Integrals involved in Gaussian filtering

Sensor fusion & nonlinear filtering

Lars Hammarstrand

INTEGRALS IN GAUSSIAN FILTERING – PREDCTION

- The prediction step in Gaussian filtering involves two integrals on the form, $\int \mathbf{g}(\mathbf{x})\mathcal{N}(\mathbf{x}, \hat{\mathbf{x}}, \mathbf{P})\, d\mathbf{x}$:

$$E_{\mathcal{N}(x;\hat{x},P)}\left\{g(x)\right\}$$

Integrals in Gaussian filter prediction

$$\hat{\mathbf{x}}_{k|k-1} = \int f(\mathbf{x}_{k-1})\mathcal{N}(\mathbf{x}_{k-1}; \hat{\mathbf{x}}_{k-1|k-1}, \mathbf{P}_{k-1|k-1})\, d\mathbf{x}_{k-1}$$

$$\mathbf{P}_{k|k-1} = \mathbf{Q}_{k-1} + \int (f(\mathbf{x}_{k-1}) - \hat{\mathbf{x}}_{k|k-1})(\cdot)^T \mathcal{N}(\mathbf{x}_{k-1}; \hat{\mathbf{x}}_{k-1|k-1}, \mathbf{P}_{k-1|k-1})\, d\mathbf{x}_{k-1}$$

- From these we can approximate

$$p(\mathbf{x}_k | y_{1:k-1}) \approx \mathcal{N}(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1})$$

INTEGRALS IN GAUSSIAN FILTERING

- The update step in Gaussian filtering involves three integrals on the form, $\int \mathbf{g}(\mathbf{x})\mathcal{N}(\mathbf{x}, \hat{\mathbf{x}}, \mathbf{P})\, d\mathbf{x}$:

Integrals in Gaussian filter update

$$\hat{\mathbf{y}}_{k|k-1} = \int h(\mathbf{x}_k)\mathcal{N}(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1})\, d\mathbf{x}_k$$

$$\mathbf{P}_{xy} = \int (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})(h(\mathbf{x}_k) - \hat{\mathbf{y}}_{k|k-1})^T \mathcal{N}(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1}) d\mathbf{x}_k$$

$$\mathbf{S}_k = \mathbf{R}_k + \int (h(\mathbf{x}_k) - \hat{\mathbf{y}}_{k|k-1})(\cdot)^T \mathcal{N}(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1})\, d\mathbf{x}_k$$

- From these we compute: $\begin{cases} \hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{P}_{xy}\mathbf{S}_k^{-1}(\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}) \\ \mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{P}_{xy}\mathbf{S}_k^{-1}\mathbf{P}_{xy}^T. \end{cases}$

INTEGRALS IN GAUSSIAN FILTERING – EXAMPLE

Polar measurements

- Suppose we observe a measurement

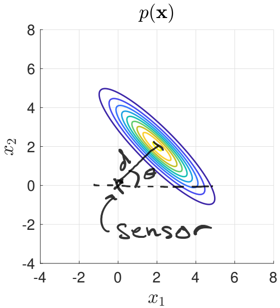
$$\mathbf{y} = \mathbf{h}(\mathbf{x}) + \mathbf{r} = \begin{bmatrix} \sqrt{x_1^2 + x_2^2} \\ \arctan\left(\frac{x_2}{x_1}\right) \end{bmatrix} + \mathbf{r}, \quad \mathbf{r} = \mathbf{0}:$$

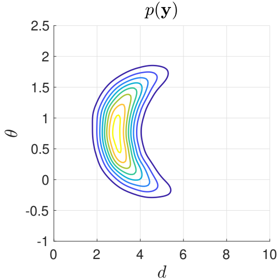
  Where our prior is

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \hat{\mathbf{x}}, \mathbf{P}) = \mathcal{N}\left(\mathbf{x}; \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 & -1.8 \\ -1.8 & 2 \end{bmatrix}\right)$$

- To predict the measurement we compute

$$\mathbb{E}\{\mathbf{y}\} = \int \mathbf{h}(\mathbf{x}) \mathcal{N}(\mathbf{x}; \hat{\mathbf{x}}, \mathbf{P})\, d\mathbf{x}$$





