# IE 521 Convex Optimization

## Lecture 1: Convex Sets

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# Outline

# Which set is different from others?

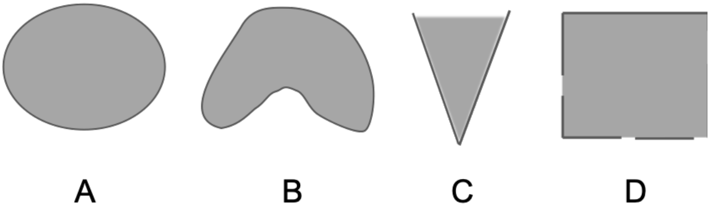



Figure: Four sets

# Which set is different from others?

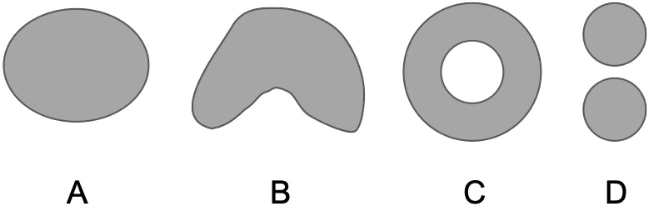



Figure: Four sets

# Interior, Closure, Boundary

Definition. Let $X$ be a nonempty set in $\mathbb{R}^n$.

- A point $x_0$ is called an <u>*interior point*</u> if $\exists r > 0$, such that $B(x_0, r) := \{x : \|x - x_0\|_2 \leq r\} \subseteq X$.
- A point $x_0$ is called a <u>*limit point*</u> if $\exists \{x_n\} \subseteq X$, such that $x_n \to x_0$ as $n \to \infty$.

Definition.

- Interior: $\text{int}(X)$ =the set of all interior point of $X$.
- Closure: $\text{cl}(X)$ =the set of all limit points of $X$.
- Boundary: $\partial(X) = \text{cl}(X)/\text{int}(X)$

Q. Let $X$ = irrationals on $[0, 1]$. What are $\text{int}(X)$ and $\text{cl}(X)$?

# Open and Closed Sets

**Definition.**

- $X$ is *closed* if $\mathrm{cl}(X) = X$;
- $X$ is *open* if $\mathrm{int}(X) = X$.

**Fact.**

- $\mathrm{int}(X) \subseteq X \subseteq \mathrm{cl}(X)$;
- $X$ is closed iff $X^c = \mathbb{R}^n / X$ is open;
- $\cap_{\alpha \in \mathcal{A}} X_\alpha$ is closed if $X_\alpha$ is closed for all $\alpha \in \mathcal{A}$.
- $\cup_{i=1}^n X_i$ is closed if $X_i$ is closed for $i = 1, \ldots, n$.

Q. If $X_1, X_2$ are closed, is $X_1 + X_2$ closed?

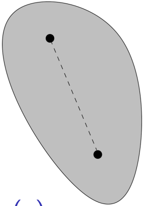
# Convex Set

## Definition

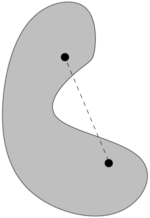
- A set $X \subseteq \mathbb{R}^n$ is convex if

$$x, y \in X, \lambda \in [0,1] \Rightarrow \lambda x + (1-\lambda) y \in X.$$

- In another words, the line segment $[x, y]$ that connects any two elements $x, y$ lies entirely in the set.



(a) convex



(b) non-convex

Figure: Examples of convex and non-convex sets

# Convex, Conic, Affine, and Linear Combinations

Definition. Given any elements $x_1, \ldots, x_k$, the ***combination*** $\lambda_1 x_1 + \ldots + \lambda_k x_k$ is called

- **Convex**: if $\lambda_i \geq 0, i = 1, \ldots, k$ and $\lambda_1 + \ldots + \lambda_k = 1$;
- **Conic**: if $\lambda_i \geq 0, i = 1, \ldots, k$;
- **Affine**: if $\lambda_1 + \ldots + \lambda_k = 1$;
- **Linear**: if $\lambda_i \in \mathbb{R}, i = 1, \ldots, k$.

