ECE 450 - Exam #4

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1 Solution

1.1 Problem A

1.1.1 The Bandpass Buttersworth Filter Design

Following the Buttersworth bandpass filter design procedure, the following $H(s)$ was obtained:

$$H(s) = \frac{s^2 3,920,400,000,000}{s^4 + s^3 2.8001 \cdot 10^6 + s^2 4.6405 \cdot 10^{12} + s 1.00805 \cdot 10^{18} + 1.296 \cdot 10^{23}}$$

The plot of this function is:

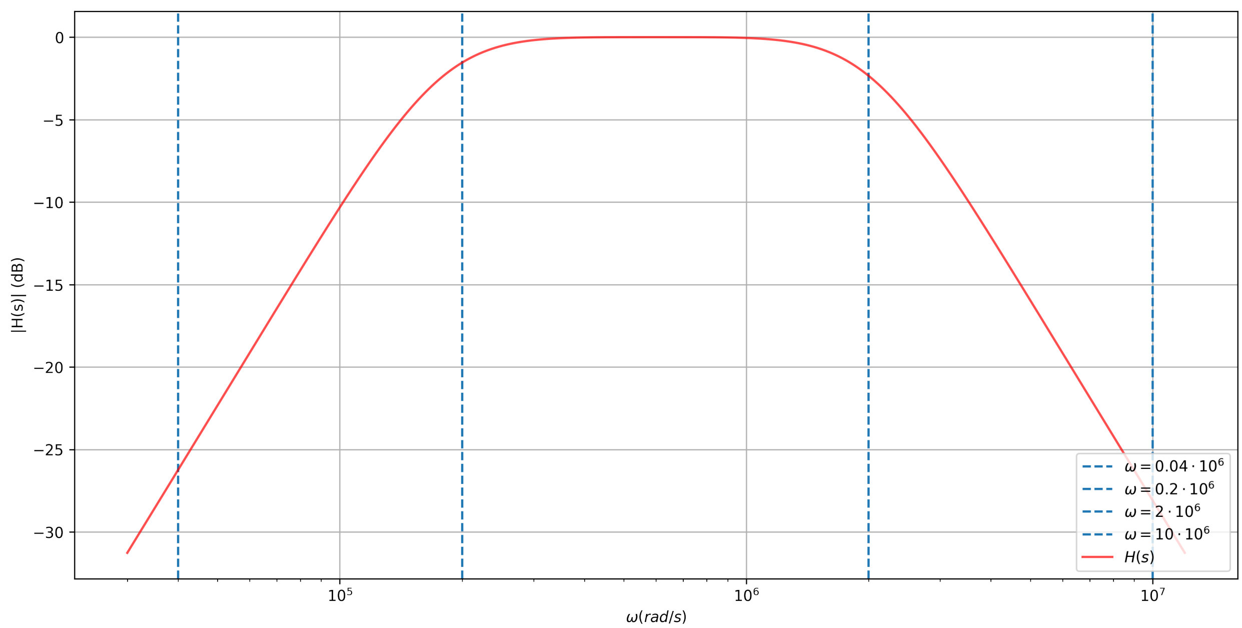


Figure 1. Bode plot of the $H(s)$ transfer function - markers on the pass and stopband frequencies.

I verified that the designed transfer function meets the given attenuation criteria, the results are:

```
Out[1]:   Omega        |H(s)|
          40000.0      -26.236057
```

200000.0	-1.542419
2000000.0	-2.339802
10000000.0	-28.077508

1.1.2 Convert $H(s)$ to $H(z)$ Using the Backwards Rectangular Approximation

$$T = 50e-9$$

$$H(z) = H(\frac{1-z^{-1}}{T}) = \frac{3.9204e12(\frac{1-z^{-1}}{50e-9})^2}{(\frac{1-z^{-1}}{50e-9})^4 + 2.8e6(\frac{1-z^{-1}}{50e-9})^3 + 4.64e12(\frac{1-z^{-1}}{50e-9})^2 + 1.01e18(\frac{1-z^{-1}}{50e-9})^1 + 1.3e23(\frac{1-z^{-1}}{50e-9})}$$

Resulting in the final $H(z)$:

$$H(z) = \frac{0.008509z^4 - 0.017019z^3 + 0.008509z^2 + 7.888609 \cdot 10^{-31}z + 9.860761 \cdot 10^{-32}}{z^4 - 3.857939z^3 + 5.584246z^2 - 3.594554z + 0.868248}$$

This results in the plot of $H(z)$:

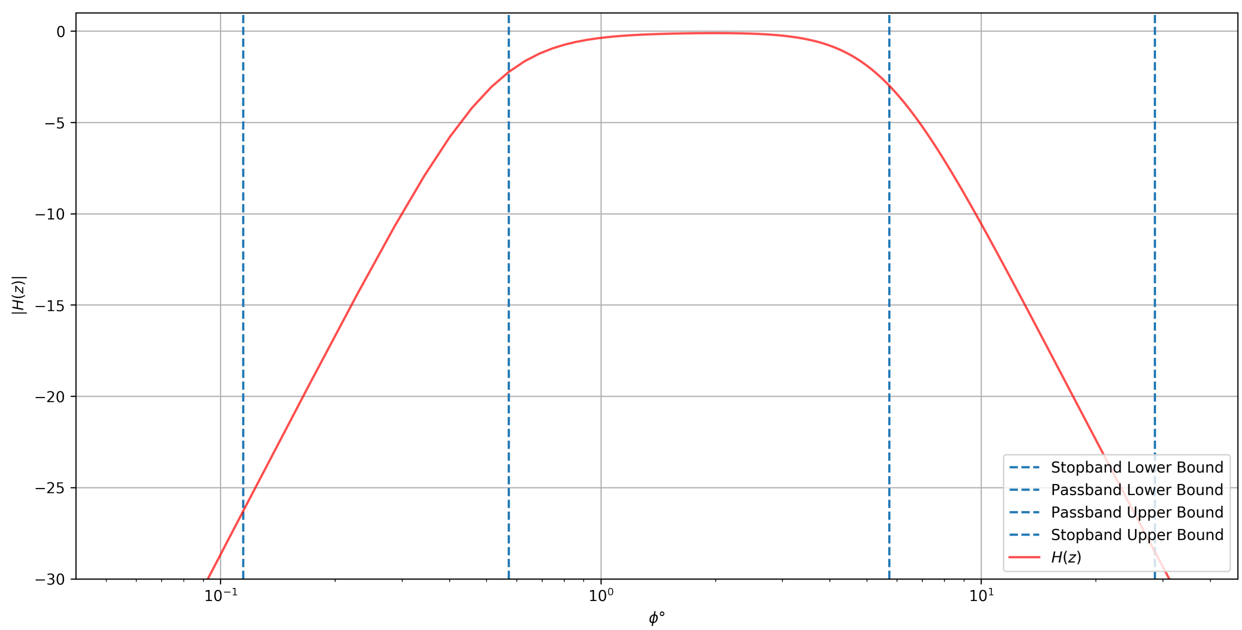


Figure 2. Bode plot of the $H(z)$ transfer function - markers on the pass and stopband angles.

