The prediction and update step in the Unscented KF and the Cubature KF

Sensor fusion & nonlinear filtering

Lars Hammarstrand

GAUSSIAN PREDICTION BY MOMENT MATCHING

$$\begin{cases} \mathbf{x}_{k-1} \big| \mathbf{y}_{1:k-1} \sim \mathcal{N}(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{P}_{k-1|k-1}) \\ \mathbf{x}_k = f(\mathbf{x}_{k-1}) + \mathbf{q}_{k-1} \end{cases} \Rightarrow \mathbf{x}_k \big| \mathbf{y}_{1:k-1} \sim \mathcal{N}(\hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1})$$

Gaussian filter prediction

$$\hat{\mathbf{x}}_{k|k-1} = \mathbb{E}\{\mathbf{x}_k \big| \mathbf{y}_{1:k-1}\} = \int \mathbf{f}(\mathbf{x}_{k-1}) \mathcal{N}(\mathbf{x}_{k-1}; \hat{\mathbf{x}}_{k-1|k-1}, \mathbf{P}_{k-1|k-1})\, d\mathbf{x}_{k-1}$$

$$\mathbf{P}_{k|k-1} = \mathrm{Cov}\{\mathbf{x}_k \big| \mathbf{y}_{1:k-1}\} = \mathbf{Q}_{k-1} + \int (\mathbf{f}(\mathbf{x}_{k-1}) - \hat{\mathbf{x}}_{k|k-1})(\cdot)^T \mathcal{N}(\mathbf{x}_{k-1}; \hat{\mathbf{x}}_{k-1|k-1}, \mathbf{P}_{k-1|k-1})\, d\mathbf{x}_{k-1}$$

- **Objective:** show how the unscented transform and the cubature rule can approximate $\hat{\mathbf{x}}_{k|k-1}$ and $\mathbf{P}_{k|k-1}$.

PREDICTION IN UKF

1. Form a set of $2n+1$ σ-points

$W_0 = 1 - n/3$

$$
\begin{aligned}
\mathcal{X}_{k-1}^{(0)} &= \hat{\mathbf{x}}_{k-1|k-1}, \\
\mathcal{X}_{k-1}^{(i)} &= \hat{\mathbf{x}}_{k-1|k-1} + \overbrace{\sqrt{\frac{n}{1-W_0}}}^{\sqrt{3}} \left(\mathbf{P}_{k-1|k-1}^{1/2}\right)_i, \quad & i = 1, 2, \ldots, n, \\
\mathcal{X}_{k-1}^{(i+n)} &= \hat{\mathbf{x}}_{k-1|k-1} - \sqrt{\frac{n}{1-W_0}} \left(\mathbf{P}_{k-1|k-1}^{1/2}\right)_i, \quad & i = 1, 2, \ldots, n, \\
W_i &= \frac{1-W_0}{2n}, = \frac{1}{6} & i = 1, 2, \ldots, 2n.
\end{aligned}
$$

2. Compute the predicted moments

$$
\begin{aligned}
\hat{\mathbf{x}}_{k|k-1} &\approx \sum_{i=0}^{2n} \mathbf{f}(\mathcal{X}_{k-1}^{(i)}) W_i \\
\mathbf{P}_{k|k-1} &\approx \mathbf{Q}_{k-1} + \sum_{i=0}^{2n} (\mathbf{f}(\mathcal{X}_{k-1}^{(i)}) - \hat{\mathbf{x}}_{k|k-1})(\cdot)^T W_i
\end{aligned}
$$

PREDICTION IN CKF

1. Form a set of $2n$ σ-points

$$\begin{aligned}
\mathcal{X}_{k-1}^{(i)} &= \hat{\mathbf{x}}_{k-1|k-1} + \sqrt{n}\left(\mathbf{P}_{k-1|k-1}^{1/2}\right)_i, \quad && i = 1, 2, \dots, n, \\
\mathcal{X}_{k-1}^{(i+n)} &= \hat{\mathbf{x}}_{k-1|k-1} - \sqrt{n}\left(\mathbf{P}_{k-1|k-1}^{1/2}\right)_i, \quad && i = 1, 2, \dots, n, \\
W_i &= \frac{1}{2n}, && i = 1, 2, \dots, 2n.
\end{aligned}$$

