Model Based Engineering

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Contents

Chapter 1

Model Based Engineering

1.1 Feedback Systems

What is a feedback system?

1.1.1 Feedback Principles

Let's quickly summarize some important feedback principles

- Control objectives that is specifications. We may have qualitative or quantitative objectives.
- Derivation of the mathematical model describing the system or the component we work on
- Design the controller
- Analyze the performance

Chapter 2

The Physics of Automotive Dynamics

In these notes, we are not particularly interested in how the combustion engine operates. In contrast, we are interested in automobiles as a macroscopic system that we would like to understand and control. In this context, we will treat the combustion engine (or any other means whatsoever producion the propulsion force) as a black box. However, we want to be able to argue upon how much force the engine system should produce such that the vehicle is able to accelerate, decelerate or travel at constant speed. In this chapter therefore, we will introduce the basic physical laws that govern the motion of the vehicle. In particular, we will consider

- Longitudinal vehicle dynamics
- Aerodynamic force $\mathbf{F}_{aero}$
- Rolling force $\mathbf{F}_{roll}$
- Gravitational force $\mathbf{F}_{grav}$

By considering the above mentioned forces, we will be able to produce a basic model that explains the longitudinal dynamics of the vehicle. In chaper 3 we will get more into modeling the subsystems of the vehicle and come up with a more refined model.

2.1 Aerodynamics & Rolling Resistance

Let us start simple and assume the motion of a vehicle in a flat road; no road gradietns. There are three types of forces that are always present in such a scenario; the propulsion force, $\mathbf{F}_p$, produced by the powertrain, the rolling force

$\mathbf{F}_{roll}$ produced by the friction of the tires and the road and the aerodynamic force $\mathbf{F}_{aero}$ which is caused due to the motion of the vehicle in the air. We can see schematically how these forces act in figure 2.1

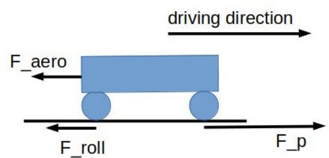

Figure 2.1: Schematics of longitudinal forces acting on vehicle.

Remark

In figure 2.1, the vectors have not been drawn to scale. Also, the propulsion force $\mathbf{F}_p$ is applied to every motorised wheel however, for simplicity we consider the net or total propulsion force acting on the vehicle. Similarly, the rolling force $\mathbf{F}_{roll}$ represents the total rolling force and not the individual force acting on every wheel. Finally, and in the same vein, the aerodynamic force, $\mathbf{F}_{aero}$, is the total force acting on the vehicle.

Remark

Forces in the other directions mainly affect the suspension and the steering of the vehicle and we will not consider them here.

It is easy to understand that $\mathbf{F}_{roll}$ will be zero when the vehicle is not moving. A commonly used model is:

$$\mathbf{F}_{roll}(\alpha) = \begin{cases} 0 & \text{if } v = 0, \\ \pm c_r m g \cos(\alpha) & \text{if } v \neq 0 \end{cases} \tag{2.1}$$

where

- α is the road slope
- m is the vehicle mass
- c_r is the rolling resistance coefficient. Typically, is about 0.01
- g is the gravitational acceleration constant

The rolling resistance force, is approximately independent of the vehicle speed and it exhibits some variation with respect to the road angle. It acts in the opposite direction of the driving. Thus, it changes sign if the vehicle is moving in the reverse direction. c_r should be as small as possible in order to keep the energy consumption low.

2.2 Questions

Question 1

Suppose a vehicle is coasting on a flat road, that is, there are no propelling or braking forces acting on the vehicle. This means the only forces that are acting on the vehicle are the rolling resistance and aerodynamic forces. Below you find diagrams describing those forces as a function of vehicle speed

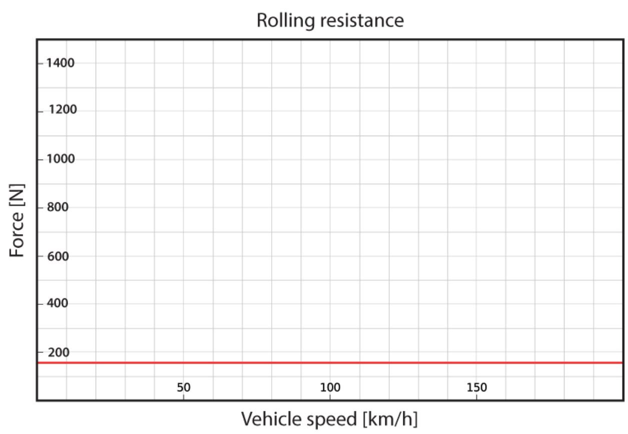

Figure 2.2: Rolling force diagram.

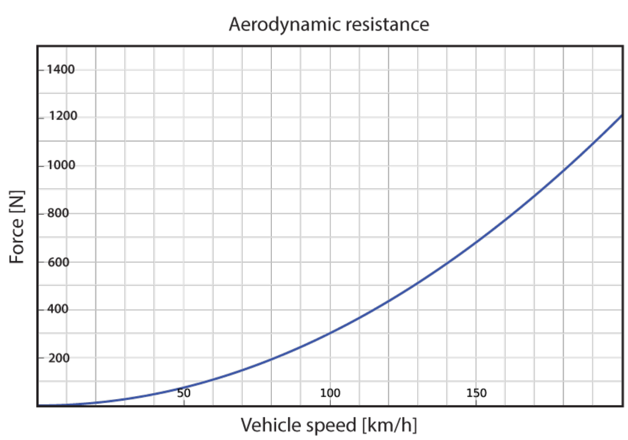

Figure 2.3: Aerodynamics force diagram.

What is the sum of the rolling resistance and aerodynamic forces (N) at 100 km/h?

Question 2

Assume the vehicle in question 1 above has the following data:

m	1600
$\alpha 4$	0.0
A_f	2.1
c_r	0.01
C_D	0.3
ρ	1.25
g	9.81

What is the running resistance on a flat road, i e the sum of the rolling resistance and the aerodynamic resistance at 50 km/h? What is the running resistance on a flat road, i e the sum of the rolling resistance and the aerodynamic resistance at 150 km/h?

Question 3

Consider the same vehicle as in Ex.1-2 above. What happens to the resisting forces if the vehicle is made 50% heavier, but all other parameters remain the same? If the vehicle front area A_f is increased 10%, but all other parameters remain the same?

Chapter 3

Modeling Automotive Systems

In this chapter, we will discuss a number of simplified mathematical models that can be used to model various aspects of an automobile. Concretely, we will cover the follwoing:

- Longitudinal vehicle dynamics
- Drivetrain modeling
- Suspension modeling
- Single track model

In doing so, we will present and apply a three phase modeling method that comprises of three steps. Namely,

- Structuring
- Constitutive relationships
- State-space model formulation

3.1 Longitudinal Dynamics

In this section we are interested in developing a model that captures the main longitudinal dynamics of the vehicle. Concretely, we are interested in how the velocity of the vehicle changes when we press or release the accelerator pedal. We will follow the three phase modeling approach.

3.1.1 Structuring

Let's assume that the vehicle consists of the powertrain and the chassis. The former produces a propulsion force $\mathbf{F}_p$ that acts on the latter. We can visualize this using a block diagram as shown in figure 3.1:

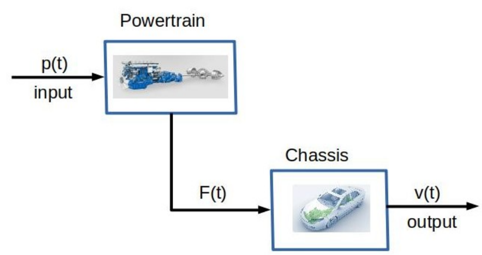

Figure 3.1: Block diagram for powertrain-chassis relationship.

The powertrain can be further broken down into several other subsystems such as:

- Engine
- Gearbox
- Wheels

This is shown in the block diagram in figure 3.2

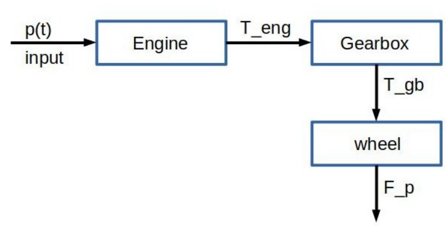

Figure 3.2: Block diagram for engine-gearbox-wheels relationship.

The advantage of such fine grained approach is that we can describe the various components individually. However, the downside is that we increase the complexity of our model. Hopefully, this increase in complexity will be reflected in an increase of accuracy of the resulting model.

Let's now start modeling the powertrain.

