Discretization of continuous time systems

Sensor fusion & nonlinear filtering

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DISCRETIZATION OVERVIEW

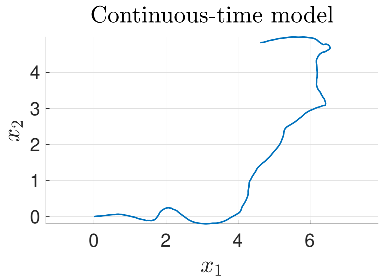
- Given a continuous-time motion model

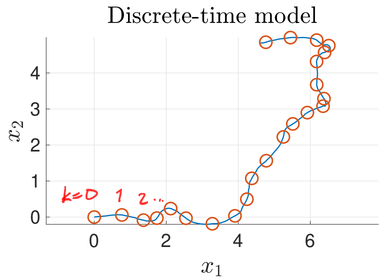
$$\dot{\mathbf{x}}(t) = \tilde{a}(\mathbf{x}(t)) + \tilde{\mathbf{q}}(t)$$

we would like to find a discrete-time motion model

$$\mathbf{x}_k = a(\mathbf{x}_{k-1}) + \mathbf{q}_{k-1}$$

where the discrete sequence is sampled from the continuous one, $\mathbf{x}_k = \mathbf{x}(kT)$.





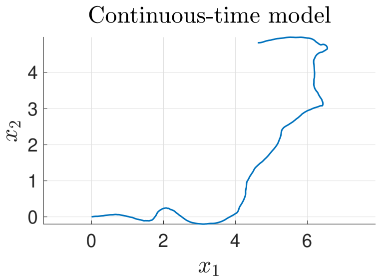
DISCRETIZATION OVERVIEW

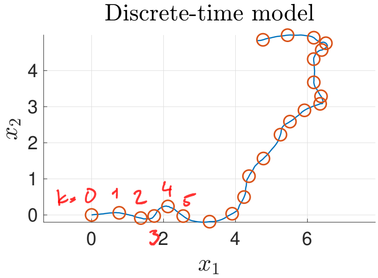
$p(x_k | x_{k-1})$ x_k x_{k-1}

Current objective

- Express distribution of $\mathbf{x}(t+T)$ given $\mathbf{x}(t)$ for different continuous time motion models.

- Two important tools:
  1. The Euler discretization method.
  2. Analytical solution of linear systems.





THE EULER METHOD

- Suppose we have a differential equation (DE)

$$\dot{\mathbf{x}}(t) = \tilde{a}(\mathbf{x}(t)) + \tilde{\mathbf{q}}(t) \qquad (1)$$

Euler method

- A simple method to approximate the solution to a DE by using

$$\dot{\mathbf{x}}(t) \approx \frac{\mathbf{x}(t+T) - \mathbf{x}(t)}{T} \qquad (2)$$

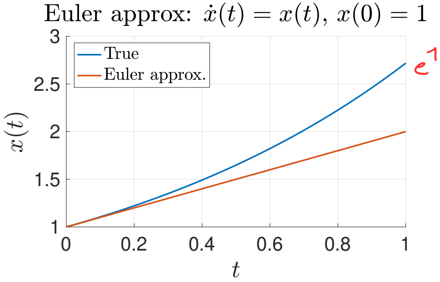
to find $\mathbf{x}(t+T)$.

Solution

$x(t+T) \approx x(t) + T\dot{x}(t)$

$= x(t) + T\tilde{a}(x(t)) + T\tilde{q}(t)$

$\Rightarrow x(1) = x(0) + 1\cdot 1 = 1 + 1 = 2$



ANALYTICAL SOLUTION OF LINEAR SYSTEMS

- Suppose we have a linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b} \tag{3}$$

  where $\mathbf{A}$ is a constant matrix and $\mathbf{b}$ is a constant vector.

- Solving for $\mathbf{x}(t + T)$:

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  1. (3) $\Leftrightarrow$ $\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{b}$

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Note: $\frac{d}{dt}e^{-\mathbf{A}t}\mathbf{x}(t) = e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t)$

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2. Multiply both sides with $e^{-\mathbf{A}t}$: $\underbrace{e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t)}_{\frac{d}{dt}e^{-At}x(t)} = e^{-\mathbf{A}t}\mathbf{b}$

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  3. Integrate both sides from t to $t+T$:

     $$\Rightarrow e^{-\mathbf{A}(t+T)}\mathbf{x}(t+T) - e^{-\mathbf{A}t}\mathbf{x}(t) = \int_t^{t+T} e^{-\mathbf{A}\tau}\mathbf{b}\, d\tau$$

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- Solving for $\mathbf{x}(t+T)$:

  1. (3) $\Leftrightarrow$ $\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{b}$ Note: $\frac{d}{dt}e^{-\mathbf{A}t}\mathbf{x}(t) = e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t)$
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  4. Multiply with $e^{\mathbf{A}(t+T)}$:

     $$\Rightarrow \mathbf{x}(t+T) = e^{\mathbf{A}T}\mathbf{x}(t) + \int_0^T e^{\mathbf{A}\tau}\, d\tau\, \mathbf{b}$$

ANALYTICAL SOLUTION OF LINEAR SYSTEMS

- For the above linear system we found that

$$\mathbf{x}(t+T) = \exp(\mathbf{A}T)\mathbf{x}(t) + \int_0^T \exp(\mathbf{A}\tau)\, d\tau \mathbf{b},$$

  but what is $\int_0^T \exp(\mathbf{A}\tau)\, d\tau$?

- By definition $\exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2} + \frac{\mathbf{A}^3 t^3}{3!} + \dots$

$$\Rightarrow \int_0^T \exp(\mathbf{A}\tau)\, d\tau = \mathbf{I}T + \frac{\mathbf{A}T^2}{2} + \frac{\mathbf{A}^2 T^3}{3!} + \dots$$

An analytical solution for linear systems

- If $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}$ then

$$\mathbf{x}(t+T) = \exp(\mathbf{A}T)\mathbf{x}(t) + \left(\mathbf{I}T + \frac{\mathbf{A}T^2}{2} + \frac{\mathbf{A}^2 T^3}{3!} + \dots\right)\mathbf{b}$$

SELF ASSESSMENT

Both the above methods have strengths and weaknesses. Some of these are correctly described below.

Check all statements that apply.

- The Euler method is easy to apply also for nonlinear systems.
- The analytical solution is an exact solution also when T is large.
- The Euler method is more accurate when T is small than when T is large.