# Rao-Blackwellized Particle Filter

Sensor fusion & nonlinear filtering

Lars Hammarstrand

# The Rao-Blackwellized particle filters

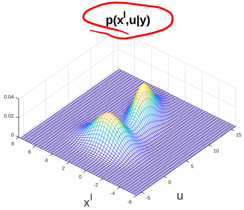


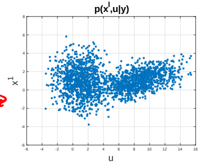
- Background:
  - particle filters are intractable in high dimensions.
  - many systems are linear in some dimensions.

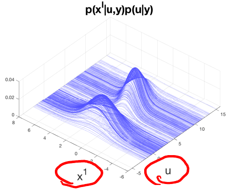
- Idea 1: *"combine a particle filter for the nonlinear states with a Kalman filter for the linear states".*

- Idea 2: If $\mathbf{x}_k = \begin{bmatrix} \mathbf{x}_k^l \\ \mathbf{u}_k \end{bmatrix}$ where $\mathbf{x}_k^l$ and $\mathbf{u}_k$ are the linear and nonlinear states:

$$p(\mathbf{x}_k^l, \mathbf{u}_{1:k} | \mathbf{y}_{1:k}) = \underbrace{p(\mathbf{x}_k^l | \mathbf{u}_{0:k}, \mathbf{y}_{1:k})}_{\text{Gaussian}} \underbrace{p(\mathbf{u}_{0:k} | \mathbf{y}_{1:k})}_{\text{Particle filter}}$$



- Assuming we have $\mathbf{x}_k = \begin{bmatrix} \mathbf{x}_k^l \\ \mathbf{u}_k \end{bmatrix}$, Rao-Blackwellized particle filters are often used for models on the form

$$\mathbf{x}_k^l = f_{k-1}^l(\mathbf{u}_{k-1}) + \mathbf{A}_{k-1}^l(\mathbf{u}_{k-1})\mathbf{x}_{k-1}^l + \mathbf{q}_{k-1}^l$$

$$\mathbf{u}_k = f_{k-1}^u(\mathbf{u}_{k-1}) + \mathbf{A}_{k-1}^u(\mathbf{u}_{k-1})\mathbf{x}_{k-1}^l + \mathbf{q}_{k-1}^u$$

$$\mathbf{y}_k = h_k(\mathbf{u}_k) + \mathbf{H}_k(\mathbf{u}_k)\mathbf{x}_k^l + \mathbf{r}_k$$

where all the noises are Gaussian.

## Bearing only tracking

- Bearing only tracking with a constant velocity motion in 2D.
  *What is $\mathbf{x}_k^l$, $\mathbf{u}_k^l$ and $\mathbf{y}_k$ in this example?*

- $\mathbf{x}_k^l$: position, $\mathbf{u}_k^l$: velocity, $\mathbf{y}_k$: bearing to target
- $\mathbf{x}_k^l$: velocity, $\mathbf{u}_k^l$: position, $\mathbf{y}_k$: bearing to target
- $\mathbf{x}_k^l$: velocity, $\mathbf{u}_k^l$: bearing to target, $\mathbf{y}_k$: position
- $\mathbf{x}_k^l$: position, $\mathbf{u}_k^l$: bearing to target, $\mathbf{y}_k$: bearing to target

## Bearing only tracking – system models

- Let us denote our state vector $\mathbf{x}_k = [\underbrace{x_k^1, x_k^2}_{u_k}, \underbrace{\dot{x}_k^1, \dot{x}_k^2}_{x_k^l}]^T$, the system models can then be written as:

$$x_k^l = \begin{bmatrix} \dot{x}_k^1 \\ \dot{x}_k^2 \end{bmatrix} = \begin{bmatrix} \dot{x}_{k-1}^1 \\ \dot{x}_{k-1}^2 \end{bmatrix} + \mathbf{q}_{k-1}^l = x_{k-1}^l - q_{k-1}^l$$

$$u_k = \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} = \begin{bmatrix} x_{k-1}^1 \\ x_{k-1}^2 \end{bmatrix} + T \begin{bmatrix} \dot{x}_{k-1}^1 \\ \dot{x}_{k-1}^2 \end{bmatrix} + \mathbf{q}_{k-1}^u = u_{k-1} + T \cdot x_k^l + q_{k-1}^u$$

$$\mathbf{y}_k = \operatorname{atan}_2(x_k^2, x_k^1) + \mathbf{r}_k$$

where $\mathbf{r}_k \sim \mathcal{N}(0, \sigma_r^2)$ and $\mathbf{q}_k = \begin{bmatrix} \mathbf{q}_k^u \\ \mathbf{q}_k^l \end{bmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} \frac{T^2}{2}\mathbf{I} \\ T\mathbf{I} \end{bmatrix} \begin{bmatrix} \sigma_q^2 & 0 \\ 0 & \sigma_q^2 \end{bmatrix} \begin{bmatrix} \frac{T^2}{2}\mathbf{I} \\ T\mathbf{I} \end{bmatrix}^T \right)$