3.1.2 Powertrain Modeling

By pressing the pedal, the engine responds by building up a torque T_{eng}. We will assume that this build up process can be adequately represented by a first order dynamics. Thus,

$$\dot{T}_{eng} = -\frac{1}{\tau}T_{eng} + \frac{k}{\tau}p \tag{3.1}$$

where

- p is the pedal position
- k is the steady state gain from the pedal position
- τ is the time constant of the engine to build up the torque

The gearbox will simply amplify the engine torque, T_{eng}, with a factor equal to the gear ratio i as given by the equation below

$$T_{gb} = iT_{eng} \tag{3.2}$$

The wheels convert the torque to force via the wheel radius r_w

$$\mathbf{F} = \frac{T_{gb}}{r_w} \tag{3.3}$$

The wheel model in equation 3.3 assumes that there is no slip between the wheels and the ground.

If we substitute the models 3.3 and 3.2 into 3.1, we obtain the following equation for $\mathbf{F}$ which from now on we will call the propulsion force and indicate it instead with $\mathbf{F}_p$

$$\frac{d\mathbf{F}_p}{dt} = -\frac{1}{\tau}\mathbf{F}_p + \frac{ki}{\tau r_w}p \tag{3.4}$$

3.1.3 Chassis Modeling

The propulsion force, $\mathbf{F}_p$, produced by the powertrain can be given as an input to the chassis model as shown schematically in figure

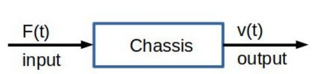

Figure 3.3: Block diagram for chassis component.

We can derive the governing equation describing the dynamics of the chassis by using Newton's law:

$$m\mathbf{a} = \mathbf{F}_T \tag{3.5}$$

where m is the vehicle mass and $\mathbf{F}_T$ is the total or net force acting on the chassis. The net force is assumed to have the following components

- $\mathbf{F}_p$ propulsion force
- $\mathbf{F}_{aero}$ aerodynamic force coming from the movement of the vehicle
- $\mathbf{F}_{grav}$ the graviational force
- $\mathbf{F}_{roll}$ the rolling resistance force

Thus,

$$\mathbf{F}_T = \sum_i \mathbf{F}_i = \mathbf{F}_p - \mathbf{F}_{aero} - \mathbf{F}_{grav} - \mathbf{F}_{roll} \tag{3.6}$$

The propulsion force is assumed to act in the direction of motion and must be large enough to overcome the three other forces in order for the vehicle to able to move. The following body diagram illustrates this.

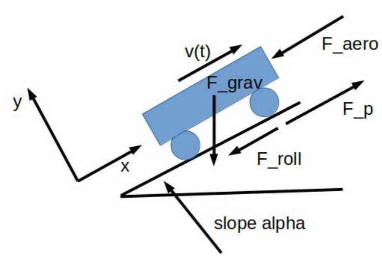

Figure 3.4: Chassis free body diagram.

Note that in the figure above the propulsion and rolling forces are only shown to act in one wheel which is not true.

Since we assume that the road has slope α, it useful to decompose the forces in the x and y direction:

$$x : m\alpha_x = F_{p,x} - F_{aero,x} - F_{grav,x} - F_{roll,x} \tag{3.7}$$
$$y : m\alpha_y = F_{p,y} - F_{aero,y} - F_{grav,y} - F_{roll,y} \tag{3.8}$$
$$\tag{3.9}$$

3.1.4 Balance Laws & Constitutive Equations

Let's now turn attention to the constitutive equations. Ideally, we would not like to have any accelerations in the y direction thus $\alpha_y = 0$. Furthermore, there in no drag force component in the in the y direction; $F_{aero,y} = 0$ whilst the drag term in the x direction is modelled via:

$$F_{aero,x} = \frac{1}{2}\rho C_D A_f v^2 \tag{3.10}$$

where

- ρ is the air density
- C_D is the drag coefficient
- $\mathbf{F}_{grav}$ the graviational force
- $\mathbf{F}_{roll}$ the rolling resistance force

From figure 3.5

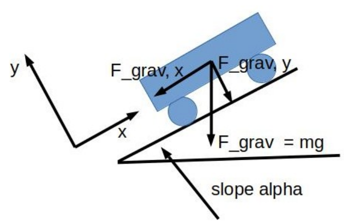

Figure 3.5: $\mathbf{F}_{grav}$ force decomposition.

we can decompose the $\mathbf{F}_{grav}$ into

$$x : F_{grav,x} = mg\sin(\alpha) \tag{3.11}$$
$$y : F_{grav,y} = mg\cos(\alpha) \tag{3.12}$$

Finally, the rolling force in the x-direction $F_{roll,x}$ is modelled as a fraction of the vehicle load:

$$F_{roll,x} = (N_1 + N_2)f \tag{3.13}$$

3.1.5 Summary Of Balance Laws & Constitutive Equations

Let's now summarize the equations and constitutive relations used for our model. Note that the acceleration $\boldsymbol{\alpha}$ is given by

$$\boldsymbol{\alpha} = \frac{d\mathbf{v}}{dt} \tag{3.14}$$

- Chassis dynamics

$$x : m\alpha_x = F_{p,x} - F_{aero,x} - F_{grav,x} - F_{roll,x} \tag{3.15}$$

$$y : 0 = N_1 + N_2 - F_{grav,y} \tag{3.16}$$

where

- $F_{aero,x} = \frac{1}{2}\rho C_D A_f v^2$
- $F_{grav,x} = mg\sin(\alpha)$
- $F_{grav,y} = mg\cos(\alpha)$
- $F_{roll,x} = (N_1 + N_2)f$ the graviational force
- N_1, N_2 are the vehicle load on the back and front wheels respectively

The propulsion force $\mathbf{F}_p$ is given by the solution of

$$\frac{d\mathbf{F}_p}{dt} = -\frac{1}{\tau}\mathbf{F}_p + \frac{ki}{\tau r_w}p \tag{3.17}$$

3.1.6 State-Space Form

We will now cast the model above into the state-space form. The first step is to select the state variables. We do so by observing what changes. We have two items here

- The propulsion force $\mathbf{F}_p$
- The velocity $\mathbf{v}$

3.2 Questions

Question 1

Consider the the longitudinal motion vehicle model

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{1}{\tau}x_1 + \frac{ki}{\tau r_w}u \\ \frac{dx_2}{dt} &= \frac{1}{m}\left(x_1 - mgf\cos(\alpha) - \frac{1}{2}\rho C_D A_f x_2^2 - mg\sin(\alpha)\right) \\ y &= x_2 \end{aligned}$$

where $x-1$ is the force at the wheels, x_2 is the vehicle velocity, u is the input signal, the accelerator pedal position, and α is the disturbance, the road slope.

Is the longitudinal vehicle dynamics model linear?

Is the longitudinal vehicle dynamics model time invariant?

Determine the required engine torque (Nm) for a vehicle acceleration of 0.3 m/s^2 on a flat road at 60 km/h!

Longitudinal slip is the relative motion between a tire and the road surface it is moving on:

$$\text{slip} = \frac{r_w\omega - v}{v} \tag{3.18}$$

which means that the wheels are spinning if slip is positive and that the wheels are skidding if slip is negative. The longitudinal vehicle dynamics model is developed under the assumption that the relative motion between the tire and the road surface is zero, i.e. is not included in the model.

If we would like to include slip into our model, how many additional states are needed in order to include longitudinal slip into the model?

One additional state is needed as we need to include the wheel speed as an additional state variable in order to capture slip as we already have velocity as one state.

When performing numerical simulations with the longitudinal vehicle dynamics model including the slip model, you might run into numerical problems. Why?

- A) Division by zero, due to zero velocity
- B) Division by zero, due to zero wheel speed
- C) Division by zero, due to zero engine torque

The slip model includes a normalization with respect to the vehicle velocity. This gives numerical problems when the velocity is zero, i.e. division by zero. Thus, option A is the correct answer.

3.3 Drivetrain Modeling

3.4 Suspension System Modeling

In this section we will derive a mathematical model for the vehicle suspension system. Concretely, we will assume a passive suspension system.

Active and Passive Suspension System

Active suspension is a type of automotive suspension that controls the vertical movement of the wheels relative to the chassis or vehicle body with an onboard system, rather than in passive suspension where the movement is being determined entirely by the road surface. Active suspensions can be generally divided into two classes:

- pure active suspensions,
- adaptive/semi-active suspensions.

