第十四周作业答案

1. 考虑下列问题：

$$\begin{aligned}
&\min\ x_1^2+x_1x_2+2x_2^2-6x_1-2x_2-12x_3\\
&s.t.\quad x_1+x_2+x_3=2\\
&\qquad\ \ -x_1+2x_2\ \ \le 3\\
&\qquad\ \ x_1,x_2,x_3\ge 0
\end{aligned}$$

求出在点 $\hat{x}=(1,1,0)^T$ 处的一个下降可行方向。

1. $\nabla f(\overline{x})=(-3,3,-12)^T, A_1=(0,0,1),\ E=(1,1,1).$

满足约束

$$\nabla f(\overline{x})^T d\le 0$$

$$A_1 d\ge 0$$

$$Ed=0.$$

都是下降可行方向.

2 考虑下列问题：

$$\begin{aligned}
&\min\ f(x)\\
&s.t.\quad g_i(x)\ge 0\ \ i=1,2,\cdots,m\\
&\qquad\ \ h_j(x)=0\ \ j=1,2,\cdots,l
\end{aligned}$$

设 $\hat{x}$ 是可行点，$I=\{i\,|\,g_i(\hat{x})=0\}$。

证明 $\hat{x}$ 为 KKT 点的充要条件是下列问题的目标函数的最优值为零：

$$\begin{aligned}
&\min\ \nabla f(\hat{x})^T d\\
&s.t.\quad \nabla g_i(\hat{x})^T d\ge 0\ \ i\in I\\
&\qquad\ \ \nabla h_j(\hat{x})^T d=0\ \ j=1,2,\cdots,l\\
&\qquad\ \ -1\le d_j\le 1\quad j=1,2,\cdots,n
\end{aligned}$$

证明：$\hat{x}$ 为 KKT 点的充要条件是存在 $w_i\ge 0(i=1,2,\cdots,m), v_j(j=1,2,\cdots,l)$ 使得

$$\begin{aligned}
&\nabla f(x)-\sum_{i=1}^{m}w_i\nabla g_i(x)-\sum_{j=1}^{l}v_j\nabla h_j(x)=0\\
&w_ig_i(x)=0, i=1,2,\cdots,m
\end{aligned}$$

令 $v_j=p_j-q_j, p_j\ge 0, q_j\ge 0, j=1,2,\cdots,l$，记

$w=(w_1,\cdots,w_m)^T, v=(v_1,\cdots,v_l)^T, p=(p_1,\cdots,p_l)^T, q=(q_1,\cdots,q_l)^T$，则

KKT 条件可以改写为

$$\nabla f(\hat{x}) - \nabla g(\hat{x})^T w - \nabla h(\hat{x})^T p + \nabla h(\hat{x})^T q = 0$$
$$w^T g(\hat{x}) = 0$$
$$w, p, q \geq 0$$

其中 $\nabla g(\hat{x})^T = \left(\nabla g_1(\hat{x}), \cdots, \nabla g_m(\hat{x})\right), \nabla h(\hat{x})^T = \left(\nabla h_1(\hat{x}), \cdots, \nabla h_l(\hat{x})\right)$，上式等价于

$$-\nabla f(\hat{x}) = \left(\nabla g(\hat{x})^T, -\nabla h(\hat{x})^T, \nabla h(\hat{x})^T\right)\begin{bmatrix} w \\ p \\ q \end{bmatrix}$$
$$w^T g(\hat{x}) = 0$$
$$w, p, q \geq 0$$

由 Farkars 定理，$\hat{x}$ 为 KKT 点的充要条件是

$$\begin{bmatrix} -\nabla g\ (\hat{x}) \\ -\nabla h\ (\hat{x}) \\ \nabla h\ (\hat{x}) \end{bmatrix} d \leq 0, -\nabla f(\hat{x})^T d > 0 \text{ 无解}$$

即 $-\nabla f(\hat{x})^T d > 0, \nabla g\ (\hat{x})^T d \geq 0, \nabla h\ (\hat{x})^T d = 0$ 无解，所以对任意满足约束问题

$$\begin{aligned} &\min \nabla f(\hat{x})^T d \\ &s.t.\quad \nabla g_i(\hat{x})^T d \geq 0 \quad i \in I \\ &\qquad\ \ \nabla h_j(\hat{x})^T d = 0 \quad j = 1, 2, \cdots, l \\ &\qquad\ \ -1 \leq d_j \leq 1 \quad j = 1, 2, \cdots, n \end{aligned}$$

的可行解 d，有 $\nabla f(\hat{x})^T d \geq 0$，但由于 $d = 0$ 是可行解，所以约束问题的目标函数最优值=0.