts) :

$$\int_{\partial\omega} \rho u.n = \int_{\omega} \nabla.(\rho u) \tag{2.22}$$

Finally :

$$\int_{\omega} \partial_t \rho + \nabla.(\rho u) = 0 \tag{2.23}$$

This is true for any ω, thus locally :

$$\partial_t \rho + \nabla.(\rho u) = 0 \tag{2.24}$$

So :

$$\frac{d}{dt}\left(\int_{\omega(t)} \rho(t,x)dx\right) = \frac{d}{dt}\left(\int_{\mathbb{R}^d} \rho(t,x)1_{\omega(0)}(x)dx\right) \tag{2.25}$$

$$= \int_{\mathbb{R}^d} \frac{d}{dt}\left(\rho 1_{\omega}\right) dx \tag{2.26}$$

Definition 1. Continuity equation

$$\partial_t \rho + \nabla \cdot \rho u = 0 \tag{2.27}$$

[1]

For material volume, we have:

Conservative equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\omega(t)} \rho(t,x)\mathrm{d}x = 0 \tag{2.28}$$

Remark. Here, u is a **given** velocity field.

[1] conservation of masse

2.1.1 Mathematical study of the problem

$$\begin{cases} \partial_t \rho + \nabla \cdot \rho u &= 0 \\ \rho(0, \cdot) &= \rho_0 \end{cases} \tag{$\mathcal{CC}$}$$

with u (smooth) given.

Case where u is constant

Assume that u is constant. In this case,

$$\partial_t \rho + u \cdot \nabla \rho + \rho \nabla \cdot u = 0 \tag{2.29}$$

. Since $\nabla \cdot u = 0$, we obtain **non conservative transport equation**

$$\partial_t \rho + u \cdot \nabla \rho = 0 \tag{2.30}$$

. Then a solution is given by

$$\rho(t, x) = \rho_0(x - tu) \tag{2.31}$$

. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_0(x - tu)) = -\nabla \rho_0(x - tu) \cdot u$$
$$\nabla_x \rho_0(x - tu) \cdot u = \nabla \rho_0(x - tu) \cdot u$$

. We see that this is in accordance with the fact that the density ρ is transported.

Case where u is not constant

$$\partial_t \rho + u \cdot \nabla \rho = \underbrace{-\rho \nabla \cdot u}_{\text{linear source term}} \tag{2.32}$$

First stage: Study of $\partial_t \rho + u \cdot \nabla \rho = 0$

Remark. $\partial_t c + u \cdot \nabla c = 0$ is also important in physics.

Assume that the fluid is actually composed of two different fluids that are miscible. The two density of these fluids are denoted by ρ_1 and ρ_2 and we have $\rho = \rho_1 + \rho_2$.

We have

$$\begin{cases} \partial_t \rho_1 + \nabla \cdot \rho_1 u &= 0 \\ \partial_t \rho_2 + \nabla \cdot \rho_2 u &= 0 \end{cases} \Rightarrow \partial_t \rho + \nabla \cdot \rho u = 0 \tag{2.33}$$

Assume that $\rho > 0$. Define the mass fraction of fluid 1 as $c = \frac{\rho_1}{\rho} = \frac{\rho_1}{\rho_1+\rho_2}$, then

$$\begin{aligned}
\partial_t c = \partial_t(\frac{\rho_1}{\rho}) &= \frac{1}{\rho}\partial_t \rho_1 - \frac{\rho_1}{\rho^2}\partial_t \rho \\
&= -\frac{1}{\rho}\nabla(\rho_1 u) + \frac{\rho_1}{\rho^2}\nabla(\rho u) \\
&= -\frac{u}{\rho}\nabla(\rho_1) - \frac{\rho_1}{\rho}\nabla \cdot u + \frac{\rho_1 u}{\rho^2}\nabla(\rho) + \frac{\rho_1}{\rho}\nabla(\cdot u) \\
&= -\frac{u}{\rho}\nabla(\rho_1) + \frac{\rho_1 u}{\rho^2}\nabla(\rho) \\
&= -u \cdot \nabla(\frac{\rho_1}{\rho}) = -u \cdot \nabla(c)
\end{aligned}$$

.

Thus, $\partial_t c + u \cdot \nabla(c) = 0$.

Idea: look for curves (in (t, x)) along which ρ is constant. Indeed, in the case where u is constant, $\rho(t, x) = \rho_0(x - ut)$ express the fact that ρ is constant along $(t, \underbrace{x + tu}_{\equiv X(t,x)})$.

$$\rho(t, x + tu) = \rho_0(x + tu - tu) = \rho_0(x)$$

does not depend on time.

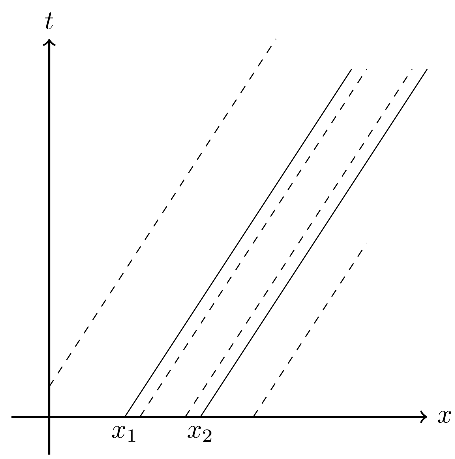


Figure 2.1: $X(t, x)$

Let us look for $X(t, x)$ such that $\rho(t, X(t, x))$ does not depend on t.

$$\begin{aligned}0 =& \frac{\mathrm{d}}{\mathrm{d}t}\rho(t, X(t,x))\\ =&\partial_t \rho(t, X(t,x)) + \nabla\rho(t, X(t,x))\cdot \partial_t X(t,x)\\ =& -u(t, X(t,x))\cdot\nabla\rho(t, X(t,x)) + \partial_t X(t,x)\nabla\rho(t, X(t,x))\end{aligned}$$

, (because $\partial_t\rho(t,x) + u(t, X(t,x))\cdot\nabla\rho(t,x) = 0 \quad \forall x, t.$)

A sufficient condition is $\partial_t X(t,x) = u(t, X(t,x)) \quad \forall t, x$. For x fixed, this is an ODE.

Consider the family of ODE:

$$\left\{\begin{array}{lcl}\partial_1 X(t,x) &=& u(t, X(t,x))\\ X(0,x) &=& x\end{array}\right. \tag{$\mathcal{C}_x$}$$

Assumptions: u is continuous with respect to t and x and globally Lipschitz-continuous with respect to x:

$$|u(t,y) - u(t,x)| \leq L|y-x| \quad \forall (t,x,y)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$$

Under these assumptions, there exist a unique global situation to ($\mathcal{C}_x$) for all $x\in\mathbb{R}^d$.

Now, if ρ is constant along X,

$$\rho(t, X(t,x)) = \rho(s, X(s,x)) = \rho(0, X(0,x)) = \rho_0(x)$$

.

Problem: If (t,x) is given, how to compute $\rho(t,x)$?

Find $y\in\mathbb{R}^d$ such that $X(t,y) = x$ then

$$\rho(t,x) = \rho(t, X(t,y)) = \rho_0(y)$$

. We have to invent the **characteristic (curve)** $x\mapsto X(t,x)$ at fixed t. Consider now $X(s,t,x)$ such that

$$\left\{\begin{array}{lcl}\partial_1 X(s,t,x) &=& u(s, X(s,t,x))\\ X(\underbrace{t}_{\text{real time}}, \underbrace{t}_{\text{Cauchy datum time}}, x) &=& x\end{array}\right. \tag{2.34}$$

The solution to this problem satisfies the **semi-group property** $X(t_3, t_2, X(t_1,x)) = X(t_3, t_1, x)$, consequence of the uniqueness of solution.

Consequence: $X(t, 0, X(0,t,x)) = X(t,t,x) = x$.

Second stage: Come back to

$$\left\{\begin{array}{lcl}\partial_t\rho + u\cdot\nabla\rho &=& 0\\ \rho(0,\cdot) &=& \rho_0\end{array}\right. \tag{$\mathcal{CNC}$}$$

.

Saying that ρ is constant along the characteristic is saying that

$$\rho(t,x) = \rho_0(X(0,t,x))$$

. Indeed, $X(t,0,X(0,t,x)) = x$, thus

$$\begin{aligned}\rho(t, X(t,0,X(0,t,x))) &= \rho(t,x)\\ \rho(0, X(0,0,X(0,t,x))) &= \rho_0(X(0,t,x))\end{aligned}$$

.

We defined characteristic curve along which **the** solution ρ has to be a constant. We define ρ as constant along these curves

$$\rho(t,x) = \rho_0(X(0,t,x))$$

.

Is this $\rho(t,x)$ solution to the Cauchy problem?

First proof. Let us prove that

$$(\frac{\mathrm{d}}{\mathrm{d}t} + u\cdot\nabla)\rho_0(X(0,t,x)) = 0$$

.

First, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_0(X(0,t,x))) = \nabla\rho_0(X(0,t,x))\partial_2 X(0,t,x)$$

.

Let us define $g(t,x) = X(t,t,x) = x = X(t,0,X(0,t,x))$. Then

$$\begin{aligned}
\partial_1 g(t,x) = 0 & \\
&= \partial_1 X(t,0,X(0,t,x)) + \nabla_3 X(t,0,X(0,t,x))\cdot\partial_2 X(0,t,x) \\
&= u(t,X(t,0,X(0,t,x))) + \nabla_3 X(t,0,X(0,t,x))\cdot\partial_2 X(0,t,x) \\
&= u(t,x) + \nabla_3 X(t,0,X(0,t,x))\cdot\partial_2 X(0,t,x)
\end{aligned}$$

Thus $\nabla_3 X(t,0,X(0,t,x))\cdot\partial_2 X(0,t,x) = -u(t,x)$.
Then, $\nabla(\rho_0(X(0,t,x))) = [\nabla_x\rho_0(X(0,t,x)))]^T\cdot\nabla_3 X(0,t,x)$
Compute $\nabla_3 X(0,t,x)$:

$$\nabla_x g(t,x) = I_d = \nabla_x X(t,0,X(0,t,x))\nabla_3 X(0,t,x))$$

Remark. Thus $\nabla_x X(t,0,X(0,t,x)) \equiv A$ is invertible and $A^{-1} = (\nabla_x X(t,0,X(0,t,x)))^{-1} = \nabla_3 X(0,t,x))$.

Thus

$$\begin{aligned}
&u(t,x)\cdot\nabla_x(\rho_0(X(0,t,x))) \\
=&u(t,x)\nabla_3 X(0,t,x)\nabla_x\rho_0(X(0,t,x)) \\
=&u(t,x)A^{-1}\nabla_x\rho_0(X(0,t,x))
\end{aligned}$$

. Finally

$$\begin{aligned}
&(\frac{\mathrm{d}}{\mathrm{d}t} + u\cdot\nabla_x)\rho_0(X(0,t,x)) \\
&= \nabla\rho_0(X(0,t,x))\partial_2 X(0,t,x) + u\nabla_x(\rho_0(X(0,t,x))) \\
&= -\nabla\rho_0(X(0,t,x))A^{-1}u(t,x) + u[(A^{-1})^T\nabla_x\rho_0(X(0,t,x)))]^T \\
&= -\nabla\rho_0(X(0,t,x))A^{-1}u(t,x) + \nabla_x\rho_0(X(0,t,x)))A^{-1}u(t,x) \\
&= 0
\end{aligned}$$

. Conclusion: $(t,x) \mapsto \rho_0(X(0,t,x)) = \rho(t,x)$ is a solution to $\partial_t \rho + u\nabla\rho = 0$. Furthur more, $\rho(0,x) = \rho_0(X(0,0,x)) = \rho_0(x)$, thus $\rho(t,x)$ satisfies the Cauchy datum. $\rho(t,x)$ is a solution to the Cauchy problem. □

Second proof. $\rho(t,x) = \rho_0(X(0,t,x))$ is constant along the characteristic.

$$\rho(t, X(t,t_0,x)) = \rho_0(X(0,t,X(t,t_0,x))) = \rho_0(X(0,t_0,x))$$

which does not depend on t.

Consequence: $\nabla_{t,x}\rho(t,x)$ is perpendicular to the tangent vector of $X(t,x)$ who is parallel to $(1, \partial_1 X)^T$. Thus,

$$\begin{aligned} &\partial_t \rho \cdot 1 + \partial_1 X \cdot \nabla_x \rho = 0 \\ \Rightarrow &\partial_t \rho + u \cdot \nabla_x \rho \end{aligned}$$

. □

Last question: is the solution to ($\mathcal{CNC}$) unique?

If $\partial_t\rho + u\cdot\nabla\rho = 0$, ρ is constant along the characteristics, thus $\rho(t,x) = \rho(0, X(0,t,x)) = \rho_0(X(0,t,x))$, the solution is unique.

Theorem 1. *Assume u is continuous with respect to (t,x) and globally Lipschitz-continuous with respect to x. The ($\mathcal{CNC}$) admits a unique global solution given by $\rho(t,x) = \rho_0(X(0,t,x))$ where the curves X are solutions to*

$$\left\{ \begin{array}{lcl} \partial_1 X(s,t,x) & = & u(s, X(x,t,x)) \\ X(t,t,x) & = & x \end{array} \right. \tag{2.35}$$

.

Come back to the continuity equation.

$$\left\{ \begin{array}{lcl} \partial_t \rho + \nabla\cdot(\rho u) & = & 0 \quad t \in \mathbb{R}\, x \in \mathbb{R}^d \\ \rho(0,\cdot) & = & \rho_0 \end{array} \right. \tag{$\mathcal{CC}$}$$

.

Remark. For the unconservative equation $\partial_t\rho + u\nabla\rho = 0$, we see that $D_t\rho = 0$. (D_t is the Lagrangian derivative)

Consider $\bar\rho(t,x) = \rho(t, X(t,0,x))$ where ρ is a solution to ($\mathcal{CC}$). We have

$$\begin{aligned} \partial_t \bar\rho(t,x) &= \partial_t \rho(t, X(t,0,x)) + \partial_1 X(t,0,x) \nabla\rho(t, X(t,0,x)) \\ &= \partial_t \rho(t, X(t,0,x)) + u(t, X(t,0,x)) \nabla\rho(t, X(t,0,x)) \\ &= -\rho(t, X(t,0,x)) \nabla u(t, X(t,0,x)) \end{aligned}$$

. Thus $\partial_t \bar{\rho}(t,x) = -\rho(t, X(t,0,x))\nabla u(t, X(t,0,x))$. Thus, $\bar{\rho}$ satisfies

$$\begin{aligned}
&\bar{\rho}(t,x) = \bar{\rho}(0,x)e^{-\int_0^t \nabla u(s,X(s,0,x))\mathrm{d}s} \\
&\bar{\rho}(0,x) = \rho(0, X(0,0,x)) = \rho(0,x) = \rho_0(x) \\
\Rightarrow\ &\bar{\rho}(t,x) = \rho_0(x)e^{-\int_0^t \nabla u(s,X(s,0,x))\mathrm{d}s}
\end{aligned}$$

. We are interested in ρ,

$$\bar{\rho}(t,x) = \rho(t, X(t,0,x)) \Leftrightarrow \rho(t,x) = \bar{\rho}(t, X(0,t,x))$$

.

Indeed,

$$\begin{aligned}
&\bar{\rho}(t,x) = \rho(t, X(t,0,x)) \quad \forall x \\
\Leftrightarrow &\bar{\rho}(t, X(0,t,y)) = \rho(t, X(t,0,X(0,t,y))) \quad \forall y \\
\Leftrightarrow &\bar{\rho}(t, X(0,t,y)) = \rho(t,y) \quad \forall y
\end{aligned}$$

.

Then

$$\rho(t,x) = \bar{\rho}(t, X(0,t,x)) = \rho_0(X(0,t,x))e^{-\int_0^t \nabla u(s,X(s,0,X(0,t,x)))\mathrm{d}s} = \rho_0(X(0,t,x))e^{-\int_0^t \nabla u(s,X(s,t,x))\mathrm{d}s} \tag{2.36}$$

.

Theorem 2. *Assume $u \in \mathcal{C}^1$ with respect to (t,x) and globally Lipschitz-continuous with respect to x. Then $(\mathcal{CC})$ has a unique global solution, given by*

$$\rho(t,x) = \rho_0(X(0,t,x))e^{-\int_0^t \nabla u(s,X(s,t,x))ds} \tag{2.37}$$

.

Remark. For $(\mathcal{CNC})$, the solution satisfies the maximum principle:

$$\min_y \rho_0(y) \leq \rho(t,x) \leq \max_y \rho_0(y)$$

.

Indeed, $\rho(t,x) = \rho_0(X(0,t,x))$.

It is the same for $(\mathcal{CC})$ when $\nabla u = 0$. In the conservative case where $\nabla u \neq 0$, the maximum principle is not satisfied.

For the $(\mathcal{CNC})$, remember that it models the evolution of a mass fraction $c = \frac{\rho_1}{\rho_1+\rho_2}$. ($c_0 \in [0,1] \Rightarrow c(t) \in [0,1]$.)

Summary of the mathematical modelling part.
Conservative transport equation.

$$\partial_t \rho + \partial_x \rho u = 0, \quad \text{(d = 1)} \tag{2.38}$$

PLIC

$$\begin{aligned} \rho(t,x) &= r(t) : \\ 0 &= \partial_t r + \partial_x r u \\ &= \partial_t r + u \partial_x r + r \partial_x u \\ &= \partial_t r + r \partial_x u \\ &= \partial_t r - r \\ \Rightarrow r(t) &= r(0) e^t = e^t \end{aligned}$$

For the (NC) equation :

$$\partial_t \rho + u \partial_x \rho = 0 \tag{2.39}$$

with $\rho(t,x) = \rho\left(0, X(0,t,x)\right)$. with $X(s,t,x) = x e^{-(s-t)}$.

2.1.2 Discretisation

<u>**GOAL**</u> : to compute approximate solutions to :

$$\partial_t \rho + \nabla . \rho u = 0 \tag{2.40}$$
$$\rho(0,.) = \rho^0 \tag{2.41}$$

and

$$\partial_t \rho + u . \nabla \rho = 0 \tag{2.42}$$
$$\rho(0,.) = \rho^0 \tag{2.43}$$

We have explicit formula for these Cauchy problem, but for more complicate problems (cf the following of the course), we will need the technique we will develop here. We will restrict to the case where $d = 1$.

