Appendices

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Contents

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A Coordinate Transformations

In order to track how a rigid body such as a seld-driving car moves, we need to know how we can express its motion using mathematical tools and notation. In this section, we will review how reference frames affect vector coordinates, compare and contrast different rotation representations, and present several reference frames

By definition a vector is a geometric object that has a magnitude and a direction. Often, we treat the concept of a vector interchangeably with the concept of vector coordinates, or the set of numbers that represent the direction and maginitude of the vector. This, however, is not necessarily correct. If we imagine that the vector is fixed in space, then its coordinates will change depending on the way in which we observe it. More precisely, the same vector quantity will have different coordinates depending on which coordinate frame or reference frame we choose to express it in.

Remark A.1. **Notation**

Let us introduce some notation that will be useful susequently. In frame a, the vector $\mathbf{r}$ has notation $\mathbf{r}_a$. Likewise in frame b, it has the coordinates $\mathbf{r}_b$. The coordinates of the vector in the two different frames are related through a rotation matrix $\mathbf{C}_{ba}$

$$\mathbf{r}_b = \mathbf{C}_{ba}\mathbf{r}_a \tag{1}$$

The rotation matrix tells us exactly how one frame is rotated with respect to the other.

Often, it will also be useful to know how the coordinates of points change as we move from one reference frame to another. For example, we may know the position of a building in some frame, and now we would like to know its position in our current vehicle frame. To compute this, we use vector addition making sure to express all of the coordinates in the same reference frame. We will use superscripts on the coordinates to indicate the start and end point of a 3D vector, again from right to left, and a subscript to indicate the frame in which this is expressed just as before. We can manipulate this expression to solve for the coordinates in the vehicle frame or an appropriate inertial frame, for example, see Figure 1.

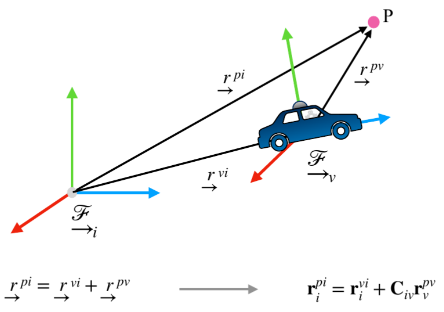

P
$\underrightarrow{r}^{pi}$
$\underrightarrow{r}^{pv}$
$\underrightarrow{r}^{vi}$
$\underrightarrow{\mathscr{F}}_v$
$\underrightarrow{\mathscr{F}}_i$
$\underrightarrow{r}^{pi} = \underrightarrow{r}^{vi} + \underrightarrow{r}^{pv}$
$\mathbf{r}_i^{pi} = \mathbf{r}_i^{vi} + \mathbf{C}_{iv}\mathbf{r}_v^{pv}$

Fig. 1: Schematic of coordinate frames.

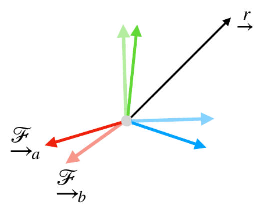

$\underrightarrow{r}$
$\underrightarrow{\mathscr{F}}_a$
$\underrightarrow{\mathscr{F}}_b$

Fig. 2: Schematic of coordinate frames.

A.1 Direction cosine matrix

A critical component of tracking reference frames is tracking their orientation or rotation with respect to some base reference frame. Rotations are particularly tricky mathematical objects and they can be the source of major bugs if not dealt with carefully and diligently. There are many different ways to represent rotations. The most common is to use a three by three rotation matrix as we've done before. This matrix defines the relationship between the basis vectors of two reference frames in terms of dot products. For this reason, it's often called the **direction cosine matrix**.

Remark A.2. **Inverse of $\mathbf{C}_{ba}$**

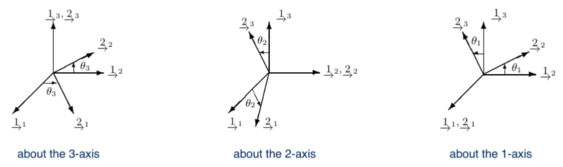
An important property to remember is that the inverse of a rotation matrix is just its transpose.

