Discretizing linear models: the transition matrix

Sensor fusion & nonlinear filtering

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FINDING THE TRANSITION MATRIX

- Suppose $\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{q}}(t)$ where $\tilde{\mathbf{A}}$ is a constant matrix.
- *How should we select* $\mathbf{A}_{k-1}$ *in* $\mathbf{x}_k = \mathbf{A}_{k-1}\mathbf{x}_{k-1} + \mathbf{q}_{k-1}$?

The Euler method

$$\mathbf{x}(t+T) \approx \mathbf{x}(t) + T(\tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{q}}(t))$$

$$\Rightarrow \quad \mathbf{A}_{k-1} = \mathbf{I} + T\tilde{\mathbf{A}}$$

Exact solution for linear systems

$$\mathbf{x}(t+T) = \exp(\tilde{\mathbf{A}}T)\mathbf{x}(t) + \int_t^{t+T} \exp(\mathbf{A}(t+T-\tau))\tilde{\mathbf{q}}(\tau)\, d\tau$$

$$\Rightarrow \quad \mathbf{A}_{k-1} = \exp(T\tilde{\mathbf{A}})$$

THE CONTINUOUS-TIME CONSTANT VELOCITY MODEL

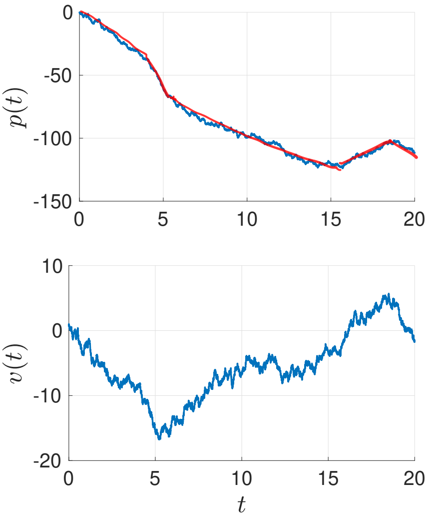
The constant velocity (CV) model

- Suppose we have a state vector $\mathbf{x}(t) = \begin{bmatrix} p(t) & v(t) \end{bmatrix}^T$ where $p(t)$ is the position and $v(t)$ the velocity in one dimension.

- The continuous time constant velocity (CV) model for this state vector is:

$$\dot{x}(t) = \begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\tilde{\mathbf{A}}} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \tilde{\mathbf{q}}(t)$$

$$\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} q_c(t) \right)$$



THE DISCRETE-TIME CONSTANT VELOCITY MODEL – TRANSITION MATRIX

- How should we select $\mathbf{A}_{k-1}$ in

$$\mathbf{x}_k = \mathbf{A}_{k-1}\mathbf{x}_{k-1} + \mathbf{q}_{k-1}$$

for the continuous-time CV model

$$\dot{\mathbf{x}}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\tilde{\mathbf{A}}} \mathbf{x}(t) + \tilde{\mathbf{q}}(t)?$$

$$\begin{bmatrix} P_k \\ V_k \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_{k-1} \\ V_{k-1} \end{bmatrix} + q_{k-1}$$

Euler method

$$\mathbf{A}_{k-1} = \mathbf{I} + T\tilde{\mathbf{A}}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + T \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}.$$

Exact solution

$$\mathbf{A}_{k-1} = \exp\left(\tilde{\mathbf{A}}T\right)$$
$$= \mathbf{I} + \tilde{\mathbf{A}}T + \tilde{\mathbf{A}}^2 T^2/2 + \cdots = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}.$$

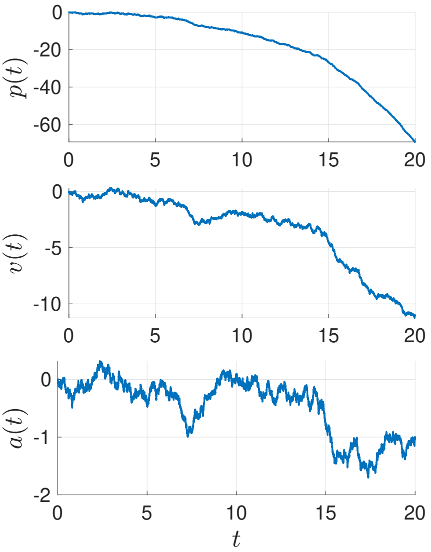
THE CONTINUOUS-TIME CONSTANT ACCELERATION MODEL

The constant acceleration (CA) model

- Suppose $\mathbf{x}(t) = \begin{bmatrix} p(t) & v(t) & a(t) \end{bmatrix}^T$ where $p(t)$, $v(t)$ and $a(t)$ are position, velocity and acceleration in one dimension.
- The continuous time constant acceleration (CA) model for this state vector is:

$$\dot{\mathbf{x}}(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\tilde{\mathbf{A}}} \mathbf{x}(t) + \tilde{\mathbf{q}}(t)$$

$$\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} q_c(t) \right.$$



THE DISCRETE-TIME CONSTANT VELOCITY MODEL – TRANSITION MATRIX

- How should we select $\mathbf{A}_{k-1}$ in

$$\mathbf{x}_k = \mathbf{A}_{k-1}\mathbf{x}_{k-1} + \mathbf{q}_{k-1}$$

for the continuous-time CA model

$$\dot{\mathbf{x}}(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\tilde{\mathbf{A}}} \mathbf{x}(t) + \tilde{\mathbf{q}}(t)?$$

Euler method

$$\mathbf{A}_{k-1} = \mathbf{I} + T\tilde{\mathbf{A}} = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}.$$

Exact solution

$$\mathbf{A}_{k-1} = \exp\left(\tilde{\mathbf{A}}T\right) = \mathbf{I} + \tilde{\mathbf{A}}T + \tilde{\mathbf{A}}^2T^2/2 + \dots$$

$$= \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}.$$

CV AND CA IN HIGHER DIMENSIONS

- In higher dimensions: assume motions in different dimensions are independent.
- The results using the exact discretization:

Constant velocity

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix}, \qquad \mathbf{A}_{k-1} = \begin{bmatrix} \mathbf{I}_n & T\mathbf{I}_n \\ \mathbf{0}_n & \mathbf{I}_n \end{bmatrix}$$

Constant acceleration

$$\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \\ \mathbf{a} \end{bmatrix}, \qquad \mathbf{A}_{k-1} = \begin{bmatrix} \mathbf{I}_n & T\mathbf{I}_n & T^2/2\mathbf{I}_n \\ \mathbf{0}_n & \mathbf{I}_n & T\mathbf{I}_n \\ \mathbf{0}_n & \mathbf{0}_n & \mathbf{I}_n \end{bmatrix}.$$

$$\tilde{A} = \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix}$$

SELF ASSESSMENT

- Suppose we have a state vector $\mathbf{x}(t) = \begin{bmatrix} p(t) & v(t) & \phi(t) \end{bmatrix}^T$ where $p(t)$ and $v(t)$ are position and velocity but $\phi(t)$ is an orientation angle in $2D$. It may then be reasonable to assume: $\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \tilde{\mathbf{q}}(t)$.

  Check the correct answer:

  - We would get
  $$\text{Euler: } \mathbf{A}_{k-1} = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{Analytical: } \mathbf{A}_{k-1} = \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}.$$
  - We would get
  $$\text{Euler: } \mathbf{A}_{k-1} = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{Analytical: } \mathbf{A}_{k-1} = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$