Last week: Chapters 15,16,17 Plus …

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This week

- Complete all your assigned work
- Monday:
  - Chapter 15 requirements
  - S^2 (one sample t-test) proofs (notice that S^2 has a different meaning in Regression)
- Wednesday, Friday:
  - Chapters 16,17
  - Quality Control
  - Product and System Reliability

The Exam

- Make sure you review Chapter 9 – we did a lot on this
- Exam will major on chapters 11-14
  - Proof
  - Examples
  - R
  - Book worked examples
- Chapter 15 – see next slides
- Chapters 16,17 will not be examined
- In addition, you learned in MATH 4753 $E(S^2) = \sigma^2$, now you have the tools to prove $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

Nonparametric Statistics

OBJECTIVE

To present some statistical tests that require fewer, or less stringent, assumptions than the methods of Chapters 8 and 10–14

CONTENTS

Sign Test for a Population Median, τ

One-Tailed Test

H_0: $\tau = \tau_0$
H_a: $\tau > \tau_0$ [or, H_a: $\tau < \tau_0$]

Test statistic:

S = Number of sample observations greater than τ_0
[or, S = Number of sample observations less than τ_0]

Two-Tailed Test

H_0: $\tau = \tau_0$
H_a: $\tau \neq \tau_0$

Test statistic:

S = larger of S_1 and S_2

where

S_1 = Number of sample observations greater than τ_0

S_2 = Number of sample observations less than τ_0

[*Note:* Eliminate observations from the analysis that are exactly equal to the hypothesized median, τ_0, and reduce the sample size accordingly.]

Observed significance level:

p-value $= P(x \geq S)$

Observed significance level:

p-value $= 2P(x \geq S)$

where x has a binomial distribution with parameters n and $p = .5$.

Rejection region: Reject H_0 if $\alpha > p$-value.

Assumption: The sample is randomly selected from a continuous probability distribution. (*Note*: No assumptions have to be made about the shape of the probability distribution.)

Wilcoxon Rank Sum Test for a Shift in Population Locations: Independent Random Samples*

Let D_1 and D_2 represent the relative frequency distributions for populations 1 and 2, respectively.

One-Tailed Test

H_0: D_1 and D_2 are identical

H_a: D_1 is shifted to the right of D_2
(or H_a: D_1 is shifted to the left of D_2)

Two-Tailed Test

H_0: D_1 and D_2 are identical

H_a: D_1 is shifted either to the left or to the right of D_2

Rank the $n_1 + n_2$ observations in the two samples from the smallest (rank 1) to the largest (rank $n_1 + n_2$). Calculate T_1 and T_2, the rank sums associated with sample 1 and sample 2, respectively. Then calculate the test statistic.

Test statistic: (one-tailed)

T_1, if $n_1 < n_2$; T_2, if $n_2 < n_1$
(Either rank sum can be used if $n_1 = n_2$.)

Test statistic: (two-tailed)

T_1, if $n_1 < n_2$; T_2, if $n_2 < n_1$
(Either rank sum can be used if $n_1 = n_2$.) We will denote this rank sum as T.

Rejection region: (one-tailed)

T_1: $T_1 \geq T_U$ [or $T_1 \leq T_L$]

T_2: $T_2 \leq T_L$ [or $T_2 \geq T_U$]

Rejection region: (two-tailed)

$T \leq T_L$ or $T \geq T_U$

where T_L and T_U are obtained from Table 15, Appendix B.

Note: Tied observations are assigned ranks equal to the average of the ranks that would have been assigned to the observations had they not been tied.

The Wilcoxon Signed Ranks Test: Matched Pairs

Let D_1 and D_2 represent the relative frequency distributions for populations 1 and 2, respectively.

One-Tailed Test

H_0: D_1 and D_2 are identical

H_a: D_1 is shifted to the right of D_2
(or H_a: D_1 is shifted to the left of D_2)

Two-Tailed Test

H_0: D_1 and D_2 are identical

H_a: D_1 is shifted either to the left or to the right of D_2

Calculate the difference within each of the n matched pairs of observations. Then rank the absolute values of the n differences from the smallest (rank 1) to the highest (rank n) and calculate the rank sum T_- of the negative differences and the rank sum T_+ of the positive differences.

