IE 521 Convex Optimization

Lecture 5: Convex Functions II

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Outline

Recap

Convex Function: $f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathrm{dom}(f)$ is convex and for any $\lambda \in [0, 1]$,

$$\forall x, y \in \mathrm{dom}(f), f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Remark. (Extended-value function) f can be extended to a function from $\mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by setting

$$f(x) = +\infty, \text{ if } x \notin \mathrm{dom}(f).$$

We say f is convex if for any $\lambda \in [0, 1]$,

$$\forall x, y, f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

General Convex Inequality

Proposition. If f is convex, then $\forall \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1$,

$$f\left(\sum_{i=1}^m \lambda_i x_i\right) \leq \sum_{i=1}^m \lambda_i f(x_i).$$

Remark. (Jensen's Inequality) Let ξ be a random variable and f be convex, then

$$f(\mathbb{E}[\xi]) \leq \mathbb{E}[f(\xi)].$$

Characterizations of Convex Functions

- Epigraph
- Level set
- One-dimensional property
- First order condition
- Second order condition

What can you tell from these sets?

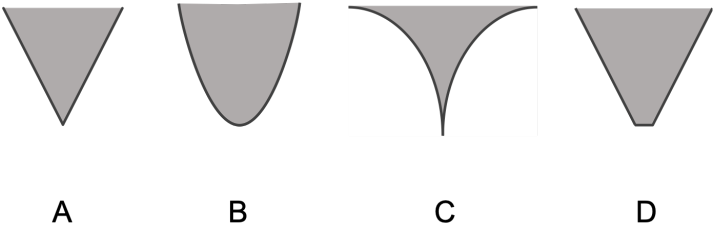

Figure: Four sets

Epigraph

Definition. The *epigraph* of a function f is

$$\text{epi}(f) = \left\{(x, t) \in \mathbb{R}^{n+1} : f(x) \le t\right\}$$

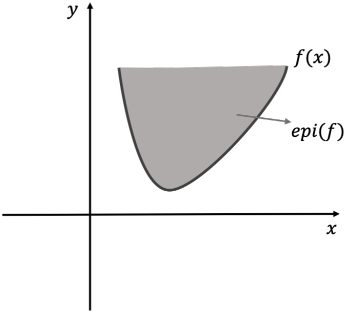

Figure: Epigraph

Proposition. f is convex iff $\text{epi}(f)$ is a convex set.

Q. Is $\text{epi}(f)$ always closed?

Epigraph

Proposition. f is convex iff $\text{epi}(f)$ is a convex set.

Proof.

- (if part) First, $\text{dom}(f)$ is convex since it is the projection of $\text{epi}(f)$. Second, since $(x_1, f(x_1)), (x_2, f(x_2)) \in \text{epi}(f)$, then

$$(\lambda x_1 + (1-\lambda)x_2, \lambda f(x_1) + (1-\lambda)f(x_2)) \in \text{epi}(f), \forall \lambda \in [0,1]$$
$$\Downarrow$$
$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2), \forall \lambda \in [0,1].$$

- (only if part) Let $(x_1, t_1), (x_2, t_2) \in \text{epi}(f)$,

$$\begin{aligned}
& f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2), \forall \lambda \in [0,1] \\
\Rightarrow\quad & f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda t_1 + (1-\lambda)t_2, \forall \lambda \in [0,1] \\
\Rightarrow\quad & \lambda(x_1, t_1) + (1-\lambda)(x_2, t_2) \in \text{epi}(f), \forall \lambda \in [0,1] \\
\Rightarrow\quad & \text{epi}(f) \text{is a convex set}
\end{aligned}$$

Epigraph

Example . If $f_i(x), i \in I$ are convex, then so is

$$g(x) := \max_{i \in I} f_i(x).$$

- Note $\mathrm{epi}(g) = \cap_{i \in I} \mathrm{epi}(f_i)$ is convex.

