Introduction to Real Business Cycle Modeling

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Introduction

- Recall the Solow growth model with stochastic TFP:

$$
\begin{aligned}
Y_t &= A_t K_t^{\alpha} && (1)\\
C_t &= (1-s)Y_t && (2)\\
Y_t &= C_t + I_t && (3)\\
K_{t+1} &= I_t + (1-\delta)K_t && (4)\\
\log A_{t+1} &= \rho \log A_t + \epsilon_{t+1}, && (5)
\end{aligned}
$$

  where $\epsilon_{t+1} \sim \mathcal{N}\left(0, \sigma^2\right)$.

- The model generates business cycle-like fluctuations in output, consumption, and investment.

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    2. Compare simulated output, consumption, and investment data with the actual data

- But keep in mind that the Solow model was not designed to explain business cycles.

Figure 1: **GDP.** The stochastic Solow growth model does a *reasonably* good job matching GDP fluctuations for the US from April 1948 to July 2018. Source: FRED.

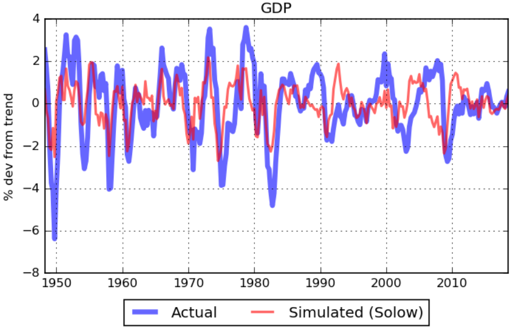

Figure 2: **Consumption.** The stochastic Solow growth model also does a *reasonably* good job matching consumption fluctuations for the US from April 1948 to July 2018. Source: FRED.

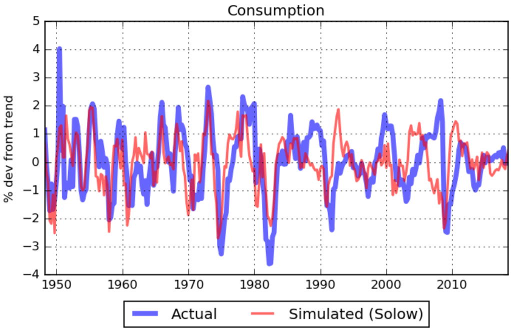

Figure 3: **Investment.** The stochastic Solow growth model *under-predicts* the magnitude of investment fluctuations for the US from April 1948 to July 2018. Source: FRED.

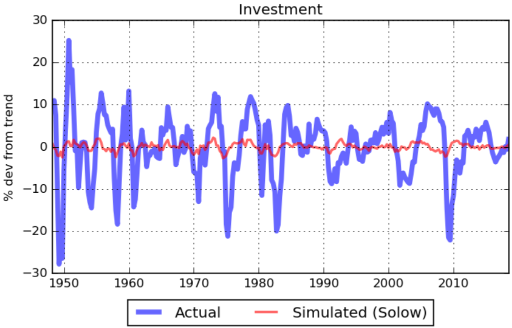

Table 1: **Standard deviations.** Actual data and data simulated from stochastic Solow model. Units are percent deviations from trend. The model under-predicts the volatility of investment in particular. Source: FRED.

	Actual Data	Simulated Data (Solow)
Output	1.62	0.93
Consumption	1.16	0.93
Investment	7.50	0.93

Table 2: **Correlations with GDP.** Actual data and data simulated from stochastic Solow model. The model over-predicts the correlation of investment and consumption with GDP. Source: FRED.

	Actual Data	Simulated Data (Solow)
Consumption	0.79	1.0
Investment	0.85	1.0

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- Summary analysis of the stochastic Solow growth model:
  - Simulated fluctuations are comparable in scale to observed fluctuations
  - Under-predicts investment volatility
  - Over-predicts consumption-GDP and investment-GDP correlation
  - No explanation for why labor fluctuates over the business cycle.
- **Objective**: Extend the model to improve performance.

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- Prescot's model is a **real business cycle** (**RBC**) model because it has no role for nominal quantities like inflation or nominal interest rates.

