Selecting the discrete time motion noise covariance

Sensor fusion & nonlinear filtering

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CONTINUOUS-TIME MOTION NOISE

- Consider a linear continuous-time DE, $\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{q}}(t)$, where $\tilde{\mathbf{q}}(t)$ is the motion noise.
- We usually assume that $\tilde{\mathbf{q}}(t)$ is a white Gaussian noise process:

$$\begin{cases} \mathbb{E}\{\tilde{\mathbf{q}}(t)\} = 0 & \text{i.e., zero mean} \\ \mathrm{Cov}\{\tilde{\mathbf{q}}(\tau_1), \tilde{\mathbf{q}}(\tau_2)\} = \delta(\tau_1 - \tau_2)\tilde{\mathbf{Q}} & \text{i.e., uncorrelated} \end{cases}$$

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- A simple example: the Wiener process $\mathbf{w}(t) = \int_0^t \tilde{\mathbf{q}}(\tau)\, d\tau$, which has zero mean and covariance

$$\begin{aligned} \mathbb{E}\{\mathbf{w}(t)\mathbf{w}(t)^T\} &= \mathbb{E}\left\{\int_0^t \tilde{\mathbf{q}}(\tau_1)\, d\tau_1 \int_0^t \tilde{\mathbf{q}}(\tau_2)^T\, d\tau_2\right\} \\ &= \int_0^t \int_0^t \underbrace{\mathbb{E}\left\{\tilde{\mathbf{q}}(\tau_1)\tilde{\mathbf{q}}(\tau_2)^T\right\}}_{\delta(\tau_1 - \tau_2)\tilde{\mathbf{Q}}} d\tau_2\, d\tau_1 = \int_0^t 1\, d\tau_1 \tilde{\mathbf{Q}} = t\tilde{\mathbf{Q}} \end{aligned}$$

DISCRETE-TIME MOTION NOISE

- Consider a linear continuous-time DE

$$\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{q}}(t),$$

  where $\tilde{\mathbf{q}}(t)$ is a white Gaussian noise process.

- We seek a discrete time motion model

$$\mathbf{x}_k = \mathbf{A}_{k-1}\mathbf{x}_{k-1} + \mathbf{q}_{k-1}, \quad \mathbf{q}_{k-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{k-1}),$$

  but how can we select $\mathbf{Q}_{k-1}$?

- Note:
  - $\mathbf{Q}_{k-1} = \mathrm{Cov}\{\mathbf{q}_{k-1}\} = \mathrm{Cov}\{\mathbf{x}_k \middle| \mathbf{x}_{k-1}\} = \mathrm{Cov}\{\mathbf{x}(t+T) \middle| \mathbf{x}(t)\}$.

EXACT SOLUTION FOR LINEAR SYSTEMS

- When $\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{q}}(t)$, then

$$\mathbf{x}(t+T) = \exp(\tilde{\mathbf{A}}T)\mathbf{x}(t) + \underbrace{\int_0^T \exp\left(\tilde{\mathbf{A}}\tau\right)\tilde{\mathbf{q}}(\tau)\, d\tau}_{q_{k-1}}.$$

$$x_k = A_{k-1}\, x_{k-1} +$$

- The discrete time noise covariance is therefore:

$$\begin{aligned}\mathbf{Q}_{k-1} &= \mathrm{Cov}\{\mathbf{x}(t+T)\,|\,\mathbf{x}(t)\} \\ &= \mathrm{Cov}\left\{\int_0^T \exp\left(\tilde{\mathbf{A}}\tau\right)\tilde{\mathbf{q}}(\tau)\, d\tau\right\} = \ldots \\ &= \int_0^T \exp\left(\tilde{\mathbf{A}}\tau\right)\tilde{\mathbf{Q}}\exp\left(\tilde{\mathbf{A}}^T\tau\right)\, d\tau\end{aligned}$$

MODIFIED EULER METHOD

- Given a DE $\dot{\mathbf{x}}(t) = \tilde{\mathbf{a}}(\mathbf{x}(t)) + \tilde{\mathbf{q}}(t)$
- We can view the Euler method as the approximation

$$\dot{\mathbf{x}}(\tau) \approx \tilde{\mathbf{a}}(\mathbf{x}(t)) + \tilde{\mathbf{q}}(t) \quad \text{for all } \tau \in [t, t+T]$$

$$\Rightarrow \; \mathbf{x}(t+T) = \mathbf{x}(t) + T(\tilde{\mathbf{a}}(\mathbf{x}(t)) + \tilde{\mathbf{q}}(t))$$