Example . If f is convex, then so is the perspective function

$$g(x, t) = t f(x/t),$$

where $\mathrm{dom}(g) = \{(x, t) : x/t \in \mathrm{dom}(f), t > 0\}$.

- Note $\mathrm{epi}(g) = P^{-1}(\mathrm{epi}(f))$ is convex, where P is the perspective mapping $(x, t, s) \to (x/t, s/t)$.
- $(x, t, s) \in \mathrm{epi}(g) \Leftrightarrow t f(x/t) \leq s \Leftrightarrow (x/t, s/t) \in \mathrm{epi}(f)$

Level Set

Definition. For any $t \in \mathbb{R}$, the *level set* of f at level t is

$$\mathrm{lev}_t(f) = \{x \in \mathrm{dom}(f) : f(x) \leq t\}.$$

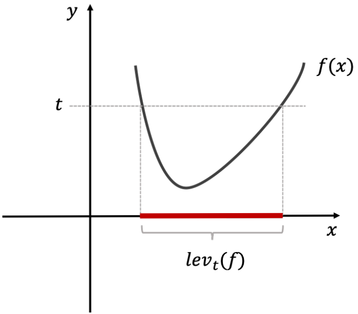

Figure: Level set

Proposition. If f is convex, then every level set is convex.

Q. Is the reverse still true?

Level Set

What are the level sets at t_1 and t_2? Are they convex?

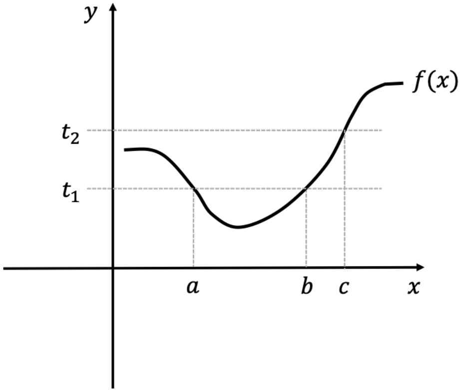

Figure: Level set

Quasi-convex Functions

Definition. f is *quasi-convex* if all level sets are convex.

Proposition. f is quasi-convex iff

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \forall \lambda \in [0,1]$$

Remark. f is quasi-concave iff

$$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}, \forall \lambda \in [0,1]$$

Example .

- $f(x) = \log(x)$ is both quasi-convex and quasi-concave;
- $f(x_1, x_2) = -x_1 x_2$ is quasi-convex on $\mathbb{R}^2_+$;
- $f(x) = \|x\|_0$ is quasi-concave on $\mathbb{R}^n_+$;
- $f(X) = \text{rank}(X)$ is quasi-concave on $\mathbb{S}^n_+$.

One-dimensional Property

Proposition. f is convex if and only is its restriction on any line is convex, i.e., $\forall x, h \in \mathbb{R}^n$,

$$\phi(t) = f(x + th)$$

is convex on its domain $\mathrm{dom}(\phi) = \{t | x + th \in \mathrm{dom}(f)\}$.

Remark. Checking convexity in $\mathbb{R}^n$ boils down to check convexity of one-dimensional function on the axis.

One-dimensional Property

Example . The negative log-determinant function

$$f(X) = -\log(\det(X))$$

is convex on $\mathbb{S}^n_{++}$.

- Sufficient to show that $\phi(t) = -\log(\det(X + tH))$ is convex for any given $X, H \in \mathbb{R}^{n\times n}$.
- Once can compute that

$$\begin{aligned}\phi(t) &= -\ln\left(\det\left(X^{\frac{1}{2}}\right)\det\left(I + tX^{-\frac{1}{2}}HX^{-\frac{1}{2}}\right)\det\left(X^{\frac{1}{2}}\right)\right)\\ &= -\sum_{i=1}^{n}\ln(1 + t\lambda_i) + \phi(0)\end{aligned}$$

where $\{\lambda_i\}_{i=1}^n$ are eigenvalues of $X^{-\frac{1}{2}}HX^{-\frac{1}{2}}$.

