IE 521 Convex Optimization

Lecture 12: Conic Programming

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5th March 2019

Outline

Extend Linear Programs to Convex Programs

- Traditional approach:

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\begin{aligned}
\min_x \quad & c^T x \\
\text{s.t.} \quad & Ax - b \geq 0
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
\min_x \quad & f_0(x) \\
\text{s.t.} \quad & f_i(x) \geq 0, i = 1, \ldots, m
\end{aligned}
$$

Extend Linear Programs to Convex Programs

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- Modern approach:

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Q. What properties should the vector inequality "$\succcurlyeq$" satisfy?

"Good" Vector Inequalities

The partial ordering "$\succcurlyeq$" on $\mathbb{R}^m$ should satisfy:

(i) Reflexibility: $a \succcurlyeq a$

(ii) Anti-symmetry: $a \succcurlyeq b$ and $b \succcurlyeq a$ implies $a = b$

(iii) Transitivity: $a \succcurlyeq b$ and $b \succcurlyeq c$ implies $a \succcurlyeq c$

(iv) Homogeneity: $a \succcurlyeq b$ and $\lambda \geq 0$ implies $\lambda a \succcurlyeq \lambda b$

(v) Additivity: $a \succcurlyeq b$ and $c \succcurlyeq b$ implies $a + c \succcurlyeq b + d$

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Remark.

$$a \succcurlyeq b \iff a - b \succcurlyeq 0 \iff a - b \in \mathcal{K}$$

where $\mathcal{K} = \{a \in \mathbb{R}^m : a \succcurlyeq 0\}$.

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$\mathcal{K}$ must be a nonempty pointed convex cone!

Example: Coordinate-wise Vector Inequality

The usual coordinate-wise vector inequality:

$$a \geq b \iff a_i - b_i \geq 0, i = 1, \ldots, m$$

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Nonnegative orthant cone:

$$\mathcal{K} = \mathbb{R}^m_+ := \{x = [x_1; \ldots; x_m] : x_i \geq 0, i = 1, \ldots, m\}$$

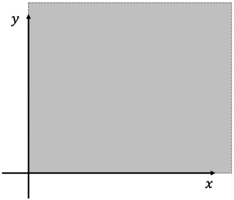

Figure: Nonnegative orthant

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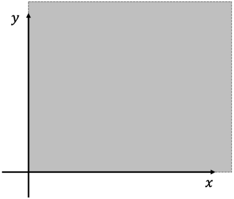

Figure: Nonnegative orthant

Remark. $\mathcal{K} = \mathbb{R}^m_+$ is also closed and has non-empty interior.

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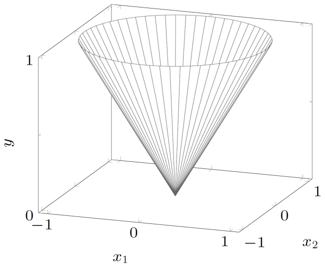
- Define a generalized inequality:

$$a \geq_{\mathcal{K}} b \iff a - b \in \mathcal{K}$$

- Strict inequality:

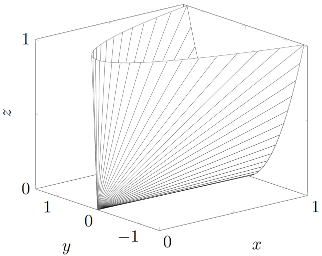
$$a >_{\mathcal{K}} b \iff a - b \in \operatorname{int}(\mathcal{K})$$

Examples



Ice cream cone

$$L = \{(x_1, x_2, y) \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} \leq y\}$$



Positive semidefinite cone

$$S = \{(x, y, z) \in \mathbb{R}^3 : \begin{bmatrix} x & y/\sqrt{2} \\ y/\sqrt{2} & z \end{bmatrix} \succeq 0\}$$

Figure from Vandenberghe lecture notes

Conic (Linear) Program

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & Ax \geq_{\mathcal{K}} b \end{aligned} \tag{CP}$$

where

- $\mathcal{K}$ is a regular cone,
- $Ax \geq_{\mathcal{K}} b$ iff $Ax - b \in \mathcal{K}$.