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- One recursion of the Rao-Blackwellized particle filter contains five steps:

| $p(\mathbf{x}^l_{k-1} \mid \mathbf{u}_{0:k-1}, \mathbf{y}_{1:k-1}) p(\mathbf{u}_{0:k-1} \mid \mathbf{y}_{1:k-1}) \rightarrow p(\mathbf{x}^l_{k} \mid \mathbf{u}_{0:k}, \mathbf{y}_{1:k}) p(\mathbf{u}_{0:k} \mid \mathbf{y}_{1:k})$ | |
|---:|:---:|
| 1) PF-pred: | $p(\mathbf{u}_{1:k-1} \mid \mathbf{y}_{1:k-1}) \rightarrow p(\mathbf{u}_{1:k} \mid \mathbf{y}_{1:k-1})$ |
| 2) KF, dyn. upd.: | $p(\mathbf{x}^l_{k-1} \mid \mathbf{u}_{1:k-1}, \mathbf{y}_{1:k-1}) \rightarrow p(\mathbf{x}^l_{k-1} \mid \mathbf{u}_{1:k}, \mathbf{y}_{1:k-1})$ |
| 3) KF-pred: | $p(\mathbf{x}^l_{k-1} \mid \mathbf{u}_{1:k-1}, \mathbf{y}_{1:k-1}) \rightarrow p(\mathbf{x}^l_{k} \mid \mathbf{u}_{1:k-1}, \mathbf{y}_{1:k-1})$ |
| 4) PF, update: | $p(\mathbf{u}_{1:k} \mid \mathbf{y}_{1:k-1}) \rightarrow p(\mathbf{u}_{1:k} \mid \mathbf{y}_{1:k})$ |
| 5) KF, meas. upd.: | $p(\mathbf{x}^l_{k} \mid \mathbf{u}_{1:k}, \mathbf{y}_{1:k-1}) \rightarrow p(\mathbf{x}^l_{k} \mid \mathbf{u}_{1:k}, \mathbf{y}_{1:k})$ |

$u_k^{(i)} = u_{k-1}^{(i)} + T \cdot x_k^l + q_{k-1}^u$

$u_k^{(i)} - u_{k-1}^{(i)} = T x_k^l + q_{k-1}^u$

$x_k^l = x_{k-1}^l + q_{k-1}^l$

$w_k^{(i)} \propto w_{k-1}^{(i)} \cdot p(y_k \mid u_k)$

- **Note:**
  - Step 2) makes use of the motion model for $\mathbf{u}_k$ to update $\mathbf{x}^l_{k-1}$.
  - The linear states are marginalized from step 1) and 4), similarly to how we normally handle noise.

## Bearing only tracking – system models

- Let us denote our state vector $\mathbf{x}_k = [x_k^1, x_k^2, \dot{x}_k^1, \dot{x}_k^2]^T$, the system models can then be written as:

$$\begin{bmatrix} \dot{x}_k^1 \\ \dot{x}_k^2 \end{bmatrix} = \begin{bmatrix} \dot{x}_{k-1}^1 \\ \dot{x}_{k-1}^2 \end{bmatrix} + \mathbf{q}_{k-1}^l$$

$$\begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} = \begin{bmatrix} x_{k-1}^1 \\ x_{k-1}^2 \end{bmatrix} + T \begin{bmatrix} \dot{x}_{k-1}^1 \\ \dot{x}_{k-1}^2 \end{bmatrix} + \mathbf{q}_{k-1}^u$$

$$\mathbf{y}_k = \mathrm{atan}_2(x_k^2, x_k^1) + \mathbf{r}_k$$

where $\mathbf{r}_k \sim \mathcal{N}(0, (\frac{\pi}{180})^2)$ & $\mathbf{q}_k = \begin{bmatrix} \mathbf{q}_k^u \\ \mathbf{q}_k^l \end{bmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} \frac{T^2}{2}\mathbf{I} \\ T\mathbf{I} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{T^2}{2}\mathbf{I} \\ T\mathbf{I} \end{bmatrix}^T\right)$

Concluding remarks:

- Rao-Blackwellized particle filters are useful to reduce the number of particles.
- These filters enable us to handle higher dimensions than normal PFs.
- They are particularly useful if Kalman gains and posterior covariances are independent of the nonlinear states
  $\Rightarrow$ sufficient to compute them one time in each recursion.