$\partial_t \rho + u \partial_x \rho = 0$

General technique : **finite differences**. Let $\Delta x > 0$ be the space step and $\Delta t > 0$ be the time step. Define

$$t^n = n \Delta t, \quad n \in \mathbb{N} \tag{2.44}$$
$$\text{and} \quad x_j = j \Delta x, \quad j \in \mathbb{Z} \tag{2.45}$$

$(\rho_j^n)_{n \in \mathbb{N}, j \in \mathbb{Z}}$ is intended to approximate $(\rho(t^n, x_j))_{n \in \mathbb{N}, j \in \mathbb{Z}}$ where ρ is the solution to :

$$\partial_t \rho + u \partial_x \rho = 0 \tag{2.46}$$
$$\rho(0,.) = \rho^0 \tag{2.47}$$

A finite difference algorithm to define (ρ_j^n) is (assuming $u = cst$) :

$$\begin{aligned}
\frac{\rho_j^{n+1}-\rho_j^n}{\Delta t}+u\frac{\rho_{j+1}^n-\rho_{j-1}^n}{2\Delta x} &= 0 && (2.48)\\
\text{or}\quad \frac{\rho_j^{n+1}-\rho_j^n}{\Delta t}+u\frac{\rho_{j}^n-\rho_{j-1}^n}{\Delta x} &= 0 && (2.49)\\
\text{or}\quad \frac{\rho_j^{n+1}-\rho_j^n}{\Delta t}+u\frac{\rho_{j+1}^n-\rho_{j}^n}{\Delta x} &= 0 && (2.50)\\
\text{or}\quad \frac{\rho_j^{n+1}-\rho_j^n}{\Delta t}+u\frac{\rho_{j+1}^{n+1}-\rho_{j-1}^{n+1}}{2\Delta x} &= 0 && (2.51)
\end{aligned}$$

Here we restrict to : **explicit** schemes : the discrete time derivative only involves the time index n (not $n+1$). It is natural to initialize the time sequence with :

$$\begin{aligned}
\rho_j^{n+1} &= \rho_j^n - u\frac{\Delta t}{\Delta x}\left(\rho_j^n-\rho_{j-1}^n\right) && (2.52)\\
\rho_j^0 &= \rho^0(x_0) && (2.53)
\end{aligned}$$

Lemma : This scheme is consistant at first order. **The consistency error** :

$$\varepsilon_j^n(\rho)=\frac{\rho(t^{n+1},x_j)-\rho(t^n,x_j)}{\Delta t}+u\frac{\rho(t^n,x_j)-\rho(t^n,x_{j-1})}{\Delta x} \tag{2.54}$$

Then, if the solution ρ is smooth enough ($\rho \in \mathcal{C}_U^2(\mathbb{R}_+\times\mathbb{R})$: the first and second derivative of ρ are uniformly bounded on $\mathbb{R}_+\times\mathbb{R}$), there exists a constant C s.t.

$$|\varepsilon_j^n| \le C(\Delta t+\Delta x), \qquad \forall n\in\mathbb{N}, \quad j\in\mathbb{Z} \tag{2.55}$$

PROOF.

$$\begin{aligned}
\frac{\rho(t^{n+1},x_j)-\rho(t^n,x_j)}{\Delta t} &= \partial_t\rho(t^n,x_j)+\underbrace{\mathcal{O}(\Delta t)}_{\text{unif. bounded w.r.t. } j \text{ and } n}\\
\frac{\rho(t^{n},x_j)-\rho(t^n,x_{j-1})}{\Delta x} &= \partial_x\rho(t^n,x_j)+\mathcal{O}(\Delta x)\\
\Rightarrow \frac{\rho(t^{n+1},x_j)-\rho(t^n,x_{j-1})}{\Delta t}+u\frac{\rho(t^{n},x_j)-\rho(t^n,x_{j-1})}{\Delta x} &= \partial_t\rho(t^n,x_j)+u\partial_x\rho(t^n,x_j)+\mathcal{O}(\Delta t+\Delta x)\\
&= 0+\mathcal{O}(\Delta t+\Delta x) \quad (\text{because}\partial_t\rho+u\partial_x\rho).
\end{aligned}$$

Thus,

$$|\varepsilon_j^n| = \mathcal{O}(\Delta t+\Delta x)\le C(\Delta t+\Delta x).$$

Theorem : Assume $\rho\in C_U^2(\mathbb{R}_+\times\mathbb{R})$ ($\rho^0\in C_U^2(\mathbb{R})$ is sufficient, because : $\rho(t,x)=\rho^0(x-ut)$.) Assume $u\ge 0$.
Assume $u\Delta t\le\Delta x$. Then, the scheme :

$$\begin{aligned}
\frac{\rho_j^{n+1}-\rho_j^n}{\Delta t}+u\frac{\rho_{j}^n-\rho_{j-1}^n}{\Delta x} &= 0 && (2.56)\\
\rho_j^0 &= \rho^0(x_j) && (2.57)
\end{aligned}$$

is convergent at the first order :

$$\exists C / |\underbrace{\rho_j^n - \rho(t^n, x_j)}_{e_j^n}| \leq C t^n (\Delta t + \Delta x) \forall n, \forall j \tag{2.58}$$

PROOF. Let's define :

$$\begin{aligned} e_j^n &= \rho_j^n - \rho(t^n, x_j) && (2.59) \\ \Rightarrow e_j^{n+1} &= \rho_j^{n+1} - \rho(t^{n+1}, x_j) && (2.60) \\ &= \rho_j^n - u\frac{\Delta t}{\Delta x}(\rho_j^n - \rho_{j-1}^n) - \rho(t^{n+1}, x_j) && (2.61) \end{aligned}$$

Recall that :

$$\varepsilon_j^n = \frac{\rho(t^{n+1}, x_j) - \rho(t^n, x_j)}{\Delta t} + u\frac{\rho(t^n, x_j) - \rho(t^n, x_{j-1})}{\Delta x} \tag{2.62}$$

Thus :

$$\begin{aligned} e_j^{n+1} &= \rho_j^n - u\frac{\Delta t}{\Delta x}\left(\rho_j^n - \rho_{j-1}^n\right) - \left(\rho(t^n, x_j) - u\frac{\Delta t}{\Delta x}(\rho(t^n, x_j) - \rho(t^n, x_{j-1}) + \Delta t \epsilon_j^n\right). && (2.63) \\ &= e_j^n - u\frac{\Delta t}{\Delta x}(e_j^n - e_{j-1}^n) - \Delta t \varepsilon_j^n && (2.64) \end{aligned}$$

(e_j^n) satisfies the same scheme as (ρ_j^n) with a source term $\Delta t \varepsilon_j^n$. So :

$$e_j^{n+1} = e_j^n \left(1 - u\frac{\Delta t}{\Delta x}\right) + e_{j-1}^n u\frac{\Delta t}{\Delta x} + \Delta t \varepsilon_j^n \tag{2.65}$$

Thus :

$$|e_j^{n+1}| \leq |e_j^n|.|1 - u\frac{\Delta t}{\Delta x}| + |e_{j-1}^n|.|u\frac{\Delta t}{\Delta x}| + \Delta t |\varepsilon_j^n| \tag{2.66}$$

Under the assumptions $u \geq 0$ and $u\Delta t \leq \Delta x$, we have :

$$|u\frac{\Delta t}{\Delta x}| = u\frac{\Delta t}{\Delta x} \tag{2.67}$$

and

$$|1 - u\frac{\Delta t}{\Delta x}| = 1 - u\frac{\Delta t}{\Delta x} \tag{2.68}$$

$$
\begin{aligned}
|e_j^{n+1}| &\leq |e_j^n|\left(1-u\frac{\Delta t}{\Delta x}\right)+|e_{j-1}^n|u\frac{\Delta t}{\Delta x}+\Delta t|\varepsilon_j^n| && (2.69)\\
&\leq \underbrace{\|e^n\|_\infty}_{\sup_{j\in\mathbb{Z}}|e_j^n|}\left(1-u\frac{\Delta t}{\Delta x}+u\frac{\Delta t}{\Delta x}\right)+\Delta t|\varepsilon_j^n| && (2.70)\\
&\leq \|e^n\|_\infty+\Delta t|\varepsilon_j^n| && (2.71)\\
&\leq \|e^n\|_\infty+\Delta t\underbrace{\|\varepsilon^n\|_\infty}_{\sup_{j\in\mathbb{Z}}|\varepsilon_j^n|} && (2.72)\\
\Rightarrow \|e^{n+1}\|_\infty &\leq \|e^n\|_\infty+\Delta t\|\varepsilon^n\|_\infty && (2.73)\\
&\leq \underbrace{\|e^0\|_\infty}_{=0}+\Delta t\sum_{k=0}^{n}\|\varepsilon^k\|_\infty \quad \text{via Discrete Gronwall Lemma} && (2.74)\\
\Rightarrow \|e^n\|_\infty &\leq \Delta t\sum_{k=0}^{n-1}\|\varepsilon^k\|_\infty && (2.75)
\end{aligned}
$$

Thanks to the **consistency** Lemma :

$$
\begin{aligned}
\|e^n\|_\infty &\leq C\Delta t\sum_{k=0}^{n-1}(\Delta t+\Delta x) && (2.76)\\
&= C\Delta t n(\Delta t+\Delta x) && (2.77)\\
&= Ct^n(\Delta t+\Delta x) && (2.78)
\end{aligned}
$$

Remarks.

- $u \geq 0$ is very important. Without this assumption, we are not able to prove the stability .
- $u\frac{\Delta t}{\Delta x} \leq 1$ also. This condition is called the Courant-Friedrichs-Lewy (CFL) (stability) condition. This expresses the fact that the **exact** solution does not cross more than one space cell in a time step.
- The estimate is not uniform in time, we are not able to bound $\sup_n \|e^n\|_\infty$ but it is a good estimate locally in time (for a fixed time).

We are working on the upwind schemes.
Upwind : the information comes from the "correct" side :

- from the left to the right if $u \geq 0$ (not convergent), (RW) scheme ;
- from the right to the left if $u \leq 0$ (convergent), (LW) scheme.

For $u \geq 0$, the scheme

$$
\frac{\rho_j^{n+1}-\rho_j^n}{\Delta t}+u\frac{\rho_j^n-\rho_{j-1}^n}{\Delta x}=0, \qquad u \geq 0 \quad (RW) \tag{2.79}
$$

does not satisfy the same stability property :

$$
\rho_j^{n+1}=\rho_j^n(1+u\frac{\Delta t}{\Delta x})+\rho_{j+1}^n(-u\frac{\Delta t}{\Delta x}) \tag{2.80}
$$

This is not a convex combination ! It is possible to prove that the scheme is not convergent.
The scheme is **not convergent**.
For $u \leq 0$, the scheme

$$\frac{\rho_j^{n+1}-\rho_j^n}{\Delta t}+u\frac{\rho_{j+1}^n-\rho_j^n}{\Delta x}=0, \qquad u\leq 0 \quad (LW) \tag{2.81}$$

is convergent.
The centered scheme is :

$$\begin{aligned}\frac{\rho_j^{n+1}-\rho_j^n}{\Delta t}+u\frac{\rho_{j+1}^n-\rho_{j-1}^n}{2\Delta x} &= 0, \quad \forall n,j && (2.82)\\ \rho_j^0 &= \rho^0(x_j) && (2.83)\end{aligned}$$

Lemma : if $\rho \in C^3_U(\mathbb{R}_+\times\mathbb{R})$,

$$\exists C, \quad |\varepsilon_j^n| \leq C(\Delta t+\Delta x^2) \tag{2.84}$$

with

$$\varepsilon_j^n=\frac{\rho(t^{n+1},x_j)-\rho(t^n,x_j)}{\Delta t}+u\frac{\rho(t^n,x_{j+1})-\rho(t^n,x_{j-1})}{2\Delta x} \tag{2.85}$$

Proof. See A.2. □

Study the stability.

$$\rho_j^{n+1}=\rho_j^n+\rho_{j-1}^n\left(u\frac{\Delta t}{2\Delta x}\right)+\rho_{j+1}^n\left(-u\frac{\Delta t}{2\Delta x}\right) \tag{2.86}$$

This is not a convex combination of the ρ_k^n !
The scheme is not stable in the norm $\|.\|_\infty$ in the sense that we do not have : $\|\rho^{n+1}\|_\infty \leq \|\rho^n\|_\infty$.
Actually, an inequality as :

$$\|\rho^{n+1}\|_\infty \leq \|\rho^n\|_\infty(1+C\Delta t) \qquad \text{(we assume this is true !)} \tag{2.87}$$

would be enough.
Indeed, it would ensure :

$$\begin{aligned}\|\rho^n\|_\infty &\leq \|\rho^0\|_\infty(1+C\Delta t)^n && (2.88)\\ &\leq \|\rho^0\|_\infty e^{Cn\Delta t} && (2.89)\\ &= \|\rho^0\|e^{Ct^n} && (2.90)\end{aligned}$$

We can prove that [2] :

$$e_j^{n+1}=e_j^n+u\frac{\Delta t}{2\Delta x}e_{j-1}^n-u\frac{\Delta t}{2\Delta x}e_{j+1}^n-\Delta t\varepsilon_j^n \tag{2.91}$$

Consequence of this is concerning the error analysis :

$$\|e^{n+1}\|_\infty \leq \|e^n\|_\infty(1+C\Delta t)+\Delta t\|\varepsilon^n\|_\infty \tag{2.92}$$

[2] See A.3

Thanks to a discrete Gronwall Lemma [3],

$$\|e^n\|_\infty \leq \|e^0\|_\infty e^{Ct^n} + \sum_{k=0}^{n-1} \|\varepsilon^k\|_\infty e^{C(t^n - t^k)} \tag{2.93}$$

and finally :

$$\|e^n\|_\infty \leq e^{Ct^n} n\Delta t \sup_n \|\epsilon^n\|_\infty \tag{2.94}$$

This **would be** a convergence estimate.
With the present scheme, it is possibleto prove that :

$$\|\rho^{n+1}\|_2 \leq \|\rho^n\|_2(1 + C\Delta t) \tag{2.95}$$

PROOF.
Let us prove this estimation :

$$\|\rho^n\|_2 = \left(\sum_j (\rho_j^n)^2\right)^{1/2} \tag{2.96}$$

We have :

$$\begin{aligned}
\left(\rho_j^{n+1}\right)^2 &= \left(\rho_j^n + u\frac{\Delta t}{2\Delta x}\rho_{j-1}^n - u\frac{\Delta t}{2\Delta x}\rho_{j+1}^n\right)^2 && (2.97)\\
&= \left(\rho_j^n + \frac{\nu}{2}\rho_{j-1}^n - \frac{\nu}{2}\rho_{j+1}^n\right)^2, \quad \text{(with } \nu = u\frac{\Delta t}{2\Delta x} \text{ the Courant number)} && (2.98)\\
&= {\rho_j^n}^2 + \frac{\nu^2}{4}{\rho_{j-1}^n}^2 - \frac{\nu^2}{4}{\rho_{j+1}^n}^2 + \nu\rho_j^n\rho_{j-1}^n - \nu\rho_j^n\rho_{j+1}^n - \frac{\nu^2}{2}\rho_{j-1}^n\rho_{j+1}^n && (2.99)\\
&\leq {\rho_j^n}^2 + \frac{\nu^2}{4}({\rho_{j-1}^n}^2{\rho_{j+1}^n}^2) + \frac{\nu^2}{4}({\rho_{j-1}^n}^2 + {\rho_{j+1}^n}^2) + \nu\rho_j^n(\rho_{j-1}^n - \rho_{j+1}^n) && (2.100)
\end{aligned}$$

(because $ab < \frac{a^2+b^2}{2}$).
Then :

$$\sum_j \left(\rho_j^{n+1}\right)^2 \leq \sum_j \left(\rho_j^n\right)^2 (1 + \nu^4) + 0 \tag{2.101}$$

$$\sum_j \left(\rho_j^{n+1}\right)^2 \leq \sum_j \left(\rho_j^n\right)^2 \underbrace{(1 + u^2\frac{\Delta t^2}{\Delta x^2})}_{1+C\Delta t} \tag{2.102}$$

If $u^2\frac{\Delta t^2}{\Delta x^2} = C\Delta t$, then :

$$u^2\Delta t = C\Delta x^2 \Rightarrow \Delta t = C\frac{\Delta x^2}{u^2}.$$

RESULTS.
If $\Delta t = C\frac{\Delta x^2}{u^2}$, one has :

$$\|\rho^{n+1}\|_2 \leq \|\rho^n\|_2(1 + C\Delta t) \tag{2.103}$$

[3]Proof in exercice, see A.4

and the scheme is convergent in the norm $\|.\|_2$. Note that this is false for the $\underbrace{\textbf{downwind}}_{(load)}$ scheme.

Remark : this scheme is not to be used because very costly due to $\Delta t \sim D\Delta x$. Other technique to prove the stability in ℓ^2 : by **Fourier Transform**.
Assume that $(\rho^n)_j \in \ell^2$. Let us define :

$$\rho^n\Delta(x) = \sum_{j\in\mathbb{Z}} \rho_j^n 1_{c_j}(x) \tag{2.104}$$

$$\text{with} \quad c_j = \left[x_j - \frac{\Delta x}{2}; x_j + \frac{\Delta x}{2}\right[\tag{2.105}$$

We have $\rho^n\Delta\ inL^2$ and $\|\rho^n\Delta\|_{L^2} = \sqrt{\Delta x}\|\rho^n\|_{\ell^2}$.
For any $f \in L^2(\mathbb{R})$, define :

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}dx \tag{2.106}$$

Thanks to the Perseval identity, $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$.
Furthermore, $f(.\hat{+}\tau)(\xi) = \hat{f}(\xi)e^{i\tau\xi}$. Consider the scheme :

$$\rho_j^{n+1} = \rho_j^n + u\frac{\Delta t}{2\Delta x}\rho_{j-1}^n - u\frac{\Delta t}{2\Delta x}\rho_{j+1}^n \tag{2.107}$$

We have :

$$\rho^{n+1}\Delta = \rho^n\Delta + u\frac{\Delta t}{2\Delta x}\rho^n\Delta(.-\Delta x) - u\frac{\Delta t}{2\Delta x}\rho^n\Delta(.+\Delta x) \tag{2.108}$$

$$\Rightarrow \|\rho^{n+1}\Delta\|_{\ell^2} = \frac{1}{\sqrt{\Delta x}}\|\rho^{n+1}\Delta\|_{L^2} \tag{2.109}$$

$$\widehat{\rho^{n+1}}\Delta(\xi) = \hat{\rho^n}\Delta(\xi) + \frac{\nu}{2}\hat{\rho^n}\Delta(\xi)e^{-i\xi\Delta x} - \frac{\nu}{2}\hat{\rho^n}\Delta(\xi)e^{i\xi\Delta x} \tag{2.110}$$

$$= \hat{\rho^n}\Delta(\xi)\left(1 - \frac{\nu}{2}(e^{i\xi\Delta x} - e^{-i\xi\Delta x})\right) \tag{2.111}$$

$$= \hat{\rho^n}\Delta(\xi)\left(1 - i\nu\sin(\xi\Delta x)\right) \tag{2.112}$$

Thus,

$$\|\rho^{n+1}\|_{\ell^2} \leq \frac{1}{\sqrt{\Delta x}}\|\hat{\rho^n}\Delta\|_{L^2}\sup_{\xi}|1 - i\nu\sin(\xi\Delta x)| \tag{2.113}$$

$$= \|\rho^n\|_{\ell^2}\sup_{\xi}|1 - i\nu\sin(\xi\Delta x)| \tag{2.114}$$

$$= \|\rho^n\|_{\ell^2}\sup_{\xi}\left(1 + \nu^2\sin(\xi\Delta x)\right)^{\frac{1}{2}} \tag{2.115}$$

We recover the condition : $\Delta t = D\Delta x^2$ in order the amplification factor $1 - i\nu\sin(\xi\Delta x)$ to be uniformly bounded (w.r.t. ξ) by $(1 + C\Delta t)$.
Exercise.
The Lax-Wendroff scheme is :

$$\rho_j^{n+1} = \rho_j^n - \frac{\nu}{2}\left(\rho_{j+1}^n - \rho_{j-1}^n\right) + \frac{\nu^2}{2}\left(\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n\right) \tag{2.116}$$

Prove that this scheme is 2^{nd} order (in Δt and Δx) consistent. Prove that the scheme is ℓ^2-stable $\left(\|\rho^{n+1}\|_{\ell^2} \leq \|\rho^n\|_{\ell^2}\right)$ (and convergent) under the CFL condition $\nu \leq 1$.

Proof. See A.5. ◻

Recall. We have convergence of the upwind scheme for $\partial_t \rho + u\partial_x \rho = 0$ under the CFL condition $\frac{u\Delta t}{\Delta x} \leq 1$. We have convergence of the centered scheme if $\Delta t \sim \Delta x^2$. However, the centered scheme is not interesting.

Q: In the case where u is not constant?

There is a natural discretisation:

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + u_j^n \frac{\rho_{j+\frac{1}{2}}^n - \rho_{j-\frac{1}{2}}^n}{\Delta x} = 0 \tag{2.117}$$

with a formula to compute $\rho_{j+\frac{1}{2}}^n$ as a function of ρ_k^n.

