Nonlinear filtering

Sensor fusion & nonlinear filtering

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GENERAL FILTERING EQUATIONS

State space models

- We mainly consider models of the type

$$
\begin{aligned}
\mathbf{x}_k &= \mathbf{f}_{k-1}(\mathbf{x}_{k-1}) + \mathbf{q}_{k-1}, &\qquad \mathbf{q}_{k-1} &\sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{k-1}) \\
\mathbf{y}_k &= \mathbf{h}_k(\mathbf{x}_k) + \mathbf{r}_k, &\qquad \mathbf{r}_k &\sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k).
\end{aligned}
$$

General/conceptual filtering solution

- Solve the filtering problem using:

Prediction: $$p(\mathbf{x}_k \mid \mathbf{y}_{1:k-1}) = \int p(\mathbf{x}_k \mid \mathbf{x}_{k-1})\, p(\mathbf{x}_{k-1} \mid \mathbf{y}_{1:k-1})\, d\mathbf{x}_{k-1}$$

Update: $$p(\mathbf{x}_k \mid \mathbf{y}_{1:k}) = \frac{p(\mathbf{y}_k \mid \mathbf{x}_k)\, p(\mathbf{x}_k \mid \mathbf{y}_{1:k-1})}{\int p(\mathbf{y}_k \mid \mathbf{x}_k)\, p(\mathbf{x}_k \mid \mathbf{y}_{1:k-1})\, d\mathbf{x}_k}$$

LINEAR AND GAUSSIAN MODELS

- An important special case is linear and Gaussian models:

$$\begin{aligned} \mathbf{x}_k &= \mathbf{A}_{k-1}\mathbf{x}_{k-1} + \mathbf{q}_{k-1}, \qquad & \mathbf{q}_{k-1} &\sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{k-1}) \\ \mathbf{y}_k &= \mathbf{H}_k\mathbf{x}_k + \mathbf{r}_k, & \mathbf{r}_k &\sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k) : \end{aligned}$$

- The Kalman filter then provides an analytical solution:

$$\text{Prediction:} \begin{cases} \hat{\mathbf{x}}_{k|k-1} = \mathbf{A}_{k-1}\hat{\mathbf{x}}_{k-1|k-1} \\ \mathbf{P}_{k|k-1} = \mathbf{A}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{A}_{k-1}^T + \mathbf{Q}_{k-1} \end{cases}$$

$$\text{Update:} \begin{cases} \hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_{k|k-1}) \\ \mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}\mathbf{S}_k\mathbf{K}_k^T \\ \vdots \end{cases}$$

NONLINEAR MODELS AND FILTERING

- Many important motion and measurement models are nonlinear.

Nonlinear models

- The coordinated turn motion model:

$$\begin{bmatrix} x_k \\ y_k \\ v_k \\ \phi_k \\ \omega_k \end{bmatrix} = \underbrace{\begin{bmatrix} x_{k-1} + \frac{2v_{k-1}}{\omega_{k-1}} \sin\left(\frac{\omega_{k-1}T}{2}\right) \cos\left(\phi_{k-1} + \frac{\omega_{k-1}T}{2}\right) \\ y_{k-1} + \frac{2v_{k-1}}{\omega_{k-1}} \sin\left(\frac{\omega_{k-1}T}{2}\right) \sin\left(\phi_{k-1} + \frac{\omega_{k-1}T}{2}\right) \\ v_{k-1} \\ \phi_{k-1} + T\omega_{k-1} \\ \omega_{k-1} \end{bmatrix}}_{f_{k-1}(x_{k-1})} + \mathbf{q}_{k-1},$$

- A sensor that measures distance:

$$y_k = \underbrace{\sqrt{x_k^2 + y_k^2}}_{h_k(x_k)} + r_k.$$

NONLINEAR FILTERING METHODS

- For nonlinear models, we have generally no analytical solution to the filtering equations.
- Common approach: Find tractable approximative solutions.
- In this course we will look at two types of nonlinear filters:
    - Gaussian filters (EKF, UKF, CKF, ...)
    - Particle filters

SELF ASSESSMENT

Why is nonlinear filtering hard? (Check all that are correct)

- The optimal filtering equations do not apply in the nonlinear setting
- We can not in general find an analytical solution to our filtering equations.
- For linear systems, it is enough to describe the posterior using the first two moments, the mean and the covariance. For nonlinear systems, we in general need infinitely many moments to describe the true posterior.
- The Markov property does not hold in the non-linear setting.