IE 521 Convex Optimization

Lecture 6: Subgradient and Subdifferential

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Outline

Question

Can you find any affine function that underestimates $f(x)$ and is tight at $x = 0$? What about when $x \neq 0$?

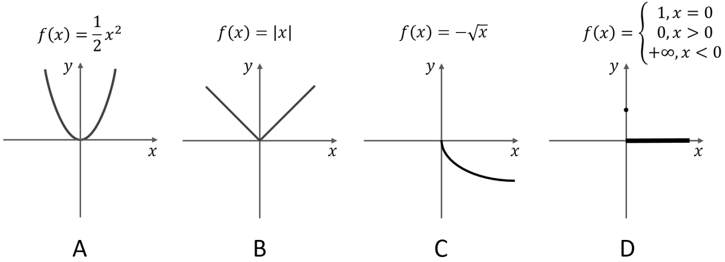


Figure: Convex Functions

Subgradient

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex.

Definition. A vector $g \in \mathbb{R}^n$ is a *subgradient* of f at a point $x_0 \in dom(f)$ if

$$f(x) \geq f(x_0) + g^T(x - x_0), \forall x.$$

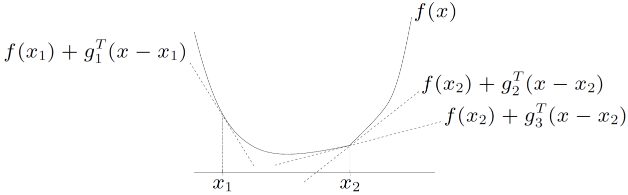


Figure: Subgradients

Definition. The set of all subgradient at x_0 is called the *subdifferential* of f at x_0 denoted as $\partial f(x_0)$.

Subgradient and Epigraph

Subgradients form supporting hyperplanes for the epigraph.

$$
\begin{aligned}
& g \in \partial f(x_0) \\
& \Leftrightarrow f(x) - g^T x \geq f(x_0) - g^T x_0, \forall x \\
& \Leftrightarrow t - g^T x \geq f(x_0) - g^T x_0, \forall (x,t) \in \mathrm{epi}(f) \\
& \Leftrightarrow \begin{bmatrix} -g \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ t \end{bmatrix} \geq \begin{bmatrix} -g \\ 1 \end{bmatrix}^T \begin{bmatrix} x_0 \\ f(x_0) \end{bmatrix}, \forall (x,t) \in \mathrm{epi}(f) \\
& \Leftrightarrow H := \left\{ (x,t) : (-g,1)^T (x,t) = (-g,1)^T (x_0, f(x_0)) \right\} \\
& \quad \text{is a supporting hyperplane of } \mathrm{epi}(f) \text{ at } (x_0, f(x_0))
\end{aligned}
$$

Examples: Differentiable Functions

Example 1. If f is differentiable at $x \in dom(f)$, then

$$\partial f(x) = \{\nabla f(x)\}.$$

Proof. Let $y = x + \epsilon d$, $g \in \partial f(x)$, then

$$\begin{aligned}
& f(x+\epsilon d) \geq f(x) + \epsilon g^T d \\
\Rightarrow\ & \frac{f(x+\epsilon d) - f(x)}{\epsilon} \geq g^T d, \forall d, \forall \epsilon \\
\Rightarrow\ & \nabla f(x)^T d \geq g^T d, \forall d, \text{ as } \epsilon \to 0 \\
\Rightarrow\ & g = \nabla f(x).
\end{aligned}$$

Examples: Simple Functions

Example 2.

(a) $f(x) = \frac{1}{2}x^2$, $\partial f(x) = x$

(b) $f(x) = |x|$, $\partial f(x) = \begin{cases} \text{sgn}(x), x \neq 0 \\ [-1, 1], x = 0 \end{cases}$.

(c) $f(x) = \begin{cases} -\sqrt{x}, x \geq 0 \\ +\infty, o.w. \end{cases}$, $\partial f(x) = \begin{cases} -\frac{1}{2\sqrt{x}}, x > 0 \\ \emptyset, x = 0 \end{cases}$.

(d) $f(x) = \begin{cases} 1, x = 0 \\ 0, x > 0 \\ +\infty, o.w. \end{cases}$, $\partial f(x) = \begin{cases} 0, x > 0 \\ \emptyset, x = 0 \end{cases}$.

Closedness of Subdifferential

Proposition. Let f be convex and $x_0 \in \mathrm{dom}(f)$. Then $\partial f(x_0)$ is convex and closed.

Proof. This is because

$$\begin{aligned}\partial f(x_0) &= \left\{g \in \mathbb{R}^n : f(x) \geq f(x_0) + g^T(x - x_0), \forall x\right\} \\ &= \cap_x \left\{g \in \mathbb{R}^n : f(x) \geq f(x_0) + g^T(x - x_0)\right\}\end{aligned}$$

is the solution to an infinite system of linear inequalities.