While adaptive suspensions only vary shock absorber firmness to match changing road or dynamic conditions, active suspensions use some type of actuator to raise and lower the chassis independently at each wheel. For more information see the Wikipedia entry: https://en.wikipedia.org/wiki/Active_suspension

The suspension system connects the vehicle body with the wheels. furthermore, it gives the whell a vertical movement possibility. A common approach to model the vehicle suspension system is to consider only one of the four wheels. The schematic of a simplified suspension system is shown in figure

For a passive suspension system, the input is road surface z_r. The outputs will be the positions of the wheels z_w and that of the chassis z_c. The block diagram is shown in figure 3.6

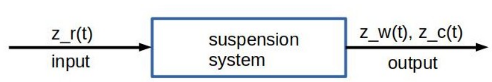

Figure 3.6: Block diagram for suspension system.

The free body diagram for the wheel mass is shown in figure 3.7

where $\mathbf{F}_t$ is the tire force, $\mathbf{F}_d$ is the suspension force and $\mathbf{F}_c$ is the chassis force. Similarly, the free body diagram for the chassis mass is shown in figure 3.7

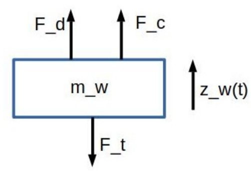

Figure 3.7: Wheel mass free body diagram.

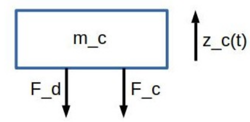

Figure 3.8: Chassis mass free body diagram.

3.5 The Bicycle Model

We would like to be able to control the vehicle; for example when following a predefined path or when making an avoidance manouver. These types of motions require control of the lateral dynamics that the longitudinal model developed in section 3.1 does not account for. We would like therefore to generalize somehow our model.

Thus, in this section, we study the kinematic bicycle model, which is often used for trajectory planning, and compare its results to a none degrees of freedom model. Modeling errors and limitations of the kinematic bicycle model are highlighted.

The three degrees of freedoom (DOFs) kinematic bicycle model, see figure 3.9, is one of the simplest models frequently used at the motion planning phase, with the belief that it is able to capture enough of the nonholonomic constraints of the actual vehicle dynamics. By constrast, even relatively simple vehicle models used for low level control can imply more than ten DOFs [2].

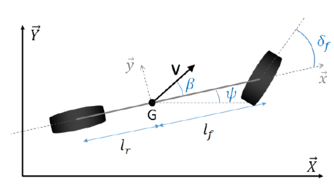

Figure 3.9: Schematic of the bicycle model.

In the kinematic bicycle model, the two front wheels (respectively the two rear wheels) of the vehicle are lumped into a unique wheel located at the center of the front axle (respectively of the rear axle) such as illustrated on Figure 1. The control inputs correspond to the acceleration $\boldsymbol{\alpha}$ and the front wheel steering angle δ of the vehicle, when assuming that only the front wheel can be steered. The kinematic bicycle model can then

Chapter 4

Linear Systems

Modeling of automotive systems (or any other system whatsoever) typically results in a mathematical description of it. This mathematical description is usually given in the form of differential equations that involve the input(s) and the output(s) of the system as well as other parameters that are of interests and affect its behavior. In this chapter, we introduce the notion of a linear system (LS) and other useful terms that will be used in subsequent chapters. We will see that a LS can be represented in different forms susch as:

- Differential equations
- Transfer functions
- State-Space models

4.1 Linear ODEs

Let's consider the ODE:

$$\frac{d^n y}{dt^n} + \alpha_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + \alpha_n y = \beta_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + \beta_n u \tag{4.1}$$

where u represents the input to the system and y represents the output. α_i and β_i are coefficients which may or may not be time dependent. In the case that the coefficients do not depend on the time variable t we have a linear time invariant (LTI) system. Otherwise the system will be time variant.

Superposition Principle

Let the linear equation:

$$Ly = F(x) \tag{4.2}$$

where L is a linear operator. If y_1 and y_2 are solution of the linear equation, then their sum will also be a solution.

Every solution to such a system can be written as the sum (because the system is linear superposition of solutions...) of a solution $y_h$to the homogeneous equation and a particular solution y_p:

$$y = y_h + y_p \tag{4.3}$$

The solution to the homogeneous solution will have the form:

$$y_h = \sum_{k=1}^{n} C_k e^{s_k t} \tag{4.4}$$

where s_k are the roots of the so called characteristic equation or polynomial (see section 4.1.1) and C_k can be determined from the initial conditions.

Now that we have established the general form of the homogeneous solution, let's turn our attention to the particular solution. In order to find the particular solution y_p, we need the input signal u. Let's assume a damped sinusoidal input signal of the following form

$$u(t) = e^{st} \tag{4.5}$$

where s is a complex variable. The particular solution is assumed to have the follwoing general form:

$$y_p = G(s)e^{st} \tag{4.6}$$

The function $G(s)$ is called the transfer function TF. Thus, the general solution of the equation 4.1 is

$$y(t) = \sum_{k=1}^{n} C_k e^{s_k t} + G(s)e^{st} \tag{4.7}$$

The first part gives the dependency due to the initial conditions. The second part is due to the input signal.

4.1.1 Characteristic Equation

In equation 4.7 the s_ks are the roots of the characteristic equation or polynomial $\alpha(s)$. This polynomial is formed from the coefficients of the system 4.1. Hence, for the system

$$\frac{d^n y}{dt^n} + \alpha_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + \alpha_n y = \beta_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + \beta_n u$$

we will have

$$\alpha(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n \tag{4.8}$$

Thus, the s_k will be the roots of the equation

$$\alpha(s) = 0 \tag{4.9}$$

Concretely, the solutions s_k are called the **poles** of the system and play a crucial role in the stability of the solution:

- If all the roots of $\alpha(s) = 0$ have $Re(s_k) < 0$ then the system is asymptotically stable
- If any of the roots of $\alpha(s) = 0$ has $Re(s_k) > 0$ then the system is unstable

Example

Consider the system

$$\dot{x} = x + u$$

Form the characteristic equation. Is the solution stable or not?

Answer

The characteristic equation of the system above is

$$\alpha(s) = s - 1 = 0$$

The only root to this equation is $s = 1$ and since $Re(s) > 0$ the system is not asymptotically stable.

4.1.2 Laplace Transformation

4.2 State-Space models

A state-space formulation of a linear system has the following form

$$\dot{x} = Ax + Bu \tag{4.10}$$
$$y = Cx + Du \tag{4.11}$$

where

- $x \in R^n$ is the state vector
- $y \in R^n$ is the output vector
- $u \in R^p$ is the input vector
- $A \in R^{n\times n}$ is the matrix describing the dynamics
- $B \in R^{n\times p}$ is the matrix describing the input
- $C \in R^{q\times n}$ is the output or sensor matrix
- $D \in R^{q\times p}$ is the direct matrix

Just like in equation 4.1, if the coefficient matrices do not depend on time, we have an LTI system.

4.3 Questions

Question 1

Consider the following model:

$$\dot{x} = x + u$$

We saw in example 4.1.1 that the solution to this system is asymptotically stable. Cast the system in the

$$\frac{dx}{dt} = Ax + Bu$$

form and argue again about its stability.

Answer

We can write the given system in the form above if we assume that

$$A = [1] \text{ and } B = [1]$$

The stability depends on the eignevalues of A and here we have only one eigenvalue which is $\lambda = 1$. Thus, since $Re(\lambda) > 0$ the system is not asymptotically stable as we concleded in example 4.1.1

Question 2

Consider the following model:

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = u$$

compute the poles of the equation and argue about the stability.

Answer

The poles of the equation are the solutions of the characteristic equation:

$$\alpha(s) = s^2 + 2s + 1 = 0$$

The discriminant of the quadratic equation is

$$\Delta = b^2 - 4ac = 0$$

Hence, the characteristic polynomial has only one real solution given by

$$s = \frac{-b}{2a} = -1$$

Thus, since $Re(s) < 0$ the system is asymptotically stable.

Question 3

Determine the poles of the following equations:

$$\begin{aligned} \text{A)} \quad & \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = u \\ \text{B)} \quad & \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = u \\ \text{C)} \quad & \frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = u \end{aligned} \tag{4.12}$$

For the the model A we have:

$$\alpha(s) = s^2 + 2s + 1 = 0$$

Thus, only one pole $s = -1$. For model B,

$$\alpha(s) = s^2 + 2s + 2 = 0$$

The discriminant $\Delta = -4 < 0$. Thus, the system has two complex solution

$$\begin{aligned} s_1 &= -1 + i \\ s_2 &= -1 - i \end{aligned} \tag{4.13}$$

For model C,

$$\alpha(s) = s^2 + 3s + 2 = 0$$

and the solutions are

$$\begin{aligned} s_1 &= -1 \\ s_2 &= -2 \end{aligned} \tag{4.14}$$

Chapter 5

Examples I

In this chapter we will present some examples in order to reinforce and clarify various topics introduced in the previous chapters

5.1 PI Cruise Controller

In this example we will analyse the model of the closed loop cruise controller. This is shown in the image below

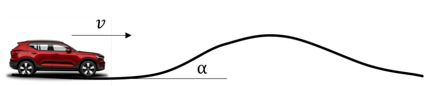

Figure 5.1: Schematics of PI close loop cruise controller.