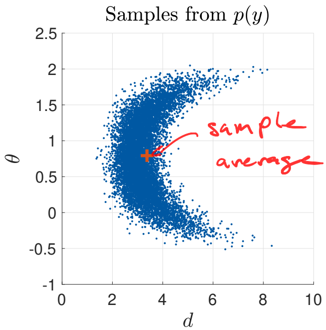
MONTE CARLO SAMPLING

The Monte Carlo method

- Generate independent and identically distributed (i.i.d.) samples, $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(N)}$, from $p(\mathbf{x})$. Approximate

$$\int g(\mathbf{x}) p(\mathbf{x})\, d\mathbf{x} \approx \frac{1}{N} \sum_{i=1}^{N} g(\mathbf{x}^{(i)}).$$

- The Monte Carlo is
  - simple to use to perform Gaussian filtering.
  - asymptotically exact!
  - seldom used (need many samples).



Cov: $g(x) = (h(x) - \hat{y})(\cdot)^T$

STOCHASTIC DECOUPLING

- Suppose $\mathbf{P}^{1/2}$ is a matrix such that $\mathbf{P}^{1/2}(\mathbf{P}^{1/2})^T = \mathbf{P}$.
- We often use the Cholesky decomposition to find $\mathbf{P}^{1/2}$. $=$ chol(P,'lower')
- If $\xi \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ then

  $E\{x\} = \hat{x}$

  $Cov\{x\} = P^{1/2} \cdot I (P^{1/2})^T = P$

$$\mathbf{x} = \hat{\mathbf{x}} + \mathbf{P}^{1/2}\xi \sim \mathcal{N}(\hat{\mathbf{x}}, \mathbf{P})$$

- By changing the variable of integration from $\mathbf{x}$ to ξ we get

$$\int \mathbf{g}(\mathbf{x})\mathcal{N}(\mathbf{x}; \hat{\mathbf{x}}, \mathbf{P})\, d\mathbf{x} = \int \underbrace{\mathbf{g}(\hat{\mathbf{x}} + \mathbf{P}^{1/2}\xi)}_{g(\xi)} \mathcal{N}(\xi; \mathbf{0}, \mathbf{I})\, d\xi.$$

Conclusion

- To perform Gaussian filtering, it is sufficient to be able to compute

$$\int \mathbf{g}(\xi)\mathcal{N}(\xi; \mathbf{0}, \mathbf{I})\, d\xi$$

SIGMA-POINT METHODS

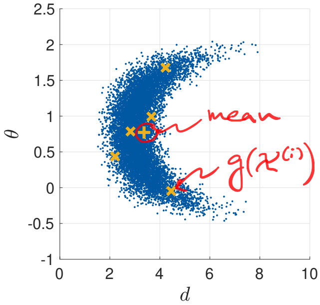
General idea

- Approximate

$$\int g(\mathbf{x})\mathcal{N}(\mathbf{x};\hat{\mathbf{x}},\mathbf{P})\,d\mathbf{x} \approx \sum_{i=1}^{N} W^{(i)} g(\mathcal{X}^{(i)})$$

where $\mathcal{X}^{(i)}$ are called σ-points and $W^{(i)}$ are weights.

- Remarks:
  - Compared to MC approximation, points selected deterministically.
  - Many σ-point methods: unscented transform, cubature rule, Gauss-Hermite quadrature, Gaussian process quadrature, marginalized transform, etc.
  - Each used in Gaussian filtering and the filters are known as UKF, CKF, GHKF, GPKF, MKF, etc.



A LOOK AHEAD

We will cover

- The unscented transform (UT)
  - The most commonly used σ-point method. Has several tuning parameters.
- The cubature rule
  - Uses one point less than UT. No tuning parameters. A simple method that often performs similar to UT.

and explain how these can be used for Gaussian filtering!

SELF ASSESSMENT

Check all statements that apply:

- The Monte Carlo method approximates expected values by sample averages.
- A σ-point method makes use of a small number of weighted random samples.
- Stochastic decoupling is about rewriting an integral with respect to a vector of correlated random variables as an integral with respect to a vector of independent random variables that have zero mean and unit variance.
- The Gauss-Hermite Kalman filter, the Unscented Kalman filter and the Cubature Kalman filter are all Gaussian filters and they only differ in how they approximate the involved integrals.