Kalman Filters

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1 Kalman Filters

Kalman filtering is an iterative process that uses a set of equations and consecutive data inputs to quickly estimat the true value; e.g. position, velocity etc., of the object being measured when the measured values contain unpredicted or random error, uncertainty or variation.

1.1 Kalaman Gain

The Kalman gain $K$is used to determine how much of the new measurements to use in order to update the new estimate. In the caluclation of the Kalman gain two quantities participate:

- Error in estimate
- Error in data measurement

Thus K is given by

$$K = \frac{E_{est}}{E_{est} + E_{meas}} \tag{1}$$

From equation 1 it follows that

$$0 \leq K \leq 1 \tag{2}$$

The Kalman gain is the used used to update the current estimate $\hat{x}_t$:

$$\hat{x}_t = \hat{x}_{t-1} + K(z - \hat{x}_{t-1}) \tag{3}$$

When $K \approx 1$ or is equal to one, the measurements we are getting a very accurate however the estimates are unstable. On the other hand when K is small, the measurements we are getting a inaccurate but the estimates are stable since the error is small. The error $E_{\hat{x}}$ in the estimate is given by

$$E_{\hat{x}_t} = (1 - K)E_{\hat{x}_{t-1}} \tag{4}$$

1.2 Calculations of Kalaman Filter

The Kalman filter iterates over the following three calculations:

- Calculate K using equation 1
- Calculate the new estimate using equation 3
- Caluclate the error in the estimate using equation 4

1.2.1 Example: Temperature Estimate

Remark 1.1. **Nonlinear Systems**

The definition above holds for nonlinear systems as well, and the results discussed here have extensions to the nonlinear case.

One of the principal uses of observers in practice is to estimate the state of a system in the presence of noisy measurements. We have not yet treated noise in our analysis, and a full treatment of stochastic dynamical systems is beyond the scope of this text. In this section, we present a brief introduction to the use of stochastic systems analysis for constructing observers. We work primarily in discrete time to avoid some of the complications associated with continuous-time random processes and to keep the mathematical prerequisites to a minimum. This section assumes basic knowledge of random variables and stochastic processes; see Kumar and Varaiya [KV86] or strm [st06] for the required material.

Consider again the LTI state-space model

$$\frac{dx}{dt} = Ax + Bu + v \quad y = Cx + Du + w \tag{5}$$

the model is augmented with additional terms representing the error or disturbance. Concretely, v is the process disturbance and w is measurements noise. Both are assumed to be normally distibuted with zero mean;

$$E[v] = 0, \quad E[vv^T] = R_v, \quad E[w] = 0, \quad E[ww^T] = R_w \tag{6}$$

Remark 1.2. **Normally distributed random variable**

A one dimensional random variable X is said to follow the normal distribution

Remark 1.3. **Nonlinear Systems**

The definition above holds for nonlinear systems as well, and the results discussed here have extensions to the nonlinear case.

R_v and R_w are the covariance matrices for the process disturbance v and the measurement noise w respectively. Furthermore, we assume that the variables v, w are not correlated i.e

$$E[vw^T] = 0 \tag{7}$$

Remark 1.4. **Corralated random variables**

Two random variables X and Y are said to be linearly correlated

The initial condition is also modeled as a Gaussian random variable

$$E[x(0)] = x_0, \quad E[x(0)x^T(0)] = P_0 \tag{8}$$

Implementation of the state-space model in a computer requires discretization. Thus the system can be written as discrete-time linear system with dynamics governed by

$$x_{t+1} = Ax_t + Bu_t + Fv_t, \quad y_t = Cx_t + w_t \tag{9}$$

Given the measurements $\{y(\tau), 0 \leq \tau \leq t\}$, we would like to find an estimate $\hat{x}_t$ that minimizes the mean square error:

$$E[(x_t - \hat{x}_t)(x_t - \hat{x}_t)^T] \tag{10}$$

Theorem 1.1. *Kalman 1961*

Consider the random process x_t with dynamics described by

$$x_{t+1} = Ax_t + Bu_t + Fv_t, \quad y_t = Cx_t + w_t$$

and noise processes and initial conditions described by 6, 7 and 8. The observer gain L that minimizes the mean square error is given by

$$L_t = AP_tC^T(R_w + CP_tC^T)^{-1}$$

where

$$P_{t+1} = (ALC)P_t(ALC)^T + R_vLR_wL^T, \quad P_0 = E[x_0x_0^T\} \tag{11}$$

A proof of this result can be found in [1]. We, note, however the following points:

- the Kalman filter has the form of a recursive filter: given mean square error P_t at time t, we can compute how the estimate and error change. Thus we do not need to keep track of old values of the output.

- Furthermore, the Kalman filter gives the estimate $\hat{x}_t$ and the error covariance P_t, so we can see how reliable the estimate is. It can also be shown that the Kalman filter extracts the maximum possible information about output data. If we form the residual between the measured output and the estimated output,

$$e_t = y_t - C\hat{x}_t \tag{12}$$

we can show that for the Kalman filter the error covariance matrix R_e is

$$R_e(i,j) = E(e_j e_k^T) = W_t \delta_{jk} \tag{13}$$

In other words, the error is a white noise process, so there is no remaining dynamic information content in the error.

Theorem 1.2. ***Kalman-Bucy 1961***

The optimal estimator has the form of a linear observer

$$\frac{d\hat{x}}{t} = A\hat{x} + Bu + L(y - C\hat{x}), \quad \hat{x}(0) = E(x(0))$$

where L is given by

$$L = PC^T R_w^{-1}$$

where P

All matrices A, B, C, R_v, R_w, P and L can be time varying. The essential condition is that the Riccati equation (8.30) has a unique positive solution.

2 Kalmans Decomposition of a Linear System

In this chapter and the previous one we have seen that two fundamental properties of a linear input/output system are reachability and observability. It turns out that these two properties can be used to classify the dynamics of a system. The key result is Kalmans decomposition theorem, which says that a linear system can be divided into four subsystems:

- Σ_{ro} which is reachable and observable
- $\Sigma_{r\bar{o}}$ which is reachable but no observable
- $\Sigma_{\bar{r}o}$ which is not reachable but is observable
- $\Sigma_{\bar{r}\bar{o}}$which is neither reachable nor observable

Thus from the input/output point of view, it is only the reachable and observable dynamics that matter.

Remark 2.1. **Kalman's decomposition for state-space**

The general case of the Kalman decomposition is more complica ted and requires some additional linear algebra; see the original pap er by Kalman, Ho, and Narendra [KHN63]. The key result is that the state space can still be decomposed into four parts, but there will be additional coupling so that the equations have the form

References

[1] Åström K. J., Murray R. M. *Feedback Systems. An Introduction for Scientists and Engineers*

[2] Philip , Florent Altche1, Brigitte dAndrea-Novel, and Arnaud de La Fortelle *The Kinematic Bicycle Model: a Consistent Model for Planning Feasible Trajectories for Autonomous Vehicles?* HAL Id: hal-01520869, https://hal-polytechnique.archives-ouvertes.fr/hal-01520869

[3] Marcos R. O., A. Maximo *Model Predictive Controller for Trajectory Tracking by Differential Drive Robot with Actuation constraints*