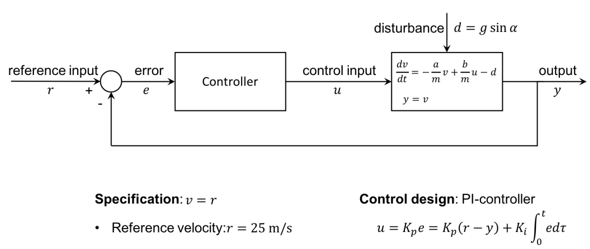

Figure 5.2: Schematics of PI close loop cruise controller.

5.1.1 Questions

Question 1:

Which features are representative for a closed loop controller?

- A) Acts on deviation
- B) No risk for instability
- C) Risk for instability
- D) Insensitive to measurement noise

Answer: A closed loop controller acts on the deviation from the reference signal in a measure-decide-act cycle. So option A is correct. Closed loop controllers are subject to instability problems; the closed loop dynamics can be shaped in such a way that the system might become unstable. So option C is also correct.

Question 2:

Which features are representative for an open loop controller?

- A) Acts on deviation
- B) No risk for instability
- C) Risk for instability
- D) Insensitive to measurement noise

Answer: In an open loop controller all control actions are preprogrammed (otherwise we have no control). Thus, option A is correct. Option B is also correct provided that the system is also stable. In this case the open loop controller will also be stable. For an open loop controller no measurements are required, so no sensitivity towards measurements (VERIFY THIS). Thus, option D is also correct.

Question 3:

Would you say that a driver is closed loop or open loop controller?

Answer: Since a driver typically aquires feedback via his/her senses and acts accordingly in order to adapt to the measured feedback, we can say that a driver is a closed loop controller.

Chapter 6

State Feedback

The state, $x(t)$, of a dynamical system is a collection of variables that permits prediction of the future development of a system. In this chapter we want to explore the idea of designing the dynamics of a system through feedback of the state. A typical feedback control system with state feedback is shown in figure 6.1

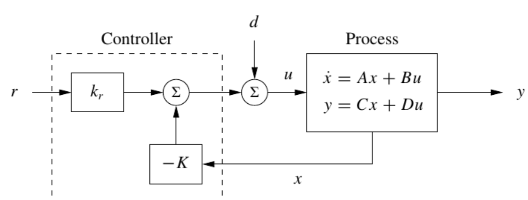

Figure 6.1: A feedback control system with state feedback. The controller uses the system state x and the reference input r to command the process through its input u. We model disturbances via the additive input d.

The goal of the feedback controller is to regulate the output of the system y such that it tracks the reference input in the presence of disturbances and also uncertainty in the process dynamics.

6.1 Overview

We have already mentioned the general form of the state-space model is

$$\dot{x} = f(x, u, d) \tag{6.1}$$
$$y = h(x, u, d) \tag{6.2}$$

both f and h can be nonlinear functions of their arguments however in this section we will assume that we are dealing with a linear system. Our aim is to design a state feedback controller. Thus, let's consider again the system

$$\dot{x} = Ax + Bu \tag{6.3}$$
$$y = Cx + Du \tag{6.4}$$

The poles of the system are given by the eigenvalues of the matrix A which are the roots of the characteristic polynomial

$$det(sI - A) = 0 \tag{6.5}$$

Feedback control is a method for shaping the dynamics of the system. This is done by modifying the eigenvalues of the system using the input u. Indeed, let's assume that u can be written as

$$u = -Kx + K_r r \tag{6.6}$$

with $K \in r^{p\times n}$, $k_r \in R^{p\times r}$ and $r \in R^r$ is the reference input. We can shape the dynamics of the system via the feedback gain K. K_r is used to change the steady state level of the system. The system dynamics becomes

$$\dot{x} = Ax + B(-Kx + K_r r) \tag{6.7}$$
$$= (A - BK)x + BK_r r \tag{6.8}$$

The control objective now is to choose K such that the closed loop dynamics $A - BK$ get the desired properties.

Remark 6.1.1. **Steady state reference gain**

The steady-state reference gain K_r does not affect the stability of the system but it does affect the steady state solution.

The gain K_r is chosen such that

$$y(t) \approx r(t), \quad \text{as} \quad t \to \infty \tag{6.9}$$

At steady state

$$\dot{x} = 0 \tag{6.10}$$

thus we can write

$$0 = (A - BK)x + BK_r r \tag{6.11}$$
$$y = Cx \tag{6.12}$$

Hence, we can write for the output y

$$y = -C(A - Bk)^{-1} BK_r r \tag{6.13}$$

If we want $y(t) \approx r(t)$, as $t \to \infty$, then K_r should be chosen as

$$K_r = -(C(A - BK)^{-1}B)^{-1} \tag{6.14}$$

The design of the controller becomes a two step process.

- shape the dynamics by choosing K
- set the steady-state level by choosing K_r

This way the reference gain depends on the feedback gain. Using the steady-state feedback gain K_r we can achieve zero steady-state error but this depends on the model parameters also. Integration can be used in order to remove the steady-state error. This can be done by introducing a new variable to remove the steady-state error:

We will illustrate how this is done assuming a SISO system.

$$\dot{z}(t) = y(t) - r(t) \tag{6.15}$$

The new state-space model becomes

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} Ax - Bu \\ y - r \end{bmatrix} = \begin{bmatrix} Ax - Bu \\ Cx - r \end{bmatrix} \tag{6.16}$$

the above can be written as

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r \tag{6.17}$$

Thus, the new controller now becomes

$$u(t) = -Kx(t) - K_I z(t) + K_r r(t) \tag{6.18}$$

6.2 Reachability

One of the fundamental properties of a control system is what set of points in the state space can be reached through the choice of a control input. It turns out that the property of reachability is also fundamental in understanding the extent to which feedback can be used to design the dynamics of a system.

Definition 6.2.1. Reachability

A linear system is reachable if for any $x_0, x_f \in R^n$ there exists a $T > 0$ and $u : [0, T] \to R$ such tha the corresponding solution satisfies $x(0) = x_0$ and $x(T) = x_f$.

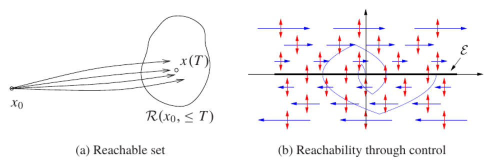

Figure 6.2: The reachable set for a control system. The set $R(x_0, \leq T)$ shown in (a) is the set of points reachable from x_0 in time less than T . The phase portrait in (b) shows the dynamics for a double integrator, with the natural dynamics drawn as horizontal arrows and the control inputs drawn as vertical arrows. The set of achievable equilibrium points is the x axis. By setting the control inputs as a function of the state, it is possible to steer the system to the origin, as shown on the sample path.

The definition of reachability addresses whether it is possible to reach all points in the state space in a transient fashion. In many applications, the set of points that we are most interested in reaching is the set of equilibrium points of the system (since we can remain at those points once we get there). The set of all possible equilibria for constant controls is given by:

$$E = x_e : Ax_e + Bu_e = 0 \ \text{ for some } \ u_e \in R \tag{6.19}$$

To find general conditions under which a linear system is reachable, we will first give a heuristic argument based on formal calculations with impulse functions.

Remark 6.2.1. Equilibrium points and reachability

We note that if we can reach all points in the state space through some choice of input, then we can also reach all equilibrium points.

6.2.1 Testing for reachability with impulse functions

When the initial state is zero, the resppnse of the system to an input $u(t)$ is given by

$$x(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \tag{6.20}$$

Let's choose the input to be an impulse function $\delta(t)$. The state then becomes

Remark 6.2.2. Impulse function $\delta(t)$

$$x_\delta = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau = \frac{dx_s}{dt} = e^{At} B \tag{6.21}$$

(Note that the state changes instantaneously in response to the impulse.) We can find the response to the derivative of an impulse function by taking the derivative of the impulse response (Exercise 5.1):

$$\frac{dx_\delta}{dt} = Ae^{At} B \tag{6.22}$$

Continuing this process and using the linearity of the system, the input

$$u(t) = \alpha_1 \delta(t) + \alpha_2 \frac{d\delta(t)}{dt} + \cdots + \alpha_n \frac{d^{n-1}\delta(t)}{dt^{n-1}} \tag{6.23}$$

gives the state

$$x(t) = \alpha_1 e^{At} B + \alpha_2 A e^{At} B + \cdots + \alpha_n A^{n-1} e^{At} B \tag{6.24}$$

On the right is a linear combination of the columns of the matrix

$$W_r = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \tag{6.25}$$

To reach an arbitrary point in the state space, we thus require that there are n linear independent columns of the matrix W_r.