Which still meets the attenuation criteria - the angles checked are a result of $\phi^\circ = \omega \cdot T \cdot \frac{180}{\pi}$.

ϕ°	$\lvert H(z)\rvert$
0.114592	-26.301946
0.572958	-2.231689
5.729578	-2.983381
28.64789	-28.535352

Clearly, the passband is not attenuated more than 3dB, and the stopband is attenuated more than 25dB.

1.2 Problem B

1.2.1 The Input Signal

For this FIR filter, I will be using the following input signal:

$$x(t) = sin(2\pi 1,000t) + sin(2\pi 10,000t) + sin(2\pi 20,000t) + sin(2\pi 50,000t)$$

The time-domain simulation of this signal is shown below, as is the fourier transform of this signal:

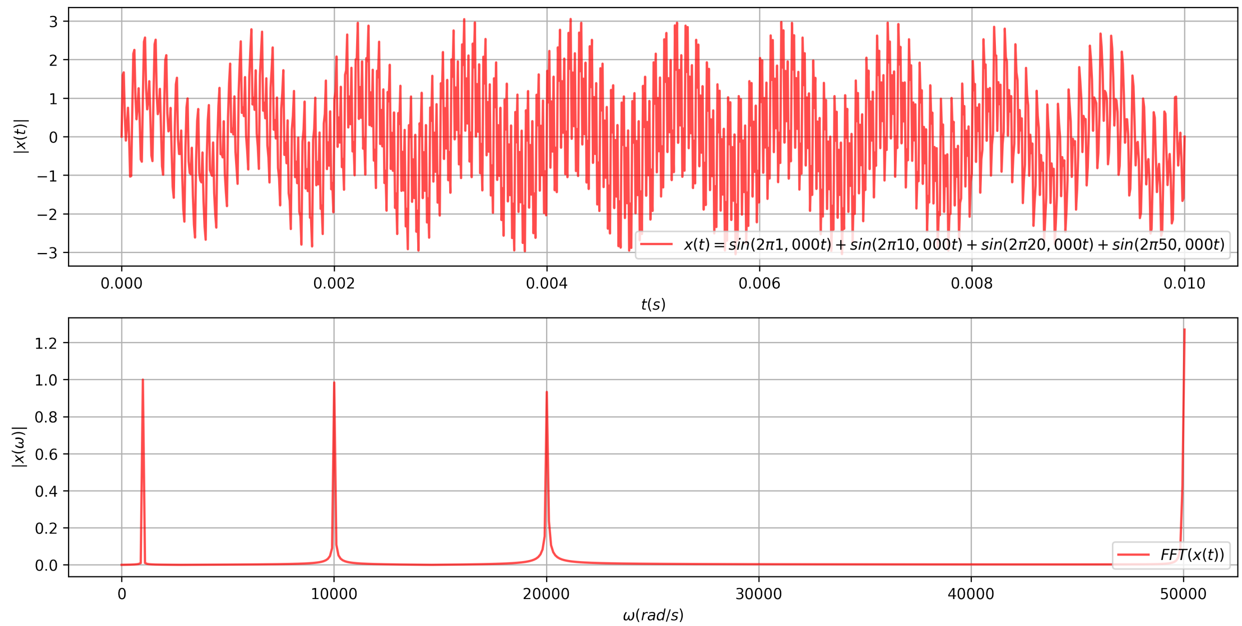


Figure 3. Time domain simulation of $x(t)$, and the FFT of that signal.

1.2.2 The Frequency-Domain Filter Design

Filtering out frequencies above the cutoff of $10^4 \frac{rad}{s}$ in the frequency plot results in the following fourier transform and time-domain signal.

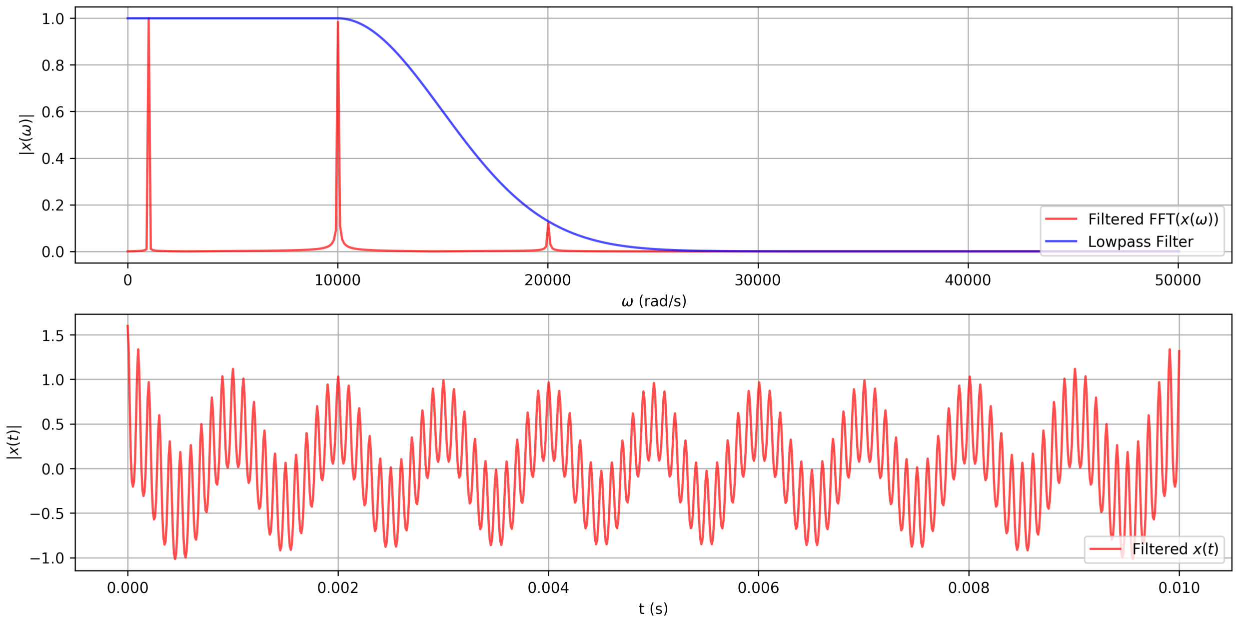


Figure 4. Lowpass filter applied to the frequency domain of the input signal and the corresponding IFFT of those filtered frequencies.