If for all j and n, $u_j^n \geq 0$, then $\rho_{j+\frac{1}{2}}^n = \rho_j^n$ is the upwind scheme and

$$\rho_j^{n+1} = \rho_j^n(1 - u_j^n \frac{\Delta t}{\Delta x}) + u_j^n \frac{\Delta t}{\Delta x}\rho_{j-1}^n \tag{2.118}$$

.

If for all j and n, $u_j^n \frac{\Delta t}{\Delta x} \leq 1$, we recover the maximum priciple:

$$|\rho_j^{n+1}| \leq \sup_k |\rho_k^n| \tag{2.119}$$

.

And the error of the scheme $e_j^n = \rho_j^n - \rho(t^n, x_j)$ satisfies

$$e_j^{n+1} = e_j^n(1 - u_j^n \frac{\Delta t}{\Delta x}) + e_{j-1}^n u_j^n \frac{\Delta t}{\Delta x} - \Delta t \varepsilon_j^n \tag{2.120}$$

.

Exercise 1.

- Prove that $|\varepsilon_j^n| \leq C(\Delta t + \Delta x)$ if ρ is smooth.
- Prove the convergence of the scheme under the CFL condition.

Proof. See A.6. ◻

Q: And when u changes sign?

Naively, we can define:

$$\rho_{j+\frac{1}{2}}^n = \begin{cases} \rho_j^n & \text{if } u_j^n \geq 0 \\ \rho_{j+1}^n & \text{if not.} \end{cases} \tag{2.121}$$

and

$$\rho_{j-\frac{1}{2}}^n = \begin{cases} \rho_{j-1}^n & \text{if } u_j^n \geq 0 \\ \rho_j^n & \text{if not} \end{cases} \tag{2.122}$$

. But then

$$u_j^n \geq 0 \Rightarrow \rho_{j+\frac{1}{2}}^n = \rho_j^n \text{ (Left winded)}$$
$$u_{j+1}^n < 0 \Rightarrow \rho_{j+\frac{1}{2}}^n = \rho_{j+1}^n \text{ (Right winded)}$$

the results are not compatibles.

Consider another idea, we write a scheme in the form

$$\rho_j^{n+1} = \rho_j^n - u_{j+\frac{1}{2}}^n \frac{\Delta t}{\Delta x}(\rho_{j+\frac{1}{2}}^n - \rho_j^n) - u_{j-\frac{1}{2}}^n \frac{\Delta t}{\Delta x}(\rho_j^n - \rho_{j-\frac{1}{2}}^n) \tag{2.123}$$

with $u_{j+\frac{1}{2}}^n = u(t^n, (j+\frac{1}{2})\Delta x)$. And we can upwind the interface value.

$$\rho_{j+\frac{1}{2}}^n = \begin{cases} \rho_j^n & \text{if } u_{j+\frac{1}{2}}^n \geq 0 \\ \rho_{j+1}^n & \text{if not} \end{cases} \tag{2.124}$$

. This leads to the scheme

$$\begin{aligned}\rho_j^{n+1} &= \rho_j^n - (u_{j+\frac{1}{2}}^n)^+ \frac{\Delta t}{\Delta x} \cdot 0 + (u_{j+\frac{1}{2}}^n)^- \frac{\Delta t}{\Delta x}(\rho_{j+1}^n - \rho_j^n) - (u_{j-\frac{1}{2}}^n)^+ \frac{\Delta t}{\Delta x}(\rho_j^n - \rho_{j-1}^n) + (u_{j-\frac{1}{2}}^n)^+ \frac{\Delta t}{\Delta x} \cdot 0 \\ &= \rho_j^n \left(1 - (u_{j+\frac{1}{2}}^n)^- \frac{\Delta t}{\Delta x} - (u_{j-\frac{1}{2}}^n)^+ \frac{\Delta t}{\Delta x}\right) + (u_{j+\frac{1}{2}}^n)^- \frac{\Delta t}{\Delta x}\rho_{j+1}^n + (u_{j-\frac{1}{2}}^n)^+ \frac{\Delta t}{\Delta x}\rho_{j-1}^n\end{aligned}$$

. Assume that $\frac{\Delta t}{\Delta x}\left[(u_{j+\frac{1}{2}}^n)^- + (u_{j-\frac{1}{2}}^n)^+\right] \leq 1$ [4], then ρ_j^{n+1} is a convexe combination of ρ_{j+1}^n, ρ_j^n and ρ_{j-1}^n, thus $|\rho_j^{n+1}| \leq \sup_k |\rho_j^n|$.

Exercise 2.

- Prove that the scheme is consistent.
- Prove that it is convergent under the CFL condition.

Proof. See A.7. □

For the conservative equation: $\partial_t \rho + \partial_x \rho u = 0$. (With u non constant, smooth and given). Natuarally, we have

$$\rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{\Delta x}(\rho_{j+\frac{1}{2}}^n u_{j+\frac{1}{2}}^n - \rho_{j-\frac{1}{2}}^n u_{j-\frac{1}{2}}^n) \tag{2.125}$$

with $u_{j+\frac{1}{2}}^n = u(t^n, (j+\frac{1}{2})\Delta x)$. And we can define the upwind interface values:

$$\rho_{j+\frac{1}{2}}^n = \begin{cases} \rho_j^n & \text{if } u_{j+\frac{1}{2}}^n \geq 0 \\ \rho_{j+1}^n & \text{if not} \end{cases} \tag{2.126}$$

.

Q: Do we have the maximum principle? NO! Because the exact solution to $\partial_t \rho + \partial_x \rho u = 0$ does not satisfy the maximum principle. ($\partial_t \rho + u\partial_x \rho = \rho \partial_x u$.) But,

$$\rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{\Delta x} u_{j+\frac{1}{2}}^n (\rho_{j+\frac{1}{2}}^n - \rho_j^n) - \frac{\Delta t}{\Delta x} u_{j-\frac{1}{2}}^n (\rho_j^n - \rho_{j-\frac{1}{2}}^n) - \frac{\Delta t}{\Delta x} \rho_j^n (u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n) \tag{2.127}$$

[4] Consider $(u_{j+\frac{1}{2}}^n)^- + (u_{j-\frac{1}{2}}^n)^+$ as a numerical approximation for $|u|$. This is coherent to the CFL condition that we've seen before.

. Thus, under the CFL condition ($\frac{\Delta t}{\Delta x}\left[(u^n_{j+\frac{1}{2}})^- + (u^n_{j-\frac{1}{2}})^+\right] \leq 1$), we have:

$$\begin{aligned}
|\rho_j^{n+1}| &\leq \sup_k |\rho_k^n| + \frac{\Delta t}{\Delta x}|\rho_j^n| \cdot |u^n_{j+\frac{1}{2}} - u^n_{j-\frac{1}{2}}| \\
&\leq \sup_k |\rho_k^n| + \Delta t|\rho_j^n| \cdot \|\partial_x u\|_\infty \\
&\leq \sup_k |\rho_k^n|(1 + \Delta t\|\partial_x u\|_\infty) \\
\Rightarrow |\rho_j^n| &\leq \sup_k |\rho_k^0| e^{n\Delta t\|\partial_x u\|_\infty} \quad \forall j
\end{aligned}$$

Remark. Compare this to the expression of the exact solution.

The error satisfies

$$|e_j^{n+1}| \leq \sup_k |e_k^n|(1 + \Delta t\|\partial_x u\|_\infty) + \Delta t|\varepsilon_j^n| \tag{2.128}$$

with

$$\begin{aligned}
\varepsilon_j^n = \frac{\rho(t^{n+1}, x_j) - \rho(t^n, x_j)}{\Delta t} &+ \frac{[u(t^n, x_{j+\frac{1}{2}})]^+\rho(t^n, x_j) - [u(t^n, x_{j+\frac{1}{2}})]^-\rho(t^n, x_{j+1})}{\Delta x} \\
&+ \frac{[u(t^n, x_{j-\frac{1}{2}})]^+\rho(t^n, x_{j-1}) - [u(t^n, x_{j-\frac{1}{2}})]^-\rho(t^n, x_j)}{\Delta x}
\end{aligned}$$

(where $x_{j+\frac{1}{2}} = (j + \frac{1}{2})\Delta x$).

Exercise 3. Prove that there exists C such that $|\varepsilon_j^n| \leq C(\Delta t + \Delta x)$.

Thus

$$|e_j^{n+1}| \leq \sup_k |e_k^n|(1 + \Delta t\|\partial_x u\|_\infty) + \Delta t C(\Delta t + \Delta x) \tag{2.129}$$

. Thanks to a discrete Gronwall Lemma,

$$\begin{aligned}
\exists C \quad |e_j^n| &\leq \sup_k |e_k^0| e^{n\Delta t\|\partial_x u\|_\infty} + \sum_{l=0}^{n-1} \Delta t C(\Delta t + \Delta x) e^{(n-l)\Delta t\|\partial_x u\|_\infty} \\
&\leq n\Delta t C(\Delta t + \Delta x) e^{n\Delta t\|\partial_x u\|_\infty}
\end{aligned}$$

. The scheme is first order convergent.

Remark. The numerical analysis of transport equation is more difficult than their analysis. This is due to the fact that there is no (simple) formula to express ρ_j^n as a function of the ρ_k^0. Precisely, there is no k such that $\rho_j^n = \rho_k^0$, although for $\partial_t \rho + u\partial = 0$, we have $\rho(t, x) = \rho(0, X(0, t, x)) = \rho(0, t)$, except when u is constant and $u\frac{\Delta t}{\Delta x} = 1$ (for the upwind scheme).

Indeed, if $u > 0$,

$$\rho_j^{n+1} = \rho_j^n - u\frac{\Delta t}{\Delta x}(\rho_j^n - \rho_{j-1}^n) = \rho_j^n - (\rho_j^n - \rho_{j-1}^n) = \rho_{j-1}^n \tag{2.130}$$

and $\rho_j^n = \rho_{j-n}^0$). In this case, the scheme is exact. ($\rho(t^n, x_j) = \rho(0, x_{j-n})$)

Transport equations in bounded domain are very complicate problems (cf. practical session). In dimension 1, consider

$$\left\{ \begin{array}{rl} \partial_t \rho + u\partial_x \rho = & 0 \\ \rho(0,x) = & \rho_0(x) \end{array} \right. \quad t > 0\, x \in [0,1] \tag{2.131}$$

. It is natural to impose the value of ρ on the inflow boundary $x = 0$ if $u > 0$; $x = 1$ if $u < 0$.

Indeed, consider that $u > 0$, then $\rho(t, x = 1) = \rho(0, 1 - ut)$. For t sufficiently small, $1 - ut \in (0,1)$, thus we cannot ask $\rho(t, x = 1)$ to be equal to a boundary datum. Conversely, $\rho(t, x = 0) = \rho(0, \underbrace{0 - ut}_{\notin(0,1)})$.

Thus in this case ($u > 0$), the problem

$$\left\{ \begin{array}{rll} \partial_t \rho + u\partial_x \rho = & 0 & t > 0\, x \in [0,1] \\ \rho(0,x) = & \rho_0(x) & x \in [0,1] \\ \rho(t,0) = & \rho_0(t) & \text{given} \end{array} \right. \tag{2.132}$$

is well posed. In dimension $d > 1$, the condition has to be imposed on $x \in \partial\Omega$ where $u \cdot \mathbf{n} > 0$.

2.1.3 Generalization of transport equation: hyperbolic Systems

The PDE system $\partial_t u + A\partial_x u = 0 \in \mathbb{R}^n$, $t \in \mathbb{R}^+$, $x \in \mathbb{R}$, where $u(t,x) \in \mathbb{R}^n$ and $A \in \mathcal{M}(\mathbb{R})$ is said **hyperbolic** if A is $\mathbb{R}$-diagonalizable; **strictly hyperbolic** if A has n distinct real eigenvalues.

Consider

$$\left\{ \begin{array}{rll} \partial_t u + A\partial_x u = & 0 & t > 0,\, x \in \mathbb{R} \\ u(0,\cdot) & = u^0 & \end{array} \right. \tag{2.133}$$

and assume the system is hyperbolic. $A = PDP^{-1}$ where D is a real diagonal matrix. We have

$$\begin{aligned} & P^{-1}(\partial_t u) + P^{-1}A\partial_x u = 0 \\ & \partial_t(P^{-1}u) + P^{-1}AP\partial_x(P^{-1}u) = 0 \\ & \partial_t(P^{-1}u) + D\partial_x(P^{-1}u) = 0 \\ \forall k \quad & \partial_t(P^{-1}u)_k + D_{kk}\partial_x(P^{-1}u)_k = 0 \\ & \Rightarrow (P^{-1}u)_k(t,x) = (P^{-1}u)_k(0, x - \lambda_k t) \\ & \Rightarrow \text{u is known} \end{aligned}$$

The PDE system $\partial_t u + \partial_x f(u) = 0 \in \mathbb{R}^n$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a given (smooth) function is said **(strictly) hyperbolic** for $u \in O \in \mathbb{R}^n$ if $\partial_t u + \nabla_u f(u) \cdot \partial_x u = 0$ is (strictly) hyperbolic for $u \in O$.

$$\nabla_u f(u) = \begin{bmatrix} \partial_1 f_1(u) & \partial_2 f_1(u) & \dots & \partial_n f_1(u) \\ \partial_1 f_2(u) & \partial_2 f_2(u) & & \vdots \\ \vdots & & \ddots & \vdots \\ \partial_1 f_n(u) & \dots & \dots & \partial_n f_n(u) \end{bmatrix} \in \mathcal{M}_n(\mathbb{R}) \tag{2.134}$$

.

Example.

- $\partial_t \rho + u\partial_x \rho = 0 \in \mathbb{R}$ is hyperbolic.
- $\partial_t u + \partial_x f(u) = 0 \in \mathbb{R}$ where $f : \mathbb{R} \to \mathbb{R}$ is given smooth is hyperbolic in $\mathbb{R}$.
- If $f(u) = \frac{u^2}{2}$, it is called **Burgers Equation**

$$\partial_t u + \partial_x \frac{u^2}{2} = 0 \Leftrightarrow \partial_t u + u\partial_x u = 0$$

.

- $$\begin{cases} \partial_t \rho + \partial_x \rho u &= 0 \in \mathbb{R} \\ \partial_t \rho u + \partial_x (\rho u^2 + p(\rho)) &= 0 \in \mathbb{R} \end{cases} \tag{2.135}$$

  where $p(\rho)$ is smooth given, is hyperbolic (in $\{(\rho, u) | \rho > 0 u \in \mathbb{R}\}$). This is **Euler system of inviscid compressible gaz** (u is velocity and ρ is mass density).

  Proof. Indeed,

  $$f : \begin{pmatrix} \rho \\ \rho u \end{pmatrix} \mapsto \begin{pmatrix} \rho u \\ \rho u^2 + p(\rho) \end{pmatrix} \tag{2.136}$$

  . Assume that $\rho > 0$. We note $q = \rho u$ then

  $$f : \begin{pmatrix} \rho \\ q \end{pmatrix} \mapsto \begin{pmatrix} q \\ \frac{q^2}{\rho} + p(\rho) \end{pmatrix} \tag{2.137}$$

  . We can calculate

  $$\nabla f(\rho, \rho u) = \begin{bmatrix} 0 & 1 \\ -u^2 + p'(\rho) & 2u \end{bmatrix} \tag{2.138}$$

  . Determination of the eigenvalues λ:

  $$\begin{aligned} & -\lambda(2u - \lambda) + u^2 - p'(\rho) = 0 \\ & \lambda^2 - 2u\lambda + u^2 - p'(\rho) = 0 \\ & \Delta = 4u^2 - 4(+u^2 - p'(\rho)) = 4p'(\rho) \\ & \Rightarrow \lambda_\pm = \frac{2u \pm 2\sqrt{p'(\rho)}}{2} = u \pm \sqrt{p'(\rho)} \in \mathbb{R} \text{ if } p'(\rho) \geq 0 \end{aligned}$$

  □

Study of scalar case Consider $\partial u + \partial_x f(u) = 0$ where $f : \mathbb{R} \to \mathbb{R}$ is a smooth given function and try to solve the Cauchy Problem with $u(0, \cdot) = u^0$ (smooth). If u is a smooth $\mathcal{C}^1$ solution, u satisfies $\partial_t u + f'(u)\partial_x u = 0$. $f'(u)$ is the transport velocity of the unknown u. The characteristic curves associated with the problem satisfy

$$\begin{cases} \partial_t X(t,x) = & f'(u(t, X(t,x))) \\ X(0,x) = & x \end{cases} \tag{2.139}$$

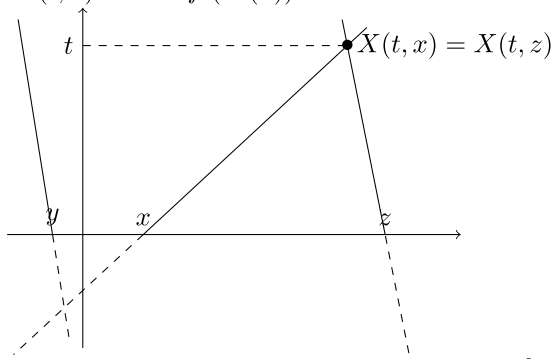
. We know that u is constant along these curves

$$u(t, X(t,x)) = u(0,x) = u^0(x)$$

The X satisfies

$$\left\{\begin{array}{rl} \partial_t X(t,x) = & f'(u^0(x)) \\ X(0,x) = & x \end{array}\right. \tag{2.140}$$

and $X(t,x) = x + tf'(u^0(x))$. The characteristics are straight lines.



In order to have $X(t,x) = X(t,z)$, we need $u^0(x) = u^0(z)$. However, $f'(u^0(x)) \neq f'(u^0(z))$. We have a contradiction.

Remark. The only assumption done is u is a smooth $\mathcal{C}^1$ solution.

Theorem 3. *Assume that $f \in \mathcal{C}^2(\mathbb{R})$, $u^0 \in \mathcal{C}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $u^{0\prime} \in L^\infty(\mathbb{R})$. Let*

$$T = \left\{\begin{array}{rl} -\frac{1}{\inf_y f''(u^0(y))u^{0\prime}(y)} & \text{if } \exists y \text{ s.t. } f''(u^0(y))u^{0\prime}(y) < 0 \\ \infty & \text{if not } (\forall y f''(u^0(y))u^{0\prime}(y) \geq 0) \end{array}\right. \tag{2.141}$$

Then,

$$\left\{\begin{array}{rll} \partial_t u + \partial_x f(u) = & 0 & t \in]0,T[\ x \in \mathbb{R} \\ u(0,x) & = u^0(x) & \end{array}\right. \tag{2.142}$$

admits a unique $\mathcal{C}^1$ maximal solution. There is no $\mathcal{C}^1$ solution for times greater than T. Furthurmore, $u(t,X(t,x)) = u^0(x) \quad \forall t$ [5] where X satisfies

$$\left\{\begin{array}{rl} \partial_t X(t,x) = & f'(u(t,X(t,x))) \\ X(0,x) = & x \end{array}\right. \tag{2.143}$$

Proof. Assume u is a bounded $\mathcal{C}^1$ solution, and define $X(t,x)$ as the solution to

$$\left\{\begin{array}{rl} \partial_t X(t,x) = & f'(u(t,X(t,x))) \\ X(0,x) = & x \end{array}\right.$$

Define $\tilde{u} = u\left(t, X(t,x)\right)$. Then,

$$\begin{array}{rcl} \partial_t \tilde{u}(t,x) & = & \partial_t u\left(t, X(t,x)\right) + \partial_t X(t,x) \partial_x u\left(t, X(t,x)\right) \\ & = & \partial_t u\left(t, X(t,x)\right) + f' \circ u\left(t, X(t,x)\right) \partial_x u\left(t, X(t,x)\right) \\ & = & \partial_t u\left(t, X(t,x)\right) + \partial_x f \circ u\left(t, X(t,x)\right) \\ & = & 0 \text{ because } u \text{ satisfies } \partial_t u + \partial_x f(u) \end{array}$$

Thus $\tilde{u}(t,x) = \tilde{u}(0,x)$, $u\left(t, X(t,x)\right) = u\left(0, X(0,x)\right) = u^0(x)$. This does not give a solution $u(t,x) \forall t, x$.
Given (t,x), are we able to find $y \in \mathbb{R}$ s.t. $X(t,y) = x$?
As $u\left(t, X(t,x)\right) = u^0(x)$, $f'\left(u\left(t, X(t,x)\right)\right) = f'\left(u^0(x)\right)$ and $X(t,x) = x + tf'\left(u^0(x)\right)$.
Given (t,x), we look for $y \in \mathbb{R}$ s.t. $x = y + tf'\left(u^0\right)$ (= **non-linear problem**).
It's a non-linear problem because we it's assume it's **not a bounded function**.
Denote $X_t(x) = x + tf'\left(u^0(x)\right)$.
We want to inverse X_t : we look for y s.t. $X_t(y) = x, y = X_t^{-1}(x)$.
X_t is a $\mathcal{C}^1$ function over $\mathbb{R}$:
$X_t'(x) = 1 + tf''\left(u^0(x)\right) {u^0}'(x)$ so $X_0'(x) = 1, \forall X_0$ is **invertible** !.
Does X_t remain one-to-one from $\mathbb{R} \mapsto \mathbb{R}$?