Linear Program (LP)

- Nonnegative Orthant:

$$\mathcal{K} = \mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_i \geq 0, i = 1, ..., m\}$$

- Linear Programs (LP):

$$\min_x \left\{ c^T x : a_i^T x - b_i \geq 0, i = 1, ..., m \right\} \qquad (LP)$$

Second Order Conic Program (SOCP)

- Lorentz Cone (second order/ice cream cone):

$$\mathcal{K} = L^{n_1} \times \cdots \times L^{n_m}$$

$$L^n = \{x \in \mathbb{R}^n : x_n \geq \sqrt{\sum_{i=1}^{n-1} x_i^2}\}$$

- Second Order Conic Program:

$$\min_x \left\{ c^T x : \| D_i x - d_i \|_2 \leq e_i^T x - f_i, i = 1, ..., m \right\} \quad (SOCP)$$

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- Here the linear map is

$$Ax - b := \left[\underbrace{D_1 x - d_1; e_1^T x - f_1}_{\mathbb{R}^{n_1}}; \dots; \underbrace{D_m x - d_m; e_m^T x - f_m}_{\mathbb{R}^{n_m}} \right]$$

Semidefinite Program (SDP)

- Positive Semidefinite Cone:

$$\mathcal{K} = S_+^m = \{X \in S^m : X \succeq 0\}$$

- Semidefinite Programs:

$$\min_x \left\{ c^T x : \sum_{i=1}^n x_i A_i - B \succeq 0 \right\} \qquad (SDP)$$

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$$\mathcal{A}(x) - b = \sum\nolimits_{i=1}^n x_i A_i - B : \mathbb{R}^n \to S^m$$

with $A_1, \ldots, A_n, B \in S^m$.

LP, SOCP, and SDP

$$\text{LP} \subseteq \text{SOCP} \subseteq \text{SDP}$$

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- $\text{LP} \subseteq \text{SOCP}$:

$$a_i^T x - b_i \geq 0 \Leftrightarrow \begin{bmatrix} 0 \\ a_i^T x - b_i \end{bmatrix} \in L^2$$

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- $\text{SOCP} \subseteq \text{SDP}$:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \in L^n \Leftrightarrow \begin{bmatrix} x_n & x_1 & x_2 & \cdots & x_{n-1} \\ x_1 & x_n & & & \\ x_2 & & x_n & & \\ \vdots & & & \ddots & \\ x_{n-1} & & & & x_n \end{bmatrix} \succeq 0$$

Schur Complement

Proposition. Let $S = \begin{bmatrix} P & Q^T \\ Q & R \end{bmatrix}$ be symmetric with $R \succ 0$.

$$S \succeq 0 \text{ if and only if } P - Q^T R^{-1} Q \succeq 0.$$

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Proof.

$$S \succeq 0 \Leftrightarrow \forall u.v : [u^T, v^T] \begin{bmatrix} P & Q^T \\ Q & R \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \geq 0$$

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Remarks

- In principle, any convex problem can be represented as a conic program.
- Other cones exist: exponential cone, power cone, copositive cone, etc.
- In practice, only the primitive cones (LP, SOCP, SDP) are used, limited by the available algorithms and solvers.
- Indeed, an extremely wide spectrum of convex problems can be converted into three standard conic programs.

Example: ℓ_2-norm minimization

$$\min_{x\in\mathbb{R}^n} \|x\|_2 \iff \begin{array}{ll}\min\limits_{x,t} & t \\ \text{s.t.} & t \geq \|x\|_2\end{array} \iff \begin{array}{ll}\min\limits_{x,t} & t \\ \text{s.t.} & \begin{bmatrix} x \\ t \end{bmatrix} \geq_{L^{n+1}} 0\end{array}$$

$$\min_{x\in\mathbb{R}^n} x^T x \iff \begin{array}{ll}\min\limits_{x,t} & t \\ \text{s.t.} & t \geq x^T x\end{array} \iff \begin{array}{ll}\min\limits_{x,t} & t \\ \text{s.t.} & \begin{bmatrix} 2x \\ t-1 \\ t+1 \end{bmatrix} \geq_{L^{n+2}} 0\end{array}$$

