IE 521 Convex Optimization

Lecture 2: Convex Geometry

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Outline

Quick Review

- Convex set
  - X is convex iff $\lambda x + (1-\lambda)y \in X, \forall x, y \in X, \lambda \in [0,1]$
- Convex hull
  - $\text{Conv}(X) = \left\{\sum_{i=1}^k \lambda_i x_i : k \in \mathbf{N}, \lambda_i \geq 0, \sum_{i=1}^k \lambda_i = 1, x_i \in X, \forall i\right\}$
- Convexity-preserving operators
  - Taking intersection, Cartesian product, summation
  - Taking affine mapping, inverse affine mapping
- Topological properties
  - For convex sets, $\text{rint}(X)$ is dense in $\text{cl}(X)$
- Representation theorem
  - Any point in the convex hull of set X with dimension d can be written as the convex combination of at most $d+1$ points in X.

Question 1

Can you find a partition of the sets whose convex hulls intersect?

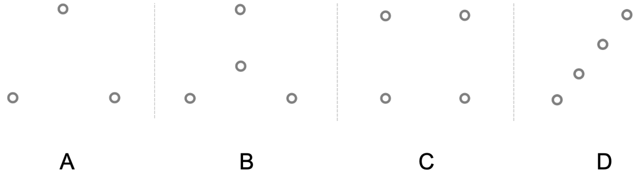

Figure: Four sets

Question 2

If I_1, I_2 and I_3 are intervals on the real line such that any two have a point in common, do all three have a point in common?

Question 3

Which group is different from others?

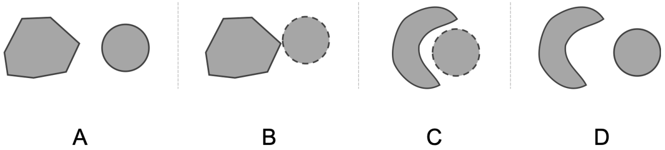

Figure: Four groups of disjoint sets

The Mathematicians

Figure: Johann Radon (1887–1956)

Figure: Eduard Helley (1884–1943)

Figure: Hermann Minkowski (1864–1909)

Radon's Theorem (J. Radon, 1921)

Theorem. Let S be a collection of N points in $\mathbb{R}^d$ with $N \geq d+2$. Then we can write $S = S_1 \cup S_2$ s.t.

$$S_1 \cap S_2 = \emptyset, \text{ and } \mathrm{Conv}(S_1) \cap \mathrm{Conv}(S_2) \neq \emptyset.$$

Remark.

- Any set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two disjoint sets whose convex hulls intersect.
- Can be used to show the VC-dimension of the class of halfspaces (linear separators) in d-dimensions is $d+1$.

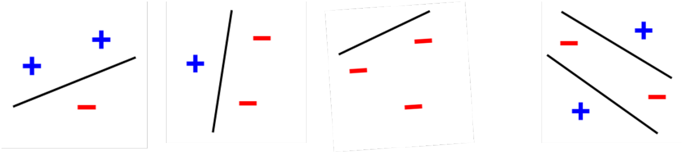

Figure: 3 points separable vs 4 points nonseparable

Proof of Radon's Theorem

- Let $S = \{x_1, ..., x_N\}$ with $N \geq d + 2$.
- Consider the linear system

$$\begin{cases} \sum_{i=1}^N \gamma_i x_i = 0 \\ \sum_{i=1}^N \gamma_i = 0 \end{cases} \quad \Rightarrow \quad \begin{array}{l} (d+1) \text{ equations} \\ N \geq (d+2) \text{ unknowns} \end{array}$$

  So there exists a non-zero solution $\gamma_1, ..., \gamma_N$.
- Let $I = \{i : \gamma_i \geq 0\}$, $J = \{j : \gamma_j < 0\}$ and $a = \sum_{i \in I} \gamma_i = -\sum_{j \in J} \gamma_j$, then

$$\sum_{i \in I} \gamma_i x_i = \sum_{j \in J} (-\gamma_j) x_j \Rightarrow \sum_{i \in I} \frac{\gamma_i}{a} x_i = \sum_{j \in J} \frac{-\gamma_j}{a} x_j$$

- The partition $S_1 = \{x_i, i \in I\}$ and $S_2 = \{x_j : j \in J\}$ gives the desired result.

Back to the Question

If I_1, I_2 and I_3 are intervals on the real line such that any two have a point in common, do all three have a point in common?

Helley's Theorem (E. Helly, 1923)

Theorem. Let $S_1, \ldots, S_N$ be a collection of convex sets in $\mathbb{R}^d$ with $N > d$. Assume every $(d+1)$ sets of them have a point in common, then all the sets have a point in common.