- If $f''\left(u^0(x)\right) {u^0}'(x) \geq 0$, then $X_t'(x) \geq 1, \forall t, x$ and X_t is one-to-one (=**bijection**).
  In this case, u is defined $\forall t \geq 0$ but not for $t < 0$!
- If not, define $m = \inf\limits_{x \in \mathbb{R}} f''\left(u^0(x)\right) {u^0}'(x)$ according to the assumptions on f and u^0. 88 Then,

$$\begin{array}{rcl} X_t(x) = 1 + tf''\left(u^0(x)\right) {u^0}'(x) & \geq & 1 + tm \qquad \text{(if } t \geq 0\text{)} \\ & \geq & \epsilon \qquad \text{(if } 1 + tm \geq \epsilon \Leftrightarrow t \leq \dfrac{\epsilon - 1}{m}\text{)} \end{array}$$

- $t < -\frac{1}{m}$, then $X_t'(x) \geq \epsilon$ for a certain $\epsilon > 0$.
  Thus X_t is one-to-one if $t < -\frac{1}{m} : \forall x \in \mathbb{R}, \exists! y / X_t(y) = x$ and $u(t,x) = u^0(y) = u^0\left(X_t^{-1}(x)\right)$. It is not possible to define a $\mathcal{C}^1$ solution after $T = -\frac{1}{m}$.
  We have

$$\begin{array}{rcl} \partial_x u(t,x) & = & {u^0}'\left(X_t^{-1}(x)\right) {X_t^{-1}}'(x) \\ & = & \dfrac{{u^0}'\left(X_t^{-1}(x)\right)}{{X_t}'\left({X_t^{-1}}'(x)\right)} \\ & = & \dfrac{{u^0}'\left(X_t^{-1}(x)\right)}{1 + tf''\left(u^0({X_t^{-1}}'(x))\right)\left({X_t^{-1}}'(x)\right)} \end{array}$$

  As $t \to T = -\frac{1}{m}$, then $\|\partial_x u(t,.)\|_\infty \to +\infty$
  (we assume that $m = \inf\limits_{x \in \mathbb{R}} f''\left(u^0(x)\right) {u^0}'(x) = f''\left(u^0(\xi)\right) {u^0}'(\xi)$).
  Then,

$$\frac{1}{1 + tf''\left(u^0(\xi)\right) {u^0}'(\xi)} \xrightarrow[t \to -\frac{1}{m}]{} +\infty.$$

$\square$

EXAMPLE :

$$f(u) = \frac{u^2}{2} \Rightarrow f'(u) = u$$

If $u \in \mathcal{C}^1$, it satisfies

$$\partial_t u + u\partial_x u = 0 \qquad \text{(Burger's equation)}$$

- If ${u^0}'(x) \geq 0 \forall x$: it's a global solution.
- If not : the $\mathcal{C}^1$ is not global and $T = -\frac{1}{\inf\limits_x {u^0}'(x)}$.

This is not satisfying. We would like to be able to consider solutions for larger times : the weak solutions (= distribution solutions).

- If u is solution to :

$$\begin{array}{rcll} \partial_t u + \partial_x f(u) & = & 0, & t > 0 \\ u(0, .) & = & u^0 & \end{array}$$

Then,

$$\begin{array}{rcl} \rho \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}) : \displaystyle\int_{\mathbb{R}_+} \int_{\mathbb{R}} \rho\partial_t u + \rho\partial_x f(u) & = & 0, \\ \Leftrightarrow \displaystyle\int_{\mathbb{R}} \underbrace{\int_{\mathbb{R}_+} \rho\partial_t u}_{-\rho(0)u(0) - \int u\partial_t\rho} + \int_{\mathbb{R}_+} \underbrace{\int_{\mathbb{R}} \rho\partial_x f(u)}_{-\int f(u)\partial_x\rho} & = & 0, \\ \Leftrightarrow -\displaystyle\int_{\mathbb{R}} u(0,x)\rho(0,x)dx - \int_{\mathbb{R}} \int_{\mathbb{R}_+} u\partial_t\rho - \int_{\mathbb{R}_+} \int_{\mathbb{R}} f(u)\partial_x\rho & = & 0 \end{array}$$

u satisfies :

$$\int_{\mathbb{R}} \int_{\mathbb{R}_+} u\partial_t\rho + f(u)\partial_x\rho dxdt = -\int_{\mathbb{R}} u^0(x)\rho(0,x)dx \quad \forall \rho \in \mathcal{C}_C^\infty(\mathbb{R}_+ \times \mathbb{R}) \qquad (D)$$

- If $u \in L^\infty$, this makes sense.

Weak Solution : We say that u is a **weak solution** to $(\mathcal{C})$ if it is a solution to (D).
Indeed, (D) admits global solutions.

Example. Discontinuous solutions and the Rankine-Hugoniot relation.
Let us look for a solution in the form $u(t,x) = u_L + (u_R - u_L)H_{\sigma t}(x)$ with $u_L \in \mathbb{R}, u_R \in \mathbb{R}, \sigma \in \mathbb{R}$ and :

$$H_\xi(x) = \left\{ \begin{array}{ll} 0 & \text{if } x \leq \xi \\ 1 & \text{if } x > \xi \end{array} \right. \qquad \text{(Heaviside function)}$$

$$\Rightarrow u(t,x) = \left\{ \begin{array}{ll} u_L & \text{if } x \leq \sigma t \\ u_R & \text{if } x > \sigma t \end{array} \right.$$

In the linear case,

$$\begin{array}{rcl} f(u) & = & au \\ \partial_t u + a\partial_x u & = & 0 \\ u(t,x) & = & u^0(x-at) \end{array}$$

and we would like to define $u^0(x-at)$ as solution to :

$$\left\{ \begin{array}{rl} \partial_t u + a\partial_x u = & 0 \\ u(0,.) = & u^0 \end{array} \right.$$

even if $u^0 \notin \mathcal{C}^1$.
If : $u^0 = u_L + (u_R - u_L)H_0(x)$, we would like to consider $u_L + (u_R - u_L)H_{at}(x)$ as a solution.

$$u(t,x) = u_L + (u_R - u_L)H_{\sigma t}(x) \qquad \forall u_L, u_R \in \mathbb{R}$$

Let's study :

$$\partial_t u + \partial_x f(u) = 0.$$

$$\begin{array}{rcl} \partial_t u & = & 0 + (u_R - u_L)(-\sigma)\delta(x-\sigma t) \\ & = & -\sigma(u_R - u_L)\delta_{\sigma t}(x). \end{array}$$

$$\partial_x f(u) = 0 + (f(u_R) - f(u_L))\delta_{\sigma t}(x).$$

Indeed,

$$f(u) = f(u_L) + (f(u_R) - f(u_L))H_{\sigma t}(x).$$

Thus,

$$\begin{array}{rcl} (-\sigma(u_R - u_L) + (f(u_R) - f(u_L))\delta_{\sigma t} & = & 0 \qquad (D') \\ \Leftrightarrow \sigma(u_R - u_L) & = & (f(u_R) - f(u_L)). \end{array}$$ **(= Rankine-Hugoniot relation)**

For any $u_L, u_R \in \mathbb{R}$, there exists such a discontinuous solution, translated at **velocity** $\sigma = \frac{f(u_R)-f(u_L)}{u_R - u_L}$ (assume $u_L \neq u_R$).

- $f(u) = au, \quad \sigma = a \quad (\Rightarrow a(t_i) = u^0(. - at))$
- $f(u) = \frac{u^2}{2}, \quad \sigma = \frac{\frac{u_R^2}{2} - \frac{u_L^2}{2}}{u_R - u_L} = \frac{u_L + u_R}{2}$. The weak formulation allows to consider more solutions... Even too much solutions !

Example.

$$f(u) = \frac{u^2}{2}, \quad u^0(x) = \left\{ \begin{array}{l} 0 \text{ (if } x < 0) \\ 1 \text{ (if } x \geq 0). \end{array} \right.$$

There are (at last) <u>2 different solutions</u> :

- $$u(t,x) = \begin{cases} 0 \text{ (if } x < \sigma t) \\ 1 \text{ (if } x \geq \sigma t). \end{cases}$$

  with $\sigma = \frac{\frac{1^2}{2} - \frac{0^2}{2}}{1-0} = \frac{1+0}{2} = \frac{1}{2}$ **(= Rankine-Hugoniot relation)**.

  $$\left(\sigma = \frac{f(u_R) - f(u_L)}{u_R - u_L} = \frac{[f(u)]}{[u]}\right)$$

- $$u(t,x) = \begin{cases} 0 & \text{(if } x < 0) \\ \frac{x}{t} & \text{(if } 0 \leq x \leq t) \\ 1 & \text{(if } x \geq t). \end{cases}$$

  Indeed u is continuous and :

  $$D_t u(t,x) = \begin{cases} x \in (-\infty, 0): \quad \partial_t 0 + \partial_x \frac{0}{2} = 0 \\ x \in [0,t): \quad \partial_t \left[\frac{x}{t}\right] + \partial_x \frac{\left(\frac{x}{t}\right)^2}{2} = -\frac{x}{t^2} + \frac{2x}{2t^2} = 0 \\ x \geq t: \quad \partial_t 1 + \partial_x \frac{1}{2} = 0. \end{cases}$$

  Here the continuity is bound so :

  $$\begin{aligned} \int_{\mathbb{R}_+} \int_{\mathbb{R}} u\partial_t\rho + \frac{u^2}{u}\partial_x\rho &= -\int_{\mathbb{R}} u^0 \rho(0,x) \\ &= -\int_{\mathbb{R}_+} \rho(0,x) \\ &= \int_{\mathbb{R}+} \int_{-\infty}^{0} u\partial_t\rho + \frac{u^2}{2}\partial_x\rho + \int_{\mathbb{R}+} \int_{0}^{t} u\partial_t\rho + \frac{u^2}{2}\partial_x\rho + \int_{\mathbb{R}+} \int_{t}^{+\infty} u\partial_t\rho + \frac{u^2}{2}\partial_x\rho \end{aligned}$$

  EXERCISE : TO PROVE THIS GO TO LECTURE NOTES !
  Well posed problem : the solution is unique and the solution depends on the data. Actually, there are an infinite number of solutions !

- For $u^0(x) = 0$, 0 is a solution but for $a > 0$:

  $$u(t,x) = \begin{cases} 0 & \text{(if } x < -\frac{a}{2}t) \\ -a & \text{(if } -\frac{a}{2}t \leq x < 0) \\ +a & \text{(if } x \leq 0 < \frac{a}{2}t) \\ 0 & \text{(if } x \geq \frac{a}{2}t) \end{cases}$$

  is a solution. These 3 discontinuities satisfy the **R-H relation** !

Ways to get rid of this **uniqueness problems** :

- Consider :

  $$\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon \partial_{xx} u^\epsilon \qquad \text{for } \epsilon > 0$$

  It is possible to prove that the solution to this problem is in $\mathcal{C}^\infty((0,+\infty)\times\mathbb{R})$ and let $\epsilon \to 0$.
  One can prove that $u^\epsilon \xrightarrow[\epsilon \to 0]{} u$ in $L^\infty\left((0,T), L^1(\mathbb{R})\right)$ for any T

  $$\left(\|y\|_{L^\infty((0,T),L^1(\mathbb{R}))} = \sup_{t\in(0,T)} \|g(t,.)\|_{L^1(\mathbb{R})}\right).$$

Thus, we define the "correct" solution to :

$$\partial_t u + \partial_x f(u) = 0$$

as the limit of (when $\epsilon \to 0$) :

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon \partial_{xx}^2 u^\epsilon.$$

- Observe that if $\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon \partial_{xx}^2 u^\epsilon$ for any $S : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^2(\mathbb{R})$ and convex ($S''(x) \geq 0 \forall x$).

$$S'(u^\epsilon)\partial_t u^\epsilon + S'(u^\epsilon) f(u^\epsilon) \partial_x u^\epsilon = \epsilon S'(u^\epsilon) \partial_{xx} u^\epsilon \tag{2.144}$$

and if $G'(u) = S'(u) f'(u), \forall u$:

$$\begin{aligned} \partial_t S(u^\epsilon) + \partial_x G(u^\epsilon) = & \quad \epsilon S'(u^\epsilon)\partial_x u^\epsilon - \epsilon \partial_x u^\epsilon \partial_x S'(u^\epsilon) & (2.145)\\ = & \quad \epsilon \partial_{xx}(S(u^\epsilon)) - \underbrace{\epsilon}_{\text{to have a sign}} \underbrace{S''(u^\epsilon)}_{\geq 0} \underbrace{(\partial_x u^\epsilon)^2}_{\geq 0 \text{ and } \xrightarrow[\epsilon \to 0]{} +\infty} . & (2.146) \end{aligned}$$

Thus, for any $\mathcal{C}^2$ and convex S, with G s.t. $G' = f'S'$ we have :

$$\partial_t S(u^\epsilon) + \partial_x G(u^\epsilon) \leq \epsilon \partial_{xx} S(u^\epsilon).$$

For any $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ s.t. $\rho(t,x) \geq 0, \forall t, x$:

$$\begin{aligned} \iint \rho(\partial_t S(u^\epsilon) + \partial_x G(u^\epsilon)) \quad & \leq \quad \epsilon \iint \rho \partial_{xx} S(u^\epsilon) \\ \Rightarrow - \int_{\mathbb{R}} S(u^\epsilon(0,x))\rho(0,x)dx - \int_{\mathbb{R}^+}\int_{\mathbb{R}} S(u^\epsilon)\partial_t \rho + G(u^\epsilon)\partial_x \rho dxdt \quad & \leq \quad \epsilon \iint S(u^\epsilon)\partial_{xx}\rho dxdt. \end{aligned}$$

Assume u^ϵ remains uniformly bounded (in L^∞) as $\epsilon \to 0$. Then, $\forall \rho, \forall S$,

$$\epsilon \iint S(u^\epsilon)\partial_{xx}\rho \xrightarrow[\epsilon \to 0]{} 0.$$

If $u^\epsilon \to u$,

$$\iint S(u)\partial_t \rho + G(u)\partial_x \rho \geq - \int_{\mathbb{R}} S(u^0)\rho(0,x)dx \tag{2.147}$$

This is the weak formulation of :

$$\partial_t S(u) + \partial_x G(u) \leq 0 \qquad (D') \tag{2.148}$$

A way to recover uniqueness for :

$$\left\{ \begin{array}{rl} \partial_t u + \partial_x f(u) = & 0 \\ u(0,.) = & u^0 \end{array} \right.$$

is to ask u to satisfy (D') for aby S convex with G s.t. $G' = f'S'$. S is called an entropy, and G is the corresponding entropy flux.

Such as weak solution, u is called an entropy solution. We can show that there exists a unique entropy solution to (C) (cf Kruzkov).

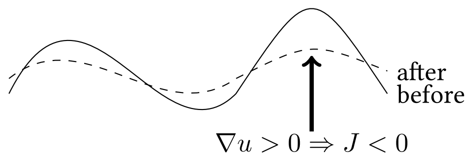


Figure 2.2: Before and after diffusion

2.2 Diffusion

$$\partial_t u = \kappa \Delta u = \kappa \sum_{i=1}^{d} \partial^2_{x_i x_i} u \tag{2.149}$$

where d is the dimension of the space.
We can also write

$$\partial_t u = \nabla \cdot J \text{ with } J = -\kappa \nabla u \tag{2.150}$$

, if κ is not constant. For example, $\kappa = \kappa(x)$ or $\kappa = \kappa(u)$. The equation is

$$\partial_t u = \nabla \cdot (-\kappa \nabla u) \tag{2.151}$$

.

Remark. κ will be assumed ≥ 0. $J = -\kappa \nabla u$ thus J is parallel to ∇u and is pointing on the opposite side. This tends to stabilize u.

2.2.1 Analysis of $\partial_t u = \kappa \partial_{xx} u$ in $(0, 1)$

$$\begin{cases} \partial_t u &= \kappa \partial_{xx} u & t > 0 \, x \in (0,1) \\ u(0,x) &= u_0(x) & x \in (0,1) \\ u(t,0) &= 0 & \\ u(t,1) &= 0 & \text{Homogeneous Dirichlet conditions} \end{cases} \tag{2.152}$$

For simplicity, assume that $u_0(0) = u_0(1) = 0$.

Remark. It is also natural to consider **homogeneous Neumann boundary conditions**.

$$\partial_x u(t,0) = 0 = \partial_x u(t,1)$$

or

$$J(t,0) = 0 = J(t,1)$$

. This is the adiabatique assumtion.

We will study this initial bounding PDE problem with the help of Fourier series. Ideas are

- $e^{-\kappa k^2\pi^2 t}\sin(k\pi x)$ is a solution to

$$\left\{\begin{array}{rcll} \partial_t u &=& \kappa\partial_{xx}u & t>0\, x\in(0,1)\\ u(0,x) &=& \sin(k\pi x) & x\in(0,1)\\ u(t,0) &=& 0 & \\ u(t,1) &=& 0 & \end{array}\right. \tag{2.153}$$

- The problem is linear, thus

$$u=\sum_{k=1}^{l} u_k e^{-\kappa k^2\pi^2 t}\sin(k\pi x)$$

is solution to

$$\left\{\begin{array}{rcll} \partial_t u &=& \kappa\partial_{xx}u & t>0\, x\in(0,1)\\ u(0,x) &=& \sum_{k=1}^{l} u_k\sin(k\pi x) & x\in(0,1)\\ u(t,0) &=& 0 & \\ u(t,1) &=& 0 & \end{array}\right. \tag{2.154}$$

.

Lemma. *$\left(\sqrt{2}\sin(k\pi x)\right)_{k\in\mathbb{N}}$ is a Hilbertian basis of $L^2(0,1)$.*

Proof. See my lecture notes. (It is a consequence of the Fejer lemma for $\left(e^{2i\pi kx}\right)_{k\in\mathbb{Z}}$.) □

Any $u\in L^2(0,1)$ can be express as

$$u(\cdot)=\sum_{k\in\mathbb{N}^*}\hat{u}(k)\sin(k\pi\cdot)$$

[6] with $\hat{u}(k)=\sqrt{2}\int_0^1 u(x)\sin(k\pi x)\mathrm{d}x$.

$$\sum_{k\in\mathbb{N}^*}\hat{u}(k)\sin(k\pi\cdot)\xrightarrow{l\to\infty, L^2} u$$

Thus if $u_0\in L^2(0,1)$,

$$u_0=\sqrt{2}\sum_{k\in\mathbb{N}^*}\hat{u_0}(k)\sin(k\pi\cdot)$$

We have a candidate to be solution

$$u(t,\cdot)=\sqrt{2}\sum_{k\in\mathbb{N}^*}e^{-\kappa k^2\pi^2 t}\hat{u_0}(k)\sin(k\pi\cdot)$$

.

