The extended Kalman filter and the Iterative extended Kalman filter

Sensor fusion & nonlinear filtering

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THE EXTENDED KALMAN FILTER (EKF)

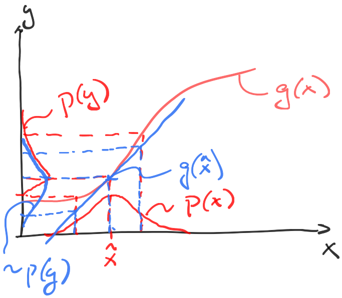
- The extended Kalman filter was developed almost immediately after the Kalman filter, in the early 1960s.

The extended Kalman filter (EKF)

- **Idea:** Linearize $\mathbf{f}_{k-1}(\mathbf{x}_{k-1})$ and $\mathbf{h}_k(\mathbf{x}_k)$ and apply the Kalman filter on the linearized system!
- Prediction: linearize $\mathbf{f}_{k-1}(\mathbf{x}_{k-1})$ around $\hat{\mathbf{x}}_{k-1|k-1}$ and use the Kalman filter prediction.
- Update: linearize $\mathbf{h}_k(\mathbf{x}_k)$ around $\hat{\mathbf{x}}_{k|k-1}$ and use the Kalman filter update.

TAYLOR SERIES EXPANSIONS

- Consider a Gaussian random variable $\mathbf{x} \sim \mathcal{N}(\hat{\mathbf{x}}, \mathbf{P})$ and a nonlinear function $\mathbf{y} = \mathbf{g}(\mathbf{x})$.



Taylor series expansions

- A first order Taylor expansion of $\mathbf{g}(\mathbf{x})$ around $\hat{\mathbf{x}}$ gives

$$\mathbf{y} \approx \mathbf{g}(\hat{\mathbf{x}}) + \underbrace{\mathbf{g}'(\hat{\mathbf{x}})}_{m \times n}\,(\mathbf{x} - \hat{\mathbf{x}})$$

where $[\mathbf{g}'(\mathbf{x})]_{ij} = \frac{\partial g_i(\mathbf{x})}{\partial x_j}$ is the Jacobian matrix of $\mathbf{g}$.

- Note that we now get

$$E\{y\} = E\{g(\hat{x}) + \underbrace{g'(\hat{x})(x-\hat{x})}_{=0}\} = g(\hat{x})$$

$$\mathbb{E}\{\mathbf{y}\} \approx \mathbf{g}(\hat{\mathbf{x}})$$

$$\mathrm{Cov}\{\mathbf{y}\} \approx \mathbf{g}'(\hat{\mathbf{x}})\,\mathbf{P}\,\mathbf{g}'(\hat{\mathbf{x}})^T.$$

$$\mathrm{Cov}\{y\} = \mathrm{Cov}\{\underbrace{g(\hat{x})}_{=0} + g'(\hat{x})(x-\hat{x})\} = g'(\hat{x})\underbrace{\mathrm{Cov}\{x-\hat{x}\}}_{P}\,g'(\hat{x})^T$$

SELF ASSESSMENT

- What is $g(\hat{\mathbf{x}})$ and $g'(\hat{\mathbf{x}})$ when

$$\mathbf{x} = \begin{bmatrix} p_1 \\ p_2 \\ v_1 \\ v_2 \end{bmatrix}, \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \text{ and } y = \sqrt{p_1^2 + p_2^2}?$$

- $g(\hat{\mathbf{x}}) = \sqrt{5}$ and $g'(\hat{\mathbf{x}}) = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 & 0 \end{bmatrix}^T$
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EKF: THE PREDICTION STEP

- We have

$$\mathbf{x}_{k-1} \big| \mathbf{y}_{1:k-1} \sim \mathcal{N}(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{P}_{k-1|k-1}).$$

- By using the approximation

$$\begin{aligned} \mathbf{x}_k &= \mathbf{f}(\mathbf{x}_{k-1}) + \mathbf{q}_{k-1} \\ &\approx \mathbf{f}(\hat{\mathbf{x}}_{k-1|k-1}) + \mathbf{f}'(\hat{\mathbf{x}}_{k-1|k-1}) \left(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1|k-1}\right) + \mathbf{q}_{k-1} \end{aligned}$$

we get:

EKF: the prediction step

$$\begin{aligned} \hat{\mathbf{x}}_{k|k-1} &= \mathbf{f}(\hat{\mathbf{x}}_{k-1|k-1}) \\ \mathbf{P}_{k|k-1} &= \underbrace{\mathbf{f}'(\hat{\mathbf{x}}_{k-1|k-1})}_{\tilde{F}} \mathbf{P}_{k-1|k-1} \underbrace{\mathbf{f}'(\hat{\mathbf{x}}_{k-1|k-1})^T}_{\tilde{F}^T} + \mathbf{Q}_{k-1} \end{aligned}$$

EKF: THE UPDATE STEP

- Similarly, if we approximate

$$\begin{aligned} \mathbf{y}_k &= \mathbf{h}(\mathbf{x}_k) + \mathbf{r}_k \\ &\approx \mathbf{h}(\hat{\mathbf{x}}_{k|k-1}) + \mathbf{h}'(\hat{\mathbf{x}}_{k|k-1}) \left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}\right) + \mathbf{r}_{k-1}, \end{aligned}$$

we get:

EKF: the update step

$$\begin{aligned} \hat{\mathbf{x}}_{k|k} &= \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \left(\mathbf{y}_k - \mathbf{h}(\hat{\mathbf{x}}_{k|k-1})\right) \\ \mathbf{P}_{k|k} &= \mathbf{P}_{k|k-1} - \mathbf{K}_k \mathbf{S}_k \mathbf{K}^T \\ \mathbf{S}_k &= \mathbf{h}'(\hat{\mathbf{x}}_{k|k-1}) \mathbf{P}_{k|k-1} \mathbf{h}'(\hat{\mathbf{x}}_{k|k-1})^T + \mathbf{R}_k \\ \mathbf{K}_k &= \mathbf{P}_{k|k-1} \mathbf{h}'(\hat{\mathbf{x}}_{k|k-1})^T \mathbf{S}_k^{-1} \end{aligned}$$