Introduction to Dynamic Optimization

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Dynamic Optimization

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- People and firms routinely make decisions that affect their future opportunities.
- For example, a household chooses how much of its income to consume today and to save for the future.

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- A person has some initial quantity of cake
- The person receives utility from consuming cake in period 0 and in period 1 only.
- Problem: what is the optimal cake consumption path through periods 0 and 1?

Preferences

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$$U(C_0, C_1) = \log C_0 + \beta \log C_1, \tag{1}$$

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- U is an *intertemporal utility function* and $\log C$ is the *flow* of utility generated each period.

- $0 \leqslant \beta \leqslant 1$ is the person's *subjective discount factor*.

Preferences

- The utility function implies indifference curves of the form:

$$C_1 = e^{\frac{\bar{U} - \log C_0}{\beta}}, \tag{2}$$

where $\bar{U}$ is some given level of utility.

- As long as $0 < \beta < 1$, the indifference curves are convex to the origin.

Budget Constraints and the Boundary Condition

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- The cake neither grows nor depreciates over time so the person faces the following budget constraints each period:

$$C_0 = K_0 - K_1 \tag{3}$$

$$C_1 = K_1 - K_2. \tag{4}$$

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- Since the person only consumes cake in periods 0 and 1, there is no value in leaving leftover cake at the end of period 1 which implies the boundary condition:

$$K_2 = 0. \tag{5}$$

Budget Constraints and the Boundary Condition

- The boundary condition allows us to eliminate K_2 from the period 1 budget constraint:

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- Notice that the implied price of C_1 in terms of C_0 is 1.

Optimization

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- The person's problem is to choose C_0, C_1, K_1, and K_2 to maximize his utility subject to the two budget constraints and the boundary condition.
- However, the problem can be simplified:
  1. Use the boundary condition to eliminate K_2 from the budget constraints.
  2. Then use the budget constraints to substitute C_0 and C_1 out of the utility function.

Optimization

- The result is an unconstrained optimization problem in K_1.

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- Take the derivative with respect to K_1 and set the derivative equal to zero to obtain the *first-order* or *optimality* condition:

$$-\frac{1}{K_0 - K_1} + \frac{\beta}{K_1} = 0. \tag{10}$$

Optimization

- Solve for K_1 to find the optimal cake holding for period 1:

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- Cake in period 1 is proportional to initial cake value.

Optimization

- Then use the budget constraints to infer optimal cake consumption each period:

$$\boxed{C_0 = \frac{1 - 2\beta}{1 - \beta} K_0} \tag{12}$$

and:

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- Notice that $C_0 + C_1 = K_0$ so all of the cake is eaten by the end.