A.2 Euler angles

Another way of representing a rotation is using three numbers called Euler angles. These angles represent an arbitrary rotation as the composition of three separate rotations about different principal axes as

$$\mathbf{C}(\theta_3, \theta_2, \theta_1) = \mathbf{C}_3(\theta_3)\mathbf{C}_3(\theta_2)\mathbf{C}_1(\theta_1) \tag{2}$$

Figure 3 shows schematically the individual rotations.



$$\mathbf{C}_3 = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{C}_2 = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \quad \mathbf{C}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

Fig. 3: Euler angles.

Euler angles are attractive in part because they are a parsimonious representation requiring only three parameters instead of nine for a full rotation matrix. Unfortunately, Euler angle representations are subject to what are called singularities. Singularities complicate state estimation because they represent particular rotations from which to Euler angles are indistinguishable. Neither quaternions, see section A.3 nor rotation matrices, section A.1, suffer from this problem at the expense of using more parameters.

A.3 Unit quaternion

A second way to represent rotations is to use something called unit quaternions. Quaternions are an interesting mathematical topic in their own right. However, it is sufficient to note that a unit quaternion can be represented as a four-dimensional vector of unit length that parameterizes a rotation about an axis defined by the vector $\mathbf{u}$, and an angle ϕ about that vector, see Figure 4.

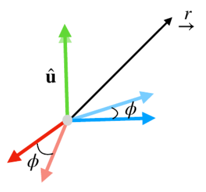


Fig. 4: The vector $\mathbf{u}$, and the angle ϕ.

Thus, we have the follwing equations

$$\mathbf{q} = \begin{bmatrix} q_w \\ \mathbf{q}_\nu \end{bmatrix} = \begin{bmatrix} \cos(\frac{\phi}{2}) \\ \hat{\mathbf{u}}\sin(\frac{\phi}{2}) \end{bmatrix}, \quad ||\mathbf{q}|| = 1 \tag{3}$$

We can convert a quaternion to a rotation matrix by using the followin algebraic expression.

$$\mathbf{r}_b = \mathbf{C}(\mathbf{q}_{ba})\mathbf{r}_a \tag{4}$$

where

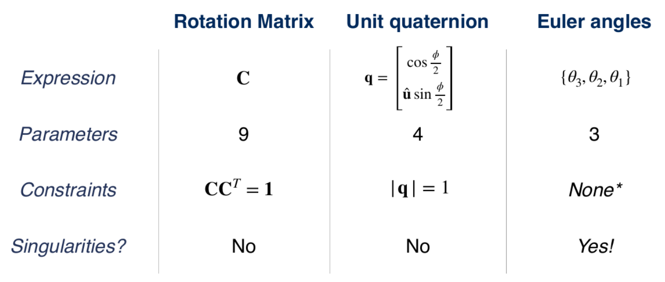
$$\mathbf{C}(\mathbf{q}) = (q_w^2 - \mathbf{q}_\nu^T\mathbf{q}_\nu)\mathbf{I} + 2\mathbf{q}_\nu\mathbf{q}_\nu^T + 2q_w[\mathbf{q}_\nu]_\times \tag{5}$$

with

$$[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \tag{6}$$

Quaternions have the advantage that they do not suffer from singularities and they only need four parameters instead of nine in order to represent 3D rotations.

Figure 5 summarizes the three approaches



	Rotation Matrix	Unit quaternion	Euler angles
Expression	$\mathbf{C}$	$\mathbf{q} = \begin{bmatrix} \cos\frac{\phi}{2} \\ \hat{\mathbf{u}}\sin\frac{\phi}{2} \end{bmatrix}$	$\{\theta_3, \theta_2, \theta_1\}$
Parameters	9	4	3
Constraints	$\mathbf{C}\mathbf{C}^T = \mathbf{1}$	$\lvert\mathbf{q}\rvert = 1$	*None**
Singularities?	No	No	*Yes!*

Fig. 5: Comparison of rotation representation approaches.

B Reference Frames

In this section, we review the concepts of coordinate frame transformations. We will start with 2D transformations and then see more on 3D transformations.