Test statistic (one-tailed):

T_-, the rank sum of the negative differences (or T_+, the rank sum of the positive differences)

Test statistic (two-tailed):

T, the smaller of T_- or T_+

Rejection region (one-tailed):

$T_- \leq T_0$ (or $T_+ \leq T_0$)

Rejection region (two-tailed):

$T \leq T_0$

where T_0 is given in Table 16 of Appendix B

(*Note:* Differences equal to 0 are eliminated and the number n of differences is reduced accordingly. Tied absolute differences receive ranks equal to the average of the ranks they would have received had they not been tied.)

The Wilcoxon Signed Ranks Test for the Median, τ, of a Single Population

One-Tailed Test

H_0: $\tau = \tau_0$

H_a: $\tau > \tau_0$ [or, H_a: $\tau < \tau_0$]

Test statistic:

T_-, the negative rank sum
[or, T_+, the positive rank sum]

Two-Tailed Test

H_0: $\tau = \tau_0$

H_a: $\tau \neq \tau_0$

Test statistic:

T, the smaller of the positive and negative rank sums, T_+ and T_-

[*Note:* The sample differences are computed as $(y_i - \tau_0)$.]

Rejection region: (one-tailed test)

$T_- \leq T_0$ [or, $T_+ \leq T_0$]

Rejection region: (two-tailed test)

$T \leq T_0$

where T_0 is found in Table 16 of Appendix B.

Assumptions:

1. A random sample of observations has been selected from the population.
2. The absolute differences $y_i - \tau_0$ can be ranked. [No assumptions must be made about the form of the population probability distribution.]
3. Differences equal to 0 are eliminated and n is reduced accordingly. Tied differences are assigned ranks equal to the average of the ranks of the tied observations.

Kruskal–Wallis H Test for Comparing k Population Probability Distributions: Completely Randomized Design

H_0: The k population probability distributions are identical

H_a: At least two of the k population probability distributions differ in location

Test statistic: $$H = \frac{12}{n(n+1)} \sum_{i=1}^{k} \frac{T_i^2}{n_i} - 3(n+1)$$

where

n_i = Number of measurements in sample i

T_i = Rank sum for sample i, where the rank of each measurement is computed according to its relative magnitude in the totality of data for the k samples

n = Total sample size $= n_1 + n_2 + \cdots + n_k$

Rejection region: $H > \chi^2_\alpha$ with $(k - 1)$ degrees of freedom

p-value: $P(\chi^2 > H_c)$

Assumptions:
1. The k samples are random and independent.
2. There are 5 or more measurements in each sample.
3. The observations can be ranked.

(*Note:* No assumptions have to be made about the shape of the population probability distributions.)

The Friedman F_r Test for a Randomized Block Design

H_0: The relative frequency distributions for the k populations are identical

H_a: At least two of the k populations differ in location (shifted either to the left or to the right of one another)

Test statistic: Rank each of the k observations within each block from the smallest (rank 1) to the largest (rank k). Calculate the treatment rank sums, $T_1, T_2, \ldots, T_k$. Then the test statistic is

$$F_r = \frac{12}{bk(b+1)} \sum T_i^2 - 3b(k+1)$$

where

b = Number of blocks employed in the experiment

k = Number of treatments

T_i = Sum of the ranks for the ith treatment

Rejection region: $F_r > \chi^2_\alpha$

p-value: $P(\chi^2 > F_r)$

where χ^2_α is based on $(k-1)$ degrees of freedom

Assumptions:

1. The k treatments were randomly assigned to the k experimental units within each block.
2. For the chi-square approximation to be adequate, either the number b of blocks or the number k of treatments should exceed 5.
3. Tied observations are assigned ranks equal to the average of the ranks that would have been assigned to the observations had they not been tied.

Theil's Test for Zero Slope in the Straight-Line Model $y = \beta_0 + \beta_1 x + \varepsilon$

One-Tailed Test

H_0: $\beta_1 = 0$

H_a: $\beta_1 > 0$ (or H_a: $\beta_1 < 0$)

Two-Tailed Test

H_0: $\beta_1 = 0$

H_a: $\beta_1 \neq 0$

Test statistic:

$$C = (-1)(\text{Number of negative } y_j - y_i \text{ differences}) + (1)(\text{Number of positive } y_j - y_i \text{ differences})$$

where y_i and y_j are the ith and jth observations ranked in increasing order of the x values, $i < j$.

