ECE 450 - Exam #3

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1 Solution

1.1 Basic Butterworth Filter

I will start by looking at a simple Butterworth Filter design that meets these criteria:

$$\omega_p \geq 400, |H_p| \geq 0.9$$

$$\omega_s \leq 200, |H_s| \leq 0.1$$

In order to implement the Butterworth design procedure, the filter must be a low-pass with a cutoff frequency of $\omega_c = 1$. I will start by guessing a center frequency of 300.

Thus, the pass and stopband frequencies need to be moved as follows:

$$\omega_p = \frac{1}{\frac{400}{300}} = 0.75, \omega_s = \frac{1}{\frac{200}{300}} = 1.5$$

Now, let's look at the order requirements for each of these two criteria:

$$n_p = \frac{log_{10}(\frac{1}{H_p^2} - 1)}{2log_{10}(\omega_p)} = \frac{log_{10}(\frac{1}{0.9^2} - 1)}{2log_{10}(0.75)} = 2.75$$

$$n_s = \frac{log_{10}(\frac{1}{H_s^2} - 1)}{2log_{10}(\omega_p)} = \frac{log_{10}(\frac{1}{0.1^2} - 1)}{2log_{10}(1.5)} = 5.67$$

In order to meet the passband criteria, I would need a third order Butterworth filter. However, I would need a sixth order Butterworth filter to pass the stopband critera. The transfer function of this would be:

$$H(s) = \frac{1}{s^6 + s^5 3.86 + s^4 7.46 + s^3 9.14 + s^2 7.46 + s3.86 + 1}$$

This sixth order low-pass filter's bode-plot is shown below, however I will look at other filter types to try and reduce the order required.

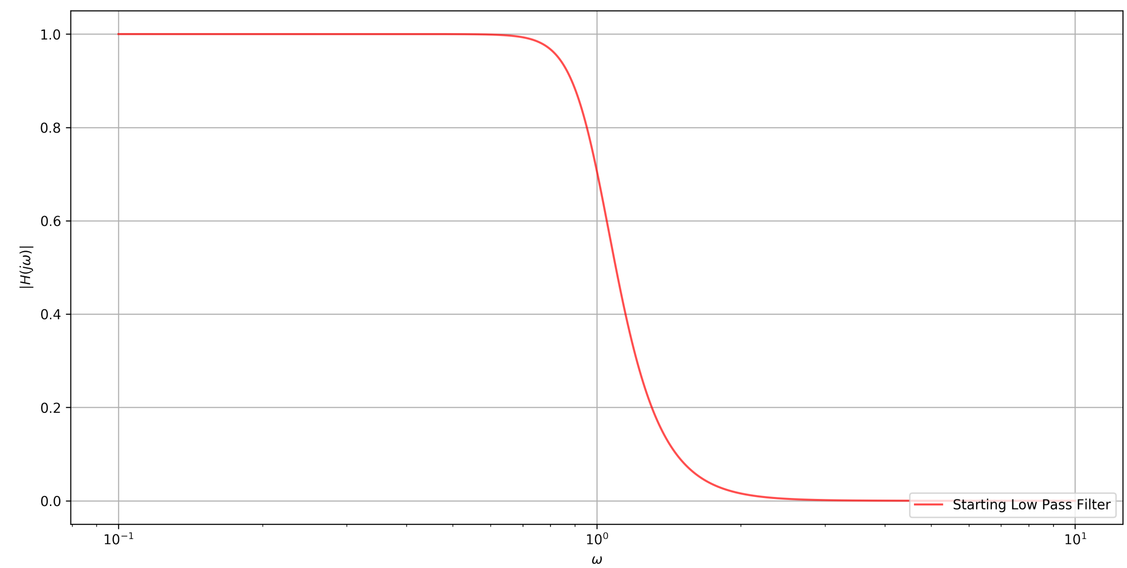


Figure 1. The Bode plot of the 6^{th} order low-pass filter.

