Output Feedback

A. Giavaras

Contents

1 Output Feedback

Chapter **??** introduced the concept of reachability. It was shown that it is possible to find a state feedback law that gives the desired closed loop eigenvalues provided that the system is reachable. Furthermore, we saw how to design controllers using the system state, $x(t)$, as feedback to out controller.

However, desigining state feedback controllers preassumes that all the states are measured. For many situations, it is highly unrealistic to assume that all the states are measured.

In this section we proceed somehow in a similar vein we can use the output $y(t)$ to modify the dynamics of the system through the use of observers. Furthermore, we will introduce the concept of observability and show that if a system is observable, it is possible to recover the state from measurements of the inputs and outputs to the system. We then show how to design a controller with feedback from the observer state.

2 Observability

For many situations, it is highly unrealistic to assume that all the states are measured. In this section we investigate how the state can be estimated by using a mathematical model and a few measurements. It will be shown that computation of the states can be carried out by a dynamical system called an **observer**, see also figure 1.

Definition 2.1. Observability A linear system is **observable** if for any $T > 0$ it is possible to determine the state of the system $x(T)$ through measurements of $y(t)$ and $u(t)$ on the interval $[0, T]$ $x(T) = x_f$.

Remark 2.1. **Nonlinear Systems**

The definition above holds for nonlinear systems as well, and the results discussed here have extensions to the nonlinear case.

Consider again the system

$$\frac{dx}{dt} = Ax + Bu \quad y = Cx + Du \tag{1}$$

where $x \in R^n$ is the state, $u \in R^p$ is the input and $y \in R^q$ the measured output.

We wish to estimate the state of the system from its inputs and outputs, as illustrated in Figure 1. In some situations we will assume that there is only one measured signal, i.e., that the signal y is a scalar and that C is a (row) vector.

This signal may be corrupted by noise n, although we shall start by considering the noise-free case. We write $\hat{x}$ for the state estimate given by the observer.

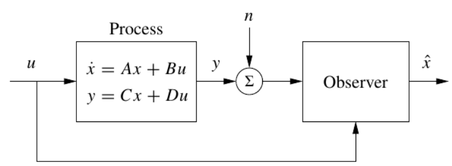

Fig. 1: Block diagram for an observer. The observer uses the process measurement y (possibly corrupted by noise n) and the input u to estimate the current state of the process, denoted $\hat{x}$.

The problem of observability is one that has many important applications, even outside feedback systems. If a system is observable, then there are no hidden dynamics inside it; we can understand everything that is going on through observation (over time) of the inputs and outputs. As we shall see, the problem of observability is of significant practical interest because it will determine if a set of sensors is sufficient for controlling a system. Sensors combined with a mathematical model can also be viewed as a virtual sensor that gives information about variables that are not measured directly. The process of reconciling signals from many sensors with mathematical models is also called sensor fusion.

3 Testing for Observability

When discussing reachability in the last chapter, we neglected the output and focused on the state. Similarly, it is convenient here to initially neglect the input and focus on the autonomous system

Remark 3.1. **Autonomous System**

$$\frac{dx}{dt} = Ax \quad y = Cx \tag{2}$$

The objective is to understand when it is possible to determine the state from observations of the output. From

$$y = Cx \tag{3}$$

we see that the output itself gives us the projection of the state x on vectors that are rows of the matrix C. The observability problem can immediately be solved if the matrix C is invertible. If the matrix is not invertible we can take the derivatives and obtain

$$\frac{dy}{dt} = C\frac{dx}{dt} = CAx \tag{4}$$

From the derivative of the output we thus get the projection of the state on vectors that are rows of the matrix CA. Proceeding in this way, we get

$$\begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x \tag{5}$$

We thus find that the state can be determined if the observability matrix

$$W_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{6}$$

has n independent rows. It turns out that we need not consider any derivatives higher than $n-1$ (this is an application of the Cayley-Hamilton theorem)

Remark 3.2. **System with inputs**

The calculation can easily be extended to systems with inputs. The state is then given by a linear combination of inputs and outputs and their higher derivatives. The observability criterion is unchanged.

Theorem 3.1. ***Observability rank condition***

A linear system of the form

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx + Du$$

is observable if and only if the observability matrix W_o is full rank.

4 Observable Canonical Form

As in the case of reachability, certain canonical forms will be useful in studying ob- servability. A linear single-input, single-output state space system is in observable canonical form if its dynamics are given by

> *Remark* 4.1. **Observable Canonical Form for Nonlinear Systems**
>
> The definition can be extended to systems with many inputs; the only difference is that the vector multiplying u is replaced by a matrix.

The characteristic polynomial for a system in observable canonical form is

$$\lambda(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1} s + \alpha_n \tag{7}$$

In order to check the observability property of a systme more formally, we can compute the observability matrix for a system in observable canonical form. This is given by

$$W_o = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ -\alpha_1 & 1 & 0 & \ldots & 0 \\ -\alpha_1^2 - \alpha_1\alpha_2 & -\alpha_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \ldots & 1 \end{bmatrix} \tag{8}$$

where * represents an entry whose exact value is not important. What is important here is the rows of this matrix are linearly independent (since it is lower triangular), and hence Wo is full rank. Hence, it is invertible.

As in the case of reachability (see section **??**), it turns out that if a system is observable then there always exists a transformation T that converts the system into observable canonical form. This is useful for proofs since it lets us assume that a system is in observable canonical form without any loss of generality. The observable canonical form may be poorly conditioned numerically.