Property 1. *Under smoothness assumption on u_0,*

$$\left\{\begin{array}{ll} \partial_t u=\kappa\partial_{xx}u & t>0\, x\in(0,1)\\ u(0,x)=\sum_{k=1}^{l} u_k\sin(k\pi x) & x\in(0,1)\\ u(t,0)=u(t,1)=0 & \end{array}\right. \tag{2.155}$$

. admits a unique solution.

[6] it is an equality in L^2

Proof. For a more complet proof, see lecture notes.

- Uniqueness: If the solution is in $L^2(0,1)$ for any time, let us denote $\hat{u}(k) = \sqrt{2}\int_0^1 u(x)\sin(k\pi x)\mathrm{d}x$. And $\partial_t u = \kappa\partial_{xx}u$ becomes $\hat{u}'(t) = \kappa\widehat{\partial_{xx}u}(k)$.

  Lemma. *Consider $g \in \mathcal{C}^2([0,1])$ such that $g(0) = g(1) = 0$. Then we have $\hat{g''}(k) = -k^2\pi^2\hat{g}(k)$.*

  Proof.

  $$\begin{aligned}
  \hat{g''}(k) &= \sqrt{2}\int_0^1 g''(x)\sin(k\pi x)\mathrm{d}x \\
  &= \sqrt{2}\left[\underbrace{[g'(x)\sin(k\pi x)]_0^1}_{=0} - k\pi\int_0^1 g'(x)\cos(k\pi x)\mathrm{d}x\right] \\
  &= -k\pi\sqrt{2}\left[\underbrace{[g(x)\cos(k\pi x)]_0^1}_{=0} + k\pi\int_0^1 g(x)\sin(k\pi x)\mathrm{d}x\right] \\
  &= -k^2\pi^2\hat{g}(k)
  \end{aligned}$$

  □

  Thus

  $$\begin{aligned}
  \hat{u}(k)'(t) &= -\kappa k^2\pi^2\hat{u}(k)(t) \\
  \Rightarrow \hat{u}(k)(t) &= \hat{u}(k)(0)e^{-\kappa k^2\pi^2 t}
  \end{aligned}$$

  .

- To prove that $u(t,x) = \sqrt{2}\sum e^{-\kappa k^2\pi^2 t}\hat{u_0}(k)\sin(k\pi x)$ is a solution, one only has to prove that its is a function and is differentiable in time, twice differentiable in space and such that it satisfies the equation $\partial_t u = \kappa\partial_{xx}u$.

Remark. Important point: $\kappa > 0$, so that $e^{-\kappa k^2\pi^2 t}$ is rapidly decreasing.

□

The same method can be applied to prove the well-posedness of

$$\begin{cases} \partial_t u = \kappa\partial_{xx}u + f \\ u(0,x) = u_0(x) \\ u(t,0) = u(t,1) = 0 \end{cases} \tag{2.156}$$

where f is a given function.

Remark.

$$\begin{cases} \partial_t u = \kappa \partial_{xx} u + f \\ u(0,x) = u_0(x) \\ \partial_x u(t,0) = \partial_x u(t,1) = 0 \end{cases} \tag{2.157}$$

can be solved with the Hilbertian basis $\left(1, \left(\sqrt{2}\cos(k\pi x)\right)_{k\in\mathbb{N}^*}\right)$.

2.2.2 The maximal principle

Theorem 4. *Consider the unique solution u to*

$$\begin{cases} \partial_t u = \kappa \partial_{xx} u + f \\ u(0,\cdot) = u_0 \textit{ smooth } u_0(0) = u_0(1) = 0 \\ u(t,0) = u(t,1) = 0 \end{cases} \tag{2.158}$$

It satifies, for any $T > 0$,

$$U = \max_{t\in[0,T]\, x\in(0,1)} u(t,x) = \max_{x\in(0,1)} u_0(x)$$

.

Proof. Idea of the proof: consider (t,x) such that $u(t,x) = \max_{s,y} u(s,y) = U$. Then $\partial_{xx} u(t,x) \leq 0$ thus $\partial_t u(t,x) \leq 0$ (u was larger before).

Rigourously, consider

$$v(t,x) = u(t,x)e^{\lambda t}$$

for $\lambda \in \mathbb{R}$. We have

$$\begin{aligned} \partial_t v &= \partial_t u e^{\lambda t} + \lambda u e^{\lambda t} \\ &= \kappa \partial_{xx} u e^{\lambda t} + \lambda v \\ &= \kappa \partial_{xx} v + \lambda v \end{aligned}$$

then

$$\begin{cases} \partial_t v = \kappa \partial_{xx} v + \lambda v \\ v(0,\cdot) = u_0 \\ v(t,0) = v(t,1) = 0 \end{cases} \tag{2.159}$$

Let us prove the theorem for v! Denote $V = \max_{t\in[0,T]\, x\in(0,1)} v(t,x)$

- Assume that $V = v(t_0,x_0)$ for $(t_0,x_0) \in (0,T) \times [0,1]$.
  - If $x_0 = 0$ or 1, $V = 0$.
  - If $x_0 \in (0,1)$, $\partial_t v(t_0,x_0) = 0 = \partial_x v(t_0,x_0)$ and $\partial_{xx} v(t_0,x_0) \leq 0$. Since we have $\partial_t v = \kappa\partial_{xx} v + \lambda v$, $\lambda v(t_0,x_0) = -\kappa \partial_{xx} v(t_0,x_0) \geq 0$. Consider $\lambda > 0$ then $v(t_0,x_0) \leq 0$. Conclusion: $V \leq 0$ if $(t_0,x_0) \in (0,T) \times [0,1]$.

- Assume that $V = v(T, x_0)$.
  - If $x_0 = 0$ or 1, $V = 0$.
  - if not, $\partial_{xx} v(T, x_0) \leq 0$ and $\partial_t v(T, x_0) \geq 0$. From $\partial_t v = \kappa \partial_{xx} v + \lambda v$, we deduce with $\lambda < 0$ that $\partial_t v(T, x_0) \geq 0$.

Conclusion: if $t_0 > 0$, $V \leq 0$. But $\max_y u_0(y) \geq 0$. Thus, the maximum value is attained at time $t = 0$. $u(t,x) = v(t,x)e^{-\lambda t}$. When $\lambda \to 0^-$, we observe that u satisfies the maximum principle. □

Another proof of the maximum priciple:

Proof. Consider $S : \mathbb{R} \to \mathbb{R}$, $S \in \mathcal{C}(\mathbb{R})$ convex.

$$S'(u)\partial_t u = \kappa S'(u)\partial_{xx} u$$
$$\partial_t S(u) = \kappa \partial_x (S(u)\partial_x u) - \underbrace{\kappa}_{\geq 0} \underbrace{s''(u)}_{\geq 0} \underbrace{\partial_x u}_{\geq 0}$$

Thus,

$$\partial_t S(u) \leq \kappa \partial_{xx}(S(u))$$

and

$$\int_0^1 \int_0^t \partial_t S(u) \mathrm{d}s \mathrm{d}x \leq \kappa \int_0^1 \int_0^t \partial_{xx} S(u) \mathrm{d}s \mathrm{d}x$$
$$\int_0^1 S(u(t,x))\mathrm{d}x - \int_0^1 S(u_0(x))\mathrm{d}x \leq \kappa \int_0^t \left[\partial_x(S(u(s,1))) - \partial_x(S(u(s,0)))\right] \mathrm{d}s$$

Choose S,

$$S(u) = \begin{cases} 0 & \text{if } u \leq M = \max_x u_0(x) \\ (u-M)^3 & \text{if } u > M \end{cases} \tag{2.160}$$

is convex and in $\mathcal{C}^2$.

With this S, $S \circ u_0 = 0$. Thus,

$$\int_0^1 S(u(t,x))\mathrm{d}x \leq \kappa \int_0^t \left[S'(\underbrace{u(s,1)}_{=0})\partial_x u(s,1) - S'(\underbrace{u(s,0)}_{=0})\partial_x u(s,0) \right] \mathrm{d}s$$

and as $S'(0) = 0$,

$$\int_0^1 S(u(t,x))\mathrm{d}x \leq 0$$

. $S \circ u$ is non negative. Thus, $S \circ u(t,x) = 0$ (almost) everywhere. Finally, $u(t,x) \leq M$ for all x. □

Remark. This proof also works for homogeneous Neumann consitions. In fact, only parts that differ is that we have

$$\int_0^1 S(u(t,x))\mathrm{d}x \leq \kappa \int_0^t \left[S'(u(s,1)) \underbrace{\partial_x u(s,1)}_{=0} - S'(u(s,0)) \underbrace{\partial_x u(s,0)}_{=0} \right] \mathrm{d}s$$

.

2.2.3 Numerical approximation

We want to approximate the solution to

$$\begin{cases} \partial_t u = \kappa \partial_{xx} u + f \\ u(0,\cdot) = u_0 \\ u(t,0) = u(t,1) = 0 \end{cases} \tag{2.161}$$

We want to compute u_j^n that will be approximate values of $u(t^n, x_j)$. Let $J \in \mathbb{N}^*$, $\Delta x = \frac{1}{J+1}$, $x_j = j\Delta x$ where $j = 0, 1, \cdots, J+1$. Let $\Delta t > 0$, $t^n = n\Delta t$, $n \in \mathbb{N}$. If u is smooth,

$$\begin{aligned} \frac{u(t^{n+1}, x_j) - u(t^n, x_j)}{\Delta t} &\sim \partial_t u(t^n, x_j) \\ \frac{u(t^n, x_j) - u(t^{n-1}, x_j)}{\Delta t} &\sim \partial_t u(t^n, x_j) \\ \frac{\partial_x u(t^n, x_{j+\frac{\Delta x}{2}}) - \partial_x u(t^n, x_{j-\frac{\Delta x}{2}})}{\Delta x} &\sim \partial_{xx} u(t^n, x_j) \\ \frac{u(t^n, x_{j+1}) - u(t^n, x_j)}{\Delta x} &\sim \partial_x u(t^n, x_{j+\frac{\Delta x}{2}}) \\ \frac{u(t^n, x_j) - u(t^n, x_{j-1})}{\Delta x} &\sim \partial_x u(t^n, x_{j-\frac{\Delta x}{2}}) \\ \text{Finally, } \frac{u(t^n, x_{j+1}) - 2u(t^n, x_j) + u(t^n, x_{j-1})}{\Delta x^2} &\sim \partial_{xx} u(t^n, x_j) \end{aligned}$$

(To be prove at the end in the consistency statement!)

We have at least two schemes:

$$\frac{u(n+1_j - u_j^n}{\Delta t} = \kappa \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + f(t^n, x_j) \tag{$\mathcal{E}$}$$

$$\frac{u(n+1_j - u_j^n}{\Delta t} = \kappa \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} + f(t^{n+1}, x_j) \tag{$\mathcal{I}$}$$

Explicit scheme

Precisely, the explicit scheme is

$$\begin{cases} \frac{u(n+1_j - u_j^n}{\Delta t} = \kappa \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + f(t^n, x_j) & n \in \mathbb{N}\, j = 1, \cdots, J \\ u_0^n = u_{J+1}^n = 0 & n \in \mathbb{N} \\ u_j^0 = u_0(x_j) & j = 1, \cdots, J \end{cases} \tag{$\mathcal{E}$}$$

Theorem 5. *Assume that $\kappa \Delta t \le \frac{\Delta x^2}{2}$. Then if (u_j^n) is defined by $(\mathcal{E})$,*

$$\sup_j |u(t^n, x_j)| \le C t^n (\Delta t + \Delta x^2)$$

if $u \in \mathcal{C}_b^{2,4}(\mathbb{R}^+, [0,1])$. (The scheme is fist order in Δt and second order in Δx convergent.)

Proof. • Consistency:

$$\varepsilon_j^n = \frac{u(t^{n+1}, x_j) - u(t^n, x_j)}{\Delta t} - \kappa \frac{u(t^n, x_{j+1}) - 2u(t^n, x_j) + u(t^n, x_{j-1})}{\Delta x^2} - f(t^n, x_j) \tag{2.162}$$

If $u \in \mathcal{C}_b^{2,4}(\mathbb{R}^+, [0,1])$, we can prove that $|\varepsilon| \leq C(\Delta t + \Delta x^2)$. Indeed,

$$\frac{u(t^{n+1}, x_j) - u(t^n, x_j)}{\Delta t} = \partial_t u(t^n, x_j) + \mathcal{O}(\Delta t)$$

$$u(t^n, x_{j+1}) = u(t^n, x_j) + \Delta x \partial_x u(t^n, x_j) + \frac{\Delta x^2}{2} \partial_{xx} u(t^n, x_j) + \frac{\Delta x^3}{6} \partial_{xxx} u(t^n, x_j) + \mathcal{O}(\Delta x^4)$$

$$u(t^n, x_{j-1}) = u(t^n, x_j) - \Delta x \partial_x u(t^n, x_j) + \frac{\Delta x^2}{2} \partial_{xx} u(t^n, x_j) - \frac{\Delta x^3}{6} \partial_{xxx} u(t^n, x_j) + \mathcal{O}(\Delta x^4)$$

$$\text{Thus, } \frac{u(t^n, x_{j+1}) - 2u(t^n, x_j) + u(t^n, x_{j-1})}{\Delta x^2} = \partial_{xx} u(t^n, x_j) + \mathcal{O}(\Delta x^2)$$

and

$$\varepsilon_j^n = \partial_t u(t^n, x_j) - \kappa \partial_{xx} u(t^n, x_j) - f(t^n, x_j) + \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2) \tag{2.163}$$

- Stability: Define

$$e_j^n = u_j^n - u(t^n, x_j) \tag{2.164}$$

. We have

$$\frac{e_j^{n+1} - e_j^n}{\Delta t} - \kappa \frac{e_{j+1}^n - 2e_j^n + e_{j-1}^n}{\Delta x^2} = -\varepsilon_j^n$$

$$e_j^{n+1} = e_j^n \left(1 - \frac{2\kappa \Delta t}{\Delta x^2}\right) + e_{j+1}^n \frac{\kappa \Delta t}{\Delta x^2} + e_{j-1}^n \frac{\kappa \Delta t}{\Delta x^2} - \Delta t \varepsilon_j^n$$

$$\text{Thus, } |e_j^{n+1}| = |e_j^n| \left(1 - \frac{2\kappa \Delta t}{\Delta x^2}\right) + |e_{j+1}^n| \frac{\kappa \Delta t}{\Delta x^2} + |e_{j-1}^n| \frac{\kappa \Delta t}{\Delta x^2} + \Delta t |\varepsilon_j^n|$$

$$\leq \max_{j=1,\cdots,J} |e_j^n| + \Delta t C(\Delta t + \Delta x^2)$$

$$\text{If } \frac{2\kappa \Delta t}{\Delta x^2} \leq 1 \leq \max_{j=1,\cdots,J} \underbrace{|e_j^0|}_{=0} + \underbrace{(n+1)\Delta t}_{t^{n+1}} C(\Delta t + \Delta x^2)$$

□

If $\Delta_t \leq \frac{\kappa \Delta_x^2}{2}$ (**very strong condition!**), the explicit scheme:

$$\begin{cases} \frac{u_j^{n+1} - u_j^n}{\Delta t} = \kappa \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} \quad n \in \mathbb{N}, \quad j = 1, \cdots, J \\ u_0^n = u_{J+1}^n = 0 \\ u_j^0 = u_0(x_j) \end{cases} \tag{$\mathcal{E}_2$}$$

(an ODE in finite dimension!) is convergent towards the solution of

$$\begin{cases} \partial_t u = \kappa \partial_{xx}^2 u \\ u(0, \cdot) = u^0 \\ u(t, 0) = u(t, 1) = 0. \end{cases} \tag{2.165}$$

Implicit scheme

Other idea: implicit schemes.

$$\begin{cases} \frac{u_j^{n+1}-u_j^n}{\Delta t} = \kappa \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + f_j^{n+1} \quad n \in \mathbb{N}, \quad j = 1, \cdots, J \\ u_0^n = u_{J+1}^n = 0 \\ u_j^0 = u_0(x_j) \end{cases} \tag{$\mathcal{I}$}$$

First question: is there a unique solution to this equation?

Generalization of the θ-scheme.

Let $\theta \in \mathbb{R}$ (at the end, $\theta \in [0,1]$) and define the scheme (θS):

$$\begin{cases} \frac{u_j^{n+1}-u_j^n}{\Delta t} - \kappa\theta \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} - \kappa(1-\theta) \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} = g_j^n, \\ u_0^n = u_{J+1}^n = 0, \quad n \in \mathbb{N}, \\ u_j^0 = u_0(x_j), \quad j = 1, \cdots, J \end{cases} \tag{θS}$$

where

$$g_j^n = f_j^{n+\frac{1}{2}} = f(t^n + \frac{\Delta_t}{2}, x_j)$$

or

$$g_j^n = \theta f_j^{n+1} + (1-\theta) f_j^n = f(\theta t^{n+1} + (1-\theta)t^n, x_j)$$

Let us denote:

$$A = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \in \mathcal{M}_J(\mathbb{R}) \quad \text{and} \quad U^n = (u_j^n)_{j=1}^J \in \mathbb{R}.$$

(θS) can be put in the matrix form:

$$\frac{1}{\Delta_t} I U^{n+1} + \kappa \frac{\theta}{\Delta_x^2} A U^{n+1} = \frac{1}{\Delta_t} I U^n + \kappa \frac{(1-\theta)}{\Delta_x^2} A U^n + F^{n+\frac{1}{2}} \tag{2.166}$$

with

$$\begin{aligned} F^{n+\frac{1}{2}} &= (f_j^{n+\frac{1}{2}})_{j=1}^J, && (2.167) \\ u_0^n = u_{J+1}^n &= 0, \quad n \in \mathbb{N}, && (2.168) \\ u_j^0 &= u_0(x_j), \quad j = 1, \cdots, J && (2.169) \end{aligned}$$

If $\left(\frac{I}{\Delta_t} + \kappa \frac{\theta}{\Delta_x^2} A\right)$ is invertible, this rewrites:

$$U^{n+1} = \left(\frac{I}{\Delta_t} + \kappa \frac{\theta}{\Delta_x^2} A\right)^{-1} \left(\frac{I}{\Delta_t} + \kappa \frac{(\theta-1)}{\Delta_x^2} A\right) U^n + F^{n+\frac{1}{2}} \tag{2.170}$$

U_j^n is not straight forwardly a convex combination ofthe u_k^n ! It is not easy to prove the discrete max principle (is it true?). Thus, it is not easy to prove the convergence in l^∞ (but it is in l^2).

Lemma. *The eigenvalues of A are:*

$$\alpha_j = 4\sin^2\left(\frac{j\pi}{2(J+1)}\right), \quad j = 1, ..., J$$

and α_j is associated with the following eigenvectors

$$v_j = \left(\sin\left(\tfrac{kj\pi}{J+1}\right)\right), \quad k = 1, ..., J.$$

Consequence.
$\left(\frac{I}{\Delta_t} + \kappa\frac{\theta}{\Delta_x^2}A\right)$ *has J different eigenvalues:*

$$\left(\frac{I}{\Delta_t} + \kappa\frac{\theta}{\Delta_x^2}\alpha_j\right)_{j=1,\dots,J} (> 0 if \theta \geq 0)\,. \tag{2.171}$$

The scheme admits a unique solution.

Remark. The eigenvalues of $-\partial^2_{x,x}$ with homogeneous Dirichlet boundary conditions on $(0,1)$ are $\lambda \in \mathbb{R}$ s.t. $\exists u \in L^2(0,1)$ s.t.

$$-\partial^2_{x,x}u = \lambda u \tag{2.172}$$
$$u(0) = u(1) = 0. \tag{2.173}$$

The eigenvectors are the $(sin(k\pi x))_{k\geq 1}$ and the associated eigenvalues are the $(k^2\pi^2)_{k\geq 1}$. Thus the eigenvalues of A (for J large) are approximations if those of $-\partial^2_{x,x}$ The scheme has a solution. Let us prove its convergence.

