

# Introduction to Dynamic Optimization

Brian C. Jenkins

University of California, Irvine

February 21, 2019

# Dynamic Optimization

- Dynamic optimization is at the heart of macroeconomic theory.

# Dynamic Optimization

- Dynamic optimization is at the heart of macroeconomic theory.
- People and firms routinely make decisions that affect their future opportunities.

# Dynamic Optimization

- Dynamic optimization is at the heart of macroeconomic theory.
- People and firms routinely make decisions that affect their future opportunities.
- For example, a household chooses how much of its income to consume today and to save for the future.

# A Two-Period Cake-Eating Problem

- A person has some initial quantity of cake

# A Two-Period Cake-Eating Problem

- A person has some initial quantity of cake
- The person receives utility from consuming cake in period 0 and in period 1 only.

# A Two-Period Cake-Eating Problem

- A person has some initial quantity of cake
- The person receives utility from consuming cake in period 0 and in period 1 only.
- Problem: what is the optimal cake consumption path through periods 0 and 1?

- The person's utility in period 0 from consuming  $C_0$  and  $C_1$  is given by:

$$U(C_0, C_1) = \log C_0 + \beta \log C_1, \quad (1)$$



- The person's utility in period 0 from consuming  $C_0$  and  $C_1$  is given by:

$$U(C_0, C_1) = \log C_0 + \beta \log C_1, \quad (1)$$

- $U$  is an *intertemporal utility function* and  $\log C$  is the *flow* of utility generated each period.

- The person's utility in period 0 from consuming  $C_0$  and  $C_1$  is given by:

$$U(C_0, C_1) = \log C_0 + \beta \log C_1, \quad (1)$$

- $U$  is an *intertemporal utility function* and  $\log C$  is the *flow* of utility generated each period.
- $0 \leq \beta \leq 1$  is the person's *subjective discount factor*.

- The utility function implies indifference curves of the form:

$$C_1 = e^{\frac{\bar{U} - \log C_0}{\beta}}, \quad (2)$$

where  $\bar{U}$  is some given level of utility.

- As long as  $0 < \beta < 1$ , the indifference curves are convex to the origin.

# Budget Constraints and the Boundary Condition

- Let  $K_t$  denote the quantity of cake available in period  $t$  with  $K_0$  given.

# Budget Constraints and the Boundary Condition

- Let  $K_t$  denote the quantity of cake available in period  $t$  with  $K_0$  given.
- The cake neither grows nor depreciates over time so the person faces the following budget constraints each period:

$$C_0 = K_0 - K_1 \tag{3}$$

$$C_1 = K_1 - K_2. \tag{4}$$

# Budget Constraints and the Boundary Condition

- A *boundary condition* imposes a limit on the endpoint of the problem.

# Budget Constraints and the Boundary Condition

- A *boundary condition* imposes a limit on the endpoint of the problem.
- Since the person only consumes cake in periods 0 and 1, there is no value in leaving leftover cake at the end of period 1 which implies the boundary condition:

$$K_2 = 0. \tag{5}$$

# Budget Constraints and the Boundary Condition

- The boundary condition allows us to eliminate  $K_2$  from the period 1 budget constraint:

$$C_0 = K_0 - K_1 \tag{6}$$

$$C_1 = K_1. \tag{7}$$



# Budget Constraints and the Boundary Condition

- The boundary condition allows us to eliminate  $K_2$  from the period 1 budget constraint:

$$C_0 = K_0 - K_1 \quad (6)$$

$$C_1 = K_1. \quad (7)$$

- Combine the two budget constraints to eliminate  $K_1$  and obtain an *intertemporal budget constraint*:

$$C_1 = K_0 - C_0. \quad (8)$$

# Budget Constraints and the Boundary Condition

- The boundary condition allows us to eliminate  $K_2$  from the period 1 budget constraint:

$$C_0 = K_0 - K_1 \tag{6}$$

$$C_1 = K_1. \tag{7}$$

- Combine the two budget constraints to eliminate  $K_1$  and obtain an *intertemporal budget constraint*:

$$C_1 = K_0 - C_0. \tag{8}$$

- Notice that the implied price of  $C_1$  in terms of  $C_0$  is 1.

- The person's problem is to choose  $C_0$ ,  $C_1$ ,  $K_1$ , and  $K_2$  to maximize his utility subject to the two budget constraints and the boundary condition.

- The person's problem is to choose  $C_0$ ,  $C_1$ ,  $K_1$ , and  $K_2$  to maximize his utility subject to the two budget constraints and the boundary condition.
- However, the problem can be simplified:

- The person's problem is to choose  $C_0$ ,  $C_1$ ,  $K_1$ , and  $K_2$  to maximize his utility subject to the two budget constraints and the boundary condition.
- However, the problem can be simplified:
  - ① Use the boundary condition to eliminate  $K_2$  from the budget constraints.

- The person's problem is to choose  $C_0$ ,  $C_1$ ,  $K_1$ , and  $K_2$  to maximize his utility subject to the two budget constraints and the boundary condition.
- However, the problem can be simplified:
  - 1 Use the boundary condition to eliminate  $K_2$  from the budget constraints.
  - 2 Then use the budget constraints to substitute  $C_0$  and  $C_1$  out of the utility function.

- The result is an unconstrained optimization problem in  $K_1$ .

$$\max_{K_1} \log(K_0 - K_1) + \beta \log K_1, \quad (9)$$

- The result is an unconstrained optimization problem in  $K_1$ .

$$\max_{K_1} \log(K_0 - K_1) + \beta \log K_1, \quad (9)$$

- Take the derivative with respect to  $K_1$  and set the derivative equal to zero to obtain the *first-order* or *optimality* condition:

$$-\frac{1}{K_0 - K_1} + \frac{\beta}{K_1} = 0. \quad (10)$$



- Solve for  $K_1$  to find the optimal cake holding for period 1:

$$K_1 = \frac{\beta}{1 - \beta} K_0 \quad (11)$$

- Solve for  $K_1$  to find the optimal cake holding for period 1:

$$K_1 = \frac{\beta}{1 - \beta} K_0 \quad (11)$$

- Cake in period 1 is proportional to initial cake value.

- Then use the budget constraints to infer optimal cake consumption each period:

$$C_0 = \frac{1 - 2\beta}{1 - \beta} K_0 \quad (12)$$

and:

$$C_1 = \frac{\beta}{1 - \beta} K_0. \quad (13)$$

- Then use the budget constraints to infer optimal cake consumption each period:

$$C_0 = \frac{1 - 2\beta}{1 - \beta} K_0 \quad (12)$$

and:

$$C_1 = \frac{\beta}{1 - \beta} K_0. \quad (13)$$

- Notice that  $C_0 + C_1 = K_0$  so all of the cake is eaten by the end.