1.2.3 Convert the Frequency-Domain Filter to an FIR Filter

The filter designed here has a very slow cutoff, barely reaching the stopband criteria. This results in an easier-to-model FIR filter.

After trial-and-error, a filter length of 12 was determined to be adequate enough. Below is the entire $h[k]$, as well as the non-shifted and shifted results of this filter.

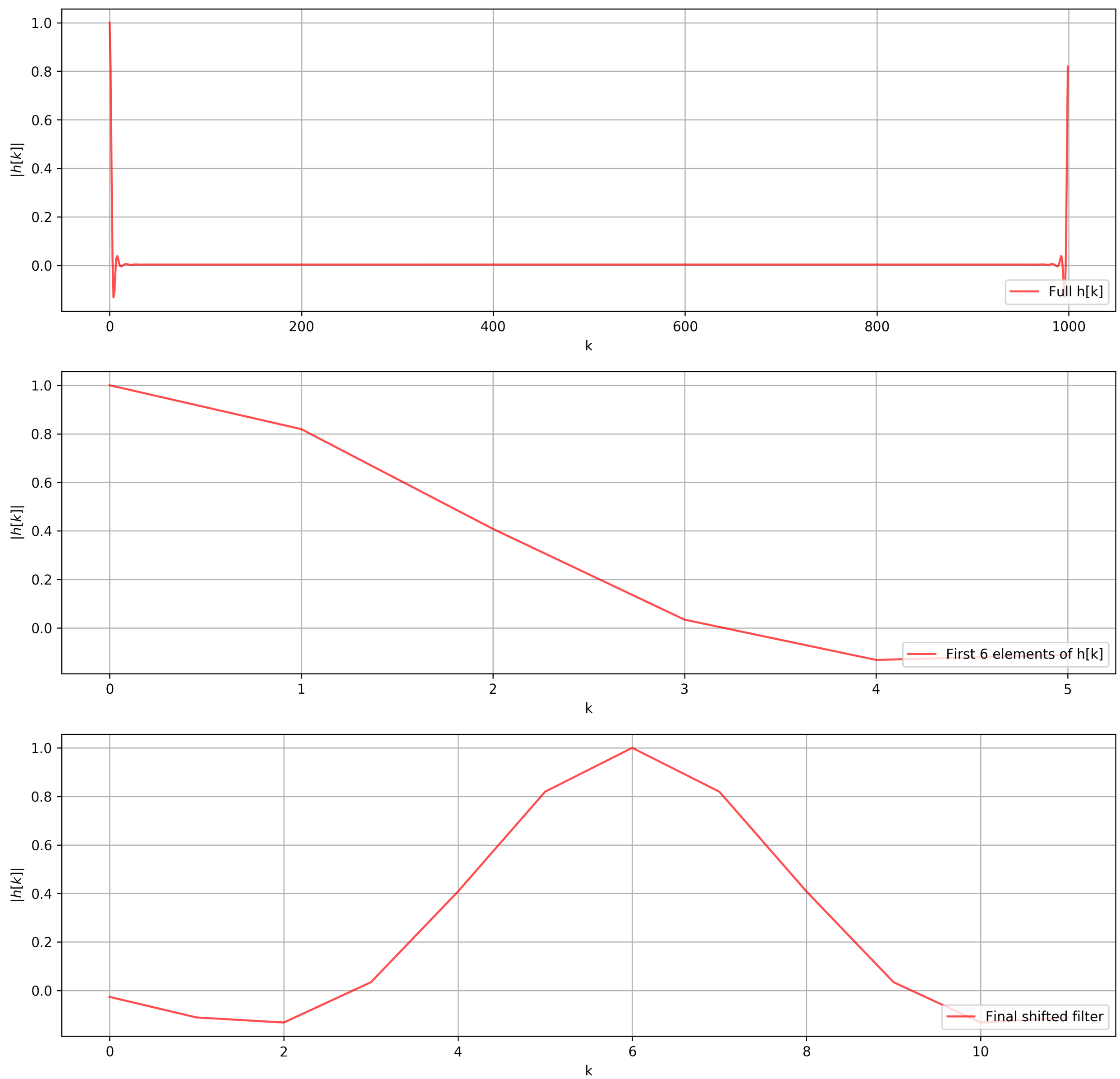


Figure 5. FIR implementation of the lowpass filter. Only 12 samples are necessary.

1.2.4 Verification of the FIR Filter

Convolving this FIR filter with the time-domain input signal results in the following input signal, as well as the new filtered FFT.

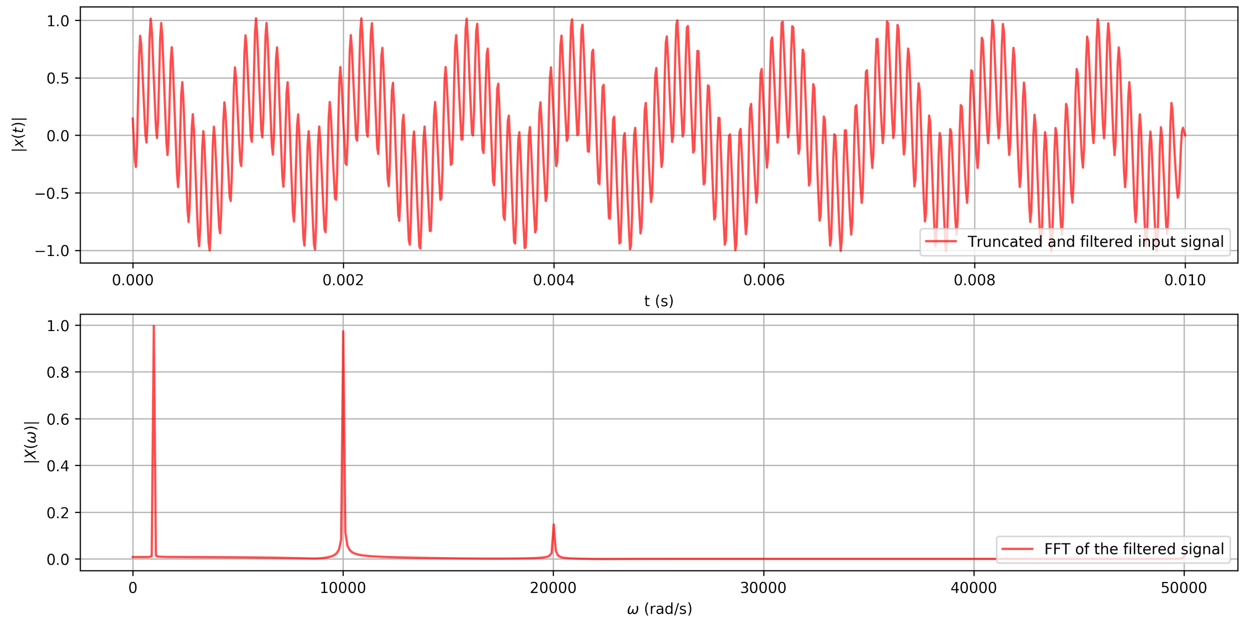


Figure 6. Filtered input signal and frequency characteristics.