Lemma. Consistency. If $u \in C_b^{3,4}(\mathbb{R}^+, (0,1))$, we have:

$$|\epsilon_j^n| \leq C\left(|1-2\theta|\Delta_t + \Delta_t^2 + \Delta_x^2\right). \tag{2.174}$$

What is ϵ_j^n ? It's the consistency error defined as:

$$\epsilon_j^n = \frac{u(t^{n+1},x_j) - u(t^n,x_j)}{\Delta_t} - \kappa\theta\frac{u(t^{n+1},x_{j+1}) - 2u(t^{n+1},x_j) + u(t^{n+1},x_{j-1})}{\Delta_x^2}$$
$$-\kappa(1-\theta)\frac{u(t^n,x_{j+1}) - 2u(t^n,x_j) + u(t^n,x_{j-1})}{\Delta_x^2} - f_j^{n+\frac{1}{2}}$$

Remark. For $\theta = \frac{1}{2}$, the scheme is second order consistent (accurate) in Δ_t and Δ_x. This scheme is called the Crank-Nicolson scheme.

Stability?

Define: $e_j^n = u_j^n - u(t^n, x_j)$. This error e_j^n satisfies:

$$\left(\frac{I}{\Delta_t} + \kappa\frac{\theta}{\Delta_x^2}A\right)E^{n+1} = \frac{I}{\Delta_t} + \left(\frac{I}{\Delta_t} + \kappa\frac{(\theta-1)}{\Delta_x^2}A\right)E^n - \epsilon^n \tag{2.175}$$

where $E^n = (e_j^n)_{j=1}^J$ and $\epsilon^n = (\epsilon_j^n)_{j=1}^J$.
This rewrites

$$\left(I + \kappa\frac{\theta\Delta_t}{\Delta_x^2}\right)E^{n+1} = \left(I + \kappa\frac{(\theta-1)\Delta_t}{\Delta_x^2}\right)E^n - \Delta_t\epsilon^n, \tag{2.176}$$

or,

$$B_\theta E^{n+1} = B_{\theta-1} E^n - \Delta_t \epsilon^n \tag{2.177}$$

with

$$B_\theta = I + \kappa \frac{\theta \Delta_t}{\Delta_x} A, \quad \forall \theta \in R \tag{2.178}$$

or even,

$$E^{n+1} = B_\theta^{-1} B_{\theta-1} E^n - \Delta_t B_\theta^{-1} \epsilon^n. \tag{2.179}$$

To prove the stability of this equation system in l^∞ is not easy. It is easier to prove it in the l^2-norm.
Let us define for $v \in \mathbb{R}^J$:

$$|||v|||_2 = \left(\frac{1}{J} \sum_{j=1}^{J} |v_j^2| \right)^{\frac{1}{2}}$$

and, in this manner, if $u(x) = 1$, if $v_j = u(x_j), j = 1, ..., J$ then: $|||v|||_2 = 1 \|u\|_{L^2}^2$ (and $\|v\|_2 = \sqrt{J}$). For $M \in \mathcal{M}_J(\mathbb{R})$, we define the norm:

$$\begin{aligned} |||M|||_2 &= \sup_{v \neq 0} \frac{|||Mv|||_2}{|||v|||_2} && (2.180) \\ &= \sup_{v \neq 0} \frac{\frac{1}{\sqrt{J}} |||Mv|||_2}{\frac{1}{\sqrt{J}} |||v|||_2} && (2.181) \\ &= \|M\|_2 && (2.182) \end{aligned}$$

We have:

$$\left|\left|\left| E^{n+1} \right|\right|\right|_2 \leq \|B_\theta^{-1} B_{\theta-1}\|_2 |||E^n|||_2 + \Delta_t \|B_\theta^{-1}\|_2 |||\epsilon^n|||_2 \tag{2.183}$$

As $B_\theta = I + \kappa \frac{\theta \Delta_t}{\Delta_x} A, \forall \theta \in R$, the eigenvalues of B_θ are

$$\left(I + \kappa \frac{\theta \Delta_t}{\Delta_x} \alpha_j \right)_j = \left(I + \kappa \frac{\theta \Delta_t}{\Delta_x} 4 \sin^2 \left(\frac{j\pi}{2(J+1)} \right) \right)_j \forall j = 1, ..., J \tag{2.184}$$

B_θ is symmetric, thus: $\|B_\theta\|_2 = \underbrace{\rho(B_\theta)}_{\text{spectral radius}} > 1$ and $\|B_\theta\|_2 = \rho(B_\theta^{-1}) < 1$.

Remark. Spectral radius. $\rho(M) = \max(|\lambda|)$, where λ are the eigenvalues. Thus:

$$\left|\left|\left| E^{n+1} \right|\right|\right|_2 \leq \|B_\theta^{-1} B_{\theta-1}\|_2 |||E^n|||_2 + \Delta_t |||\epsilon^n|||_2 \tag{2.185}$$

Moreover, as

$$\begin{aligned} |\epsilon_j^n| &\leq C(|1 - 2\theta|\Delta_t + \Delta_t^2 + \Delta_x^2) && (2.186) \\ \Rightarrow |||\epsilon^n|||_2 &\leq C(|1 - 2\theta|\Delta_t + \Delta_t^2 + \Delta_x^2) && (2.187) \end{aligned}$$

thanks to the modification of the l^2-norm !
Finally, under some smoothness assumptions on u,

$$\left|\left|\left| E^{n+1} \right|\right|\right| \leq \|B_\theta^{-1} B_{\theta-1}\|_2 |||E^n|||_2 + \Delta_t C(|1 - 2\theta|\Delta_t + \Delta_t^2 + \Delta_x^2) \tag{2.188}$$

Remark. Do not right :
$$\|B_\theta^{-1}B_{\theta-1}\|_2 \leq \|B_\theta^{-1}\|_2\|B_{\theta-1}\|_2,$$
it's true but not good for estimates!

Lemma. • *If* $\theta \in [\frac{1}{2}, 1]$, $\|B_\theta^{-1}B_{\theta-1}\|_2 \leq 1$

- *If* $\theta \in [0, \frac{1}{2})$, $\|B_\theta^{-1}B_{\theta-1}\|_2 \leq 1$
  for any J *if and only if* $\Delta_t \leq \frac{\Delta_x^2}{2\kappa(1-2\theta)}$

Remark. • For $\theta \in [\frac{1}{2}, 1]$, there is no stability condition.

- For $\theta < \frac{1}{2}$, there is a restrictive stability condition
- For $\theta = 0$, we recover the condition previously obtained for the explicit scheme
- For $\theta = \frac{1}{2}$, there is no stability condition (and the scheme is second order accurate!).

Proof. $B_\theta^{-1}B_{\theta-1}$ is symmetric, because B_θ^{-1} and $B_{\theta-1}$ commute: they have the same eigenvectors: the V_j (eigenvectors of A). Indeed, the eigenvectors of $\left(I + \kappa\lambda\frac{\Delta_t}{\Delta_x^2}A\right)$ are the eigenvectors of A for any λ. Thus:
$$\|B_\theta^{-1}B_{\theta-1}\|_2 = \rho\left(B_\theta^{-1}B_{\theta-1}\right)$$
The eigenvalues of $B_\theta^{-1}B_{\theta-1} = L_\theta$ are :
$$\lambda_j = \frac{1 + \kappa(\theta-1)\frac{\Delta_t}{\Delta_x^2}\alpha_j}{1 + \kappa\theta\frac{\Delta_t}{\Delta_x^2}\alpha_j} \quad \text{(associated to the } V_j\text{)}$$
L_θ is called the amplification metrix of the scheme.
We have:
$$\lambda_j = \frac{1 + 4\kappa(\theta-1)\frac{\Delta_t}{\Delta_x^2}\sin^2\left(\frac{j\pi}{2(J+1)}\right)}{1 + 4\kappa\theta\frac{\Delta_t}{\Delta_x^2}\sin^2\left(\frac{j\pi}{2(J+1)}\right)} \quad j = 1, ..., J$$
Let us define:
$$g(x) = \frac{1 + 4\kappa(\theta-1)\frac{\Delta_t}{\Delta_x^2}x}{1 + 4\kappa\theta\frac{\Delta_t}{\Delta_x^2}x}, \quad x \in [0,1]$$
We have: $\|L_\theta\|_2 \leq \|g\|_{L^\infty(0,1)}$ and $\|g\|_{L^\infty} = \lim\limits_{J\to\infty}\|L_\theta\|_2$.
We consider $\theta \in [0,1]$. g is decreasing (over $[0,1]$), and $\|g\|_\infty = \max\left(|g(0)|, |g(1)|\right)$ with $g(0) = 1 \leq 1$, $|g(1)| = \left|\frac{1+4\kappa(\theta-1)\frac{\Delta_t}{\Delta_x^2}x}{1+4\kappa\theta\frac{\Delta_t}{\Delta_x^2}x}\right|$.
We look for conditions under which $|g(1)| \leq 1$. The inequality writes:
$$\begin{aligned} \left|1 + 4\kappa(\theta-1)\frac{\Delta_t}{\Delta_x^2}x\right| &\leq \left|1 + 4\kappa\theta\frac{\Delta_t}{\Delta_x^2}x\right| \\ &= 1 + 4\kappa\theta\frac{\Delta_t}{\Delta_x^2}x \end{aligned}$$

This rewrites:

$$-1-4\kappa\theta\frac{\Delta_t}{\Delta_x^2} \leq 1+4\kappa(\theta-1)\frac{\Delta_t}{\Delta_x^2} \leq 1+4\kappa\theta\frac{\Delta_t}{\Delta_x^2}$$

$$\begin{cases} -4\kappa\frac{\Delta_t}{\Delta_x^2} \leq 0 \\ \text{and} \\ -2-8\kappa\theta\frac{\Delta_t}{\Delta_x^2} \leq -4\kappa\frac{\Delta_t}{\Delta_x^2} \end{cases} \tag{$\mathcal{I}$}$$

$$-4\kappa(\theta-\frac{1}{2})\frac{\Delta_t}{\Delta_x^2} \leq 1$$

This is true if and only if:

- $\theta \geq \frac{1}{2}$
- or $2\kappa(1-2\theta)\frac{\Delta_t}{\Delta_x^2} \leq 1$ (with $\theta \leq \frac{1}{2}$ obviously!)

This is to say, if and only if:

- $\theta \geq \frac{1}{2}$
- or $\Delta_t \leq \frac{\Delta_x^2}{2\kappa(1-2\theta)}$ (with $\theta \leq \frac{1}{2}$ obviously!)

□

Theorem 6. *Assume:* $u \in C_b^{3,4}(\mathbb{R}^+, (0,1))$. *Then, if* $\theta \geq \frac{1}{2}$ *or if* $\Delta_t \leq \frac{\Delta_x^2}{2\kappa(1-2\theta)}$, $\exists c$ *s.t.*

$$|||E^n|||_2 \leq Ct^n\left(|1-2\theta|\Delta_t + \Delta_t^2 + \Delta_x^2\right)$$

Remark. This is not a convergence result. This is not that clear as $\Delta_t, \Delta_x \xrightarrow[J\to\infty]{} O^+$. But $|||.|||_2$ depends on J! Is this theorem a result that tells the convergence of the discrete solution towards the exact one ? In order to compare the exact solution to the discrete one, we have to "put" them in the same space. Usually, this is done by putting the exact solution in the discrete space. But this space has a dimension that varies (here, with respect to Δ_t and Δ_x).
The right manner to compare the exact solution and the discrete one is to put the discrete one into the space of the exact one: for example, by interpolation:

$$(u_j^n) \xrightarrow{\text{reconstruction operator}} u_{\Delta_t,\Delta_x}(t,x)$$

where for example:

- $u_{\Delta_t,\Delta_x} = u_j^n \forall n, j$
- u_{Δ_t,Δ_x} is linear with respect to t and x on any interval $[t^n, t^{n+1}]$ and $[x_j, x_{j+1}]$.

With the modified l^2-norm we have defined, we see that:

$$\begin{aligned} \| u_{\Delta_t,\Delta_x}(t,.) - u(t,.) \|_{L^2(0,1)} &\leq C \sup_{n, t^n \leq t} \| U^n - U(t^n) \|_2 \\ \text{where } U^n &= (u_j^n)_{j=1}^J \\ \text{and } U(t^n) &= (u(t^n, x_j))_{j=1}^J \end{aligned}$$

(where C does not depend on J!)

Remark. Final remark on the heat equation.

1. It is possible to prove that the l^∞ convergence of the θ-scheme but under a condition

$$\Delta_t \leq C \Delta_x^2$$

even for $\theta \geq \frac{1}{2}$ (it involves entropy functions, cf correction par mail???? ***)

2. The solution to

$$\begin{cases} \partial_t u = \kappa \partial^2_{xx} u \\ u(0,\cdot) = u^0 \\ u(t,0) = u(t,1) = 0. \end{cases}$$

is very smooth (even if u^0 only belongs to $L^2(0,1)$).
$u \in C^\infty((0,+\infty) \times (0,1))$, and even $u(t,.)$ is analytic $\forall t > 0$ (= in space $\forall t > 0$). (cf exercice ***)
The heat equation is not local, there is no dependence cone: $u(t,x)$ depends on all the values of $u(0,.)$ (for any $t > 0$). Indeed, consider $u^0 = \mathbb{1}_{[\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon]}$. Let $t > 0$ small. If there exists a dependence cone, then, for $x = \frac{1}{4}$, we can find $t > 0$ s.t. $u(t, \frac{1}{4})$ only depends on:

$$\left\{ u^0(x), x \in \left[\frac{1}{4} - ct, \frac{1}{4} + ct \right] \right\} = \{0\}$$

thus $u(\frac{1}{4}, t) = 0$. This remains true for any $x \in \left[\frac{1}{4} - \eta, \frac{1}{4} + \eta\right]$ for one $\eta > 0$.
As $u(t,.)$ is analytic, $u(t,.) \equiv 0$, which is **false**.

3. For homogeneous Neumann conditions: replace the first line of A: $\begin{pmatrix} 1 & -1 & 0 & \dots & 0 \end{pmatrix}$ by a larger one: $\begin{pmatrix} -1 & 2 & -1 & 0 & \dots & 0 \end{pmatrix}$ (because $u_0 = u_1$). Indeed:

$$\begin{aligned} \partial_x u(x=0) &= 0 \\ \Leftrightarrow \frac{u_1 - u_0}{\Delta_x} &= 0 \\ \Rightarrow u_1 = u_0 & \end{aligned}$$

The same for the last line, $\begin{pmatrix} 0 & \dots & 0 & -1 & 1 \end{pmatrix}$ is replaced by a larger ine: $\begin{pmatrix} 0 & \dots & 0 & -1 & 2 & -1 \end{pmatrix}$ (because $u_J = u_{J+1}$). Indeed:

$$\begin{aligned} \partial_x u(x=1) &= 0 \\ \Leftrightarrow \frac{u_{J+1} - u_J}{\Delta_x} &= 0 \\ \Rightarrow u_J = u_{J+1} & \end{aligned}$$

.

Chapter 3

Perfect fluid

[1]

On considère un fluide occupant $\mathbb{R}^d$ ($d = 1, 2, 3$) ou $\Omega \in \mathbb{R}^d$ de densité $\rho(t, x)$ (densité volumique de masse) at time $t \in \mathbb{R}$, position $x \in \mathbb{R}^d$ and velocity $u(t, x) \in \mathbb{R}^d$.

We will derive equations on ρ and u (at least) allowing to know (compute) $\rho(t, x)$ and $u(t, x)$ knowing some initial conditions, e.g. $\rho(0, x)$ and $u(0, x)$ for all x.

The first physical principle we can use to obtain an equation is the conservation of mass.

$$\partial_t \rho + \nabla(\rho u) = 0 \quad (t, x) \in \mathbb{R} \times \Omega \tag{3.1}$$

Indeed, let $\omega(t)$ be a material volume.

$$\omega(t) = \left\{ x \in \mathbb{R}^d | \exists y \in \mathbb{R}^d \text{ s.t. } \begin{array}{l} \cdot \quad y = \omega(0) \\ \cdot \quad x = X(t, y) \end{array} \text{ where } \left\{ \begin{array}{l} \partial_t X(t, y) = u(t, X(t, y)) \\ X(o, y) = y \end{array} \right\} \right. \tag{3.2}$$

Then

$$\begin{aligned} 0 =& \frac{\mathrm{d}}{\mathrm{d}t} \int_{\omega(t)} \rho(t, y) \mathrm{d}y \\ =& \int_{\omega(t)} \partial_t \rho(t, y) \mathrm{d}y + \int_{\partial\omega(t)} \rho u \cdot n \mathrm{d}y \\ =& \int_{\omega(t)} \partial_t \rho + \int_{\omega(t)} \nabla \cdot (\rho u) \\ =& \int_{\omega(t)} (\partial_t \rho + \nabla \cdot (\rho u)) \\ {}^2 \Rightarrow& \partial_t \rho + \nabla \cdot (\rho u) = 0 \in \mathbb{R} \end{aligned}$$

But, here u is not a given vector field. There lacks at least une equation (in $\mathbb{R}^d$). (We have $(\rho, u) \in \mathbb{R}^{d+1}$ as unknowns.)

Newton's second law provides another equation.

$$m\dot{u} = m\ddot{x}(t) = F$$

[1] Fluides parfaits

for a particle with constant mass, position x, submitted to its force F.

Applying this to material volume $\omega(t)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\int_{\omega(t)} \rho(t,y)u(t,y)\mathrm{d}y}_{\text{impulse of } \omega(t)} = \text{ forces acting on } \omega(t) \tag{3.3}$$

.

Forces acting on $\omega(t)$

- external volume forces f: (for example, the gravity force)
- thers is no internal forces thanks to Newton's third law: action-reaction principle.
- surface forces f_σ acting on $\partial\omega(t)$ thus we have

$$\int_{\omega(t)} \partial_t(\rho u) + \int_{\partial\omega(t)} (\rho u)(u \cdot n)\mathrm{d}\sigma = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\omega(t)} (\rho u) = \int_{\omega(t)} f + \int_{\partial\omega(t)} f_\sigma$$

. [3]

Furthermore, we have

Lemma.

$$\int_{\partial\omega(t)} (\rho u)(u \cdot n) = \int_{\omega(t)} \nabla \cdot (\rho u \otimes u) \tag{3.4}$$

where

$$\rho u \otimes u = \rho(u \otimes u)$$
$$u \otimes u(t,x) \in \mathcal{M}_d(\mathbb{R}) \textit{ with } (u \otimes u)_{i,j} = u_i u_j$$

If M is a matrix field then $\nabla \cdot M$ is the vector field where the i-th component is the divergence of the corresponding line of M.

$$(\nabla \cdot M)_i = \sum_{j=1}^{d} \partial_j M_{i,j}$$

. Thus, $(\nabla \cdot \rho u \otimes u)_i = \sum_{j=1}^{d} \partial_j(\rho u_i u_j)$.

Proof. Let us prove 3.4. For any $i = 1, \cdots, d$,

$$(\int_{\partial\omega(t)} (\rho u)(u \cdot n))_i = \int_{\partial\omega(t)} \rho u_i(u \cdot n) = \int_{\partial\omega(t)} \nabla(\rho u_i u) = \int_{\partial\omega(t)} \sum_{j=1}^{d} \partial_j(\rho u_i u_j) = (\int_{\omega(t)} \nabla \cdot (\rho u \otimes u))_i$$

. □

[3] $(\rho u)(u \cdot n)$ is the vector with component $(\rho u)_i \cdot (u \cdot n) \quad i = 1, \cdots, d$.