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  1. The **baseline** RBC model described in these slides:
     - **No labor supply choice**: household only chooses how much to consume and save.
     - **Centralized**: Household makes all production and allocation decisions. I.e., no firms, markets, or prices
  2. In the next lecture, we'll add labor choice to the baseline model

Centralized RBC Model without Labor

- A *representative household* lives for an infinite number of periods.

- The *expected present value of lifetime utility* to the household from consuming $C_0, C_1, C_2, \ldots$ is denoted by U_0:

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$$U_0 = \log(C_0) + \beta E_0 \log(C_1) + \beta^2 E_0 \log(C_2) + \cdots \quad (6)$$

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$$\begin{aligned} U_0 &= \log(C_0) + \beta E_0 \log(C_1) + \beta^2 E_0 \log(C_2) + \cdots && (6) \\ &= E_0 \sum_{t=0}^{\infty} \beta^t \log(C_t), && (7) \end{aligned}$$

- $0 < \beta < 1$ is the household's *subjective discount factor*. (For a quarterly model: $\beta \approx 0.99$ makes usually sense)

- E_0 denotes the *expectation with respect to all information available as of date 0*.

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- Production in period t:

$$F(A_t, K_t) = A_t K_t^{\alpha} \tag{8}$$

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- Capital depreciates at the constant rate δ per period.

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&\vdots && (13)
\end{aligned}
$$

- Express the constraints more concisely as

$$
C_t + K_{t+1} = A_t K_t^{\alpha} + (1-\delta)K_t, \qquad (14)
$$

for $t = 0, 1, 2, 3, \ldots$

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- We'll solve the problem for period 0 and then generalize the solution to apply to all periods $0, 1, 2, \ldots$

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- In period 0, the household solves:

$$\max_{C_0, K_1} E_0 \sum_{t=0}^{\infty} \beta^t \log(C_t) \tag{15}$$

$$\text{s.t.} \quad C_t + K_{t+1} = A_t K_t^{\alpha} + (1-\delta) K_t$$

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- The problem can be written as a choice of K_1 only:

$$\max_{K_1} \; E_0 \sum_{t=0}^{\infty} \beta^t \log(A_t K_t^{\alpha} + (1-\delta) K_t - K_{t+1}) \tag{16}$$

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- Note:

$$E_0 \sum_{t=0}^{\infty} \beta^t \log(AK_t^{\alpha} + (1-\delta)K_t - K_{t+1})$$
$$= \log(A_0 K_0^{\alpha} + (1-\delta)K_0 - K_1)$$
$$+ \beta E_0 \log(A_1 K_1^{\alpha} + (1-\delta)K_1 - K_2)$$
$$+ [\text{terms independent of } K_1] \tag{17}$$

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$$+ [\text{terms independent of } K_1] \tag{17}$$

- So:

$$\frac{\partial}{\partial K_1} U_0 = -\frac{1}{A_0 K_0^\alpha + (1-\delta)K_0 - K_1} + \beta E_0 \left[\frac{\alpha A_1 K_1^{\alpha-1} + 1 - \delta}{A_1 K_1^\alpha + (1-\delta)K_1 - K_2} \right] \tag{18}$$

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- Therefore the first-order condition for the optimal choice of K_1 is:

$$\frac{1}{A_0 K_0^{\alpha} + (1-\delta)K_0 - K_1} = \beta E_0 \left[\frac{\alpha A_1 K_1^{\alpha-1} + 1 - \delta}{A_1 K_1^{\alpha} + (1-\delta)K_1 - K_2} \right] \tag{19}$$

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$$\frac{1}{\underbrace{A_0 K_0^{\alpha} + (1-\delta)K_0 - K_1}_{C_0}} = \beta E_0 \left[\frac{\alpha A_1 K_1^{\alpha-1} + 1 - \delta}{\underbrace{A_1 K_1^{\alpha} + (1-\delta)K_1 - K_2}_{C_1}} \right] \tag{19}$$

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- Or more concisely:

$$\frac{1}{C_0} = \beta E_0 \left[\frac{\alpha A_1 K_1^{\alpha-1} + 1 - \delta}{C_1} \right] \tag{20}$$

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- So given $K_0 > 0$ and A_0, the equilibrium paths for consumption, capital, and TFP are described described by:

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- Computing numeric values for consumption and capital is not trivial. Recall the Euler equation:

$$\frac{1}{C_t} = \beta E_t \left[\frac{\alpha A_{t+1} K_{t+1}^{\alpha-1} + 1 - \delta}{C_{t+1}} \right] \tag{24}$$

- Consumption at date t depends on the *expectation* of consumption at date $t+1$ which in turn depends on the expectation of consumption at date $t+2$ and so on.