2. Compute the predicted moments

$$\hat{\mathbf{x}}_{k|k-1} \approx \sum_{i=1}^{2n} \mathbf{f}(\mathcal{X}_{k-1}^{(i)}) W_i$$

$$\mathbf{P}_{k|k-1} \approx \mathbf{Q}_{k-1} + \sum_{i=1}^{2n} (\mathbf{f}(\mathcal{X}_{k-1}^{(i)}) - \hat{\mathbf{x}}_{k|k-1})(\cdot)^T W_i.$$

SELF ASSESSMENT

If a σ-point method approximates

$$\mathbb{E}\{\mathbf{h}(\mathbf{x})\} = \int \mathbf{h}(\mathbf{x})\mathcal{N}(\mathbf{x}; \hat{\mathbf{x}}, \mathbf{P})\, d\mathbf{x} \approx \sum_i W_i \mathbf{h}(\mathcal{X}^{(i)})$$

then $\mathbb{E}\{\mathbf{h}(\mathbf{x})(\mathbf{f}(\mathbf{x}) + \mathbf{x})\}$ would be approximated as

- $\sum_i \mathbf{h}(\mathcal{X}^{(i)})W_i \times \left(\sum_i W_i\mathbf{f}(\mathcal{X}^{(i)}) + W_i\mathcal{X}^{(i)})\right)$
- $\sum_i \mathbf{h}(\mathcal{X}^{(i)})W_i \times \left(\sum_i W_i\mathbf{f}(\mathcal{X}^{(i)}) + \sum_i W_i\mathcal{X}^{(i)})\right)$
- $\mathbf{h}(\sum_i \mathcal{X}^{(i)}W_i) \times \left(\mathbf{f}(\sum_i W_i\mathcal{X}^{(i)}) + \sum_i W_i\mathcal{X}^{(i)})\right)$
- $\sum_i W_i\mathbf{h}(\mathcal{X}^{(i)})(\mathbf{f}(\mathcal{X}^{(i)}) + \mathcal{X}^{(i)})$

Check the correct answer.

GAUSSIAN UPDATE BY MOMENT MATCHING

$$\begin{cases} \mathbf{x}_k \middle| \mathbf{y}_{1:k-1} \sim \mathcal{N}(\hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1}) \\ \mathbf{y}_k = h(\mathbf{x}_k) + \mathbf{r}_k, \end{cases} \Rightarrow \begin{cases} \hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{P}_{xy}\mathbf{S}_k^{-1}(\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}) \\ \mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{P}_{xy}\mathbf{S}_k^{-1}\mathbf{P}_{xy}^T. \end{cases}$$

Gaussian filter update

$$\hat{\mathbf{y}}_{k|k-1} = \int \mathbf{h}(\mathbf{x}_k)\mathcal{N}(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1})\, d\mathbf{x}_k$$

$$\mathbf{P}_{xy} = \int (\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1})(\mathbf{h}(\mathbf{x}_k) - \hat{\mathbf{y}}_{k|k-1})^T \mathcal{N}(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1})\, d\mathbf{x}_k$$

$$\mathbf{S}_k = \mathbf{R}_k + \int (\mathbf{h}(\mathbf{x}_k) - \hat{\mathbf{y}}_{k|k-1})(\cdot)^T \mathcal{N}(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1})\, d\mathbf{x}_k$$

- Objective: show how $\hat{\mathbf{y}}_{k|k-1}$, $\mathbf{P}_{xy}$ and $\mathbf{S}_k$ can be computed using the unscented transform and the cubature rule.

