Nonlinear motion models

Sensor fusion & nonlinear filtering

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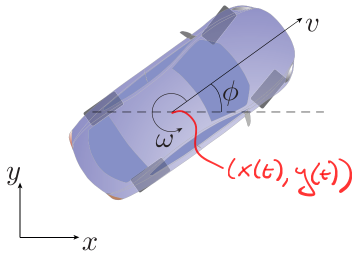
A COORDINATED TURN MODEL

Continuous-time coordinated turn model

- Assumptions:
  - Heading $\phi(t)$ is described by a CV model.
  - Velocity is a Wiener process.
- State vector and motion model are summarized as:

$$\underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{v}(t) \\ \dot{\phi}(t) \\ \dot{\omega}(t) \end{bmatrix}}_{\dot{\mathbf{x}}(t)} = \underbrace{\begin{bmatrix} v(t)\cos(\phi(t)) \\ v(t)\sin(\phi(t)) \\ 0 \\ \omega(t) \\ 0 \end{bmatrix}}_{\tilde{\mathbf{a}}(\mathbf{x}(t))} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{\Gamma}} \underbrace{\begin{bmatrix} q_c^v(t) \\ q_c^\omega(t) \end{bmatrix}}_{\mathbf{q}_c(t)}$$



THE EULER METHOD

- According to the (modified) Euler method:

$$\mathbf{x}(t+T) \approx \mathbf{x}(t) + T\tilde{\mathbf{a}}(\mathbf{x}(t)) + \int_t^{t+T} \tilde{\mathbf{q}}(\tau)\, d\tau.$$

- From this we obtain the discrete time motion model

$$\mathbf{x}_k = \mathbf{x}_{k-1} + T\tilde{\mathbf{a}}(\mathbf{x}_{k-1}) + \mathbf{q}_{k-1} \Leftrightarrow \begin{bmatrix} x_k \\ y_k \\ v_k \\ \phi_k \\ \omega_k \end{bmatrix} = \begin{bmatrix} x_{k-1} + T v_{k-1} \cos(\phi_{k-1}) \\ y_{k-1} + T v_{k-1} \sin(\phi_{k-1}) \\ v_{k-1} \\ \phi_{k-1} + T\omega_{k-1} \\ \omega_{k-1} \end{bmatrix} + \mathbf{q}_{k-1},$$

where $\mathbf{q}_{k-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{k-1})$ and

$$\mathbf{Q}_{k-1} = T\tilde{\mathbf{Q}} = T\boldsymbol{\Gamma} \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_\omega^2 \end{bmatrix} \boldsymbol{\Gamma}^T = \operatorname{diag}\left(0, 0, T\sigma_v^2, 0, T\sigma_\omega^2\right)$$

EXACT SOLUTION TO DETERMINISTIC CT

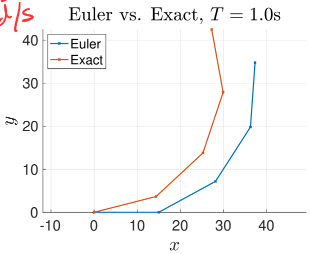
$V_0 = 15\,m/s$
$\omega_0 = .5\ rad/s$

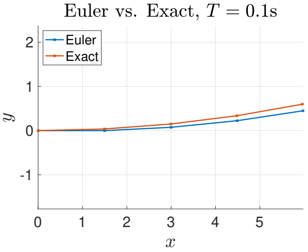
- It turns out that it is possible to solve the deterministic part analytically.
- Combined with modified Euler for the noise we get:

$$\begin{bmatrix} x_k \\ y_k \\ v_k \\ \phi_k \\ \omega_k \end{bmatrix} = \begin{bmatrix} x_{k-1} + \frac{2v_{k-1}}{\omega_{k-1}} \sin\left(\frac{\omega_{k-1}T}{2}\right) \cos\left(\phi_{k-1} + \frac{\omega_{k-1}T}{2}\right) \\ y_{k-1} + \frac{2v_{k-1}}{\omega_{k-1}} \sin\left(\frac{\omega_{k-1}T}{2}\right) \sin\left(\phi_{k-1} + \frac{\omega_{k-1}T}{2}\right) \\ v_{k-1} \\ \phi_{k-1} + T\omega_{k-1} \\ \omega_{k-1} \end{bmatrix} + \mathbf{q}_{k-1},$$

where

$\mathbf{q}_{k-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{k-1})$ and $\mathbf{Q}_{k-1} = \mathrm{diag}(0, 0, 0, T\sigma_v^2, 0, T\sigma_\omega^2)$.





DIRECT DISCRETIZATION OF THE NOISE

- In order to simplify the derivation of the noise covariance, it is common to assume a simpler distribution for $\tilde{\mathbf{q}}(t)$.

- Idea: assume that the noise $\tilde{\mathbf{q}}(t)$ is piecewise constant between samples, i.e., that it is constant in every interval, $[0, T]$, $[T, 2T]$, etc.
  $\rightsquigarrow$ This greatly simplifies the derivation of $\mathbf{Q}_{k-1}$ in many examples.

- **Covariance of $\tilde{\mathbf{q}}(t)$?**
  Assume that $\tilde{\mathbf{q}}(t) \sim \mathcal{N}(\mathbf{0}, \tilde{\mathbf{Q}}/T)$.

  Why divide by T? Note that $\int_0^T \tilde{\mathbf{q}}(t)\, dt = T\tilde{\mathbf{q}}(t)$ then has covariance $\mathbb{E}\{T\tilde{\mathbf{q}}(t)\tilde{\mathbf{q}}(t)^T T\} = T\tilde{\mathbf{Q}}$.

DISCRETIZED LINEARIZATION

- Solving a nonlinear differential equation is not easy!

Linearized motion model

- We can linearize $\tilde{\mathbf{a}}$ about some estimate $\hat{\mathbf{x}}(t)$:

$$\dot{\mathbf{x}}(\tau) \approx \tilde{\mathbf{a}}(\hat{\mathbf{x}}(t)) + \tilde{\mathbf{a}}'(\hat{\mathbf{x}}(t))\,(\mathbf{x}(\tau) - \hat{\mathbf{x}}(t)) + \tilde{\mathbf{q}}(\tau), \quad \tau \in [t, t+T].$$

- Suppose that $\tilde{\mathbf{q}}(\tau)$ is also constant for $\tau \in (t, t+T)$.
- An accurate approximation of $\mathbf{x}(t+T)$ is obtained using:

An analytical solution for linear systems

- If $\dot{\mathbf{x}}(\tau) = \tilde{\mathbf{A}}\mathbf{x}(\tau) + \mathbf{b}$ for $\tau \in [t, t+T]$, then

$$\mathbf{x}(t+T) = \exp(\tilde{\mathbf{A}}T)\mathbf{x}(t) + \left(\mathbf{I}T + \frac{\tilde{\mathbf{A}}T^2}{2} + \frac{\tilde{\mathbf{A}}^2T^3}{3!} + \dots\right)\mathbf{b}$$

SELF ASSESSMENT

The coordinated turn (CT) model is nonlinear and therefore harder to handle than for instance a CV or a CA model. We should thus only use it if yields better predictions. Here is a question to check if you understand some of the differences between CV and CT:

1. Without noise, the CT and CV models always describe motions along straight lines.
2. Without noise, the CT model describes the motion along a circle whereas the CV model describes the motion along a straight line.
3. Without noise, the CT model cannot describe motions along a straight line since it has a parameter ω that describes the yaw rate (that is, how fast the object is turning).

Check all that apply.