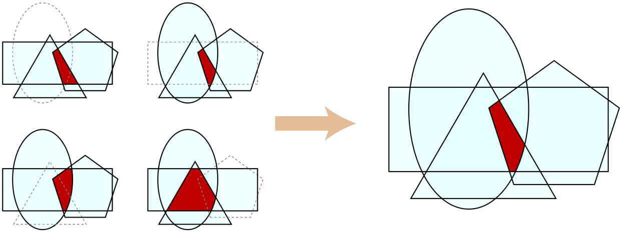

Figure: Four convex sets in $\mathbb{R}^2$

Q. Does the theorem still hold if we relax $N = \infty$?
Q. Does the theorem still hold if we relax $(d+1)$ sets to d sets?

Helley's Theorem (E. Helly, 1923)

Theorem. Let $S_1, ..., S_N$ be a collection of convex sets in $\mathbb{R}^d$ with $N > d$. Assume every $(d+1)$ sets of them have a point in common, then all the sets have a point in common.

Remark.

- Not true for infinite collection:
  - E.g. $S_i = [i, \infty), \cap_{i=1}^{+\infty} S_i = \emptyset$
- Not true if reduce $(d+1)$ sets to d sets.

Corollary. Let $\{S_\alpha\}$ be any collection of compact convex sets in $\mathbb{R}^d$. If every $(d+1)$ sets have a point in common, then all sets have a points in common.

Proof of Helley's Theorem

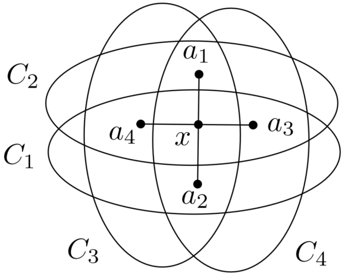

Figure: Illustration of $N = 4$, $d = 2$

Proof of Helley's Theorem

By induction on N.

- Base case: $N = d + 1$, obviously true.
- Induction step: Assume the collection of $N(\geq d+1)$ sets have common point if every $(d+1)$ of them have common point. Show this holds for $N+1$ sets.
- From the assumption, $\exists \{x_1, x_2, ..., x_{N+1}\}$ such that $x_i \in S_1 \cap ... \cap S_{i-1} \cap S_{i+1} \cap ... \cap S_{N+1} \neq \emptyset$.
- By Radon's theorem, we can split it into two disjoint sets, $\{x_1, \ldots, x_k\}$ and $\{x_{k+1}, \ldots, x_N\}$, and

$$\text{Conv}(\{x_1, ..., x_k\}) \cap \text{Conv}(\{x_{k+1}, ..., x_{N+1}\}) \neq \emptyset.$$

- Let $z \in \text{Conv}(\{x_1, ..., x_k\}) \cap \text{Conv}(\{x_{k+1}, ..., x_{N+1}\})$. It can be shown that $z \in S_1 \cap ... \cap S_{N+1}$ (why?).

Application of Helley's Theorem

Baby Theorem Let X contain a finite set of points in the plane, such that every three of them are contained in a disk of radius 1. Then X is contained in a disk of radius 1.

Jung's Theorem. Let X contain a finite set of points in the plane, such that any two of them have distance no greater than 1. Then X is contained in a disk of radius $1/\sqrt{3}$.

Jung's Theorem. Let $X \subset \mathbb{R}^n$ be a compact set such that any two of them has Euclidean distance no greater than 1. Then X is contained in a ball with radius $\sqrt{\frac{n}{2(n+1)}}$.

Application of Helley's Theorem

Question. Consider the optimization problem

$$p_* = \min_{x \in \mathbb{R}^{10}} g_0(x), \quad \text{s.t. } g_i(x) \leq 0, i = 1, ..., 521.$$

- Suppose $\forall t \in \mathbb{R}$, $X_0 = \{x \in \mathbb{R}^{10} : g_0(x) \leq t\}$ is convex, $X_i = \{x \in \mathbb{R}^{10} : g_i(x) \leq 0\}$ is convex.
- How many constraints can you drop without affecting the optimal value?

Other Applications of Helley's Theorem

Helleys theorem is a very fundamental result in convex geometry and can be applied to show many results.

- The centerpoint theorem
- Farkas Lemma
- Sion-Kakutani Theorem
- Chebyshev approximation

Back to the Question

When can we separate two sets?