5 State Estimation

State feedback control design, as explained in t chapter **??** requires that we have access to the complete state vector. However, measuring the complete state vector is not always feasible (for example a sensor may not be available at all) and it may also be expensive. In this section we will introduce the idea of state estimation as a way to get access to the state variables.

> *Remark* 5.1. **Soft Sensors**
>
> The concept of using software instead of sensors to access the quantity we are interested in, is referred to as soft sensors in the automotive industry.

Recall that the idea of state feedback control was to modify the eigenvalues of the system under consideration by using the input

$$u = -Kx + K_r r \tag{9}$$

However, it can be seen that this requires the state vector x. The idea of state estimation is to design something called an observer that tries to provide an estimate say $\hat{x}$ of the state vector x. The observer is fed with the same input as the real plant. The goal of the observer is to provide somehow a replica of the true state vector. Thus, we would like to have the error $\tilde{x}$ between the two quantities to be zero. Namely,

$$\tilde{x} = x - \hat{x} = 0 \tag{10}$$

In this section, we want to construct a dynamical system of the form

$$\frac{d\hat{x}}{dt} = F\hat{x} + Gu + Hy \tag{11}$$

where u and y are the input and output of the original system and $\hat{x} \in R^n$ is an estimate of x with the property

$$\hat{x}(t) \rightarrow x(t), \quad \text{as} \quad t \rightarrow \infty \tag{12}$$

Let's consider again the system

$$\frac{dx}{dt} = Ax + Bu \quad y = Cx + Du \tag{13}$$

assume further that D is zero. Assuming that the input u is known, then an estimate of the state x is given by

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu \tag{14}$$

We would like to know the properties of this estimate; how far is from the exact state? The estimation error is [1]

$$\tilde{x} = x - \hat{x} \tag{15}$$

substituting into equation 14 we find that

$$\frac{d\tilde{x}}{dt} = A\tilde{x} \tag{16}$$

We already know that the behavior of this system depends on the eigenvalues of A. Concretely, if matrix A has all its eigenvalues in the left half-plane, the error $\tilde{x}$ will go to zero, and hence equation 14 is a dynamical system whose output converges to the state of the system 13.

The observer given by equation 14 uses only the process input u; the measured signal does not appear in the equation. We must also require that the system be stable, and essentially our estimator converges because the state of both the observer and the estimator are going zero. This is not very useful in a control design context since we want to have our estimate converge quickly to a nonzero state so that we can make use of it in our controller. We will therefore attempt to modify the observer so that the output is used and its convergence properties can be designed to be fast relative to the systems dynamics. This version will also work for unstable systems.

Let's now consider the following observer

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{17}$$

The term $L(y - C\hat{x})$ provides feedback and hence the observer in 18 can be seen as a generalization of the observer in 14. The supplied feedback is proportional to the difference between the observed output and the output predicted by the observer. Substituting the expression for the error $\tilde{x}$ we arraive at

$$\frac{d\tilde{x}}{dt} = (A - LC)\tilde{x} \tag{18}$$

If the matrix L can be chosen in such a way that the matrix $A - LC$ has eigenvalues with negative real parts, the error $\tilde{x}$ will go to zero. The convergence rate is determined by an appropriate selection of the eigenvalues.

Remark 5.2. **Observer design and state feedback duality**

Notice the similarity between the problems of finding a state feedback and finding the observer. State feedback design by eigenvalue assignment is equivalent to finding a matrix K so that $A - BK$ has given eigenvalues. Designing an observer with prescribed eigenvalues is equivalent to finding a matrix L so that $A - LC$ has given eigenvalues. Since the eigenvalues of a matrix and its transpose are the same we can establish the following equivalences:

$$A \leftrightarrow A^T, \quad B \leftrightarrow C^T, \quad K \leftrightarrow L^T, \quad W_r \leftrightarrow W_o^T \tag{19}$$

The observer design problem is the dual of the state feedback design problem.

Theorem 5.1. ***Observer design by eigenvalue assignment***

Consider the system

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx$$

with one input and one output. Let the characteristic polynomial related to matrix A be

$$\lambda(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1}s + \alpha_n$$

If the systme is observable, then the dynamical system

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x})$$

is an observer for the system with L chosen as

$$L = W_o^{-1}\tilde{W}_o \begin{bmatrix} p_1 - \alpha_1 \\ p_2 - \alpha_2 \\ \vdots \\ p_n - \alpha_n \end{bmatrix}$$

The matrices W_o and $\tilde{W}_o$ are given by

The resulting observer error $\tilde{x}$ is governed by a differential equation that has the following characteristic polynomial

$$p(s) = s^n + p - 1s^{n-1} + \ldots + p_n$$

The dynamical system (7.10) is called an observer for (the states of) the system (7.9) because it will generate an approximation of the states of the system from its inputs and outputs. This form of an observer is a much more useful form than the one given by pure differentiation in equation (7.3).

6 Questions

Question 1

What is the main reason for using an estimator in feedback control?

- A) There are process disturbances in the model

- B) We are measuring the wrong quantities
- C) It is too expensive or impractical to measure each state variable correct

Answer

Option C is the correct answer.

References

[1] Åström K. J., Murray R. M. *Feedback Systems. An Introduction for Scientists and Engineers*

[2] Philip , Florent Altche1, Brigitte dAndrea-Novel, and Arnaud de La Fortelle *The Kinematic Bicycle Model: a Consistent Model for Planning Feasible Trajectories for Autonomous Vehicles?* HAL Id: hal-01520869, https://hal-polytechnique.archives-ouvertes.fr/hal-01520869

[3] Marcos R. O., A. Maximo *Model Predictive Controller for Trajectory Tracking by Differential Drive Robot with Actuation constraints*