

Introduction to Dynamic Optimization

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Dynamic Optimization

A Two-Period Cake-Eating Problem

- A person has some initial quantity of cake
- The person receives utility from consuming cake in period 0 and in period 1 only.
- Problem: what is the optimal cake consumption path through periods 0 and 1?

- The person's utility in period 0 from consuming C_0 and C_1 is given by:

$$U(C_0, C_1) = \log C_0 + \beta \log C_1, \quad (1)$$

- U is an *intertemporal utility function* and $\log C$ is the *flow* of utility generated each period.
- $0 \leq \beta \leq 1$ is the person's *subjective discount factor*.

- The utility function implies indifference curves of the form:

$$C_1 = e^{\frac{\bar{U} - \log C_0}{\beta}}, \quad (2)$$

where \bar{U} is some given level of utility.

- As long as $0 < \beta < 1$, the indifference curves are convex to the origin.

Budget Constraints and the Boundary Condition

- Let K_t denote the quantity of cake available in period t with K_0 given.
- The cake neither grows nor depreciates over time so the person faces the following budget constraints each period:

$$C_0 = K_0 - K_1 \tag{3}$$

$$C_1 = K_1 - K_2. \tag{4}$$

Budget Constraints and the Boundary Condition

- A *boundary condition* imposes a limit on the endpoint of the problem.
- Since the person only consumes cake in periods 0 and 1, there is no value in leaving leftover cake at the end of period 1 which implies the boundary condition:

$$K_2 = 0. \tag{5}$$

Budget Constraints and the Boundary Condition

- The boundary condition allows us to eliminate K_2 from the period 1 budget constraint:

$$C_0 = K_0 - K_1 \tag{6}$$

$$C_1 = K_1. \tag{7}$$

- Combine the two budget constraints to eliminate K_1 and obtain an *intertemporal budget constraint*:

$$C_1 = K_0 - C_0. \tag{8}$$

- Notice that the implied price of C_1 in terms of C_0 is 1.

- The person's problem is to choose C_0 , C_1 , K_1 , and K_2 to maximize his utility subject to the two budget constraints and the boundary condition.
- However, the problem can be simplified:
 - 1 Use the boundary condition to eliminate K_2 from the budget constraints.
 - 2 Then use the budget constraints to substitute C_0 and C_1 out of the utility function.

- The result is an unconstrained optimization problem in K_1 .

$$\max_{K_1} \log(K_0 - K_1) + \beta \log K_1, \quad (9)$$

- Take the derivative with respect to K_1 and set the derivative equal to zero to obtain the *first-order* or *optimality* condition:

$$-\frac{1}{K_0 - K_1} + \frac{\beta}{K_1} = 0. \quad (10)$$

- Solve for K_1 to find the optimal cake holding for period 1:

$$K_1 = \frac{\beta}{1 - \beta} K_0 \quad (11)$$

- Cake in period 1 is proportional to initial cake value.

- Then use the budget constraints to infer optimal cake consumption each period:

$$C_0 = \frac{1 - 2\beta}{1 - \beta} K_0 \quad (12)$$

and:

$$C_1 = \frac{\beta}{1 - \beta} K_0. \quad (13)$$

- Notice that $C_0 + C_1 = K_0$ so all of the cake is eaten by the end.