# Convex Sets, Cones, Affine and Linear Subspaces

## Definition.

- A set is *convex* if all convex combinations of its elements are in the set;
- A set is a *convex cone* if all conic combinations of its elements are in the set;
- A set is a *affine subspace* if all affine combinations of its elements are in the set;
- A set is a *linear subspace* if all linear combinations of its elements are in the set.

Clearly, a linear subspace is always a convex cone; a convex cone is always a convex set.

Note: Cones vs. Convex cones.

# Convex, Conic, Affine Hulls

**Definition.** Given any set $X$, we define

- *Convex hull* of $X$:

  $\mathrm{Conv}(X) =$ set of all convex combinations of points in $X$.

- *Conic hull* of $X$:

  $\mathrm{Cone}(X) =$ set of all conic combinations of points in $X$.

- *Affine hull* of $X$:

  $\mathrm{Aff}(X) =$ set of all affine combinations of points in $X$.

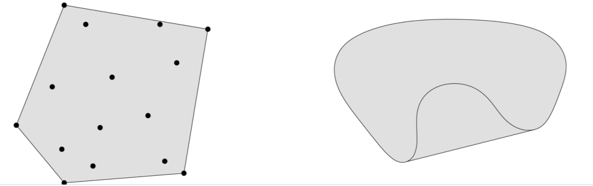


Figure: Examples of convex hulls

# Properties of Convex Sets

**Proposition.**

1. A convex hull is always convex.
2. If $X$ is convex, then $\text{Conv}(X) = X$.
3. For any set $X$, $\text{Conv}(X)$ is the smallest convex set that contains $X$.

# Examples of Convex Sets

Example 1. Simple sets:

- *Hyperplane*: $\{x \in \mathbb{R}^n : a^T x = b\}$
- *Halfspace*: $\{x \in \mathbb{R}^n : a^T x \leq b\}$
- *Affine space*: $\{x \in \mathbb{R}^n : Ax = b\}$
- *Polyhedron*: $\{x \in \mathbb{R}^n : Ax \leq b\}$
- *Simplex*: $\{x \in \mathbb{R}^n : x \geq 0, \sum_{i=1}^n x_i = 1\}$.

Example 2. Euclidean balls:

$$\{x \in \mathbb{R}^n : \|x - a\|_2 \leq r\}.$$

Example 3. Ellipsoid:

$$\{x \in \mathbb{R}^n : (x-a)^T Q (x-a) \leq r^2\}$$

where $Q \succ 0$ and is symmetric.

# Examples of Convex Cones

Example 1. Positive Orthant:

$$\{x \in \mathbb{R}^n : x \geq 0\}$$

Example 2. Norm cones:

$$\{(x, t) \in \mathbb{R}^{n+1} : \|x\|_2 \leq t\}$$

Example 3. Positive semidefinite matrices:

$$\mathbb{S}^n_+ := \{X \in \mathbb{S}^n : X \succeq 0\}$$

# Operations that Preserves Convexity

## Intersection

- If $X_\alpha, \alpha \in \mathcal{A}$ are convex sets, then so is

$$\cap_{\alpha \in \mathcal{A}} X_\alpha.$$

## Cartesian product:

- If $X_i \subseteq \mathbb{R}^{n_i}, i = 1, \ldots, k$ are convex, then so is

$$X_1 \times \cdots \times X_k.$$

## Weighted summation:

- If $X_i \subseteq \mathbb{R}^n, i = 1, \ldots, k$ convex, then so is

$$\alpha_1 X_1 + \cdots + \alpha_k X_k.$$

# Operations that Preserve Convexity

## Affine image:

- If $X \subseteq \mathbb{R}^n$ is a convex set and $\mathcal{A}(x) : x \mapsto Ax + b$ is an affine mapping from $\mathbb{R}^n$ to $\mathbb{R}^k$, then so is

$$\mathcal{A}(X) := \{Ax + b : x \in X\}.$$

- *Proof:*
  Let $y_1, y_2 \in \mathcal{A}(X) \Rightarrow \exists x_1, x_2 \in X$ such that $y_1 = Ax_1 + b$ and $y_2 = Ax_2 + b$. For $\lambda \in [0, 1]$,

$$\lambda y_1 + (1 - \lambda) y_2 = A(\lambda x_1 + (1 - \lambda) x_2) + b \in \mathcal{A}(X)$$

  because $\lambda x_1 + (1 - \lambda) x_2 \in X$.

