IE 521 Convex Optimization

Lecture 3: Separation Theorems

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Outline

Quick Review

- Radon's Theorem
  - Any set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two disjoint sets whose convex hulls intersect.
- Helley's Theorem
  - If every $(d+1)$ of the sets from a collection of n sets in $\mathbb{R}^d$ intersect $(n > d)$, then the whole collection of sets intersect.

Question

What's in common and what's different?

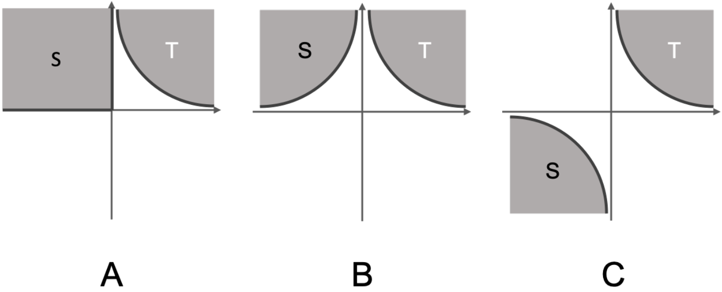


Figure: Separation of sets

Separation of Sets

Definition. Let S and T be two nonempty convex sets in $\mathbb{R}^n$. A hyperplane $H = \{x \in \mathbb{R}^n : a^T x = b\}$ with $a \neq 0$ is said to separate S and T if $S \cup T \not\subset H$ and

$$S \subset H^- = \left\{x \in \mathbb{R}^n : a^T x \leq b\right\}$$

$$T \subset H^+ = \left\{x \in \mathbb{R}^n : a^T x \geq b\right\}$$

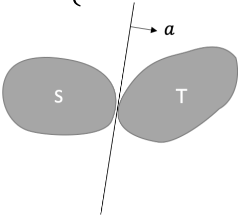


Figure: Separation of two sets

- Separation is equivalent to say

$$\sup_{x\in S} a^T x \leq \inf_{x\in T} a^T x \text{ and } \inf_{x\in S} a^T x < \sup_{x\in T} a^T x.$$

Strict Separation of Sets

Definition. Let S and T be two nonempty convex sets in $\mathbb{R}^n$. A hyperplane $H = \left\{x \in \mathbb{R}^n : a^T x = b\right\}$ with $a \neq 0$ is said to <u>strictly separate S and T</u> if

$$S \subset H^{--} = \left\{x \in \mathbb{R}^n : a^T x < b\right\}$$

$$T \subset H^{++} = \left\{x \in \mathbb{R}^n : a^T x > b\right\}$$

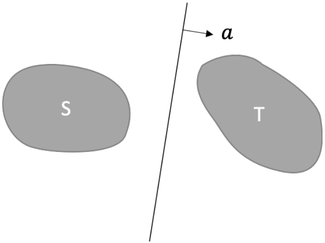


Figure: Strict Separation of two sets

Strong Separation of Sets

Definition. Let S and T be two nonempty convex sets in $\mathbb{R}^n$. A hyperplane $H = \{x \in \mathbb{R}^n : a^T x = b\}$ with $a \neq 0$ is said to strongly separate S and T if there exits $b' < b < b''$ such that

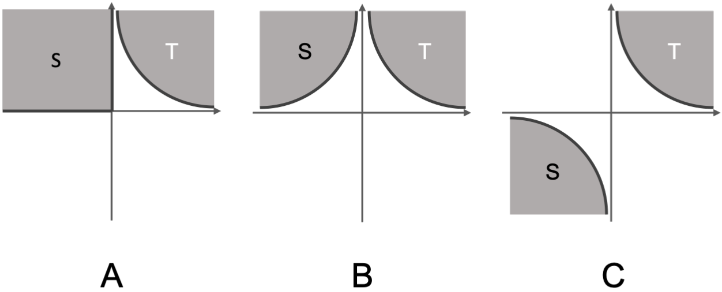
$$S \subset \left\{x \in \mathbb{R}^n : a^T x \leq b'\right\}$$
$$T \subset \left\{x \in \mathbb{R}^n : a^T x \geq b''\right\}$$

Remark.

- Strong separation $\Longrightarrow$ strict separation.
- Strict separation $\not\Longrightarrow$ strong separation.
- Strong separation is equivalent to say

$$\sup_{x \in S} a^T x < \inf_{x \in T} a^T x.$$

Example



In all examples, S and T are two closed and convex sets

(A) S and T are separated, but not strictly;

(B) S and T are strictly separated, but not strongly;

(C) S and T are strongly separated.