- Solving the problem requires numerical methods like those employed in the `linearsolve` Python package.

Figure 4: **Baseline RBC model without labor.** Impulse responses to a one percent shock to TFP in period 5.

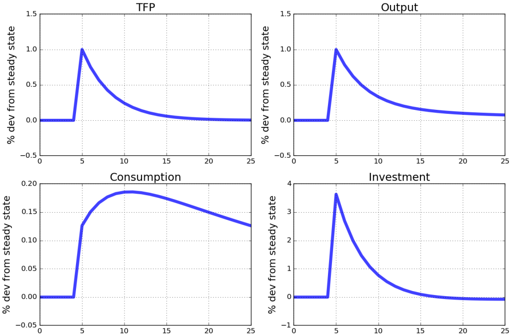

Figure 5: **GDP.** Like the stochastic Solow model, the baseline RBC model without labor does a *reasonably* good job matching GDP fluctuations for the US. Source: FRED.

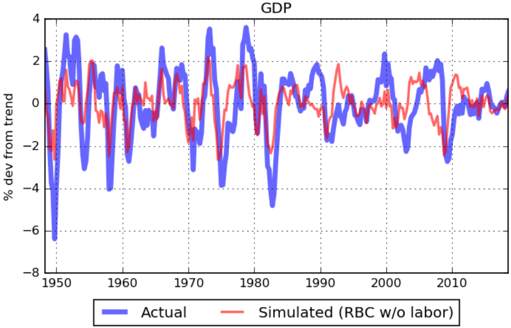

Figure 6: **Consumption.** In contrast to the stochastic Solow model, the baseline RBC model without labor *under-predicts* the magnitude of consumption fluctuations for the US. Source: FRED.

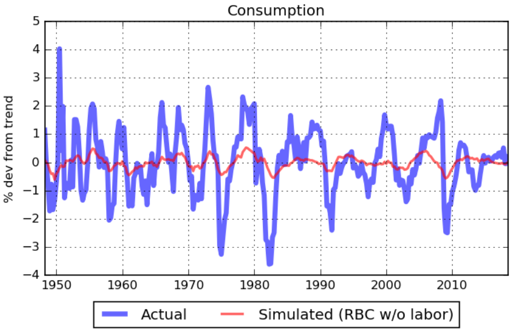

Figure 7: **Investment.** The baseline RBC model does a *reasonably* good job matching the magnitude of investment fluctuations better than the stochastic Solow model. Source: FRED.

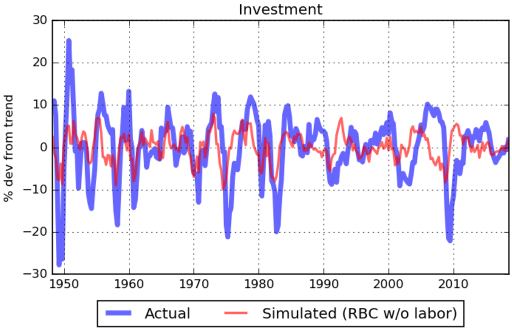

Table 3: **Standard deviations.** Actual data and simulated data. Units are percent deviations from trend. Source: FRED.

	Actual Data	Solow	RBC w/o Labor
Output	1.62	0.93	0.94
Consumption	1.16	0.93	0.22
Investment	7.50	0.93	3.39

Table 4: **Correlations with GDP.** Actual data and simulated data. Source: FRED.

	Actual Data	Solow	RBC w/o Labor
Consumption	0.79	1.0	0.62
Investment	0.85	1.0	0.99

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- **Next**: Add a labor-leisure tradeoff to the household's problem.

References

Prescott, Edward C., "Theory Ahead of Business Cycle Measurement," *Federal Reserve Bank of Minneapolis Quarterly Review*, Fall 1986, *10* (4), 9–22.