Existence of Subgradient

Theorem. Let f be convex and $x_0 \in \text{rint}(\text{dom}(f))$. Then $\partial f(x_0)$ is nonempty and bounded.

Remark. The reverse is also true. If $\forall x_0 \in \text{dom}(f), \partial f(x_0)$ is non-empty, and $\text{dom}(f)$ is convex, then f is convex.

Proof. Let $g \in \partial f(x_0)$ and $x_0 = \lambda x + (1-\lambda)y$, we have

$$\begin{cases} f(x) \geq f(x_0) + g^T(x - x_0) \\ f(y) \geq f(x_0) + g^T(y - x_0) \end{cases}$$

$$\Rightarrow \lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y)$$

Proof of Existence and Boundedness

- **(Nonempty)** By separation theorem, $\exists \alpha = (s, \beta) \neq 0$,

$$s^T x + \beta t \geq s^T x_0 + \beta f(x_0), \forall (x, t) \in \text{epi}(f)$$

We must have $\beta > 0$ (why?). Setting $g = -\beta^{-1} s$,

$$f(x) \geq f(x_0) + g^T (x - x_0), \forall x$$

- **(Bounded)** Suppose $\partial f(x_0)$ is unbounded, i.e. $\exists g_k \in \partial f(x_0)$, s.t. $\| g_k \|_2 \to \infty$, as $k \to \infty$. Let $x_k = x_0 + \delta \frac{g_k}{\|g_k\|_2} \in \text{dom}(f)$. By convexity,

$$f(x_k) \geq f(x_0) + g_k^T (x_k - x_0) = f(x_0) + \delta \| g_k \|_2 \to \infty.$$

Contradicts with the continuity of f over $\text{int}(\text{dom}(f))$.

Monotonicity

Proposition. The subdifferential of a convex function f is a *monotone operator*, i.e.,

$$(u-v)^T(x-y) \geq 0, \forall x, y, u \in \partial f(x), v \in \partial f(y).$$

Proof.

By definition, we have

$$\begin{cases} f(y) \geq f(x) + u^T(y-x) \\ f(x) \geq f(y) + v^T(x-y) \end{cases}$$

Combining the two inequalities leads to the monotonicity.

Directional Derivative

Definition. The ***directional derivative*** of a function f at x along direction d is

$$f'(x;d) = \lim_{\delta \to 0^+} \frac{f(x+\delta d) - f(x)}{\delta}.$$

Remark.

- If f is differentiable, then $f'(x;d) = \nabla f(x)^T d$.
- $f'(x;d) = \phi'(0^+)$, where $\phi(\alpha) = f(x+\alpha d)$.
- $f'(x;d) = \inf_{t>0} \left(tf(x+d/t) - tf(x)\right)$ is convex in d (why?).
- $f'(x;d)$ defines a lower bound on f on direction d: $f(x+\alpha d) \geq f(x) + \alpha f'(x;d), \forall \alpha \geq 0$.

Descent Direction

Definition. The direction d is called a <u>*descent direction*</u> if

$$f'(x; d) < 0.$$

- If f is differentiable, then $d = -\nabla f(x)$ is a descent direction, except when it is zero.

Q. Is negative subgradient always a descent direction?

Descent Direction

- Negative subgradient may not be a descent direction.

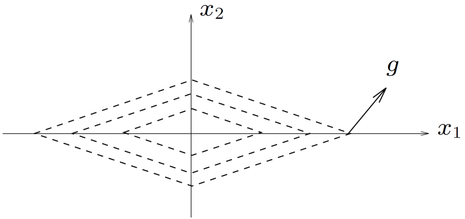


Figure: Contours of function $f(x_1, x_2) = |x_1| + 2|x_2|$

  - At $x = (1, 0)$, $\partial f(x) = \{(1, a) : a \in [-2, 2]\}$.
  - Consider $g = (1, 0)$, $d = -g$ is a descent direction.
  - Consider $g = (1, 2)$, $d = -g$ is not a descent direction.
- Note: let $g_* = \operatorname{argmin}_{g \in \partial f(x)} \{\|g\|_2^2\}$, then $d = -g_*$ is a descent direction if $g_* \neq 0$.