Separation Hyperplane Theorem

Theorem. Let S and T be two nonempty convex sets. Then S and T can be separated <u>if and only if</u>

$$\mathrm{rint}(S) \cap \mathrm{rint}(T) = \emptyset.$$

Corollary. Let S be a nonempty convex set and $x_0 \in \partial S$. There exists a supporting hyperplane $H = \left\{x : a^T x = a^T x_0\right\}$ with $a \neq 0$ such that

$$S \subset \left\{x : a^T x \leq a^T x_0\right\}, \text{ and } x_0 \in H.$$

Proof of Separation Theorem

S and T can be separated iff $\mathrm{rint}(S) \cap \mathrm{rint}(T) = \emptyset$.

- **Necessity.**
  - S and T are separated implies that for some $a \neq 0$,
    $$\sup_{x\in S} a^T x \leq \inf_{x\in T} a^T x.$$
  - If $z \in \mathrm{rint}(S) \cap \mathrm{rint}(T)$, then
    $$z = \mathrm{argmax}_{x\in S}\{a^T x\} = \mathrm{argmin}_{x\in T}\{a^T x\}.$$
  - The linear function $f(x) = a^T x$ has to be constant on both S and T, i.e., $S \cap T \subset H$. (why?)

Proof of Separation Theorem

S and T can be separated iff $\text{rint}(S) \cap \text{rint}(T) = \emptyset$.

- **Sufficiency.** Based on constructive steps:
    1. Separation of a convex set S and $x_0 \notin \text{cl}(S)$ (key step);
    2. Separation of a convex set S and $x_0 \notin \text{rint}(S)$;
    3. Separation of 0 and $\text{rint}(S) - \text{rint}(T)$;
    4. Separation of S and T.

Separation of Convex Set and A Point Outside

Proposition. Let S be convex and closed, $x_0 \notin S$. Then x_0 and S can be separated.

Proof. Define
$$d(\{x_0\}, S) := \inf\{\|x_0 - x\|_2 : x \in S\}$$
$$\mathrm{proj}(x_0) := \mathrm{argmin}_{x\in S}\{\|x_0 - x\|_2\}$$

Then $d(\{x_0\}, S) > 0$ and $\mathrm{proj}(x_0)$ exists and is unique (why?). The hyperplane

$$H := \left\{x : a^T x = b\right\}, a = x_0 - \mathrm{proj}(x_0), b = a^T x_0 - \frac{||a||_2}{2}$$

separates x_0 and S, i.e. $a^T x < b$, $\forall x \in S$, $a^T x_0 > b$. (why?)

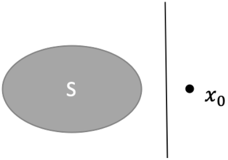


Figure: Separation of a convex set and a point

Strong Separation Hyperplane Theorem

Theorem. Let S and T be two nonempty convex sets. Then S and T can be strongly separated if and only if

$$\text{dist}(S, T) := \inf\{\|s - t\|_2 : s \in S, t \in T\} > 0.$$

In particular, if $S - T$ is closed and $S \cap T = \emptyset$, then S and T can be strongly separated.

Remarks

- "$S - T$ is closed" is only a sufficient condition for strong (strict) separation, not a necessary condition.
- Even if both S and T are closed convex, $S - T$ might not be closed, and they might not even be strictly separated.
- When both S and T are closed convex, $S \cap T = \emptyset$ and at least one of them is bounded, then $S - T$ is closed, and S and T can be strongly separated.

Proof of Strong Separation Theorem

S and T can be strongly separated iff $\text{dist}(S, T) > 0$.

- **Necessity.** If S and T are strongly separated, then $\exists a \neq 0 : \alpha := \sup_{x \in S} a^T x < \inf_{y \in T} a^T y := \beta$.
  Hence, $\forall x \in S, y \in T$:
  $$\|a\|_2 \|y - x\|_2 \geq a^T(y - x) \geq \beta - \alpha \Rightarrow \|y - x\|_2 \geq \frac{\beta - \alpha}{\|a\|_2}.$$

- **Sufficiency.** Suppose $r := \text{dist}(S, T) > 0$, then $(S - T)$ and $B(0, r)$ are two disjoint convex sets. By Separation Theorem, $\exists a \neq 0$,
  $$\sup_{z \in S - T} a^T z = \sup_{x \in S, y \in T} a^T (x - y) \leq \inf_{z \in B(0,r)} a^T z < 0.$$

Feasibility of Linear System

Example . Show that the following system have no solution.

$$\begin{cases} x_1 - x_2 + 2x_3 & \leq 0 & \cdots \times 3 \\ -x_1 + x_2 - x_3 & \leq 0 & \cdots \times 5 \\ 2x_1 - x_2 + 3x_3 & \leq 0 & \cdots \times 3 \\ 4x_1 - x_2 + 10x_3 & > 0 & \end{cases}$$

$3 \times Eq.(1) + 5 \times Eq.(2) + 3 \times Eq.(3) \Rightarrow 4x_1 - x_2 + 10x_3 \leq 0.$
Note the system

$$\begin{cases} y_1 - y_2 + 2y_3 & = 4 \\ -y_1 + y_2 - y_3 & = -1 \\ 2y_1 - y_2 + 3y_2 & = 10 \\ y_1, y_2, y_3 & \geq 0 \end{cases}$$ has a solution $y = (3, 5, 3)$.