Thus, we have

$$\int_{\omega(t)} \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) = \int_{\omega(t)} f + \int_{\partial\omega(t)} f_\sigma \tag{3.5}$$

.
Let $x \in \Omega$ (in the fluid!). Let $n \in \mathbb{R}^d$ be a normed vector, $\varepsilon > 0$ and $D(x, n, \varepsilon)$ be the disk centered at x, orthogonal to n with surface ε.

Assume that the force $F(x, n, \varepsilon)$ exerted by the fluid on the side where n points over $D(x, n, \varepsilon)$

- is parallel to n and
- is equivalent, as $\varepsilon \to 0$, to $\varepsilon p(x) n$.

$$\frac{F(x, n, \varepsilon)}{\varepsilon} \to_{\varepsilon \to 0^+} -p(x)n$$

Then the fluid is called **perfect**, and the scalar $p(x)$ is called its **pressure**.

In a perfect fluid with pressure p(x),

$$\int_{\partial\omega(t)} f_\sigma = \int_{\partial\omega(t)} -pn \tag{3.6}$$

. Assume that the fluid is perfect with pressure $p(t, x)$. We have

$$\int_{\omega(t)} \partial_t(\rho u) + \nabla(\rho u \otimes u) = \int_{\omega(t)} f - \int_{\partial\omega(t)} pn = \int_{\omega(t)} f - \underbrace{\int_{\omega(t)} \nabla p}_{\int_{\omega(t)} \nabla \cdot (pI)}$$

. Thus, for any $\omega(t)$,

$$\int_{\omega(t)} (\partial_t(\rho u) + \nabla(\rho u \otimes u) + \nabla p - f) = 0 \tag{3.7}$$

and

$$\partial_t(\rho u) + \nabla(\rho u \otimes u) + \nabla p = f \in \mathbb{R}^d \tag{3.8}$$

or

$$\partial_t(\rho u) + \nabla(\rho u \otimes u + pI) = f \in \mathbb{R}^d \tag{3.9}$$

.

Usually, f is known. For example,

Example. If f is the gravity force (assumed to be constant), $f = \rho g$. And

$$\begin{cases} \partial_t \rho + \nabla(\rho u) = 0 \\ \partial_t(\rho u) + \nabla(\rho u \otimes u) + \nabla p = \rho g \end{cases} \tag{3.10}$$

is a system of $d + 1$ equations and $d + 2$ unknowns.

Ways to close the system

- Assume that $p = p(\rho)$ **Barotropic gaz**

  Example. – **Isothermal gaz**: $p = C\rho$.

  – **Adiabatic gaz**: $p = C\rho^\gamma$ with $\gamma > 1$.

  In this case, if $d = 1$ and neglecting external forces, we have

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = 0 \end{cases} \tag{3.11}$$

  This is a hyperbolic system, if p is an increasing function of ρ. We also remark that

$$\partial_t \begin{pmatrix} \rho \\ \rho u \end{pmatrix} + \partial_x F(\rho, \rho u) = 0 \quad \text{with } F(\rho, \rho u) = \begin{pmatrix} \rho u \\ \rho u^2 + p(\rho) \end{pmatrix} \tag{3.12}$$

$$\underbrace{\Leftrightarrow}_{\text{if the solution is smooth}} \quad \partial_t \begin{pmatrix} \rho \\ \rho u \end{pmatrix} + \underbrace{\nabla_{\rho,\rho u} F(\rho, \rho u)}_{2\times 2 \text{ matrix, } \mathbb{R}\text{-diagonalizable}} \partial_x \begin{pmatrix} \rho \\ \rho u \end{pmatrix} = 0 \tag{3.13}$$

- If the pressure also depends on the temperature T, we have

$$\begin{cases} \partial_t\rho + \nabla(\rho u) = 0 \\ \partial_t(\rho u) + \nabla(\rho u \otimes u) + \nabla p(\rho, T)I = 0 \end{cases} \tag{3.14}$$

  . We need another equation. What is usually done is is to define the total energy

$$e(t, x) = \frac{1}{2}|u|^2 + \varepsilon$$

  where ε is the internal energy. It is commmonly assumed that $\varepsilon = c_v T$ (c_v is known.) We have a physical priciple **conservation of total energy**

$$\partial(\rho e) + \nabla(\rho e u + \underbrace{pu}_{\text{power of pressure}}) = \underbrace{fu}_{\text{power of external forces}} \tag{3.15}$$

  then

$$p = p(\rho, T) = p(\rho, \frac{\varepsilon}{c_v}) = p(\rho, \frac{e - \frac{|u|^2}{2}}{c_v}) = \tilde{p}(\rho, u, e)$$

  and the system is closed.

  Example.

$$p = \tilde{p}(\rho, u, e) = (\gamma - 1)\rho\varepsilon = (\gamma - 1)\rho(e - \frac{|u|^2}{2})$$

$$\Leftrightarrow PV = nRT$$

  (**Ideal gas law**)

  With $\gamma > 1$, the system is hyperbolic and has three different eigenvalues.

Remark. $d = 1$, **Isobaric(compressible) gas** with $p = p(\rho)$.

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = 0 \end{cases} \tag{3.16}$$

If ρ, u are smooth, we have

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \rho \partial_t u + \underbrace{u \partial_t \rho + u \partial_x \rho u}_{=0} + \rho u \partial_x u + \partial_x p = 0 \end{cases} \tag{3.17}$$

thus

$$\rho(\quad \underbrace{\partial_t u + u \partial_x u}_{\text{terms of Burger's equation}} \quad + \partial_x p = 0$$

.

- Assume **incompressibility** of the fluid: the volume of a material volume $\omega(t)$ does not depend on time.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\int_{\omega(t)} 1 \mathrm{d}x) = 0 = \int_{\omega(t)} \underbrace{\partial_t 1}_{=0} + \int_{\partial\omega(t)} 1 u \cdot n = \int_{\omega(t)} \nabla \cdot u$$

. Thus $\nabla \cdot u = 0$. ($\nabla_x u(t, x) = 0 \, \forall t, x$.) Finally,

$$\begin{cases} \partial_t \rho + \nabla(\rho u) = 0 \\ \partial_t(\rho u) + \nabla(\rho u \otimes u) + \nabla p = 0 \\ \nabla \cdot u = 0 \end{cases} \tag{3.18}$$

.

Property 2. In a perfect incompressible fluid with pressure p, density f and velocity u, we have

$$\begin{cases} \partial_t \rho + u \nabla \rho = 0 \\ \partial_t(\rho u) + \nabla(\rho u \otimes u) + \nabla p = f \\ \nabla \cdot u = 0 \end{cases} \tag{3.19}$$

. *where $(u \cdot \nabla)$ is defined as*

$$((u \cdot \nabla)v)_i = \sum_{j=1}^{d} u_j \partial_j v_i = (\nabla v \cdot u)_i$$

thus $((u \cdot \nabla)u)_i = \sum_{j=1}^{d} u_j \partial_j u_i$.

Proof. First equation:

$$\partial_t \rho + \nabla(\rho u) = 0 = \partial_t \rho + u \nabla \rho + \rho \underbrace{\nabla u}_{=0}$$

Second equation: First remark that if $d = 1$, this writes

$$\rho(\partial_x u + u \partial_x u) + \partial_x p = f$$

that we already derived. We start with

$$\partial_t(\rho u) + \nabla \rho u \otimes u + \nabla p = f$$

. For the fisrt term, $\partial_t(\rho u) = \rho\partial_t u + u\partial_t \rho$. Let us decompose $\nabla \cdot \rho u \otimes u$

$$\begin{aligned}
(\nabla \rho u \otimes u)_i &= \sum_{j=1}^{d} \partial_j(\rho u_i u_j) \\
&= \sum_{j=1}^{d} \rho u_j \partial_j u_i + \sum_{j=1}^{d} u_i \partial_j(\rho u_j) \\
&= (\rho(u \cdot \nabla)u)_i + u_i \sum_{j=1}^{d} \partial_j(\rho u_j) \\
&= (\rho(u \cdot \nabla)u)_i + (\nabla \cdot (\rho u)u)_i \\
\Rightarrow \nabla \rho u \otimes u &= \rho(u \cdot \nabla)u + \nabla \cdot (\rho u)u
\end{aligned}$$

Thus,

$$\begin{aligned}
f &= \partial_t(\rho u) + \nabla \rho u \otimes u + \nabla p \\
&= \rho\partial_t u + \underbrace{u\partial_t \rho + u\nabla(\rho u)}_{=0} + \rho(u \cdot \nabla)u + \nabla p \\
&= \rho(\partial_t u + u\nabla u) + \nabla p
\end{aligned}$$

□

Remark. If $\rho \neq 0$, the impulse equation is often written as

$$\partial_t u + (u \cdot \nabla)u + \frac{\nabla p}{\rho} = \frac{f}{\rho} = g \quad \text{if } f = \rho g$$

.

Remark. The equation $\nabla u = 0$ plays a particular role. It is not an evolution equation.
This can be seen as a constraint on the fluid. From this point of view, p can be understood as a Lagrange multiplier associated with this constraint. We will see it a bit more precisely in similar context: Stokes equation.

Reminder on a previous property. In a perfect incompressible fluid with pressure p, density f and velocity u, we have

$$\begin{cases} \partial_t \rho + u\nabla\rho = 0 \\ \partial_t(\rho u) + \nabla(\rho u \otimes u) + \nabla p = f \\ \nabla \cdot u = 0 \end{cases} \tag{3.20}$$

. What if $p = p(\rho)$? (barotropic) ($\Leftrightarrow$ Saint Venant system but not the same meaning)

$$\left\{\begin{array}{l} \partial_t h + u\nabla h = 0 \\ \partial_t(hu) + \nabla(hu \otimes u) + \nabla g\frac{h^2}{2} = -hgz' \end{array}\right. \tag{3.21}$$

. Assume the incompressibility of the fluid $\left(\frac{d}{dt}\int_{w(t)} 1 = 0\right) \to \nabla.u = 0$

Lemma. $\nabla\rho u \otimes u = \rho(u.\nabla)u + u\nabla.(\rho u)$ *if* (ρ, u) *is smooth !*

Property 3. *In a perfect incompressible fluid, we have:*

$$\left\{\begin{array}{l} \partial_t\rho + u\nabla\rho = 0 \\ \rho\left(\partial_t u + (u.\nabla)u\right) + \nabla p = f \\ \nabla \cdot u = 0 \end{array}\right. \tag{3.22}$$

.

Remark. $\rho(t,x) = \rho(0, X(0,t,x))$ where

$$\left\{\begin{array}{l} \partial_1 X(s,t,x) = u(s,t,x) \\ X(t,t,x) = x \end{array}\right. \tag{3.23}$$

.

Property 4. *In a perfect incompressible homogeneous fluid we have* $\rho = \rho_0$ *(constant !) and*

$$\partial_t + (u.\nabla)u + \frac{\nabla\rho}{\rho_0} = \frac{f}{\rho_0} = g$$

if $f = \rho_0 g$

Remark. If $\rho(0,.)$ is constant, then $\rho(t,.)$ is the same constant because of the characteristics formula (here the incompressibility is necessary!)

Remark. In dimension 1, $\nabla u = \partial_x u = 0 \Rightarrow u$ is constant. Not interesting!
In dimension ≥ 2, $\nabla.u = 0$ is not trivial.

3.1 Examples of solution to this system

3.1.1 Archimede's law

Consider a perfect incompressible homogeneous fluid at rest submitted to the gravity $f = \rho\vec{g}$ with $\vec{g} = g.\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$.
(We have:

$$\begin{array}{rcl} \partial_t\rho + \underbrace{\nabla(\rho u)}_{=0} & = & 0 \\ \underbrace{\partial_t\rho u + \nabla\rho u \otimes u}_{=0} + \nabla p & = & \rho\vec{g} \end{array}$$

) The fluid equation becomes

$$\nabla p = \rho_0 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \tag{3.24}$$

$\Rightarrow p = c - \rho_0 g z = \rho_0 g(z_0 - z)$ (=the pressure).
Consider a solid represented by $\theta \subset \mathbb{R}^3$ in the fluid. The force exerted by the fluid on the solid is, by definition,

$$F = \int_{\partial\theta} -p\vec{n} \, \mathrm{d}\sigma \tag{3.25}$$

(scalar$\times\vec{n}$ if orthogonal surface).

$$F = \int_{\partial\theta} -p\vec{n} \, \mathrm{d}\sigma = \int_{\partial\theta} \rho g(z - z_0)\vec{n} \, \mathrm{d}\sigma \tag{3.26}$$

$$F = \int_{\theta} \nabla \left((x, y, z) \mapsto \rho_0 g(z - z_0)\right) = \int_{\theta} \begin{pmatrix} 0 \\ 0 \\ \rho_0 g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \rho_0 g.\mathrm{vol}(\theta) \end{pmatrix} \tag{3.27}$$

(Archimede's law)
Prove the same without the assumption that the fluid is homogeneous but only assuming that is satisfied: $\rho = \rho(z)$. We can assume gravity force is a function of z.

3.1.2 First bernoulli's law

We consider a perfect incompressible stationnary (= independant of time)+ homogeneous fluid. Assume that the massic forces derive froma potential V:

$$f == -\rho\nabla V \tag{3.28}$$

(Remember that for a point particle submitted to a force $F = -\nabla V$, we obtained $\frac{d}{dt}\left(\frac{m|u|^2}{2} + V\right) = 0$ (the kinetic energy) where V is a constant.)
Then, $\frac{|u|^2}{2} + \frac{p}{\rho} + V$ is constant along the streamlines (= trajectories in this case = characteristic curves).

Proof. A streamline is a solution of $Y(\tau)$ of $Y'(\tau) = u(Y(\tau))$. Consider a function $G(Y(\tau))$ (in the following we will take $G(Y(\tau)) = \frac{1}{2}|u(Y(\tau))|^2 + \frac{p(Y(\tau))}{\rho} + V(Y(\tau))$ and will prove that $G(Y(\tau_1)) = G(Y(\tau_2)) \forall \tau_1, \tau_2$).

$$\begin{aligned} G(Y(\tau_2)) - G(Y(\tau_1)) &= \int_{\tau_1}^{\tau_2} \frac{d}{d\tau} G \circ Y(\tau) \, \mathrm{d}\tau \\ &= \int_{\tau_1}^{\tau_2} \nabla G(Y(\tau)).Y'(\tau) \, \mathrm{d}\tau \\ &= \int_{\tau_1}^{\tau_2} \nabla G(Y(\tau)).u(Y(\tau)) \, \mathrm{d}\tau \end{aligned}$$

Let us prove that $\quad \nabla G(Y(\tau)).u(Y(\tau)) = 0 \quad$ if $G(x) = \frac{1}{2}|u(x)|^2 + \frac{p(x)}{\rho} + V(x)$.
The fluid equation is:

$$(u.\nabla)u + \frac{\nabla p}{\rho} = -\nabla V$$

Thus:

$$(u.\nabla)u + \nabla\left(\frac{p}{\rho_0}\right) + \nabla V = 0$$

We will show that:

$$\nabla\left(\frac{|u|^2}{2}\right).u = \left((u.\nabla)u\right).u$$

Thus we will obtain:

$$\begin{aligned}
\nabla G.u &= \left((u-\nabla)\right).u + \nabla\left(\frac{p}{\rho}\right).u + \nabla V.u = 0.u = 0 \\
\nabla\left(\frac{|u|^2}{2}\right).u &= \sum_{i=1}^{d} \nabla\left(\frac{|u|^2}{2}\right)_i u_i \\
&= \frac{1}{2}\sum_i \partial_i\left(\sum_{j=1}^{d} u_j^2\right) u_i \\
&= \sum_i \sum_j u_j\left(\partial_i u_j\right) u_i \\
&= \sum_j u_j\left(\sum_i u_i \partial_i u_j\right) \\
&= \sum_j u_j\left((u.\nabla)u\right)_j \\
&= (u.\nabla u).u
\end{aligned}$$

Actually, $\rho = \rho_0$ is not necessary.
$\frac{\nabla p}{\rho} \neq \nabla\left(\frac{p}{\rho}\right)$ if ρ is not constant, but $\frac{\nabla p}{\rho}.u = \nabla\left(\frac{p}{\rho}\right).u$. Indeed,

$$\begin{aligned}
\nabla\left(\frac{p}{\rho}\right) &= \frac{\nabla p}{\rho} - \frac{p}{\rho^2}\nabla\rho \\
\text{thus}\quad \nabla\left(\frac{p}{\rho}\right).u &= \frac{\nabla p}{\rho}.u - \frac{p}{\rho^2}\nabla\rho.u \\
&= \frac{\nabla p}{\rho}.u \\
\Rightarrow \nabla\left(\frac{p}{\rho}\right) &= \frac{\nabla p}{\rho}
\end{aligned}$$

because $\underbrace{\partial_t\rho}_{=0\text{ (stationnary)}} + \underbrace{\rho\nabla.u}_{=0\text{ (incompressibility)}} + u.\nabla\rho = 0 \qquad \Rightarrow u.\nabla\rho = 0$ $\square$

3.1.3 Second Bernoulli's law

Consider a perfect incompressible homogeneous potential flow:

$$\exists\rho : (t,x) \mapsto \rho(t,x) \quad \text{s.t.} \quad u(t,x) = \nabla_x\rho(t,x) \tag{3.29}$$

(in the following ”curl” = ”rotationnel”!)

Remark. If $u = \nabla\rho$, curl$(u) =$ curl$(\nabla\rho) = \nabla \wedge u = 0$.

If $d = 3$, $\nabla \wedge u = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix}$.

If $d = 2$, $u = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \Rightarrow \nabla \wedge u = \begin{pmatrix} 0 \\ 0 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix}$.

Usually, if $d = 2$, ”$\nabla \wedge u \equiv \partial_1 u_2 - \partial_2 u_1$”.

In a simply connected set, the inverse is true. If u is a smooth curl free vector field, $\exists \rho$ a scalar field s.t. $u = \nabla\rho$ Assume that $\rho f = -\rho\nabla V$. Then the quantity:

$$\partial_t \rho + \frac{1}{2}|u|^2 + \frac{p}{\rho} + V = a(t) \tag{3.30}$$

This quantity does not depend on x.

Proof. We have

$$\rho(\partial_t u + (u.\nabla)u) + \nabla p = -\rho\nabla V$$

and

$$\text{curl}(u) = 0 \Rightarrow \partial_i u_j = \partial_i u_i \forall i, j$$

Thus

$$\nabla \frac{|u|^2}{2} = (u.\nabla)u$$

Indeed,

$$\begin{aligned} \left(\nabla \frac{|u|^2}{2}\right)_i &= \frac{1}{2}\partial_i \left(\sum_j u_j^2\right) \\ &= \frac{1}{2}\sum_j \partial_i \left(u_j^2\right) \\ &= \sum_i u_j \partial_i u_j \\ &= \sum_j u_j \partial_j u_i \\ &= ((u.\nabla)u)_i \end{aligned}$$

Thus, the momentum equation writes:

$$\partial_t(\nabla\rho) + \nabla\frac{|u|^2}{2} + \frac{\nabla p}{\rho_0} + \nabla V = 0$$

and finally:

$$\nabla\left(\partial_t \rho + \frac{|u|^2}{2} + \frac{p}{\rho_0} + V\right) = 0$$

□

Remark. If the flow is moreover stationary,

$$\partial_t \rho + \frac{|u|^2}{2} + \frac{p}{\rho_0} + V = 0$$

Remark. On the equivalence: $u = \nabla \rho \Leftrightarrow \mathrm{curl}(u) = 0$ for smooth u.

1. In dimension 1, any $u(x)$ is a gradient: the gradient of $x \mapsto \int_0^x u(y)\,\mathrm{d}y = \rho(x)$.
2. If $\mathrm{curl}(u) = 0$, u is a gradient, if $d \geq 2$:
   - Assume u is defined on $\mathbb{R}^2$;
   - Define $\rho(x) = \int\limits_0^{|x|} u\left(s\frac{x}{|x|}\right) . \frac{x}{|x|}\,\mathrm{d}s$ for $x \neq 0$.