Observed significance level:

$$p\text{-value} = \begin{cases} P(x \geq C) \text{ for } H_a\text{: } \beta_1 > 0 \\ P(x \leq C) \text{ for } H_a\text{: } \beta_1 < 0 \end{cases}$$

Observed significance level:

p-value $= 2\min(p_1, p_2)$

where

$p_1 = P(x \geq C)$

$p_2 = P(x \leq C)$

where the values of $P(x \geq C) = P(x \leq -C)$ are given in Table 18 of Appendix B.

Assumptions: The random error ε is independent.

Key Formulas

Test	Test Statistic	Large-Sample Approximation	
Sign	S = number of sample measurements greater than (or less than) hypothesized median, τ_0	$Z = \dfrac{S - .5n}{.5\sqrt{n}}$	842
Wilcoxon rank sum	T_1 = rank sum of sample 1 or T_2 = rank sum of sample 2	$Z = \dfrac{T_1 - \dfrac{n_1(n_1 + n_2 + 1)}{2}}{\sqrt{\dfrac{n_1 n_2(n_1 + n_2 + 1)}{12}}}$	845, 850
Wilcoxon signed ranks	T_- = negative rank sum or T_+ = positive rank sum	$Z = \dfrac{T_+ - \dfrac{n(n+1)}{4}}{\sqrt{\dfrac{n(n+1)(2n+1)}{24}}}$	854, 855
Kruskal–Wallis	$H = \dfrac{12}{n(n+1)}\sum \dfrac{T_j^2}{n_j} - 3(n+1)$		860
Friedman	$F_r = \dfrac{12}{bk(k+1)}\sum T_j^2 - 3b(k+1)$		865
Spearman rank correlation (shortcut formula)	$r_s = 1 - \dfrac{6\sum d_i^2}{n(n^2 - 1)}$ where d_i = difference in ranks of ith observations for samples 1 and 2		871
Theil's Zero slope	$C = (-1)$(number of negative $y_i - y_j$ differences) $+ (1)$ (Number of positive $y_i - y_j$ differences)		873

LANGUAGE LAB

Symbol	Description
τ (tau)	Population median
S	Test statistic for sign test (see Key Formulas)
T_i	Sum of ranks of observations in sample i
T_L	Critical lower Wilcoxon rank sum value
T_U	Critical upper Wilcoxon rank sum value
T_+	Sum of ranks of positive differences of paired observations
T_-	Sum of ranks of negative differences of paired observations
T_0	Critical value of Wilcoxon signed ranks test
H	Test statistic for Kruskal–Wallis test (see Key Formulas)
F_r	Test statistic for Friedman test (see Key Formulas)
r_s	Spearman's rank correlation coefficient (see Key Formulas)
ρ (rho)	Population correlation coefficient
C	Test statistic for Theil's zero slope test (see Key Formulas)

Chapter Summary Notes

- **Distribution-free tests**—do not rely on assumptions about the probability distribution of the sampled population
- **Nonparametrics**—distribution-free tests that are based on **rank statistics**
- *One-sample* nonparametric test for the population median—**sign test**
- Nonparametric test for *matched pairs*—**Wilcoxon rank test**
- Nonparametric test for a *completely randomized design*—**Kruskal–Wallis test**
- Nonparametric test for a *randomized block design*—**Friedman test**
- Nonparametric test for *rank correlation*—**Spearman's test**
- Nonparametric test for *zero slope*—**Theil's *C* test**

To review chapter 15 for the exam

- Go through the above mentioned tests
  - Know what a non-parametric test is
  - Know the name of the test and what parametric test it replaces.
  - Know the NULL for each test.
  - One caveat – You will need to know how the sign test works from a theoretical and more complete perspective

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

Proof 1

Proof 2

Identity:

$$\sum_i (\boldsymbol{x} - \boldsymbol{x}_i)^2 = \sum_i \boldsymbol{x}_i^2 - \boldsymbol{n}\bar{\boldsymbol{x}}^2$$

Projection theory