To ensure this finite filter accomplishes the desired frequency-attenuation, looking at the FFT's magnitude is required. I've highlighted the magnitudes of the input signal's frequencies.

$\omega \frac{rad}{s}$	$\lvert FFT(X(\omega))\rvert$
1001.001001	0.996236
10010.01001	0.973437
20020.02002	0.147796
50050.05005	0.017076

The passband frequencies, those below 10^4 radians per second, are attenuated less than 15%, while those above the stopband frequency of $2 \cdot 10^4$ radians per second are attenuated more than 15%.

1.2.5 Explicit Magnitudes of $h[k]$

For this 12-unit long filter, the magnitudes of $h[k]$ are shown below:

k	$h[k]$
0	-0.026368
1	-0.111207
2	-0.131925
3	-0.034091
4	-0.408149
5	0.819260
6	1.000000
7	0.819260
8	0.408149

```
9     0.034091
10    -0.131925
11    -0.111207
```

2 Code Appendix

```
In [2]: import numpy as np
        import seaborn as sns
        import pandas as pd
        import matplotlib.pyplot as plt
        from scipy import signal as sig
        from control import margin, tf
```

2.0.1 Generic function to convolve any number of equations

```
In [3]: def convolve_all(values):
            temp_conv = values[0]
            if len(values) > 1:
                for next_val in values[1:]:
                    temp_conv = np.convolve(temp_conv, next_val)

            return temp_conv
```

2.0.2 Generic function to create plots

```
In [757]: # Color list for multiple lines on each subplot
          colors = ["red", "blue", "green", "gray", "purple", "orange"]

          # Generic Function to create a plot
          def create_plot(x, y, xLabel=["X-Values"], yLabel=["Y-Values"], title=[("Plot", )],
                          size=(18, 14), logx=False, y_lines=[[np.nan]], y_line_labels=[[""]], y
              plt.figure(figsize=size, dpi=300)
              for c, (x_vals, y_vals, x_labels, y_labels, titles, lines, line_labels, lims) in e
                  plt.ylim(lims[0], lims[1])
                  for line, label in zip(lines, line_labels): # Add each vertical line to this p
                      plt.axvline(x=line, linestyle='--', label=label)
                  for c2, (y_v, t) in enumerate(zip(y_vals, titles)):
                      plt.subplot(len(x), 1, c + 1)
                      # Add a plot to the subplot, use transparency so they can both be seen
                      plt.plot(x_vals, y_v, label=t, color=colors[c2], alpha=0.70)
                      plt.ylabel(y_labels)
                      plt.xlabel(x_labels)
                      plt.grid(True)
                      plt.legend(loc='lower right')
                      if logx:
                          plt.xscale("log")

              plt.show()
```

2.0.3 Generic function to generate the Bode plots of a transfer function

```
In [13]: def magnitude_phase_response(num, den, omega_range, omega_step=10, gain_num=None, gain_
             if isinstance(gain_num, (np.ndarray, list)) and isinstance(gain_den, (np.ndarray, l
                 num = convolve_all([num, gain_num])
                 den = convolve_all([den, gain_den])

             system = sig.lti(num, den)
             w, h_mag, h_phase = sig.bode(system, np.arange(omega_range[0], omega_range[1], omeg
             _, phase_margin, _, crossover_w = margin(h_mag, h_phase, w)

             return w, h_mag, h_phase, phase_margin, crossover_w
```

2.0.4 Generic function to generate an *nth* order Buttersworth Filter

```
In [19]: def buttersworth_tf(order):
             pole_list = [np.sin((np.pi * (2 * k - 1))/(2 * order)) + complex(0, np.cos((np.pi *
             s_pole_list = [[1, pole] for pole in pole_list]

             return convolve_all(s_pole_list)
```

2.0.5 Generic function to shift a given filter to a new center frequency

```
In [238]: def shift_filter(num, den, new_center):
              new_num = np.pad(num, (len(den) - len(num), 0), 'constant')
              new_den = den

              for i in range(len(new_den)):
                  new_num[i] *= (new_center ** i)
                  new_den[i] *= (new_center ** i)

              return np.real(new_num), np.real(new_den)
```

2.0.6 Generic function to convert a given lowpass filter to a bandpass filter

```
In [348]: # This function assumes the numerator is just a constant, no s-terms
          def convert_to_bandpass(num, den, w_c):
              num = np.trim_zeros(num, 'f') # Trim front-leading zeros
              den = np.trim_zeros(den, 'f')
              num = np.concatenate((num, [0] * (len(den) - 1))) # Shift num. by s^n
              temp_den = np.zeros((len(den) - 1) * 2 + 1)
              for power, constant in enumerate(den[::-1]):
                  if power is 0:
                      temp_den[-len(den)] = constant # Add in non-s term, shifted by s^n
                      continue

                  # Get product of the (s^2+wc^2)^n for this current power of s^n
                  non_shifted = np.multiply(constant, convolve_all([[1, 0, w_c ** 2] for _ in ra
                  # Shift by the multiplied s^n/s^n
```

```
            shifted = np.pad(non_shifted, (0, len(den)-power-1), mode='constant')
            # Add to running total den.
            temp_den = np.add(temp_den, np.pad(shifted, (len(temp_den)-len(shifted), 0), m

        return np.real(num), np.real(temp_den)
```

2.0.7 Generic function to create a plot of a z-domain transfer function

```
In [734]: def z_plot(num, den, T, phi_range=[None, np.pi]):
              phi_range[0] = T if phi_range[0] is None else phi_range[0]
              phi = np.arange(phi_range[0], phi_range[1], T)
              angles = angles = [np.exp(complex(0, angle)) for angle in phi]

              # Loop through all angles, calculate that angles H(z)
              h_z = []
              for z in angles:
                  num_sum, den_sum = 0, 0
                  for z_pow, num_val in enumerate(num):
                      num_sum += num_val * z ** (len(num) - z_pow)
                  for z_pow, den_val in enumerate(den):
                      den_sum += den_val * z ** (len(den) - z_pow)
                  h_z.append(num_sum / den_sum)

              return np.multiply(180 / np.pi, phi), 20 * np.log10(h_z)
```