1.2 Lowpass Chebychev II Filter

In an attempt to reduce the order required to pass the given criteria, I will redesign the filter being implemented as a Chebychev type II filter. A type II filter is required, despite being more complicated to design, because we are given the design restriction of having a monotonic pass-band – a characteristic of only a type II filter.

Using the previous ω_p, ω_s, H_p, and H_s values, the design procedure of the Chebychev II filter can be followed, and results in the following:

$$\begin{aligned} \epsilon &= 0.101 & n &= 5 \\ \alpha &= 19.950 & a &= 0.645 \\ b &= 1.185 \end{aligned}$$

The minimum order calculation here reveals that the order of this filter is indeed lower than that of the simple Butterworth filter shown in **Section 1**. Starting by just designing the low-pass Chebychev II Filter, the following transfer function is found:

$$H_{LP}(s) = \frac{s^4 0.50 - s^3 1.27 \cdot 10^{-31} + s^2 2.01 - s 1.69 \cdot 10^{-31} + 1.61}{s^5 + s^4 3.25 + s^3 5.15 + s^2 5.41 + s 3.304 + 1.61}$$

$$H_{LP}(s) \approx \frac{s^4 0.50 + s^2 2.01 + 1.61}{s^5 + s^4 3.25 + s^3 5.15 + s^2 5.41 + s 3.304 + 1.61}$$

This transfer function results in the following plot:

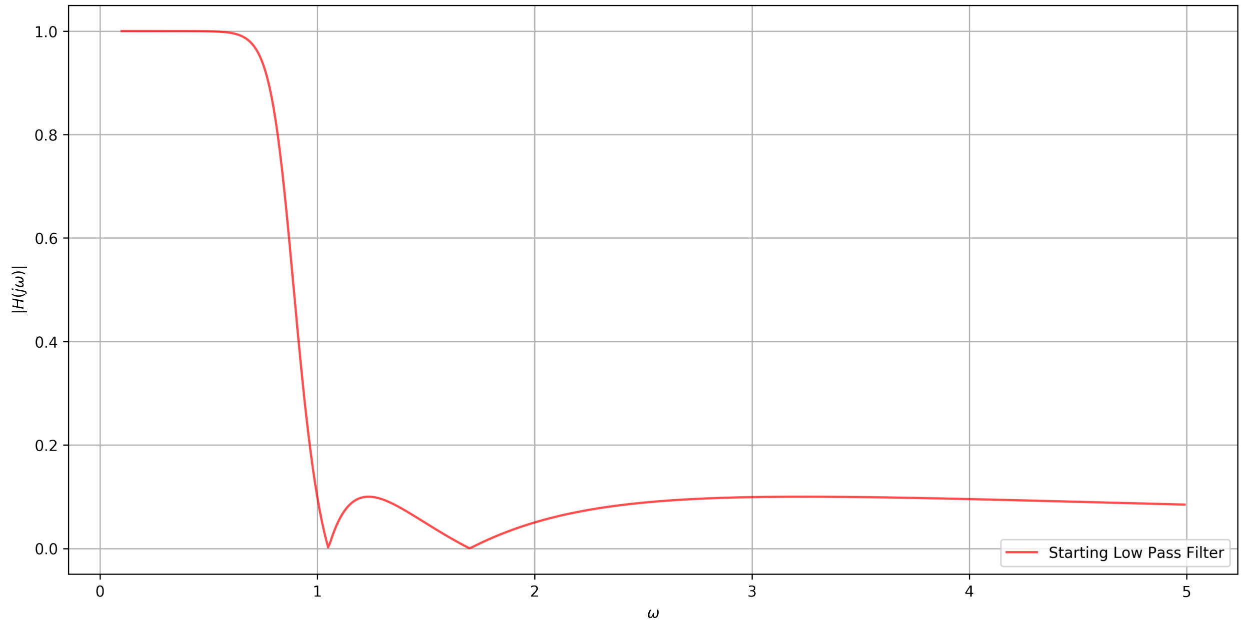


Figure 2. The Bode plot of the lowpass 5^{th} order Chebychev II filter.