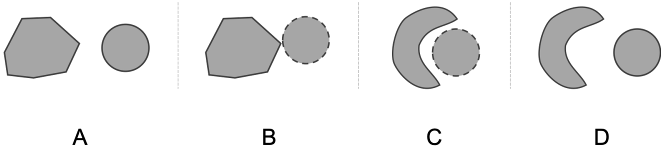

Figure: Four groups of disjoint sets

Separation of Sets

Definition. Let S and T be two nonempty convex sets in $\mathbb{R}^n$. A hyperplane $H = \{x \in \mathbb{R}^n : a^T x = b\}$ with $a \neq 0$ is said to separate S and T if $S \cup T \not\subset H$ and

$$S \subset H^- = \left\{x \in \mathbb{R}^n : a^T x \leq b\right\}$$

$$T \subset H^+ = \left\{x \in \mathbb{R}^n : a^T x \geq b\right\}$$

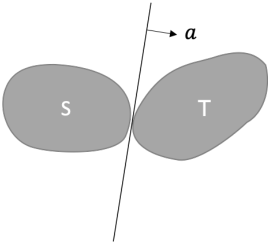

Figure: Separation of two sets

Strict Separation of Sets

Definition. Let S and T be two nonempty convex sets in $\mathbb{R}^n$. A hyperplane $H = \left\{x \in \mathbb{R}^n : a^T x = b\right\}$ with $a \neq 0$ is said to <u>strictly separate S and T</u> if

$$S \subset H^{--} = \left\{x \in \mathbb{R}^n : a^T x < b\right\}$$

$$T \subset H^{++} = \left\{x \in \mathbb{R}^n : a^T x > b\right\}$$

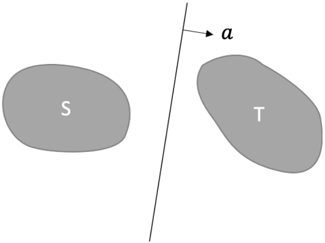

Figure: Strict Separation of two sets

Strong Separation of Sets

Definition. Let S and T be two nonempty convex sets in $\mathbb{R}^n$. A hyperplane $H = \{x \in \mathbb{R}^n : a^T x = b\}$ with $a \neq 0$ is said to <u>strongly separate S and T</u> if there exits $b' < b < b''$ such that

$$S \subset \left\{x \in \mathbb{R}^n : a^T x \leq b'\right\}$$
$$T \subset \left\{x \in \mathbb{R}^n : a^T x \leq b''\right\}$$

Remark.

- Strict separation does not necessarily imply strong separation.
- Strong separation is equivalent to say

$$\sup_{x \in S} a^T x < \inf_{x \in T} a^T x.$$

Separation Hyperplane Theorem

Theorem. Let S and T be two nonempty convex sets. Then S and T can be separated if and only if

$$\mathrm{rint}(S) \cap \mathrm{rint}(T) = \emptyset.$$

Supporting Hyperplane Theorem

Theorem. Let S be a nonempty convex set and $x_0 \in \partial S$. Then there exists a hyperplane $H = \left\{x : a^T x = a^T x_0\right\}$ with $a \neq 0$ such that

$$S \subset \left\{x : a^T x \leq a^T x_0\right\}, \text{ and } x_0 \in H.$$

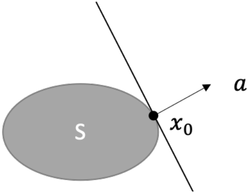

Figure: Supporting hyperplane

- This follows directly from the previous theorem.
- Such a hyperplane is called a <u>supporting hyperplane</u>.

Strict Separation Hyperplane Theorem I

Theorem. Let S be closed and convex and $x_0 \notin S$, Then there exists a hyperplane that strictly separates x_0 and S.

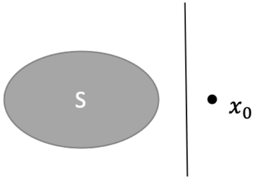

Figure: Strict separation

- Closedness of the set is crucial here.
- Separating hyperplane can be constructed based on the projection.

Strict Separation Hyperplane Theorem II

Theorem. Let S and T be two nonempty convex sets and $S \cap T = \emptyset$. If $S - T$ is closed, then S and T can be strictly separated.

Remark.

- Even if both S and T are closed convex, $S - T$ might not be closed, and they might not be strictly separated.
- When both S and T are closed convex, $S \cap T = \emptyset$ and at least one of them is bounded, then $S - T$ is closed, and S and T can be strictly separated

References

- Boyd & Vandenberghe, Chapter 2.5
- Ben-Tal & Nemirovski, Chapter 1.2.2-1.2.6