Remark 6.2.3. **The reachability matrix** W_r

The matrix W_r is called the reachability matrix.

6.2.2 Testing for reachability with the convolution equation

However, an input $u(t)$ consisting of a sum of impulse functions and their derivatives is a very violent signal. To see that an arbitrary point can be reached with smoother signals we can make use of the convolution equation. Assuming that the initial condition is zero, the state of a linear system is given by

$$x(t) = \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau = \int_0^t e^{A(\tau)} Bu(t-\tau)d\tau \tag{6.26}$$

It follows from the theory of matrix functions, specifically the Cayley-Hamilton theorem (see Exercise 6.10), that

Remark 6.2.4. The Cayley-Hamilton theorem

$$e^{A\tau} = I\alpha_0(\tau) + A\alpha_1(\tau) + \cdots + A^{n-1}\alpha_{n-1}(\tau) \tag{6.27}$$

where α_i are scalar functions, and we find that

$$x(t) = B\int_0^t \alpha_0(\tau)u(t-\tau)d\tau + AB\int_0^t \alpha_1(\tau)u(t-\tau)d\tau + \cdots + A^{n-1}B\int_0^t \alpha_{n-1}(\tau)u(t-\tau)d\tau \tag{6.28}$$

Again we observe that the right-hand side is a linear combination of the columns of the reachability matrix W_r given by equation (6.3). This basic approach leads to the following theorem.

Theorem 6.2.1. *Reachability rank condition*

A linear system is reachable if and only if the reachability matrix W_r is invertible.

6.2.3 Reachable Canonical Form

It is often convenient to change coordinates and write the dynamics of the system in the transformed coordinates

$$z = Tx \tag{6.29}$$

One application of a change of coordinates is to convert a system into a canonical form in which it is easy to perform certain types of analysis.

Remark 6.2.5. **Reachable Canonical Form**

A linear state space system is in reachable canonical form, if its dynamics are given by

$$\frac{dz}{dt} = Az + Bu \tag{6.30}$$

$$y = Cz + du \tag{6.31}$$

The characteristic polynomial for a system in reachable canonical form is given

$$\lambda(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n \tag{6.32}$$

The reachability matrix also has a relatively simple structure:

We now consider the problem of changing coordinates such that the dynamics of a system can be written in reachable canonical form. Let A, B represent the dynamics of a given system and $\tilde{A}, \tilde{B}$ be the dynamics in reachable canonical form. Suppose that we wish to transform the original system into reachable canonical form using a coordinate transformation $z = Tx$. As shown in the last chapter, the dynamics matrix and the control matrix for the transformed system are

$$\tilde{A} = TAT^{-1}, \quad \tilde{B} = TB \tag{6.33}$$

The reachability matrix for the transformed system then becomes

$$\tilde{W}_r = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix} \tag{6.34}$$

We can transforming each element individually and thus get the reachability matrix fro the transformed system

$$\tilde{W}_r = TW_r \tag{6.35}$$

However, we know that W_r is invertible. Therefore we can solve for the transformation T that takes the system into reachable canonical form:

$$T = \tilde{W}_r W_r^{-1} \tag{6.36}$$

Theorem 6.2.2. *Reachable canonical form*

Let A and B be the dynamics and control matrices for a reachable system. Then there exists a transformation

$$z = Tx$$

such that in the transformed coordinates the dynamics and control matrices are in the reachable canoinical form and the characteristic polynomial for A is given by

$$det(sI - A) = 0$$

One important implication of this theorem is that for any reachable system, we can assume without loss of generality that the coordinates are chosen such that the system is in reachable canonical form. This is particularly useful for proofs, as we shall see later in this chapter. However, for high-order systems, small changes in the coefficients a_i (that is the coefficients of the characteristic polynomial) can give large changes in the eigenvalues. Hence, the reachable canonical form is not always well conditioned and must be used with some care.

6.3 Stabilization by State Feedback

In this section, we will assume that the system to be controlled is described by a linear state model and has a single input (for simplicity). The feedback control law will be developed step by step using a single idea: the positioning of closed loop eigenvalues in desired locations.

Figure 6.3 has already been shown at the begining of this chapter. It shows a diagram of a typical control system using state feedback.

The full system consists of the process dynamics, which we take to be linear, the controller elements K and k_r , the reference input (or command signal) r and process disturbances d. The goal of the feedback controller is to regulate the output of the system y such that it tracks the reference input in the presence of disturbances and also uncertainty in the process dynamics.

An important element of the control design is the performance specification. The simplest performance specification is that of stability: in the absence of any disturbances, we would like the equilibrium point of the system to be asymptotically stable. More sophisticated performance specifications typically involve

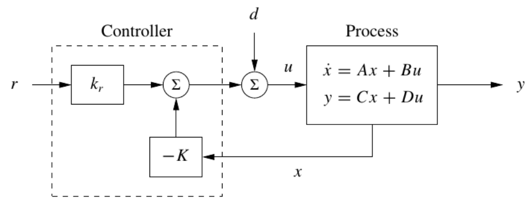

Figure 6.3: A feedback control system with state feedback. The controller uses the system state x and the reference input r to command the process through its input u. We model disturbances via the additive input d.

giving desired properties of the step or frequency response of the system, such as specifying the desired rise time, overshoot and settling time of the step response. Finally, we are often concerned with the disturbance attenuation properties of the system: to what extent can we experience disturbance inputs d and still hold the output y near the desired value?

Consider a system described by the linear differential equation

$$\frac{dx}{dt} = Ax + Bu \quad y = Cx + Du \tag{6.37}$$

6.4 Exercises

Exercise 1

Consider the system

$$\dot{x} = Ax + Bu \tag{6.38}$$
$$y = Cx \tag{6.39}$$

controlled by a state feedback controller:

$$u = -Kx \tag{6.40}$$

What are the sizes of the matrices A, B, C, and K if we have one input signal, three states and two output signals.

Answer

Since the state has three components then A must be 3×3. As we have only one input but the result of Bu must be added to the result of Ax then B must be 3×1. Since the output has two components then C must be 2×3. Finally, K must be 1×3.

Exercise 2

Consider again the system in the previous question. Assume now that but now the system has two input signals, three states and one output signal. What are the sizes of the involved matrices?

Answer

Using the same line of reasoning, since the state has three components then A must be 3×3. As we have only two input but the result of Bu must be added to the result of Ax then B must be 3×2. Since the output has two components then C must be 1×3. Finally, K must be 2×3.

Exercise 3

Consider again the system in the previous question. Assume now that but now the system has two input signals, four states and one output signal. What are the sizes of the involved matrices?

Answer

Using the same line of reasoning, since the state has three components then A must be 4×4. As we have only two input but the result of Bu must be added to the result of Ax then B must be 4×2. Since the output has two components then C must be 1×4. Finally, K must be 2×4.

Exercise 4

Consider the mechanical system shown in Figure 6.4

The system is described by the following state-space model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -c/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u \tag{6.41}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{6.42}$$

where x_1 is the position of the mass, x_2 is the velocity of the mass and u is the input signal, the force. The output y is the position. Assume that the desired closed loop characteristic polynomial is

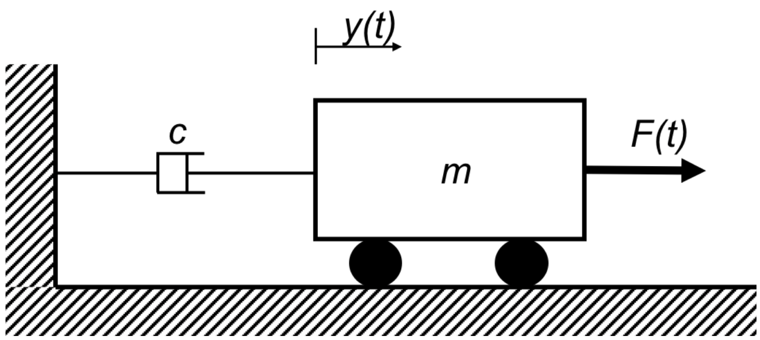

Figure 6.4: Schematic of mechanical system

$$p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 \tag{6.43}$$

and the state feedback controller is

$$u = -Kx \tag{6.44}$$

Determine the elements of $K = [k_{11}, k_{12}]$

Answer

By substituting the expression for u into the equation describing the state dynamics, we can write

$$\dot{x} = (A - BK)x \tag{6.45}$$

We can now calculate the characteristic polynomial via

$$det(sI - (A - BK)) = 0 \tag{6.46}$$

This gives us a second order polynomial

$$s^2 + ((c + k_{12})/m)s + k_{11}/m \tag{6.47}$$

by matching the coefficients with the desired closed loop characteristic polynomial, we get

$$k_{12} = 2\zeta m\omega_n - c, \quad k_{11} = m\omega_n^2 \tag{6.48}$$

Exercise 5

Consider the mechanical system shown in Figure 6.4 described by the same steady-state model. Assume that the feedback controller is

$$u = -Kx + K_r r \tag{6.49}$$

where r is the reference signal. We want to determine the steady state reference gain K_r such that the steady-state level is 1.