UPDATE IN UKF

$N(x_k; \hat{x}_{k|k-1}, P_{k|k-1})$

1. Form a set of $2n+1$ σ-points

$$\mathcal{X}_k^{(0)} = \hat{\mathbf{x}}_{k|k-1}, \quad W_i = \frac{1 - W_0}{2n}, \; i > 1,$$

$$\mathcal{X}_k^{(i)} = \hat{\mathbf{x}}_{k|k-1} + \sqrt{\frac{n}{1 - W_0}} \left(\mathbf{P}_{k|k-1}^{1/2}\right)_i, \qquad i = 1, 2, \ldots, n,$$

$$\mathcal{X}_k^{(i+n)} = \hat{\mathbf{x}}_{k|k-1} - \sqrt{\frac{n}{1 - W_0}} \left(\mathbf{P}_{k|k-1}^{1/2}\right)_i, \qquad i = 1, 2, \ldots, n,$$

2. Compute the desired moments

$$\hat{\mathbf{y}}_{k|k-1} \approx \sum_{i=0}^{2n} \mathbf{h}(\mathcal{X}_k^{(i)}) W_i$$

$$\mathbf{P}_{xy} \approx \sum_{i=0}^{2n} \left(\mathcal{X}_k^{(i)} - \hat{\mathbf{x}}_{k|k-1}\right) \left(\mathbf{h}\left(\mathcal{X}_k^{(i)}\right) - \hat{\mathbf{y}}_{k|k-1}\right)^T W_i$$

$$\mathbf{S}_k \approx \mathbf{R}_k + \sum_{i=0}^{2n} \left(\mathbf{h}\left(\mathcal{X}_k^{(i)}\right) - \hat{\mathbf{y}}_{k|k-1}\right) (\cdot)^T W_i$$

UPDATE IN CKF

1. Form a set of $2n$ σ-points

$$\mathcal{X}_k^{(i)} = \hat{\mathbf{x}} + \sqrt{n}\left(\mathbf{P}_{k|k-1}^{1/2}\right)_i, \qquad i = 1, 2, \ldots, n,$$

$$\mathcal{X}_k^{(i+n)} = \hat{\mathbf{x}} - \sqrt{n}\left(\mathbf{P}_{k|k-1}^{1/2}\right)_i, \qquad i = 1, 2, \ldots, n,$$

$$W_i = \frac{1}{2n}, \qquad i = 1, 2, \ldots, 2n.$$

2. Compute the desired moments

$$\hat{\mathbf{y}}_{k|k-1} \approx \sum_{i=1}^{2n} \mathbf{h}(\mathcal{X}_k^{(i)}) W_i$$

$$\mathbf{P}_{xy} \approx \sum_{i=1}^{2n} \left(\mathcal{X}_k^{(i)} - \hat{\mathbf{x}}_{k|k-1}\right)(\mathbf{h}(\mathcal{X}_k^{(i)}) - \hat{\mathbf{y}}_{k|k-1})^T W_i$$

$$\mathbf{S}_k \approx \mathbf{R}_k + \sum_{i=1}^{2n} (\mathbf{h}(\mathcal{X}_k^{(i)}) - \hat{\mathbf{y}}_{k|k-1})(\cdot)^T W_i$$

SELF ASSESSMENT

If a σ-point method is exact for any polynomial up to degree 9 and $\mathbf{h}(\mathbf{x}_k)$ is a polynomial of degree 3, then the σ-point method would help us to:

- Prove that the posterior distribution, $p(\mathbf{x}_k \middle| \mathbf{y}_{1:k})$ is Gaussian.
- Compute the cross covariance $\mathbf{P}_{xy}$ exactly.
- Compute the posterior mean, $\hat{\mathbf{x}}_{k|k}$, exactly.
- Compute the predicted measurement $\hat{\mathbf{y}}_{k|k-1}$ exactly.

Check all that apply.