# Operations that Preserve Convexity

Inverse affine image:

- If $X \subseteq \mathbb{R}^n$ is a convex set and $\mathcal{A}(y) : y \mapsto Ay + b$ is an affine mapping from $\mathbb{R}^k$ to $\mathbb{R}^n$, then so is

$$\mathcal{A}^{-1}(X) := \{y : Ay + b \in X\}.$$

- *Proof:* self-exercise.

Example. The solution set of linear matrix inequality:

$$\{x | x_1 A_1 + \cdots + x_k A_k \preceq B\}$$

where $A_i, B$ are positive semidefinite matrices.

# Nice Properties of Convex Sets

Proposition. Let $X$ be convex with nonempty interior. Then

- If $x_0 \in \text{int}(X)$ and $x \in \text{cl}(X)$, then $[x_0, x) \in \text{int}(X)$.
- Moreover, $\text{int}(X)$ is dense in $\text{cl}(X)$.

Remark. In general, $\text{int}(X)$ and $\text{cl}(X)$ can differ dramatically.

- If $X =$ irrationals on $[0, 1]$, $\text{int}(X) = \emptyset, \text{cl}(X) = [0, 1]$.

Q. What happens if $X$ is convex but $\text{int}(X) = \emptyset$?

# Nice Properties of Convex Sets

Definition. **(Relative Interior and Dimension)**

- $\text{rint}(X) = \{x : \exists r > 0, \text{s.t. } B(x,r) \cap \text{Aff}(X) \subseteq X\}$
- $\dim(X) = \dim(\text{Aff}(X))$

Fact. For a convex and nonempty set, $\text{rint}(X) \neq \emptyset$.

Proposition. Let $X$ be a nonempty convex set. Then

a) $\text{int}(X), \text{cl}(X), \text{rint}(X)$ are convex
b) $x_0 \in \text{rint}(X), x \in \text{cl}(X) \Rightarrow [x_0, x) \in \text{rint}(X), \forall \lambda \in (0,1]$
c) $\text{cl}(\text{rint}(X)) = \text{cl}(X)$
d) $\text{rint}(\text{cl}(X)) = \text{rint}(X)$

# Question

Suppose there are 100 different kinds of herbal tea, everyone of them is a blend of 25 herbs. Donald wants a particular mixture of all herbal teas with equal proportions. What's the least number of teas he should buy?

# Carathéodory Representation Theorem

Theorem. Let $X \subseteq \mathbb{R}^n$ be non empty and $\dim(X) = d \leq n$. Every point $x \in \text{Conv}(X)$ is a convex combination of at most $(d+1)$ points, i.e.

$$\text{Conv}(X) = \left\{ \sum_{i=1}^{d+1} \lambda_i x_i : x_i \in X, \lambda_i \geq 0, \sum_{i=1}^{d+1} \lambda_i = 1 \right\}.$$

*Proof:* Suppose the minimal representation of $x \in \text{Conv}(X)$ has $m \geq d+1$ terms, $x = \sum_{i=1}^m \alpha_i x_i$, where $\alpha_i \geq 0, \sum_{i=1}^m \alpha_i = 1$.The system of linear equations

$$\begin{cases} \sum_{i=1}^m \delta_i x_i = 0 \\ \sum_{i=1}^m \delta_i = 0 \end{cases}$$

has non trivial solution.

Rewrite $x = \sum_{i=1}^m (\alpha_i - t\delta_i) x_i$. Let $\lambda_i(t) = (\alpha_i - t\delta_i), \forall i$, we have $\sum \lambda_i(t) = 1$. Let $t_* = \min\left\{\frac{\alpha_i}{\delta_i}, \delta_i > 0\right\} := \frac{\alpha_j}{\delta_j}$ , then $\lambda_i(t_*) > 0, \forall i \neq j$ and $\lambda_j(t_*) = 0$. This leads to a smaller representation of $x$.

# References


- Boyd & Vandenberghe, Chapter 2.1-2.3
- Ben-Tal & Nemirovski, Chapter 1.1