Answer

The reference gain K_r is given by

$$K_r = -(C(A - BK)^{-1}B)^{-1} \tag{6.50}$$

Thus, by substituting the values for k_{11} and k_{12} and performing the calculations, we arrive at

$$K_r = m\omega_n^2 \tag{6.51}$$

Exercise 6

Consider the closed loop system with the controller designed as in the previous exercise. Determine the transfer function for closed loop system, from the reference signal to the output. How many zeros does the system have?

Answer

The transfer function can be computed as

$$G_{ry}(s) = C(sI - A + BK)^{-1}BK_r$$

substituting the relevant quantities we arrive at

$$G_{ry}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

the number of zeros is 0 as the numerator is constant.

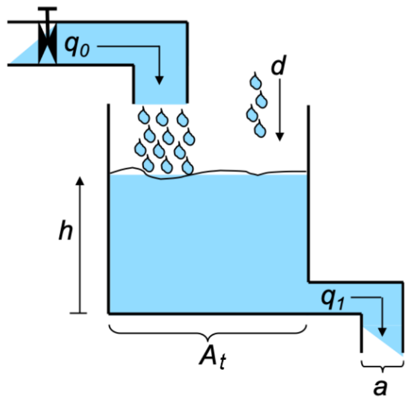

Figure 6.5: Watertank schematics.

Exercise 7

Consider the tank system as shown in figure 6.5. The watertank is modeled by

$$\begin{aligned} \dot{h} &= \frac{1}{A_t} q_0 + \frac{1}{A_t} d - \frac{\alpha}{A_t}\sqrt{2gh} \\ q_1 &= \alpha\sqrt{2gh} \end{aligned}$$

where is h the level in the tank, q_0 is the inflow, d is the disturbance and q_1 is the outflow. Linearize the tank system around the equilibrium point $(h_e, q_{0e}, d_e) = (h_0, q_0, 0)$. Determine the matrices A, B, C and H , and corresponding to the model:

$$\begin{aligned} \Delta\dot{h} &= A\Delta h + B\Delta q_0 + H\Delta d \\ \Delta q_1 &= C\Delta h \end{aligned}$$

Answer

Let's consider the following functions

$$\begin{aligned} f(h, q_0, d) &= \frac{1}{A_t} q_0 + \frac{1}{A_t} d - \frac{\alpha}{A_t}\sqrt{2gh} \\ l(h, q_0, d) &= \alpha\sqrt{2gh} \end{aligned}$$

The linearized system can be determined as:

$$\begin{aligned}\Delta\dot{h} &= \frac{\partial f}{\partial h}\Delta h + \frac{\partial f}{\partial q_0}\Delta q_0 + \frac{\partial f}{\partial d}\Delta d \\ \Delta q_1 &= \frac{\partial l}{\partial h}\Delta h\end{aligned}$$

The partial derivatives can be expressed as

$$\frac{\partial f}{\partial h} = -\frac{\alpha\sqrt{2g}}{2A_t\sqrt{h}}, \quad \frac{\partial f}{\partial q_0} = \frac{1}{A_t}, \quad \frac{\partial f}{\partial d} = \frac{1}{A_t}, \quad \frac{\partial l}{\partial h} = \frac{\alpha\sqrt{2g}}{2\sqrt{h}}$$

Inserting the equilibrium point gives:

$$\begin{aligned}\Delta\dot{h} &= -\frac{\alpha\sqrt{2g}}{2A_t\sqrt{h_0}}\Delta h + \frac{1}{A_t}\Delta q_0 + \frac{1}{A_t}\Delta d \\ \Delta q_1 &= \frac{\alpha\sqrt{2g}}{2\sqrt{h_0}}\Delta h\end{aligned}$$

Thus, the matrices are

$$A = -\frac{\alpha\sqrt{2g}}{2A_t\sqrt{h_0}}, \quad B = \frac{1}{A_t}, \quad H = \frac{1}{A_t}, \quad C = \frac{\alpha\sqrt{2g}}{2\sqrt{h_0}}$$

Exercise 8

A leaf spring system is shown in figure 6.6

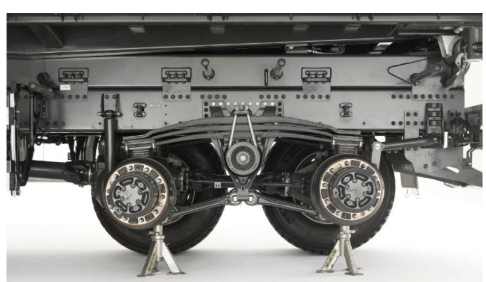

Figure 6.6: Leaf spring system.

The system can be described using the following nonlinear model:

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k_1 x_1}{m} - \frac{k_2 x_1^3}{m}\end{bmatrix}$$

Three equilibrium points was considered for x_{1e} namely 0 and $\pm\sqrt{-k_1/k_2}$. Linearize the model around the equilibrium point

$$(x_{1e}, x_{2e}) = (\sqrt{-k_1/k_2}, 0)$$

Determine system matrix A Express the matrix elements using the system parameter symbols k_1, k_2 and m.

Answer

We can use the functions f_1, f_2 to condense the right hand side.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k_1 x_1}{m} - \frac{k_2 x_1^3}{m} \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

The linearized system can be determined as:

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_1}{m} - \frac{3k_2 x_{1e}^2}{m} & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

Evaluation of the partial derivatives at the equilibrium point gives

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{2k_1}{m} & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

Thus,

$$A = \begin{bmatrix} 0 & 1 \\ \frac{2k_1}{m} & 0 \end{bmatrix}$$

Exercise 9

Consider the following longitudinal vehicle nonlinear model :

$$\frac{dx_1}{dt} = \frac{1}{\tau}x_1 + \frac{ki}{\tau r_w}u$$

$$\frac{dx_2}{dt} = \frac{1}{m}(x_1 - mgf\cos d_1 - \frac{1}{2}\rho C_D a_f x_2^2 - mgsind_1)$$

$$y = x_2$$

where x_1 is the force at the wheels, x_2 is the vehicle velocity, u is the input signal, the accelerator pedal position, and d_1 is the disturbance, the road slope.

This problem is about determining the equilibrium point $(x_{1e}, x_{2e}, u_e, d_{1e})$ that corresponds to the condition when vehicle is traveling with a velocity of $15m/s$ on a flat road.

Answer

The equilibrium point is determined by setting the derivatives equal to zero:

$$0 = -\frac{1}{\tau}x_1 + \frac{ki}{\tau r_w}u$$
$$0 = \frac{1}{m}\left(x_1 - mgf\cos d_1 - \frac{1}{2}\rho C_D A_f x_2^2 - mg\sin d_1\right)$$
$$y = x_2$$

From the specification of the exercise we get $(x_{2e} = 15)$ and $(d_{1e} = 0)$. Using the algebraic equations the other two equilibrium points can be determined as $x_{1e} = \frac{1}{2}\rho C_D A_f x_2^{2e} + mgf$ and $u_e = \frac{r_w}{ki}$, $x_{1e} = \frac{r_w \rho C_D A_f x_{2e}^2}{2ki}$. If we insert numerical values we get: $(x_{1e}, x_{2e}, u_e,\quad d_{1e}) = (3216, 15, 0.2010, 0)$.