Directional Derivative and Subdifferential

Theorem. Let f be convex and $x \in \text{int}(\text{dom}(f))$, then

$$f'(x;d) = \max_{g \in \partial f(x)} g^T d$$

Proof

- Easy to show $f'(x; d) \geq \max_{g \in \partial f(x)} g^T d$.
- Suffice to show that $\exists \tilde{g} \in \partial f(x)$, s.t. $f'(x; d) \leq \tilde{g}^T d$.
  - Let $\tilde{g}$ be a subgradient of $f'(x; d)$ at d.
  - For any $v, \lambda \geq 0$:

$$\begin{aligned} f(x + \alpha v) - f(x) &\geq \alpha f'(x; v) \\ &= f'(x; \alpha v) \\ &\geq f'(x; d) + \tilde{g}^T(\alpha v - d). \end{aligned}$$

  - Setting $\alpha = \infty$ implies $f(x + v) - f(x) \geq f'(x; v) \geq \tilde{g}^T v$; thus $\tilde{g} \in \partial f(x)$.
  - Setting $\alpha = 0$ implies $f'(x; d) \leq \tilde{g}^T d$.

Calculus of Subgradients

Assume $x \in \text{int}(\text{dom}(h))$.

- Conic combination: Let $h(x) = \beta_1 f_1(x) + \beta_2 f_2(x)$ with $\beta_1, \beta_2 \geq 0$,

$$\partial h(x) = \beta_1 \partial f_1(x) + \beta_2 \partial f_2(x).$$

- Affine transformation: Let $h(x) = f(Ax + b)$,

$$\partial h(x) = A^T \partial f(Ax + b).$$

- Pointwise maximum: Let $h(x) = \max_{i=1,\dots,m} f_i(x)$,

$$\partial h(x) = \text{Conv}\left\{\partial f_i(x) | f_i(x) = h(x)\right\}.$$

- Pointwise supreme: Let $h(x) = \max_{\alpha \in \mathcal{A}} f_\alpha(x)$,

$$\partial h(x) = \text{cl}\left(\text{Conv}\left\{\partial f_\alpha(x) | f_\alpha(x) = h(x)\right\}\right).$$

Weak Calculus

- Maximization: $f(x) = \max_{y \in Y} \phi(x, y)$, where $\phi(x, y)$ is convex in x for any $y \in Y$.
  - Find $\hat{y} \in \operatorname{argmax}_{y \in Y} \phi(x, y)$.
  - $g \in \partial\phi(x, \hat{y})$ is a subgradient of $f(x)$.
- Minimization: $f(x) = \min_{y \in Y} \phi(x, y)$, where $\phi(x, y)$ is convex in (x, y) and Y is convex.
  - Find $\hat{y} \in \operatorname{argmin}_{y \in Y} \phi(x, y)$.
  - $g \in \partial\phi(x, \hat{y})$ is a subgradient of $f(x)$.
- Composition: $f(x) = F(f_1(x), \ldots, f_m(x))$, where $F(y_1, \ldots, y_m)$ is non-decreasing and convex.
  - Find $(d_1, \ldots, d_m) \in \partial F(y_1, \ldots, y_m)|_{y_i = f_i(x), i=1,\ldots,m}$.
  - Find $g_i \in \partial f_i(x), i = 1, \ldots, m$
  - $g = \sum_{i=1}^m d_i g_i$ is a subgradient of $f(x)$.

Example: Piecewise Linear Function

Example 3. Consider a single period inventory system. The cost $f(x)$ at inventory level x given demand d is

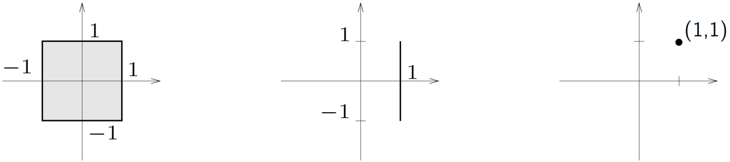
$$f(x) = h \cdot \max(x - d, 0) + p \cdot \max(d - x, 0).$$

The subgradient of $f(x)$ is

$$\partial f(x) = \begin{cases} h, & x > d \\ [-p, h], & x = d \\ -p, & x < d \end{cases}.$$

Example: ℓ_1-Norm

Example 5. $f(x) = \|x\|_1 = \max_{s \in \{-1,1\}^d} \{s^T x\}$



$\partial f(x)$ at $x = (0,0)$ at $x = (1,0)$ at $x = (1,1)$

Figure: Subgradient of $f(x) = \|x\|_1$ on $\mathbb{R}^2$

Example: general norm

Example 6. $f(x) = \|x\|$, here $\|\cdot\|$ is an arbitrary norm

$$\partial f(x) = \{g : g^T x = \|x\| \text{ and } \|g\|_* \leq 1\}.$$

- $\|\cdot\|_*$ is the dual norm: $\|y\|_* = \max_{x:\|x\|\leq 1} y^T x$.
- In particular, $\partial f(0) := \{g : \|g\|_* \leq 1\}$.

References

- Ben-Tal & Nemirovski, Chapter 2.6
- Bertsekas, Nedich, & Ozdaglar, Chapter 4