The distribution of s^2

Results needed.

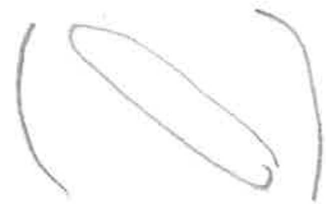
Spectral Decomposition Theorem

Suppose A is an $n \times n$ symmetric matrix, then:

$$A = \lambda_1 e_1 e_1' + \lambda_2 e_2 e_2' + \dots + \lambda_n e_n e_n'$$

where λ_i are eigen values and e_i are eigen vectors such that $e_i' e_j = 0$ when $i \neq j$ & $e_i' e_j = 1$ when $i = j$. We call such vectors 'orthonormal'

$$tr(A) = \sum_{i=1}^{n} a_{ii} = \text{sum of diagonal elements}$$



a) $tr(cA) = c\,tr(A)$

b) $tr(A \pm B) = tr(A) \pm tr(B)$

c) $tr(AB) = tr(BA)$

d) $tr(B^{-1}AB) = tr(A)$ [not used but included]

e) $tr(AA') = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2$ [not used but included]

If c is a vector $var(c'X) = c'\Sigma c$ where $\Sigma = cov(X)$

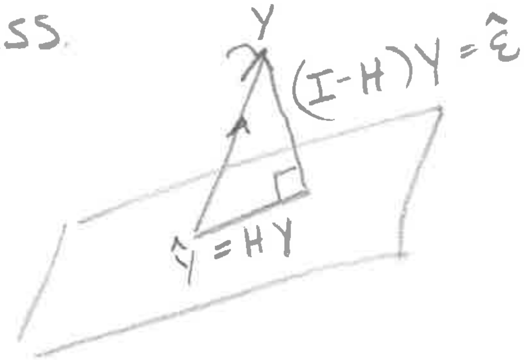
If $Z = CX$, C is a matrix then $cov(Z) = C\,cov(X)\,C'$

$$tr(A) = \sum_{i=1}^{n} \lambda_i$$

The distribution of s^2

$$\hat{\varepsilon}'\hat{\varepsilon} = (Y-\hat{Y})'(Y-\hat{Y}) = \sum_i \hat{\varepsilon}_i^2$$ — this is the RSS.

$$= [(I-H)Y]'(I-H)Y$$



$Y = X\beta + \varepsilon$

$$\hat{\varepsilon}'\hat{\varepsilon} = [(I-H)Y]'(I-H)(X\beta+\varepsilon)$$

$$(I-H)X = \emptyset$$

$$= [(I-H)Y]'(0 + (I-H)\varepsilon)$$

$$= [(I-H)(X\beta+\varepsilon)]'((I-H)\varepsilon)$$

$$= [(I-H)\varepsilon]'(I-H)\varepsilon$$

$$= \varepsilon'[(I-H)'](I-H)\varepsilon \qquad (I-H)' = I-H, \quad (I-H)^2 = I-H$$

$$= \varepsilon'(I-H)\varepsilon$$

Look at $I-H$.

$$tr(I-H) = tr(I) - tr(H)$$

$$tr(H) = tr(X(X'X)^{-1}X') = tr((X'X)^{-1}X'X) = tr(I_{K+1})$$

$$tr(I-H) = n - (K+1)$$

But $tr(I-H) = \sum_{i=1}^{n} \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ and since $I-H$ is idempotent $(I-H)^2 = I-H$, λ_i's are $0, 1$ so $n-(K+1)$ are 1s and $K+1$ are 0's

Distribution of s^2

$$\hat{\varepsilon}'\hat{\varepsilon} = \varepsilon'(I-H)\varepsilon$$

$$= \varepsilon'(\lambda_1 e_1 e_1' + \lambda_2 e_2 e_2' + \dots + \lambda_n e_n e_n')\varepsilon$$ spectral decomposition

But $k+1$ of the eigenvalues are zero – choose those to be last $\quad \lambda_i = 0 \ \forall i > n-(k+1)$

$$\therefore \hat{\varepsilon}'\hat{\varepsilon} = \varepsilon'(\lambda_1 e_1 e_1' + \dots + \lambda_{n-(k+1)} e_{n-(k+1)} e_{n-(k+1)}')\varepsilon$$

$$= \lambda_1 \varepsilon' e_1 e_1' \varepsilon + \dots + \lambda_{n-(k+1)} \varepsilon' e_{n-(k+1)} e_{n-(k+1)}' \varepsilon$$

$$= \lambda_1 V_1^2 + \dots + \lambda_{n-(k+1)} V_{n-(k+1)}^2$$

$$= V_1^2 + \dots + V_{n-(k+1)}^2, \quad \lambda_i = 1$$

Where $V_i = e_i'\varepsilon$; $\operatorname{var}(V_i) = e_i' \operatorname{cov}\varepsilon\, e_i = e_i'\sigma^2 I e_i = \sigma^2$

$\operatorname{cov}(V_i, V_j) = e_i' \operatorname{cov}(\varepsilon) e_j = e_i'\sigma^2 I e_j = 0 \quad i \neq j$

So, since $V_i = e_i'\varepsilon$, $\varepsilon \sim N$ $\therefore V_i \sim N$

Since $\operatorname{cov}(V_i, V_j) = 0$ $\quad V_i$ is indept V_j

$$\therefore \hat{\varepsilon}'\hat{\varepsilon} = \sum_{i=1}^{n-(k+1)} V_i^2 .$$

Let $Z_i = \dfrac{V_i - E(V_i)}{\sigma_{V_i}} = \dfrac{V_i - 0}{\sigma}$, $Z_i \sim N(0,1)$

$\therefore V_i = \sigma Z_i$

$$\text{So } \hat{\varepsilon}'\hat{\varepsilon} = \sum_{i=1}^{n-(k+1)} \sigma^2 Z_i^2 = \sigma^2 \sum_{1}^{n-(k+1)} Z_i^2$$

But $s^2 = \dfrac{\hat{\varepsilon}'\hat{\varepsilon}}{n-(k+1)}$, so $n-(k+1)\, s^2 = \sigma^2 \sum_{i=1}^{n-(k+1)} Z_i^2$

$$\chi^2_{n-(k+1)} = \sum Z_i^2$$

$$\chi^2_{n-(k+1)} = \frac{n-(k+1)\, s^2}{\sigma^2}$$