Example: Convex Quadratic Programming

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$$\min_{x \in \mathbb{R}^n} \quad x^T Q x + q^T x$$

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$$\iff \quad \min_{x,t} \quad t$$
$$\text{s.t.} \quad \begin{bmatrix} 2L^T x \\ t - q^T x - 1 \\ t - q^T x + 1 \end{bmatrix} \geq_{L^{n+2}} 0$$

Example: Spectral Norm Minimization

Let $A_1, \dots, A_n$ be symmetric matrix in $\mathbb{R}^{p\times p}$.

$$\min_{x\in\mathbb{R}^n} \quad \lambda_{\max}\left(\sum_{i=1}^n x_i A_i\right)$$

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$$\text{s.t.} \quad t\cdot I_p - \sum_{i=1}^n x_i A_i \succeq 0$$

Example: Spectral Norm Minimization

Let $A_1, \ldots, A_n$ be general matrix in $\mathbb{R}^{p\times q}$.

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$$\Longleftrightarrow \quad \min_{x,t} \quad t$$

$$\text{s.t.} \quad \begin{bmatrix} t\cdot I_p & \sum_{i=1}^n x_i A_i \\ \sum_{i=1}^n x_i A_i^T & t\cdot I_q \end{bmatrix} \succeq 0$$

Example: Sparse Group Lasso

$$\min_w \left\{ \|Xw - y\|_2^2 + \lambda \sum_{i=1}^{p} \|w_i\|_2 \right\}$$

- $X \in \mathbb{R}^{m\times n}, y \in \mathbb{R}^m$ are the input data
- $w = [w_1; \ldots ; w_p] \in \prod_{i=1}^{p} \mathbb{R}^{n_i}$ is the decision variable.

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$$\begin{aligned} \min_{w,t_0,t_1,\ldots,t_p} \quad & t_0 + \lambda \sum_{i=1}^{p} t_i \\ \text{s.t.} \quad & \begin{bmatrix} 2(Xw-y) \\ t_0 - 1 \\ t_0 + 1 \end{bmatrix} \geq_{L^{m+2}} 0 \qquad (SOCP) \\ & \begin{bmatrix} w_i \\ t_i \end{bmatrix} \geq_{L^{n_i+1}} 0, \quad \forall i = 1, \ldots, p. \end{aligned}$$

Example: Robust Linear Program

- Recall the robust linear program

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & a_i^T x \le b_i, \forall a_i \in \mathcal{E}_i, i = 1, \dots, m \end{aligned}$$

where $\mathcal{E}_i = \{\bar{a}_i + P_i u : \|u\|_2 \le 1\}$.

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where $\mathcal{E}_i = \{\bar{a}_i + P_i u : \|u\|_2 \leq 1\}$.

- This is equivalent to the convex program

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, i = 1, \dots, m \end{aligned}$$

which is an SOCP.

Example: Robust Portfolio Selection

- Assume that returns $r_i, i = 1, ..., n$ are exactly known. To maximize the return leads to

$$\max_x \quad r^T x, \text{ s.t. } x \geq 0, \; \mathbf{1}^T x = 1$$

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- Now assume the returns are known within an ellipsoid:

$$\mathcal{U} = \left\{ \hat{r} + \rho \hat{\Sigma}^{1/2} u : \ \|u\|_2 \leq 1 \right\}$$

where $\hat{r}$ and $\hat{\Sigma}$ are the empirical mean and covariance. The robust portfolio problem is

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- This is equivalent as an SOCP:

$$\max_x \; \hat{r}^T x - \rho \|\hat{\Sigma}^{1/2} x\|_2, \text{ s.t. } x \geq 0, \; \mathbf{1}^T x = 1$$

References

- Ben-Tal & Nemirovski (2013), Chapter 1.4