   We have $u = \nabla \rho$. Indeed: $u = \nabla \rho \Leftrightarrow u.e = \nabla \rho . e \forall e$.

   - if $e = x$, $\nabla \rho(x).x = u(x).x$
   - if $e \neq x$ and $e \neq 0$, we define $y = x + e$ s.t.

$$\rho(y) = \int\limits_0^{|y|} u\left(s\frac{y}{|y|}\right) . \frac{y}{|y|}\,\mathrm{d}s \tag{3.31}$$

We have also
$$\rho(y) = \rho(x) + \int\limits_0^{|e|} u\left(x + \frac{se}{|e|}\right) . \frac{e}{|e|}\,\mathrm{d}s \tag{3.32}$$

where x is a vector, s a scalar and e a vector!
Indeed,

$$\int\limits_0^{|x|} u\left(s\frac{x}{|x|}\right) . \frac{x}{|x|}\,\mathrm{d}s + \int\limits_0^{|e|} u\left(x + \frac{se}{|e|}\right) . \frac{e}{|e|}\,\mathrm{d}s + \int\limits_0^{|y|} u\left(y + s\frac{-y}{|y|}\right)\,\mathrm{d}s = 0$$

because: $\mathrm{curl}(u) = 0$.
Thus $\nabla \rho(x).e = u.e$.
u has to be smooth enough inside the triangle ($\Rightarrow$no hole in the domain!). The domain has to be simply connected.

Remark. In a simply connected set, if u is smooth enough:

$$\{\nabla u = 0\} \Leftrightarrow \{\exists v / u = \nabla \wedge v\}$$

Remark. On boundary conditions for potential incompressible flows. If $u = \nabla \rho$ and $\nabla . u = 0$, $\Delta \rho = 0$. **Elliptic equation:** if the pb is set in an open set $\Omega \subset \mathbb{R}^d$, it is natural to impose boundary conditions for ρ of type:

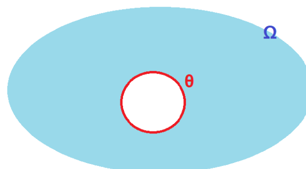
$$\rho = \rho_0 \text{ on } \partial\Omega \quad \text{(Dirichlet)}$$
$$(\nabla \rho . n =) \quad \partial_n \rho = g \text{ on } \partial\Omega \quad \text{(Neumann)}$$

What does it mean for $u = \nabla f$ (cas particulier)? For Neumann conditions, we have $u.n = g$ (way to impose the normed velocity). If $g = 0$ (homogeneous NC), $u.n = 0$ (non penetration condition).
For DC, imposing $\rho_{=}\rho_0$ on $\partial\Omega$

- $\Leftrightarrow$ imposing $\nabla\rho.\tau = \nabla\rho_0.\tau$ (if ρ_0 can be defined in $\mathbb{R}^d$) for any tangent vector τ.
- $\Leftrightarrow$ imposing $u.\tau = \nabla\rho_0.\tau$ (Dirichlet BC are a way to impose the tangential velocity).

3.1.4 D'Alembert condition

Consider θ a smooth open set in $\mathbb{R}^3$ (typically $\theta = B(0,1)$ or $B(0,R)$ with$R \to +\infty$) and $\theta \subset \Omega$, Ω has to be large. Consider a perfect incompressible homogeneous fluid in $\Omega \setminus \theta$ (θ is like an obstacle!)



in a stationary potential flow:

$$\exists \rho(x)/u(x) = \nabla\rho(x) \tag{3.33}$$

Assume that θ is impenetrable:

$$\partial_n \rho = 0 \text{ on } \partial\theta$$

and that $u = U$ "at infinity" where U is a constant field.
The problem rewrites:

$$\begin{cases} u = \nabla\rho \\ \Delta\rho = 0 \\ \partial_n\rho = 0 \quad \text{on } \partial\theta \partial_n\rho = U.n \quad \text{on } \partial\Omega \end{cases} \tag{3.34}$$

We want to compute the force F exerted by the fluid on θ:

$$F = \int_{\partial\theta} -p.n \tag{3.35}$$

Thanks to Bernoulli's second law,

$$\partial_t\rho + \frac{|u|^2}{2} + \frac{p}{\rho} = C \tag{3.36}$$

$$p = \rho_0 C - \frac{\rho_0}{2}|u|^2. \tag{3.37}$$

$$\text{Thus,} \qquad F_U = -\int_{\partial\theta} \rho_0 C.n + \frac{\rho_0}{2}\int_{\partial\theta} |u|^2.n \tag{3.38}$$

$$= \underbrace{-\int_{\partial\theta} \nabla(\rho_0 C)}_{=0} + \frac{\rho_0}{2}\int_{\partial\theta} |u|^2.n \tag{3.39}$$

$$= \frac{\rho_0}{2}\int_{\partial\theta} |u|^2.n \tag{3.40}$$

Consider now the problem where we replace U with $-U$. The solution is $-\rho$. Thus:

$$F_{-U} = -F_U$$

If θ is symmetric to a plane that is orthogonal to U (if $\theta = B(0,1)$). Thus $F_U = 0$.

Chapter 4

Viscous Newtonian Fluids

Definition 2. Let $x \in \mathbb{R}^d$ (occupied by a fluid), $n \in \mathbb{R}^d$, $\quad \epsilon > 0$.
Let $D(x, n, \epsilon)$ be the disk with center x, orthogonal to n, with surface ϵ.
The fluid is said to have a **stress tensor** $\sigma(x) \in \mathcal{M}_d(\mathbb{R})$ if $\frac{F(x,n,\epsilon)}{\epsilon} \xrightarrow[\epsilon \to 0]{} \sigma(x)n$ where $F(x, n, \epsilon)$ is the force exerted by the fluid on the side where n points on $D(x, n, \epsilon)$ (the force density is linear with respect to n).

Example. In a perfect fluid, $\sigma(x) = -p(x)I$.
In a fluid that has a stress tensor, the force exerted on a surface S by the fluid on the side of n is $\int_S \sigma.n$.
We will try to express σ as a function of this flow to derive a new fluid model $\sigma = \text{function}(p, \nabla u)$. (grâce au gradient de la vélocité, il y a une force F le long de l'obstacle!)

Appendices

Appendix A

Answers to exercises

A.1

A.1

Proof. Since we have
$$V^{int} = \sum_{i<j} V(|x_j - x_i|)$$
. Then

The notation in course are strange. □

A.2

Lemma : if $\rho \in C^3_U(\mathbb{R}_+ \times \mathbb{R})$,
$$\exists C, \quad |\varepsilon^n_j| \leq C(\Delta t + \Delta x^2) \tag{A.1}$$
with
$$\varepsilon^n_j = \frac{\rho(t^{n+1}, x_j) - \rho(t^n, x_j)}{\Delta t} + u\frac{\rho(t^n, x_{j+1}) - \rho(t^n, x_{j-1})}{2\Delta x} \tag{A.2}$$

Proof. With a Taylor expansion, we have:
$$\rho(t^{n+1}, x_j) - \rho(t^n, x_j) = \Delta t \partial_t \rho(t^n, x_j) + \mathcal{O}(\Delta t^2)$$
$$\rho(t^n, x_{j+1}) - \rho(t^n, x_j) = \Delta x \partial_x \rho(t^n, x_j) + \frac{\Delta x^2}{2} \partial_x^2 \rho(t^n, x_j) + \mathcal{O}(\Delta x^3)$$
$$\rho(t^n, x_{j-1}) - \rho(t^n, x_j) = -\Delta x \partial_x \rho(t^n, x_j) + \frac{\Delta x^2}{2} \partial_x^2 \rho(t^n, x_j) + \mathcal{O}(\Delta x^3)$$

By substracting the last two relations, we obtain:

$$\rho(t^n, x_{j+1}) - \rho(t^n, x_{j-1}) = 2\Delta x \partial_x \rho(t^n, x_j) + \mathcal{O}(\Delta x^3)$$

Thus,

$$\begin{aligned}\varepsilon_j^n &= \partial_t \rho(t^n, x_j) + \mathcal{O}(\Delta t) + u\partial_x \rho(t^n, x_j) + \mathcal{O}(\Delta x^2) \\ &= \mathcal{O}(\Delta t + \Delta x^2)\end{aligned}$$

Then we can conclude. □

A.3

Verification.

$$\begin{aligned}e_j^{n+1} &= \rho_j^n - u\frac{\Delta t}{2\Delta x}\left(\rho_{j+1}^n - \rho_{j-1}^n\right) - \left(\rho(t^n, x_j) - u\frac{\Delta t}{2\Delta x}(\rho(t^n, x_{j+1}) - \rho(t^n, x_{j-1}) + \Delta t \varepsilon_j^n\right). \\ &= e_j^n - u\frac{\Delta t}{2\Delta x}(e_{j+1}^n - e_{j-1}^n) - \Delta t \varepsilon_j^n\end{aligned}$$

□

A.4 Proof of discrete Gronwall Lemma

Proof. Remark that $1 + \Delta t \leq e^{C\Delta t}$ (since $C\Delta t$ is positive), then we can prove the implication by recurrence. □

A.5 Lax-Wendroff scheme

Exercise.
The Lax-Wendroff scheme is :

$$\rho_j^{n+1} = \rho_j^n - \frac{\nu}{2}\left(\rho_{j+1}^n - \rho_{j-1}^n\right) + \frac{\nu^2}{2}\left(\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n\right) \tag{A.3}$$

Prove that this scheme is 2^{nd} order (in Δt and Δx) consistent. Prove that the scheme is ℓ^2-stable $\left(\|\rho^{n+1}\|_{\ell^2} \leq \|\rho^n\|_{\ell^2}\right)$ (and convergent) under the CFL condition $\nu \leq 1$.

Proof. **Consistency:**

We define

$$\Delta t\varepsilon_j^n = \rho(t^{n+1},x_j)-\rho(t^n,x_j)+\frac{\nu}{2}\left(\rho(t^n,x_{j+1})-\rho(t^n,x_{j-1})\right)-\frac{\nu^2}{2}\left(\rho(t^n,x_{j+1})-2\rho(t^n,x_j)+\rho(t^n,x_{j-1})\right)$$

With a Taylor expansion, we have:

$$\begin{aligned}
\rho(t^{n+1},x_j)-\rho(t^n,x_j) &= \Delta t\partial_t\rho(t^n,x_j)+\frac{\Delta t^2}{2}\partial_t^2\rho(t^n,x_j)\mathcal{O}(\Delta t^3)\\
\rho(t^n,x_{j+1})-\rho(t^n,x_j) &= \Delta x\partial_x\rho(t^n,x_j)+\frac{\Delta x^2}{2}\partial_x^2\rho(t^n,x_j)+\mathcal{O}(\Delta x^3)\\
\rho(t^n,x_{j-1})-\rho(t^n,x_j) &= -\Delta x\partial_x\rho(t^n,x_j)+\frac{\Delta x^2}{2}\partial_x^2\rho(t^n,x_j)+\mathcal{O}(\Delta x^3)
\end{aligned}$$

By summing and substracting the last two relations, we obtain:

$$\begin{aligned}
\rho(t^n,x_{j+1})-\rho(t^n,x_{j-1}) &= 2\Delta x\partial_x\rho(t^n,x_j)+\mathcal{O}(\Delta x^3)\\
\rho(t^n,x_{j+1})-2\rho(t^n,x_j)+\rho(t^n,x_{j-1}) &= \Delta x^2\partial_x^2\rho(t^n,x_j)+\mathcal{O}(\Delta x^3)
\end{aligned}$$

By deriving the equation with respect to t and x, we will also have:

$$\partial_t^2\rho = u^2\partial_x^2\rho$$

. Thus,

$$\begin{aligned}
\Delta t\varepsilon_j^n &= \Delta t\partial_t\rho(t^n,x_j)+\frac{\Delta t^2}{2}\partial_t^2\rho(t^n,x_j)+\mathcal{O}(\Delta t^3)+\frac{\nu}{2}\left(2\Delta x\partial_x\rho(t^n,x_j)+\mathcal{O}(\Delta x^3)\right)-\frac{\nu^2}{2}\left(\Delta x^2\partial_x^2\rho(t^n,x_j)\mathcal{O}(\Delta x^3)\right)\\
&= \Delta t\left(\partial_t\rho(t^n,x_j)+u\partial_x\rho(t^n,x_j)\right)+\frac{\Delta t^2}{2}\partial_t^2\rho(t^n,x_j)-\frac{u^2\Delta t^2}{2}\partial_x^2\rho(t^n,x_j)+\mathcal{O}(\Delta t^3)+\mathcal{O}(\Delta t\Delta x^2)+\mathcal{O}(\Delta t^2\Delta x)\\
\varepsilon_j^n &= \mathcal{O}(\Delta t^2)+\mathcal{O}(\Delta x^2)+\mathcal{O}(\Delta t\Delta x)
\end{aligned}$$

Therefore, the scheme is consistent of order 2.

ℓ^2 stability:

Assume that $\rho_j^n \in \ell^2$ so we can define $\rho_\Delta^n \in L^2$ with the same definition as before. Then, we have

$$\rho_\Delta^{n+1} = \rho_\Delta^n - \frac{\nu}{2}\left(\rho_\Delta^n(\cdot+\Delta x)-\rho_\Delta^n(\cdot-\Delta x)\right)+\frac{\nu^2}{2}\left(\rho_\Delta^n(\cdot+\Delta x)-2\rho_\Delta^n+\rho_\Delta^n(\cdot-\Delta x)\right)$$

Then

$$
\begin{aligned}
\|\rho^{n+1}\| &= \frac{1}{\sqrt{\Delta x}}\|\rho_\Delta^{n+1}\|_{\ell^2} = \frac{1}{\sqrt{\Delta x}}\|\hat{\rho_\Delta^{n+1}}\|_{L^2} \\
&= \frac{1}{\sqrt{\Delta x}}\|\hat{\rho_\Delta^n} - \frac{\nu}{2}\hat{\rho_\Delta^n}\left(e^{i\Delta x\xi} - e^{-i\Delta x\xi}\right) + \frac{\nu^2}{2}\hat{\rho_\Delta^n}\left(e^{i\Delta x\xi} + e^{-i\Delta x\xi} - 2\right)\|_{L^2} \\
&= \frac{1}{\sqrt{\Delta x}}\|\hat{\rho_\Delta^n}\|_{L^2}\| - \frac{\nu}{2}\left(e^{i\Delta x\xi} - e^{-i\Delta x\xi}\right) + \frac{\nu^2}{2}\left(e^{i\frac{\Delta x\xi}{2}} - e^{-i\frac{\Delta x\xi}{2}}\right)^2\| \\
&= \frac{1}{\sqrt{\Delta x}}\|\hat{\rho_\Delta^n}\|_{L^2}\|1 - i\nu\sin(\Delta x\xi) - 2\nu^2\sin^2(\frac{\Delta x\xi}{2})\| \\
&= \frac{1}{\sqrt{\Delta x}}\|\hat{\rho_\Delta^n}\|_{L^2}\|(1 - 2\nu^2\sin^2(\frac{\Delta x\xi}{2}))^2 + \nu^2\sin^2(\Delta x\xi)\| \\
&= \frac{1}{\sqrt{\Delta x}}\|\hat{\rho_\Delta^n}\|_{L^2}\|[1 - \nu^2 + \nu^2(1 - 2\sin^2(\frac{\Delta x\xi}{2})]^2 + \nu^2\sin^2(\Delta x\xi)\| \\
&= \frac{1}{\sqrt{\Delta x}}\|\hat{\rho_\Delta^n}\|_{L^2}\|(1-\nu^2)^2 + 2(1-\nu^2)\nu^2\cos(\Delta x\xi) + \nu^4\cos^2(\Delta x\xi) + \nu^2\sin^2(\Delta x\xi)\| \\
&= \frac{1}{\sqrt{\Delta x}}\|\hat{\rho_\Delta^n}\|_{L^2}\|(1-\nu^2)^2 + 2(1-\nu^2)\nu^2 + \nu^2\| \text{ since } \mu^2 < 1 \Rightarrow \mu^4 < \mu^2 \\
&= \frac{1}{\sqrt{\Delta x}}\|\hat{\rho_\Delta^n}\|_{L^2} = \|\rho^n\|
\end{aligned}
$$

Thus $\|\rho^{n+1}\| \leq \|\rho^n\|$, and we can conclude that the scheme is ℓ^2 stable. □

A.6

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Exercise 4.
- Prove that $|\varepsilon_j^n| \leq C(\Delta t + \Delta x)$ if ρ is smooth.
- Prove the convergence of the scheme under the CFL condition.

Proof. With a Taylor expansion, we have:

$$
\begin{aligned}
\rho(t^{n+1}, x_j) - \rho(t^n, x_j) &= \Delta t\partial_t\rho(t^n, x_j) + \mathcal{O}(\Delta t^2) \\
\rho(t^n, x_j) - \rho(t^n, x_{j-1}) &= \Delta x\partial_x\rho(t^n, x_j) + \mathcal{O}(\Delta x^2)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\varepsilon_j^n &= \partial_t\rho(t^n, x_j) + \mathcal{O}(\Delta t) + u\partial_x\rho(t^n, x_j) + \mathcal{O}(\Delta x) \\
&= \mathcal{O}(\Delta t + \Delta x)
\end{aligned}
$$

Thus, we have the result.

Moreover, under the CFL condition, we have:

$$\begin{aligned}
|e_j^{n+1}| &\leq |e_j^n|(1-u_j^n\frac{\Delta t}{\Delta x}) + |e_{j-1}^n|u_j^n\frac{\Delta t}{\Delta x} + \Delta t|\varepsilon_j^n| \\
&\leq \|e^n\| + \Delta t\|\varepsilon^n\| \\
\Rightarrow \|e^n\| &\leq \|e^0\| + \Delta t\sum_{i=0}^{n-1}\|\varepsilon^n\| \\
&= n\Delta t C(\Delta t + \Delta x)
\end{aligned}$$

. So the scheme is convergent. □

A.7

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Exercise 5.

- Prove that the scheme is consistent.
- Prove that it is convergent under the CFL condition.

Proof. The scheme is

$$\begin{aligned}
\rho_j^{n+1} &= \rho_j^n + (u_{j+\frac{1}{2}}^n)^-\frac{\Delta t}{\Delta x}(\rho_{j+1}^n - \rho_j^n) - (u_{j-\frac{1}{2}}^n)^+\frac{\Delta t}{\Delta x}(\rho_j^n - \rho_{j-1}^n) \\
&= \rho_j^n\left(1 - (u_{j+\frac{1}{2}}^n)^-\frac{\Delta t}{\Delta x} - (u_{j-\frac{1}{2}}^n)^+\frac{\Delta t}{\Delta x}\right) + (u_{j+\frac{1}{2}}^n)^-\frac{\Delta t}{\Delta x}\rho_{j+1}^n + (u_{j-\frac{1}{2}}^n)^+\frac{\Delta t}{\Delta x}\rho_{j-1}^n
\end{aligned}$$

. Thus,

$$\begin{aligned}
\varepsilon_j^n &= \frac{\rho(t^{n+1},x_j) - \rho(t^n,x_j)}{\Delta t} - (u_{j+\frac{1}{2}}^n)^-\frac{(\rho(t^n,x_{j+1}) - \rho(t^n,x_j)}{\Delta x}) + (u_{j-\frac{1}{2}}^n)^+\frac{(\rho(t^n,x_j) - \rho(t^n,x_{j-1}))}{\Delta x} \\
&= \partial_t\rho(t^n,x_j) + \mathcal{O}(\Delta t) + \partial_x\rho(t^n,x_j) + \mathcal{O}(\Delta x) \\
&= \mathcal{O}(\Delta t + \Delta x)
\end{aligned}$$

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□

¡++¿