2.0.8 Generic function to convert a given frequency to it's z-domain angle

```
In [738]: def convert_z_angle(omega, T):
              return omega * T * 180 / np.pi
```

2.1 Problem A

2.1.1 Obtain the transfer function $H_{BP}(s)$

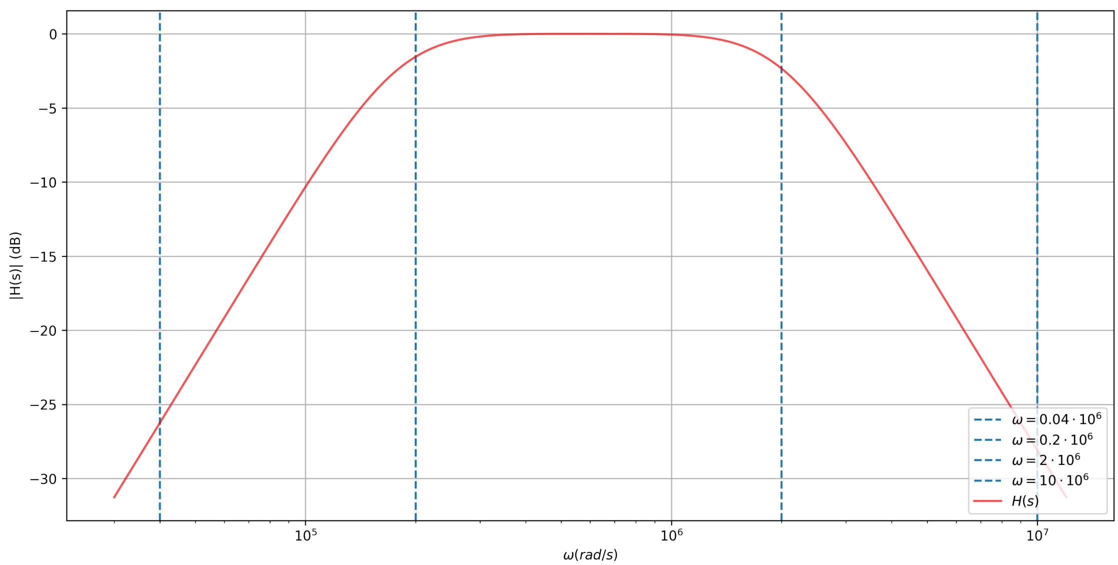
```
In [349]: # Define the variables of this problem
          passband_low, passband_high = 0.2e6, 2e6
          stopband_low, stopband_high = 0.04e6, 10e6
          h_p_db, h_s_db = -3, -25

          bandwidth = (passband_high - passband_low) * 1.1 # Expand the bandwidth a bit
          center_freq = .6e6 # Trial and error found this to accomplish the desireable attenuati
          h_p, h_s = 10 ** (0.05 * h_p_db), 10 ** (0.05 * h_s_db)
          w_p, w_s = 1, (stopband_high - stopband_low) / (center_freq)

          # Find the number of poles needed
          n = int(np.ceil(np.log10(1 / (h_s ** 2) - 1) / (2 * np.log10(w_s))))

          # Generate the low-pass filter with a corner frequency of 1
          num_lp = [1]
```

```
    den_lp = buttersworth_tf(n)

    # Transform to a low-pass filter at the desired bandwidth
    num_lp_shifted, den_lp_shifted = shift_filter(num_lp, den_lp, bandwidth)

    # Transform to a bandpass filter with G_lp((s^2+center^2)/s)
    num_bp, den_bp = convert_to_bandpass(num_lp_shifted, den_lp_shifted, center_freq)
    print ("Num: ", num_bp, "\nDen: ", np.real(den_bp))
```

```
Num:  [3920400000000              0              0]
Den:  [1.00000000e+00 2.80014285e+06 4.64040000e+12 1.00805143e+18
 1.29600000e+23]
```

2.1.2 Plot the transfer function $H_{BP}(s)$

```
In [728]: w, h_mag, _, _, _ = magnitude_phase_response(num_bp, den_bp, [0.03e6, 12e6], omega_ste
          create_plot([w], [(h_mag, )], ["$\omega (rad/s)$"], ["|H(s)| (dB)"], [("$H(s)$", )], l
                      y_lines=[[stopband_low, passband_low, passband_high, stopband_high]],
                      y_line_labels=[["$\omega = 0.04\cdot 10^6$", "$\omega = 0.2\cdot 10^6$",
                                      "$\omega = 2\cdot 10^6$", "$\omega = 10\cdot 10^6$"]], siz
```



2.1.3 Verify $H(s)$ meets the given attenuation criteria

```
In [343]: df = pd.DataFrame(np.transpose([w, h_mag]), columns=["$\omega$", "$|H(s)|$"])
          df[(df["$\omega$"]==passband_low) | (df['$\omega$'] == passband_high) |\
             (df['$\omega$'] == stopband_low) | (df['$\omega$'] == stopband_high)]
```

```
Out[343]:          $\omega$   $|H(s)|$
          10       40000.0 -26.236057
          170     200000.0  -1.542419
          1970   2000000.0  -2.339802
          9970  10000000.0 -28.077508
```