The Celebrated Farkas' Lemma

Theorem. (Farkas' Lemma) Let $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$. Exactly one of the following sets must be empty:

(i) $\{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$;

(ii) $\left\{y \in \mathbb{R}^m : A^T y \leq 0, b^T y > 0\right\}$.

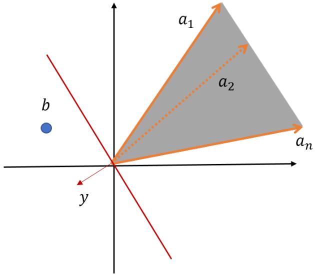
- System (i) and (ii) are often called *strong alternatives*, i.e. exactly one of them must be feasible.
- This is an example of "theorem on alternatives".

Figure: Gyula Farkas (1847–1930)

Geometric View of Farkas' Lemma

- Let $A = [a_1|a_2|...|a_n]$, define the cone $\text{Cone}\{a_1, ..., a_n\} = \{\sum_{i=1}^n x_i a_i : x_i \geq 0, i = 1, ..., n\}$

$$
\begin{aligned}
&\{Ax = b, x \geq 0\} \text{ is infeasible} \\
\Longleftrightarrow\, & b \notin \text{Cone}\{a_1, ..., a_n\} \\
\Longrightarrow\, & \exists y, y^T a_i \leq 0, \forall i, y^T b > 0
\end{aligned}
$$



- Farkas' lemma can be regarded as a special case of the separation theorem.

Proof of Farkas' Lemma

- System (i) feasible $\Rightarrow$ system (ii) infeasible.
  Otherwise, $0 < b^T y = (Ax)^T y = x^T(A^T y) \leq 0$.

- System (i) infeasible $\Rightarrow$ system (ii) feasible.
  Let $C = \text{Cone}\{a_1, ..., a_n\}$.
  Then C is convex and closed. And $b \notin C$.
  - By the separation theorem, b and C can be strongly separated, i.e. $\exists y \neq 0 \in \mathbb{R}^m, \gamma \in \mathbb{R}$, s.t.

$$y^T z \leq \gamma, \forall z \in C, y^T b > \gamma.$$

  - Since $0 \in C$, we have $\gamma \geq 0$.
  - Show that $\gamma = 0$. Suppose $\gamma > 0$, and $\exists z_0 \in C$ such that $y^T z_0 > 0$, then $y^T(\alpha z_0) > \gamma$ for α large enough.
  - Since $a_1, .., a_n \in C$, we have $y^T a_i \leq 0$, $\forall i = 1, ..., m$, i.e., $A^T y \leq 0$.

Remarks

- The closedness of the cone $\text{Cone}\{a_1, ..., a_n\}$ is crucial here. Note that in general, when S is not a finite set, $\text{Cone}(S)$ is not always closed.
- Farkas' Lemma can also be proved by Fourier-Motzkin elimination.
- Result can be generalized to convex inequalities.

Variant of Farkas' Lemma

Theorem. Exactly one of the following two sets must be empty:

(i) $\{x \in \mathbb{R}^n : Ax \leq b\}$

(ii) $\left\{y \geq 0 : A^T y = 0, b^T y < 0\right\}$

Theorem. Exactly one of the following two sets must be empty:

(i) $\{x \in \mathbb{R}^n : Ax = b\}$

(ii) $\left\{y \in \mathbb{R}^m : A^T y = 0, b^T y \neq 0\right\}$

Duality of Linear Program

Consider the primal and dual pair of linear programs

$$(P) \quad \begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad\qquad (D) \quad \begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y \leq c \end{aligned}$$

Theorem. **(LP Duality)** If (P) has a finite optimal value, then so does (D) and the two values equal each other.

Proof: Homework Exercise.

Who introduced LP duality?

Figure: Leonid Kantorovich (1912–1986)

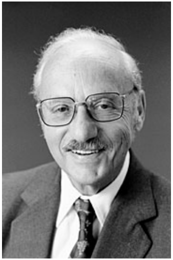


Figure: George Dantzig (1914–2005)

Figure: John von Neumann (1903–1957)

References

- Boyd & Vandenberghe, Chapter 2.5
- Ben-Tal & Nemirovski, Chapter 1.2.5-1.2.6