The coordinate frames we use are right-handed frames by convention, and we'll use a few standard coordinate frames. The global world or inertial coordinate frame, is a fixed reference frame attached to the earth. Often, we'll represent this coordinate frame as East North Up, ENU, relative to a reference point nearby. Or Earth-Centered Earth Fixed, ECEF, as is used in GNSS systems.

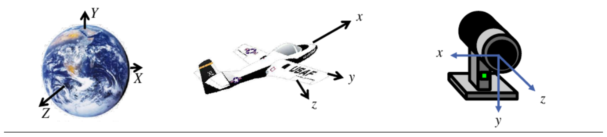


Fig. 6: Schematic of coordinate frames.

The next frame we'll talk about is the body frame, which is placed in some key location on the body of the vehicle. For example, the center of gravity (CoG) of a vehicle or the center point of the rear axle. This frame is moving and rotating with respect to the fixed inertial frame as the vehicle moves about.

Finally, the sensor frame is a coordinate frame which attaches to each sensor describing the coordinates used for the sensor output. In many robotics applications, we need to attach several coordinates to a moving system and also represent elements from these frames in the inertial frame. To do this, we need to transform variables from one coordinate frame to the other. For example, from the body frame to the inertial. Even a simple two wheeled robot with a single sensor has three such frames to consider, while the self-driving car can have dozens.

To maintain a consistent representation of sensor data for perception, we need to be able to transform information between coordinate frames. Let's take a look at a simple transformation example for a velocity vector of a car.

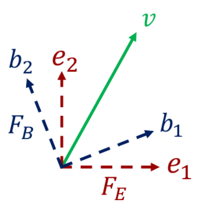


Fig. 7: Schematic of vector rotation.

Generally, kinematic variables such as velocity are represented in the form of a vector, with both magnitude and direction. In figure 7, the vector v is presented with a green arrow in a two-dimensional coordinate frame. We have two coordinate frames displayed here. The body frame, defined by axes b_1 and b_2, and the inertial frame defined by axes e_1 and e_2, both in the 2D plane.

Let's assume the two coordinate frames; frame e and frame b, have the same fixed origin. But frame b is rotated by some angle θ relative to frame e. We can then define the rotational matrices C_{EB}, which transforms vectors from the frame b to the frame e and C_{BE} which projects the frame e onto frame b using the angle θ as shown.

$$C_{EB} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}, \quad C_{BE} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{7}$$

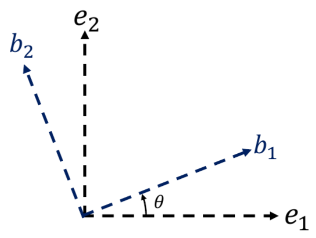


Fig. 8: Rotation angle θ.

Now, let's extend our example to include a translation. Here, we see a two-wheeled robot and we'd like to represent the position of a point P observed by the robot in the robot body frame b, with respect to the inertial frame e. The position of the robot with respect to the inertial frame is (x, y), and the orientation of the robot once again is θ.

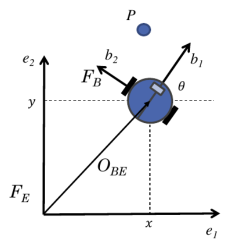


Fig. 9: Rotation angle θ.

The following equations relate the location of point P in the body coordinates P_B and the inertial frame P_E. Note that in general to transform one point from one coordinate to the other coordinate frame, body to inertial and vice versa, requires two terms. The translation of the origin O_{BE} and O_{EB} in this case, and the rotation of the axis C_{EB} and C_{BE}. Finally, we can summarize the transformation between two coordinate frames using homogeneous coordinates, which lead to a compact matrix multiplication to apply the transformation.

$$P_B = C_{EB}(\theta)P_E + O_{EB} \tag{8}$$

where O_{EB} is the translation term expressed in body frame. Similarly,

$$P_E = C_{BE}(\theta)P_B + O_{BE} \tag{9}$$

We extend our location vector to include (x, y), and one and can then transform from body to inertial coordinates using P inertial is C_{EB} and O_{EB} times P in the body frame.

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