1.3 Highpass Chebychev II Filter

Now I will need to change the starting lowpass transfer function to a highpass filter. This transition will change the bandpass to monotonic, and is accomplished with:

$$H_{HP}(s) = H_{LP}(\frac{1}{s})$$

This highpass filter is the following:

$$H_{HP}(s) = \frac{s^5 1.61 - s^4 1.69 \cdot 10^{-31} + s^3 2.01 - s^2 1.27 \cdot 10^{-31} + s0.50}{s^5 1.61 + s^4 3.30 + s^3 5.41 + s^2 5.15 + s3.25 + 1}$$

$$H_{HP}(s) \approx \frac{s^5 1.61 + s^3 2.01 + s0.50}{s^5 1.61 + s^4 3.30 + s^3 5.41 + s^2 5.15 + s3.25 + 1}$$

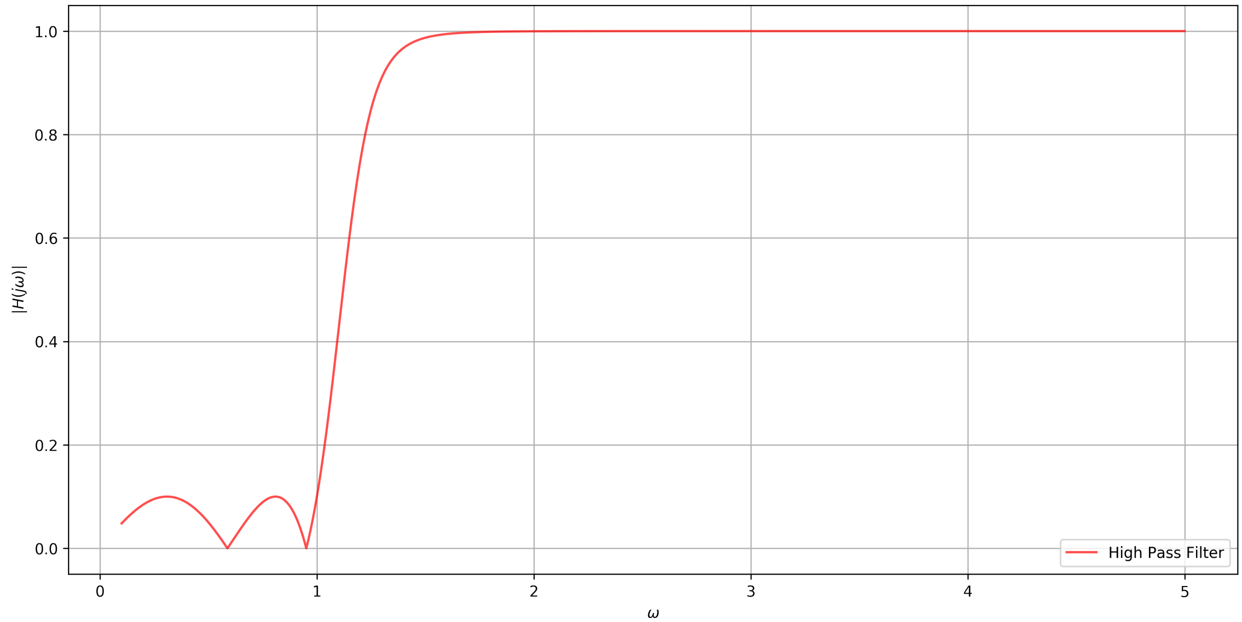


Figure 3. The Bode plot of the highpass 5^{th} order Chebychev II filter.