First Order Condition

Proposition. Assume f is differentiable, then f is convex if and only if $\mathrm{dom}(f)$ is convex and

$$f(x) \geq f(x_0) + \nabla f(x_0)^T(x - x_0), \forall x, x_0.$$

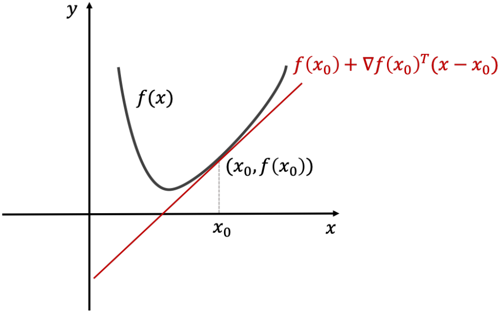

Figure: First-order condition

Remark. $(\nabla f(x_0), -1)$ is a supporting hyperplane of $\mathrm{epi}(f)$.

First Order Condition

Proposition. Assume f is differentiable, then f is convex if and only if $\text{dom}(f)$ is convex and

$$f(x) \geq f(x_0) + \nabla f(x_0)^T(x - x_0), \forall x, x_0.$$

Proof.

- (if part) $\forall \lambda \in [0,1]$, let $x_0 = \lambda x + (1-\lambda)y$, then

$$f(x) \geq f(x_0) + \nabla f(x_0)^T(x - x_0)$$
$$f(y) \geq f(x_0) + \nabla f(x_0)^T(y - x_0)$$
$$\Rightarrow \lambda f(x) + (1-\lambda) f(y) \geq f(x_0) = f(\lambda x + (1-\lambda)y).$$

- (only if part) By convexity, we have for all $\lambda \in [0,1]$,

$$f((1-\lambda)x_0 + \lambda x) \leq (1-\lambda) f(x_0) + \lambda f(x)$$
$$\Rightarrow \frac{f(x_0 + \lambda(x - x_0))}{\lambda} \leq \frac{f(x_0)}{\lambda} + f(x) - f(x_0)$$
$$\Rightarrow f(x) \geq f(x_0) + \frac{f(x + \lambda(x - x_0)) - f(x_0)}{\lambda}, \forall \lambda \in [0,1]$$
$$\Rightarrow f(x) \geq f(x_0) + \nabla f(x_0)^T(x - x_0), \text{ letting } \lambda \to 0$$

Second Order Condition

Proposition. Assume f is twice-differentiable, then f is convex if and only if dom(f) is convex and

$$\nabla^2 f(x) \succeq 0, \forall x \in dom(f).$$

Example .

- Quadratic function: $f(x) = \frac{1}{2}x^T Qx + b^T x + c$ is convex if and only if $Q \succeq 0$.
- Log-sum-exp: $f(x) = \log(\sum_{i=1}^n e^{x_i})$ is convex on $\mathbb{R}^n$.
- Geometric mean: $f(x) = (\prod_{i=1}^n x_i)^{1/n}$ is concave on $\mathbb{R}^n_{++}$.

Second Order Condition

Proposition. Assume f is twice-differentiable, then f is convex if and only if dom(f) is convex and

$$\nabla^2 f(x) \succeq 0, \forall x \in dom(f).$$

Proof.

- (if part) Any one dimensional restriction

  $\phi(t) = f(x+th)$ is convex since $\phi''(t) = h^T \nabla^2 f(x+th) h \geq 0.$

  Hence, f is convex.

- (only if part) $\forall x, h, \phi(t) = f(x+th)$ is convex on the axis

  $$\phi''(t) = h^T \nabla^2 f(x+th) h \geq 0$$

  $\Rightarrow \ \phi''(0) = h^T \nabla^2 f(x) h \geq 0, \forall h.$ Hence $\nabla^2 f(x) \succeq 0.$

Continuity of Convex Functions

Theorem. If f is convex, f is continuous on $\text{rint}(dom(f))$.