2.1.4 Convert $H(s)$ to $H(z)$ using the backwards rectangular approximation

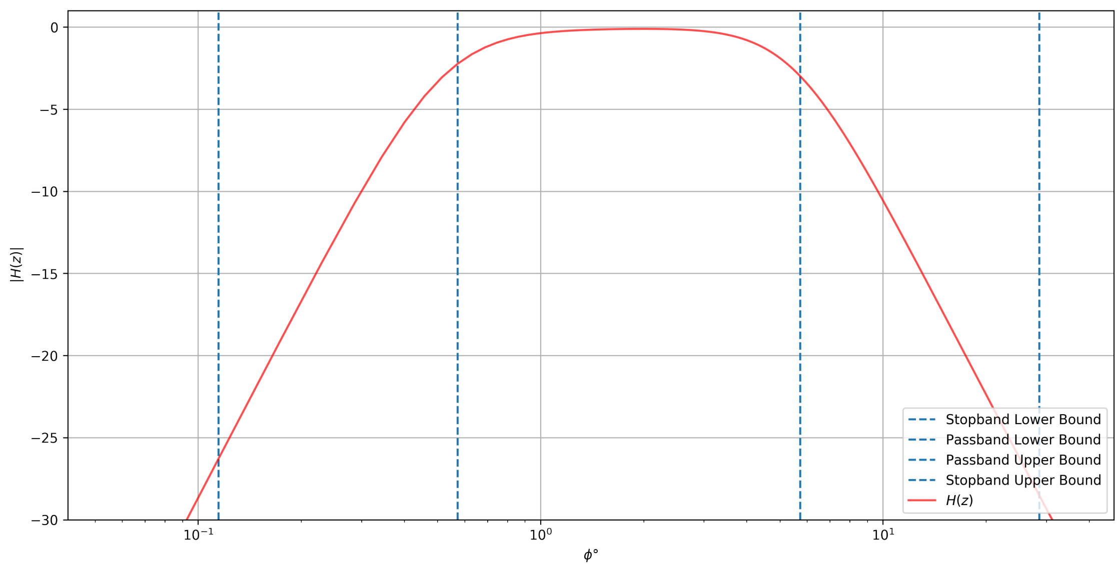
T = 50e-9

$$H(z) = H(\frac{1 - z^{-1}}{T}) = \frac{3.9204e12(\frac{1-z^{-1}}{50e-9})^2}{(\frac{1-z^{-1}}{50e-9})^4 + 2.8e6(\frac{1-z^{-1}}{50e-9})^3 + 4.64e12(\frac{1-z^{-1}}{50e-9})^2 + 1.01e18(\frac{1-z^{-1}}{50e-9})^1 + 1.3e23(\frac{1-z^{-1}}{50e-9})}$$

Resulting in the final $H(z)$:

$$H(z) = \frac{0.008509z^4 - 0.017019z^3 + 0.008509z^2 + 7.888609e{-31}z + 9.860761e{-32}}{z^4 - 3.857939z^3 + 5.584246z^2 - 3.594554z + 0.868248}$$

```
In [788]: # Sample period
          T = 50e-9
          # The H(z) transfer function
          num = [0.008509703539690870822, -0.017019407079381741644, 0.008509703539690870822,
                 7.8886090522101180541e-31, 9.8607613152626475676e-32]
          den = [1, -3.8579392997545856723, 5.5842464777865698716,
                 -3.5945549738519009091, 0.86824849910120098173]
          # Get the significant angles
          stopband_low_angle, stopband_high_angle = convert_z_angle(stopband_low, T), convert_z_
          passband_low_angle, passband_high_angle = convert_z_angle(passband_low, T), convert_z_

          # Plot the z-domain transfer function
          x, y = z_plot(num, den, 0.001, [0.00, 0.6])
          x, y = np.real(x), np.real(y)
          create_plot([x], [(y, )], ["$\phi°$"], ["$|H(z)|$"], [("$H(z)$", )], logx=True,
                      y_lines=[[stopband_low_angle, stopband_high_angle,passband_low_angle, pass
                      y_line_labels=[['Stopband Lower Bound', 'Passband Lower Bound',
                                      'Passband Upper Bound','Stopband Upper Bound']], ylim=[[-3
```



2.1.5 Verify the backwards rectangular approximation is still valid

```
In [786]: df = pd.DataFrame(np.transpose([x, y]), columns=["$\phi°$", "$|H(z)|$"])
          display(df.iloc[(df["$\phi°$"] - stopband_low_angle).abs().argsort()[:1]])
          display(df.iloc[(df["$\phi°$"] - passband_low_angle).abs().argsort()[:1]])
          display(df.iloc[(df["$\phi°$"] - passband_high_angle).abs().argsort()[:1]])
          display(df.iloc[(df["$\phi°$"] - stopband_high_angle).abs().argsort()[:1]])
```

```
    $\phi°$   $|H(z)|$
2  0.114592 -26.301946


     $\phi°$  $|H(z)|$
10  0.572958 -2.231689


      $\phi°$  $|H(z)|$
100  5.729578 -2.983381


      $\phi°$   $|H(z)|$
500  28.64789 -28.535352
```

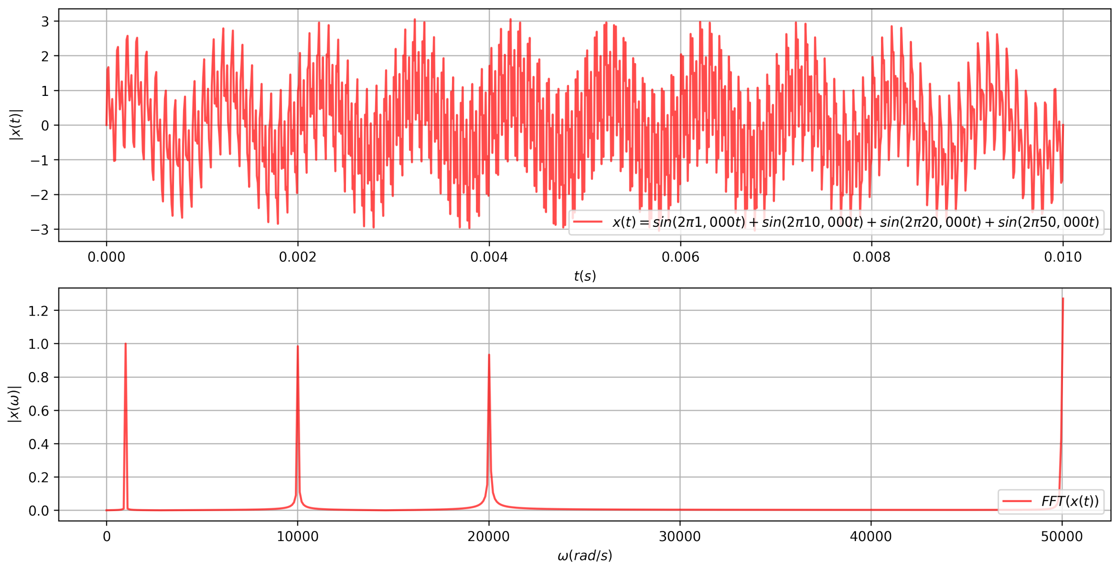
2.2 Problem B

Passband, $\omega \leq 10^4$ and is attenuated $0.85 \leq |H| \leq 1$.
Stopband, $\omega \geq 2 \cdot 10^4$ and is annenuated $0 \leq |H| \leq 0.15$.