1.4 Shifted Highpass Chebychev II Filter

The final action necessary is to move the highpass filter to the desired center frequency. Back in **Section 1.1**, an arbitrary center frequency of 300 $\frac{rad}{s}$ was chosen. The shift will be performed as follows:

$$H_f(s) = H_{HP}(\frac{s}{300})$$

The final transfer function of this filter is thus:

$$H_f(s) = \frac{s^5 1.61 - s^4 1.69 \cdot 10^{-31} + s^3 2.01 - s^2 1.27 \cdot 10^{-31} + s0.50}{s^5 1.61 + s^4 9.91 \cdot 10^2 + s^3 4.87 \cdot 10^5 + s^2 1.39 \cdot 10^8 + s2.64 \cdot 10^{10} + 2.43 \cdot 10^{12}}$$

$$H_f(s) \approx \frac{s^5 1.61 + s^3 2.01 + s0.50}{s^5 1.61 + s^4 9.91 \cdot 10^2 + s^3 4.87 \cdot 10^5 + s^2 1.39 \cdot 10^8 + s2.64 \cdot 10^{10} + 2.43 \cdot 10^{12}}$$

The bode plot of this figure results in the following:

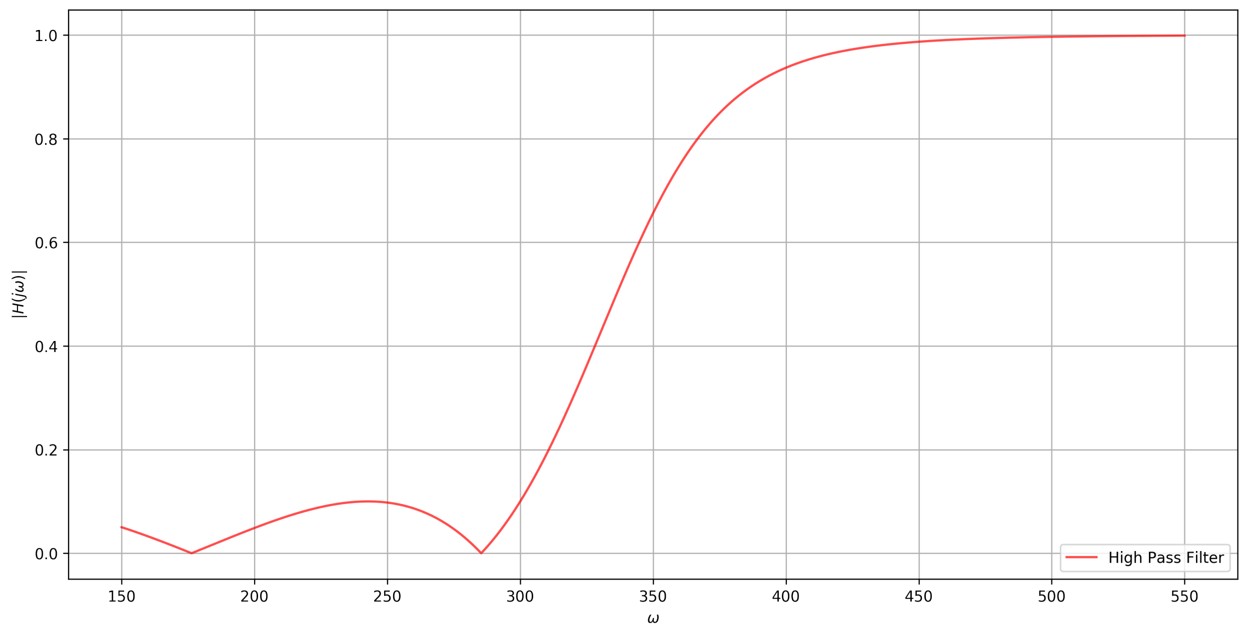


Figure 4. The Bode plot of the final highpass 5^{th} order Chebychev II filter; centered around $\omega_c = 300$

To verify specific characteristics of the filter at the given frequency criteria, I will look at the response magnitude at ω of 200 and 400 $\frac{rad}{s}$. The passband needs to stay above 0.9, and the stopband needs to stay below 0.1.