Exercise 10

Determine the partial derivatives needed for linearizing the the system

$$\frac{dx_1}{dt} = \frac{1}{\tau}x_1 + \frac{ki}{\tau r_w}u = f_1(x_1, x_2, u, d_1)$$
$$\frac{dx_2}{dt} = \frac{1}{m}(x_1 - mgf\cos d_1 - \frac{1}{2}\rho C_D a_f x_2^2 - mgsind_1) = f_2(x_1, x_2, u, d_1)$$
$$y = x_2 = h(x_1, x_2, u, d_1)$$

around the equilibrium point $(x_{1e}, x_{2e}, u_e, d_{1e})$. The linearized system can be determined as

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} \Delta u + \begin{bmatrix} \frac{\partial f_1}{\partial d_1} \\ \frac{\partial f_2}{\partial d_2} \end{bmatrix} \Delta d_1$$
$$\Delta y = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} \frac{\partial h}{\partial u} \end{bmatrix} \Delta u + \begin{bmatrix} \frac{\partial h}{\partial d_1} \end{bmatrix} \Delta d_1 \tag{6.52}$$

Answer

We will have:

$$\frac{\partial f_1}{\partial x_1} = \frac{\partial}{\partial x_1}(-\frac{1}{\tau}x_1 + \frac{ki}{\tau r_w}u) = -\frac{1}{\tau}$$
$$\frac{\partial f_1}{\partial x_2} = \frac{\partial}{\partial x_2}(-\frac{1}{\tau}x_1 + \frac{ki}{\tau r_w}u) = 0$$
$$\frac{\partial f_1}{\partial u} = \frac{\partial}{\partial u}(-\frac{1}{\tau}x_1 + \frac{ki}{\tau r_w}u) = \frac{ki}{\tau r_w}$$
$$\frac{\partial f_1}{\partial d_1} = \frac{\partial}{\partial d_1}(-\frac{1}{\tau}x_1 + \frac{ki}{\tau r_w}u) = 0$$
$$\frac{\partial f_2}{\partial x_1} = \frac{\partial}{\partial x_1}\frac{1}{m}\left(x_1 - mgf\cos d_1 - \frac{1}{2}\rho C_D A_f x_2^2 - mg\sin d_1\right) = \frac{1}{m}$$
$$\frac{\partial f_2}{\partial x_2} = \frac{\partial}{\partial x_2}\frac{1}{m}\left(x_1 - mgf\cos d_1 - \frac{1}{2}\rho C_D A_f x_2^2 - mg\sin d_1\right) = -\frac{\rho C_D A_f x_2}{m}$$
$$\frac{\partial f_2}{\partial u} = \frac{\partial}{\partial u}\frac{1}{m}\left(x_1 - mgf\cos d_1 - \frac{1}{2}\rho C_D A_f x_2^2 - mg\sin d_1\right) = 0$$
$$\frac{\partial f_2}{\partial d_1} = \frac{\partial}{\partial d_1}\frac{1}{m}\left(x_1 - mgf\cos d_1 - \frac{1}{2}\rho C_D A_f x_2^2 - mg\sin d_1\right) = gf\sin d_{1e} - g\cos d_{1e}$$
$$\frac{\partial h}{\partial x_1} = \frac{\partial}{\partial x_1}x_2 = 0$$
$$\frac{\partial h}{\partial x_2} = \frac{\partial}{\partial x_2}x_2 = 1$$
$$\frac{\partial h}{\partial u} = \frac{\partial}{\partial u}x_2 = 0$$
$$\frac{\partial h}{\partial d_1} = \frac{\partial}{\partial d_1}x_2 = 0$$

Exercise 11

The linearized longitudinal vehicle dynamics model for a vehicle traveling at $20m/sec$ on a flat road is given by the following state-space model

The number of zeros is 0 as the numerator is constant.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1.25 & 0\\ 0.000005 & -0.0024\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}20000\\ 0\end{bmatrix}u + \begin{bmatrix}0\\ -9.82\end{bmatrix}d_1 \tag{6.53}$$
$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{6.54}$$

where x_1 is the force at the wheels, x_2 is the vehicle velocity, u is the input signal and d_1 is the disturbance of the road.

Design a state feedback cruise controller using pole placement on the form

$$u = -Kx + k_r r \tag{6.55}$$

such that the closed loop system's characteristic polynomial becomes

$$p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 \tag{6.56}$$

where $\omega_n = 0.6$ and $\zeta = 1/\sqrt{2}$. The reference gain k_r should be chosen such that the steady state gain is 1. The disturbance d_1 can be considered to be 0.

Chapter 7

Output Feedback

Chapter 6 introduced the concept of reachability. It was shown that it is possible to find a state feedback law that gives the desired closed loop eigenvalues provided that the system is reachable. Furthermore, we saw how to design controllers using the system state, $x(t)$, as feedback to out controller.

However, desigining state feedback controllers preassumes that all the states are measured. For many situations, it is highly unrealistic to assume that all the states are measured.

In this section we proceed somehow in a similar vein we can use the output $y(t)$ to modify the dynamics of the system through the use of observers. Furthermore, we will introduce the concept of observability and show that if a system is observable, it is possible to recover the state from measurements of the inputs and outputs to the system. We then show how to design a controller with feedback from the observer state.

7.1 Observability

For many situations, it is highly unrealistic to assume that all the states are measured. In this section we investigate how the state can be estimated by using a mathematical model and a few measurements. It will be shown that computation of the states can be carried out by a dynamical system called an **observer**, see also figure 7.1.

Definition 7.1.1. Observability A linear system is **observable** if for any $T > 0$ it is possible to determine the state of the system $x(T)$ through measurements of $y(t)$ and $u(t)$ on the interval $[0, T]$ $x(T) = x_f$.

Remark 7.1.1. **Nonlinear Systems**

The definition above holds for nonlinear systems as well, and the results discussed here have extensions to the nonlinear case.

Consider again the system

$$\frac{dx}{dt} = Ax + Bu \quad y = Cx + Du \tag{7.1}$$

where $x \in R^n$ is the state, $u \in R^p$ is the input and $y \in R^q$ the measured output.

We wish to estimate the state of the system from its inputs and outputs, as illustrated in Figure 7.1. In some situations we will assume that there is only one measured signal, i.e., that the signal y is a scalar and that C is a (row) vector. This signal may be corrupted by noise n, although we shall start by considering the noise-free case. We write $\hat{x}$ for the state estimate given by the observer.

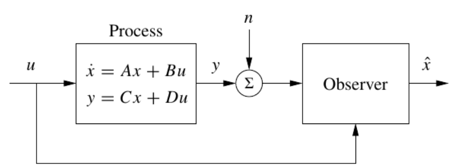

Figure 7.1: Block diagram for an observer. The observer uses the process measurement y (possibly corrupted by noise n) and the input u to estimate the current state of the process, denoted $\hat{x}$.

The problem of observability is one that has many important applications, even outside feedback systems. If a system is observable, then there are no hidden dynamics inside it; we can understand everything that is going on through observation (over time) of the inputs and outputs. As we shall see, the problem of observability is of significant practical interest because it will determine if a set of sensors is sufficient for controlling a system. Sensors combined with a mathematical model can also be viewed as a virtual sensor that gives information about variables that are not measured directly. The process of reconciling signals from many sensors with mathematical models is also called sensor fusion.

7.2 Testing for Observability

When discussing reachability in the last chapter, we neglected the output and focused on the state. Similarly, it is convenient here to initially neglect the input and focus on the autonomous system

Remark 7.2.1. **Autonomous System**

$$\frac{dx}{dt} = Ax \quad y = Cx \tag{7.2}$$

The objective is to understand when it is possible to determine the state from observations of the output. From

$$y = Cx \tag{7.3}$$

we see that the output itself gives us the projection of the state x on vectors that are rows of the matrix C. The observability problem can immediately be solved if the matrix C is invertible. If the matrix is not invertible we can take the derivatives and obtain

$$\frac{dy}{dt} = C\frac{dx}{dt} = CAx \tag{7.4}$$

From the derivative of the output we thus get the projection of the state on vectors that are rows of the matrix CA. Proceeding in this way, we get

$$\begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x \tag{7.5}$$

We thus find that the state can be determined if the observability matrix

$$W_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{7.6}$$

has n independent rows. It turns out that we need not consider any derivatives higher than $n-1$ (this is an application of the Cayley-Hamilton theorem)

Remark 7.2.2. **System with inputs**

The calculation can easily be extended to systems with inputs. The state is then given by a linear combination of inputs and outputs and their higher derivatives. The observability criterion is unchanged.

Theorem 7.2.1. ***Observability rank condition***

A linear system of the form

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx + Du$$

is observable if and only if the observability matrix W_o is full rank.

7.3 Observable Canonical Form

As in the case of reachability, certain canonical forms will be useful in studying ob- servability. A linear single-input, single-output state space system is in observable canonical form if its dynamics are given by

Remark 7.3.1. **Observable Canonical Form for Nonlinear Systems**

The definition can be extended to systems with many inputs; the only difference is that the vector multiplying u is replaced by a matrix.

The characteristic polynomial for a system in observable canonical form is

$$\lambda(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1} s + \alpha_n \tag{7.7}$$

In order to check the observability property of a systme more formally, we can compute the observability matrix for a system in observable canonical form. This is given by

$$W_o = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ -\alpha_1 & 1 & 0 & \ldots & 0 \\ -\alpha_1^2 - \alpha_1\alpha_2 & -\alpha_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \ldots & 1 \end{bmatrix} \tag{7.8}$$

where * represents an entry whose exact value is not important. What is important here is the rows of this matrix are linearly independent (since it is lower triangular), and hence Wo is full rank. Hence, it is invertible.