$$T = 10e - 6$$

2.2.1 The input signal

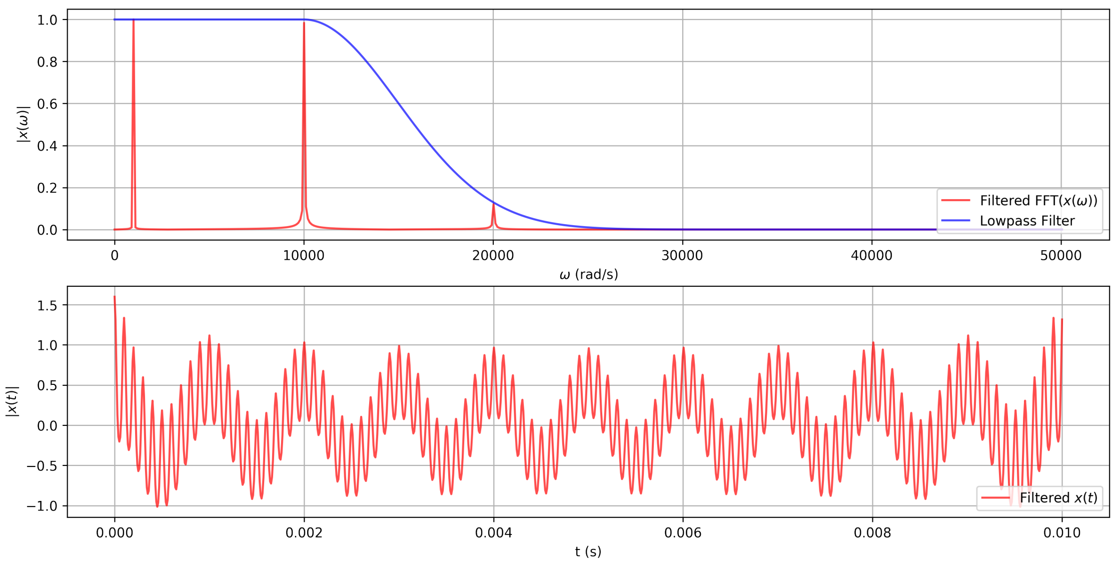
```
In [650]: pi2 = 2 * np.pi
          T, N = 10e-6, 1000
          # The time domain we'll plot against
          t = np.linspace(0, N*T, N)
          # The input signal has frequencies of 1,000 Hz, 10,000Hz, 20,000Hz, and 50,000Hz
          s = np.sin(pi2*1e3*t) + np.sin(pi2*1e4*t) + np.sin(pi2*2e4*t) + np.sin(pi2*5e4*t)
          # The FFT of the input signal - ignoring the mirrored frequencies
          fft = np.fft.fft(s) * 2 / N
          f = np.linspace(0, 1 / T, N)

          create_plot([t, f[:N//2+1]], [(s, ), (np.abs(fft)[:N//2+1], )],
                      ["$t (s)$", "$\omega (rad/s)$"], ["$|x(t)|$", "$|x(\omega)|$"],
                      [("$x(t)=sin(2\pi 1,000t)+sin(2\pi 10,000t)+sin(2\pi 20,000t)+sin(2\pi 50,
                       ("$FFT(x(t))$", )], y_lines=[[np.nan], [np.nan]], y_line_labels=[[""], ["
```



2.2.2 Design the filter

```
In [651]: cutoff_freq = 10 ** 4
          factor = 4650 # A higher factor means a slower decay - 4650 barely does it
          # Design the filter to be zeros and then decaying-exponential after the cutoff frequen
          filter_f = [1. if freq <= cutoff_freq else np.exp(-0.5 * (((freq - cutoff_freq) / fact
          # Apply the filter to the input signal's FFT
          filtered_fft = np.multiply(np.abs(fft), filter_f)
          # Take the IFFT of the filtered signal's FFT
          filtered_t = np.fft.ifft(filtered_fft).real * N / 2

          create_plot([f[:N//2+1], t], [(filtered_fft[:N//2+1], filter_f[:N//2+1]), (filtered_t,
```

```
                ["$\omega$ (rad/s)", "t (s)"], ["$|x(\omega)|$", "$|x(t)|$"],
                [("Filtered FFT($x(\omega)$)", "Lowpass Filter"), ("Filtered $x(t)$", )],
                y_lines=[[np.nan], [np.nan]], y_line_labels=[[""], [""]], size=(14, 7))
```

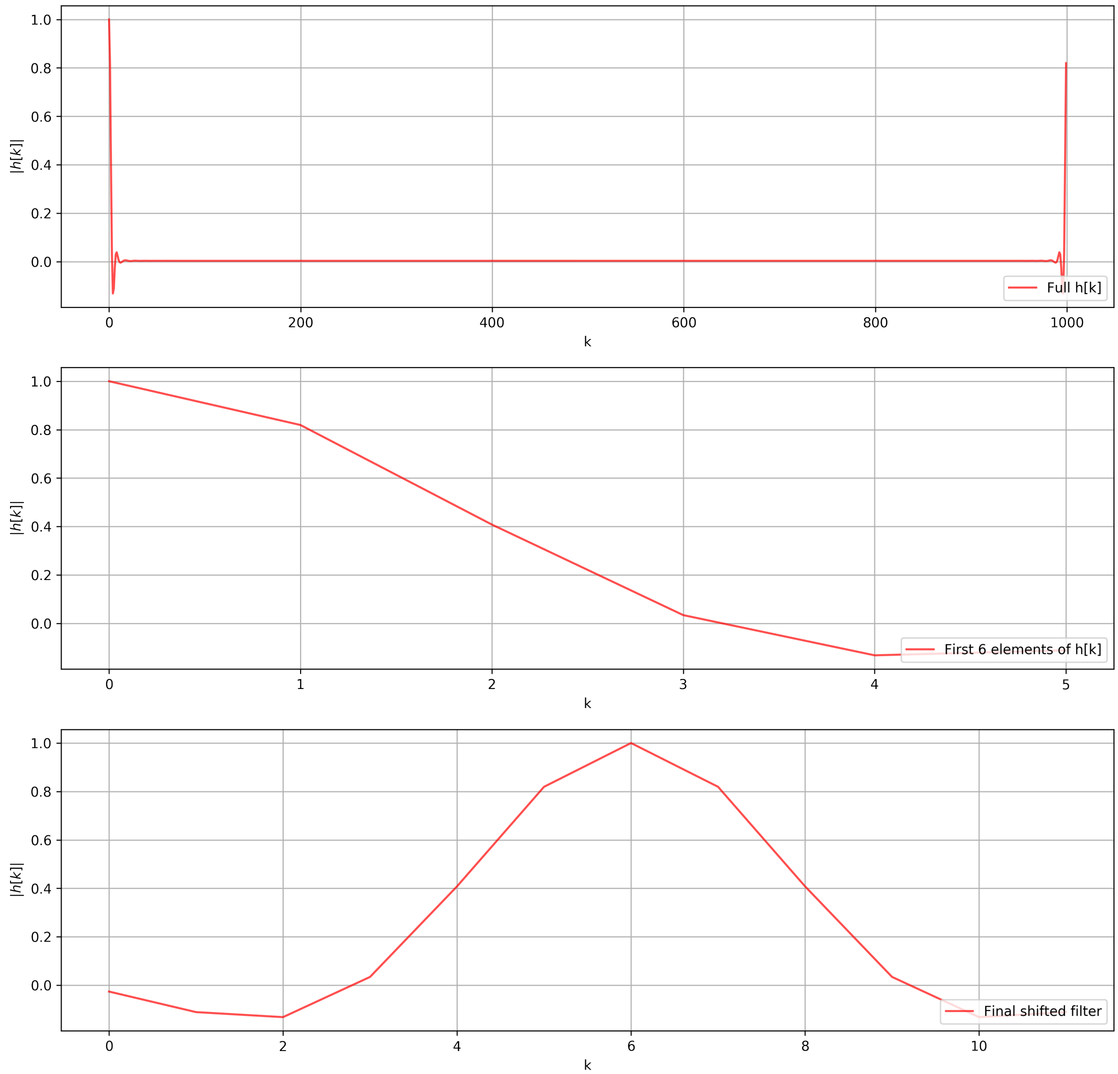



```
In [630]: df = pd.DataFrame(np.transpose([f, filtered_fft]), columns=["f", "x"])
          max(df[df['f'] >= 2e4]['x'])
```

```
Out[630]: 0.0915604036855752
```