Modified Euler

- In the modified Euler method, we use $\dot{\mathbf{x}}(\tau) \approx \tilde{\mathbf{a}}(\mathbf{x}(t)) + \tilde{\mathbf{q}}(\tau)$,

$$\begin{aligned} \Rightarrow \quad \mathbf{x}(t+T) &= \mathbf{x}(t) + \int_t^{t+T} \dot{\mathbf{x}}(\tau)\, d\tau \\ &= \mathbf{x}(t) + \underbrace{\int_t^{t+T} \tilde{\mathbf{a}}(\mathbf{x}(t))\, d\tau}_{T\tilde{\mathbf{a}}(\mathbf{x}(t))} + \int_t^{t+T} \tilde{\mathbf{q}}(\tau)\, d\tau. \end{aligned}$$

$$\Rightarrow \mathrm{Cov}\{\mathbf{x}(t+T) \,|\, \mathbf{x}(t)\} \approx T\tilde{\mathbf{Q}}.$$

TWO METHODS TO SELECT $\mathbf{Q}_{K-1}$

- We have proposed two methods to select the noise covariance matrix.

For linear continuous time systems

- When $\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{q}}(t)$, we can use

$$\mathbf{Q}_{k-1} = \int_0^T \exp\left(\tilde{\mathbf{A}}\tau\right)\tilde{\mathbf{Q}}\exp\left(\tilde{\mathbf{A}}^T\tau\right)\, d\tau$$

For nonlinear continuous time systems

- For linear and nonlinear systems $\dot{\mathbf{x}}(t) = \tilde{\mathbf{a}}(\mathbf{x}(t)) + \tilde{\mathbf{q}}(t)$, we can use

$$\mathbf{Q}_{k-1} = T\tilde{\mathbf{Q}}$$

THE CONSTANT VELOCITY MODEL

- In many cases, the motion noise is zero on some of the state variables.
- Using a matrix $\boldsymbol{\Gamma}$, we can then express the motion noise as

$$\tilde{\mathbf{q}}(t) = \boldsymbol{\Gamma}\mathbf{q}_c(t)$$

where $\mathbf{q}_c(t)$ is the motion noise in some dimensions.

$$\Rightarrow \quad \tilde{\mathbf{Q}} = \boldsymbol{\Gamma}\mathbf{Q}_c\boldsymbol{\Gamma}^T$$

- The constant velocity model is a good example:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\tilde{\mathbf{A}}} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\boldsymbol{\Gamma}} q_c(t) \Rightarrow \quad \tilde{\mathbf{Q}} = \overset{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\boldsymbol{\Gamma}} Q_c \overset{\begin{bmatrix} 0 & 1 \end{bmatrix}}{\boldsymbol{\Gamma}^T} = \begin{bmatrix} 0 & 0 \\ 0 & Q_c \end{bmatrix}$$

SELF ASSESSMENT, PART 1

Things we know about CV and the modified Euler method:

- For linear and nonlinear systems $\dot{\mathbf{x}}(t) = \tilde{\mathbf{a}}(\mathbf{x}(t)) + \tilde{\mathbf{q}}(t)$, we can use $\mathbf{Q}_{k-1} = T\tilde{\mathbf{Q}}$.
- The constant velocity model can be described as:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} q_c(t) \Rightarrow \tilde{\mathbf{Q}} = \begin{bmatrix} 0 & 0 \\ 0 & Q_c \end{bmatrix}$$

- The modified Euler method thus suggests that we use

  - $\mathbf{Q}_{k-1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,
  - $\mathbf{Q}_{k-1} = \begin{bmatrix} 0 & 0 \\ 0 & T^2 \end{bmatrix} Q_c$,
  - $\mathbf{Q}_{k-1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$,
  - $\mathbf{Q}_{k-1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} Q_c T$

  Check the correct statement.

SELF ASSESSMENT, PART 2

Summary of CV and the exact solution for linear systems:

- When $\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{q}}(t)$, we can use $\mathbf{Q}_{k-1} = \int_0^T \exp\left(\tilde{\mathbf{A}}\tau\right) \tilde{\mathbf{Q}} \exp\left(\tilde{\mathbf{A}}^T\tau\right)\, d\tau$
- We know that for the constant velocity model we have:

$$\Rightarrow \tilde{\mathbf{Q}} = \begin{bmatrix} 0 & 0 \\ 0 & Q_c \end{bmatrix}, \exp(\tilde{\mathbf{A}}\tau) = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}, \exp(\tilde{\mathbf{A}}^T\tau) = \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix}.$$

The exact solution for linear systems is

- $\mathbf{Q}_{k-1} = \begin{bmatrix} T^3 & T^2 \\ 0 & T \end{bmatrix} Q_c$
- $\mathbf{Q}_{k-1} = \begin{bmatrix} T^2 & T \\ T & 1 \end{bmatrix} Q_c$
- $\mathbf{Q}_{k-1} = \begin{bmatrix} T^3/3 & T^2/2 \\ T^2/2 & T \end{bmatrix} Q_c$
- $\mathbf{Q}_{k-1} = \begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix} Q_c$

Check the correct statement.