Remark.

- Convex functions are "almost" everywhere continuous;
- f needs not to be continuous on $\text{dom}(f)$. E.g.,

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & x > 0 \\ +\infty, & o.w. \end{cases}$$

Corollary. Let f be convex and $X \subseteq \text{rint}(\text{dom}(f))$ be a closed, bounded set. Then f is bounded on X.

Continuity of Convex Functions

Proof. W.l.o.g, assume $\dim(\mathrm{dom}(f)) = n$, $0 \in int(dom(f))$ and $\{x : \|x\|_2 \le 1\} \subseteq dom(f)$. Consider the continuity at point 0. Let $\{x_k\} \to 0$ with $\|x_k\|_2 \le 1$.

(a) $\limsup_{k\to\infty} f(x_k) \le f(0)$. Observe
$x_k = (1 - \|x_k\|_2) \cdot 0 + \|x_k\|_2 \cdot y_k$, where $y_k = \frac{x_k}{\|x_k\|_2} \in dom(f)$.
By convexity of f, we have

$$f(x_k) \le (1 - \|x_k\|_2) \cdot f(0) + \|x_k\|_2 \cdot f(y_k)$$

Therefore, $\limsup_{k\to\infty} f(x_k) \le f(0)$.

(b) $\liminf_{k\to\infty} f(x_k) \ge f(0)$. Observe
$0 = \frac{1}{\|x_k\|_2+1} x_k + \frac{\|x_k\|_2}{\|x_k\|_2+1} z_k$, where $z_k = -\frac{x_k}{\|x_k\|_2} \in dom(f)$.
By the convexity of f, we have

$$f(0) \le \frac{1}{\|x_k\|_2 + 1} f(x_k) + \frac{\|x_k\|_2}{\|x_k\|_2 + 1} f(z_k)$$

Therefore, $\liminf_{k\to\infty} f(x_k) \ge f(0)$.

Local Lipscthiz Continuity of Convex Functions

Theorem. Let f be convex and $X \subseteq \text{rint}(\text{dom}(f))$ be a closed, bounded set. Then f is Lipschitz continuous on X.

Proof: Skipped.

Remark. . All three conditions are essential (1) closedness (2) boundedness (3)relative interior

- $f(x) = 1/x$, $X = (0, 1]$, not Lipschitz continuous
- $f(x) = x^2$, $X = \mathbb{R}$, not Lipschitz continuous
- $f(x) = -\sqrt{x}$, $X = [0, 1]$, not Lipschitz continuous

Closed Functions

Definition. A function is <u>*closed*</u> if epi(f) is a closed set.

Remark.

- Any continuous function is closed.
- A closed function is not necessarily continuous.

Proposition. The following are equivalent:

(i) $f(x)$ is closed;

(ii) $f(x)$ is lower-semicontinuous (l.s.c), i.e.,

$$\forall x \in \mathbb{R}^n, \{x_k\} \to x, f(x) \leq \liminf_{k\to\infty} f(x_k).$$

(iii) Every level set is closed.

Closed Convex Functions

Definition. A function is *closed convex* if $\text{epi}(f)$ is a closed and convex set.

- A convex is closed if it is lower-semicontinuous.

Example .

- The indicator function $I_C(x) = \begin{cases} +\infty, & x \notin C \\ 0, & x \in C \end{cases}$ is closed convex if C is a closed convex set;
- $f(x) = \begin{cases} 1, & x = 0 \\ 0, & x > 0 \end{cases}$ is convex but not closed.

Remark. A closed convex function f can be viewed as the pointwise supremum of all affine minorants of f (affine functions that underestimate f.

References

- Boyd & Vandenberghe, Chapter 3.1
- Ben-Tal & Nemirovski, Chapter 2.1-2.4