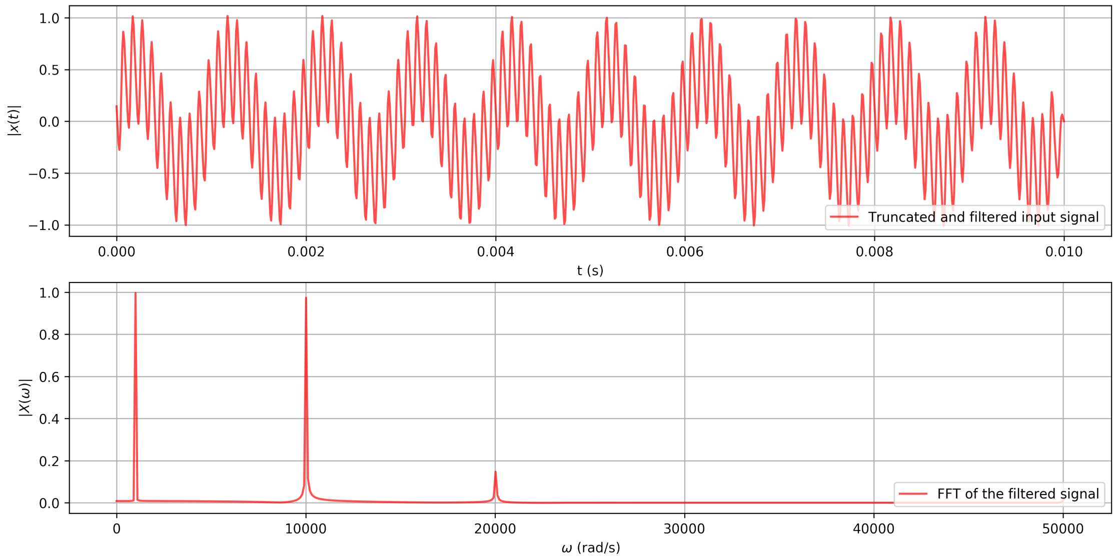
2.2.3 Convert the filter from the frequency filter to an FIR filter

```
In [724]: fir_length = 8
          # Take the IFFT of the above filter designed above
          filter_fir = np.fft.ifft(filter_f).real * N / 2
          # Normalize the filter
          max_val = max(filter_fir)
          filter_fir = [val / max_val for val in filter_fir]
          # Create the shifted and shortened filter
          final_filter = np.concatenate([filter_fir[-fir_length:], filter_fir[:fir_length]])

          create_plot([np.arange(len(filter_fir)), np.arange(fir_length), np.arange(fir_length *
                      [(filter_fir, ), (filter_fir[:fir_length], ), (final_filter, )],
                      ["k", "k", "k"], ["$|h[k]|$", "$|h[k]|$", "$|h[k]|$"],
                      [("Full h[k]", ), ("First {} elements of h[k]".format(fir_length), ), ("Fi
                      size=(14, 14), y_lines=[[np.nan], [np.nan], [np.nan]], y_line_labels=[[""]
```



2.2.4 Verify the FIR filter works

```
In [725]: # Convolve the FIR filter with the input signal
          final_filtered_t = np.convolve(final_filter, s)
          # Shift to ignore the 'startup' phase, and normalize the function
          truncated_filtered_t = final_filtered_t[(fir_length * 2)-1:] / 1.5 / sum(abs(final_fil
          # Take the FFT of the filtered signal for verification
          final_filtered_fft = np.fft.fft(truncated_filtered_t) * 4 / N

          create_plot([t, f[:N//2+1]], [(truncated_filtered_t, ), (np.abs(final_filtered_fft)[:N
                      ["t (s)", "$\omega$ (rad/s)"], ["$|x(t)|$", "$|X(\omega)|$"],
                      [("Truncated and filtered input signal", ), ("FFT of the filtered signal",
                      size=(14, 7), y_lines=[[np.nan], [np.nan]], y_line_labels=[[""], [""], [""
```



2.2.5 Look at the magnitudes of the input signals to verify their attenuation

```
In [726]: df = pd.DataFrame(np.transpose([f[:N//2+1], np.abs(final_filtered_fft)[:N//2+1]]),
                            columns=["Frequency", "Magnitude of FFT(X($\omega$))"])
          display(df.iloc[((df['Frequency'] - 1000).abs().argsort())[:1]])
          display(df.iloc[((df['Frequency'] - 10000).abs().argsort())[:1]])
          display(df.iloc[((df['Frequency'] - 20000).abs().argsort())[:1]])
          display(df.iloc[((df['Frequency'] - 50000).abs().argsort())[:1]])
```

	Frequency	Magnitude of FFT(X(ω))
10	1001.001001	0.971841

	Frequency	Magnitude of FFT(X(ω))
100	10010.01001	0.934444

	Frequency	Magnitude of FFT(X(ω))
200	20020.02002	0.086387

	Frequency	Magnitude of FFT(X(ω))
500	50050.05005	0.001171

2.2.6 The explicit amplitudes of $h[k]$

```
In [727]: pd.DataFrame(np.transpose([final_filter]), columns=['$h[k]$']).rename_axis('$k$')
```

Out[33]:

k	$h[k]$
0	-0.026368
1	-0.111207
2	-0.131925
3	-0.034091
4	-0.408149
5	0.819260
6	1.000000
7	0.819260
8	0.408149
9	0.034091
10	-0.131925
11	-0.111207