As in the case of reachability (see section 6.2.3), it turns out that if a system is observable then there always exists a transformation T that converts the system into observable canonical form. This is useful for proofs since it lets us assume that a system is in observable canonical form without any loss of generality. The observable canonical form may be poorly conditioned numerically.

7.4 State Estimation

State feedback control design, as explained in t chapter 6 requires that we have access to the complete state vector. However, measuring the complete state vector is not always feasible (for example a sensor may not be available at all) and it may also be expensive. In this section we will introduce the idea of state estimation as a way to get access to the state variables.

> *Remark* 7.4.1. **Soft Sensors**
>
> The concept of using software instead of sensors to access the quantity we are interested in, is referred to as soft sensors in the automotive industry.

Recall that the idea of state feedback control was to modify the eigenvalues of the system under consideration by using the input

$$u = -Kx + K_r r \tag{7.9}$$

However, it can be seen that this requires the state vector x. The idea of state estimation is to design something called an observer that tries to provide an estimate say $\hat{x}$ of the state vector x. The observer is fed with the same input as the real plant. The goal of the observer is to provide somehow a replica of the true state vector. Thus, we would like to have the error $\tilde{x}$ between the two quantities to be zero. Namely,

$$\tilde{x} = x - \hat{x} = 0 \tag{7.10}$$

In this section, we want to construct a dynamical system of the form

$$\frac{d\hat{x}}{dt} = F\hat{x} + Gu + Hy \tag{7.11}$$

where u and y are the input and output of the original system and $\hat{x} \in R^n$ is an estimate of x with the property

$$\hat{x}(t) \rightarrow x(t), \quad \text{as} \quad t \rightarrow \infty \tag{7.12}$$

Let's consider again the system

$$\frac{dx}{dt} = Ax + Bu \quad y = Cx + Du \tag{7.13}$$

assume further that D is zero. Assuming that the input u is known, then an estimate of the state x is given by

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu \tag{7.14}$$

We would like to know the properties of this estimate; how far is from the exact state? The estimation error is [1]

$$\tilde{x} = x - \hat{x} \tag{7.15}$$

substituting into equation 7.14 we find that

$$\frac{d\tilde{x}}{dt} = A\tilde{x} \tag{7.16}$$

We already know that the behavior of this system depends on the eigenvalues of A. Concretely, if matrix A has all its eigenvalues in the left half-plane, the error $\tilde{x}$ will go to zero, and hence equation 7.14 is a dynamical system whose output converges to the state of the system 7.13.

The observer given by equation 7.14 uses only the process input u; the measured signal does not appear in the equation. We must also require that the system be stable, and essentially our estimator converges because the state of both the observer and the estimator are going zero. This is not very useful in a control design context since we want to have our estimate converge quickly to a nonzero state so that we can make use of it in our controller. We will therefore attempt to modify the observer so that the output is used and its convergence properties can be designed to be fast relative to the systems dynamics. This version will also work for unstable systems.

Let's now consider the following observer

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{7.17}$$

The term $L(y - C\hat{x})$ provides feedback and hence the observer in 7.18 can be seen as a generalization of the observer in 7.14. The supplied feedback is proportional to the difference between the observed output and the output predicted by the observer. Substituting the expression for the error $\tilde{x}$ we arraive at

$$\frac{d\tilde{x}}{dt} = (A - LC)\tilde{x} \tag{7.18}$$

If the matrix L can be chosen in such a way that the matrix $A - LC$ has eigenvalues with negative real parts, the error $\tilde{x}$ will go to zero. The convergence rate is determined by an appropriate selection of the eigenvalues.

Remark 7.4.2. **Observer design and state feedback duality**

Notice the similarity between the problems of finding a state feedback and finding the observer. State feedback design by eigenvalue assignment is equivalent to finding a matrix K so that $A - BK$ has given eigenvalues. Designing an observer with prescribed eigenvalues is equivalent to finding a matrix L so that $A - LC$ has given eigenvalues. Since the eigenvalues of a matrix and its transpose are the same we can establish the following equivalences:

$$A \leftrightarrow A^T, \quad B \leftrightarrow C^T, \quad K \leftrightarrow L^T, \quad W_r \leftrightarrow W_o^T \tag{7.19}$$

The observer design problem is the dual of the state feedback design problem.

Theorem 7.4.1. ***Observer design by eigenvalue assignment***

Consider the system

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx$$

with one input and one output. Let the characteristic polynomial related to matrix A be

$$\lambda(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1} s + \alpha_n$$

If the systme is observable, then the dynamical system

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

is an observer for the system with L chosen as

$$L = W_o^{-1}\tilde{W}_o \begin{bmatrix} p_1 - \alpha_1 \\ p_2 - \alpha_2 \\ \vdots \\ p_n - \alpha_n \end{bmatrix}$$

The matrices W_o and $\tilde{W}_o$ are given by

The resulting observer error $\tilde{x}$ is governed by a differential equation that has the following characteristic polynomial

$$p(s) = s^n + p - 1s^{n-1} + \ldots + p_n$$

The dynamical system (7.10) is called an observer for (the states of) the system (7.9) because it will generate an approximation of the states of the system from its inputs and outputs. This form of an observer is a much more useful form than the one given by pure differentiation in equation (7.3).

7.5 Questions

Question 1

What is the main reason for using an estimator in feedback control?

- A) There are process disturbances in the model
- B) We are measuring the wrong quantities
- C) It is too expensive or impractical to measure each state variable correct

Answer

Option C is the correct answer.

Chapter 8

Kalman Filters

One of the principal uses of observers in practice is to estimate the state of a system in the presence of noisy measurements. We have not yet treated noise in our analysis, and a full treatment of stochastic dynamical systems is beyond the scope of this text. In this section, we present a brief introduction to the use of stochastic systems analysis for constructing observers. We work primarily in discrete time to avoid some of the complications associated with continuous-time random processes and to keep the mathematical prerequisites to a minimum. This section assumes basic knowledge of random variables and stochastic processes; see Kumar and Varaiya [KV86] or strm [st06] for the required material.

Consider again the LTI state-space model

$$\frac{dx}{dt} = Ax + Bu + v \quad y = Cx + Du + w \tag{8.1}$$

the model is augmented with additional terms representing the error or disturbance. Concretely, v is the process disturbance and w is measurements noise. Both are assumed to be normally distibuted with zero mean;

$$E[v] = 0, \quad E[vv^T] = R_v, \quad E[w] = 0, \quad E[ww^T] = R_w \tag{8.2}$$

R_v and R_w are the covariance matrices for the process disturbance v and the measurement noise w respectively. Furthermore, we assume that the variables v, w are not correlated i.e

$$E[vw^T] = 0 \tag{8.3}$$

The initial condition is also modeled as a Gaussian random variable

$$E[x(0)] = x_0, \quad E[x(0)x^T(0)] = P_0 \tag{8.4}$$

Implementation of the state-space model in a computer requires discretization. Thus the system can be written as discrete-time linear system with dynamics governed by

$$x_{t+1} = Ax_t + Bu_t + Fv_t, \quad y_t = Cx_t + w_t \tag{8.5}$$

Given the measurements $\{y(\tau), 0 \leq \tau \leq t\}$, we would like to find an estimate $\hat{x}_t$ that minimizes the mean square error:

$$E[(x_t - \hat{x}_t)(x_t - \hat{x}_t)^T] \tag{8.6}$$

Theorem 8.0.1. ***Kalman 1961***

Consider the random process x_t with dynamics described by

$$x_{t+1} = Ax_t + Bu_t + Fv_t, \quad y_t = Cx_t + w_t$$

and noise processes and initial conditions described by 8.2, 8.3 and 8.4. The observer gain L that minimizes the mean square error is given by

$$L_t = AP_tC^T(R_w + CP_tC^T)^{-1}$$

where

$$P_{t+1} = (ALC)P_t(ALC)^T + FR_vF^T + LR_wL^T, \quad P_0 = E[x_0x_0^T\} \tag{8.7}$$

A proof of this result can be found in [1]. We, note, however the following points:

- the Kalman filter has the form of a recursive filter: given mean square error P_t at time t, we can compute how the estimate and error change. Thus we do not need to keep track of old values of the output.
- Furthermore, the Kalman filter gives the estimate $\hat{x}_t$ and the error covariance P_t, so we can see how reliable the estimate is. It can also be shown that the Kalman filter extracts the maximum possible information about output data. If we form the residual between the measured output and the estimated output,

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