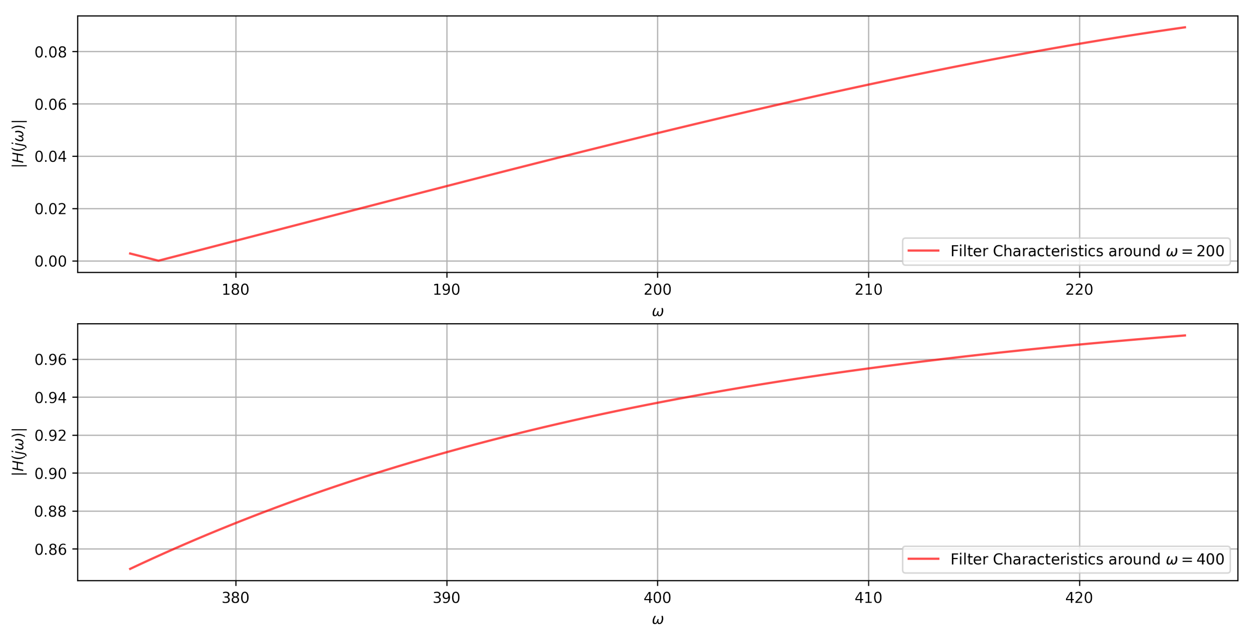


Figure 5. Zoomed in filter characteristics at $\omega = 200, 400$.

As the plots show, both of the given criteria are met.

1.5 Time-Domain Simulations

I will simulate this filter's response to sinusoidal inputs of two frequencies – 200 and 400 radians per second. The filter should attenuate the 200 radians per second input signal to less than 10%, and the 400 radians per second signal should barely be attenuated at all.

Below is the time-domain response of these two signals:

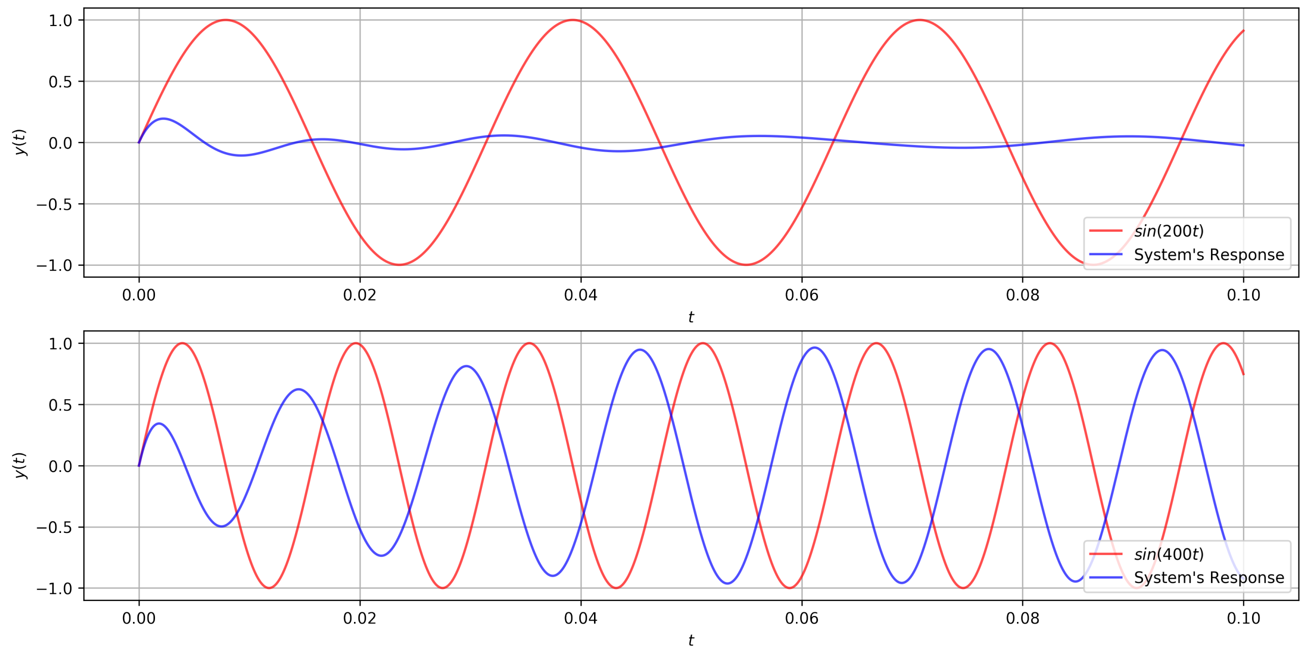


Figure 6. Time domain response of the two input signals.

2 Code Appendix

2.1 Package Imports

```
In [1]: import numpy as np
        import seaborn as sns
        import pandas as pd
        import matplotlib.pyplot as plt
        from scipy import signal as sig
        from control import margin, tf
```

2.2 Generic function to convolve any number of equations

```
In [2]: def convolve_all(values):
            temp_conv = values[0]
            if len(values) > 1:
                for next_val in values[1:]:
                    temp_conv = np.convolve(temp_conv, next_val)

            return temp_conv
```

2.3 Generic function to plot the responses of a system

```
In [3]: # Color list for multiple lines on each subplot
        colors = ["red", "blue", "green", "gray", "purple", "orange"]
        step_size = 0.005

        # Generic Function to create a plot
        def create_plot(x, y, xLabel=["X-Values"], yLabel=["Y-Values"],
                        title=[("Plot", )], num_rows=1, size=(18, 14), logx=False):
            plt.figure(figsize=size, dpi=300)
            for c, (x_vals, y_vals, x_labels, y_labels, titles) in enumerate(zip(x, y, xLabel, y
                for c2, (y_v, t) in enumerate(zip(y_vals, titles)):
                    plt.subplot(num_rows, 1, c + 1)
                    # Add a plot to the subplot, use transparency so they can both be seen
                    plt.plot(x_vals, y_v, label=t, color=colors[c2], alpha=0.70)
                    plt.ylabel(y_labels)
                    plt.xlabel(x_labels)
                    plt.grid(True)
                    plt.legend(loc='lower right')
                    if logx:
                        plt.xscale("log")

            plt.show()
```

2.4 Generic function shift a given filter to a new center frequency

```
In [4]: def shift_filter(num, den, new_center):
            new_num = np.pad(num, (len(den) - len(num), 0), 'constant')
```

```
    new_den = den

    for i in range(len(new_den)):
        new_num[i] *= (new_center ** i)
        new_den[i] *= (new_center ** i)

    return new_num, new_den
```

2.5 Generic function to convert a low-pass filter to a high-pass filter

```
In [5]: def low_to_high(num, den):
            return np.flip(np.pad(num, (len(den)-len(num), 0), 'constant')), np.flip(den)
```

Let's start with a simple Butterworth Filter design that meets these criteria:

$$\omega_p \geq 400, |H_p| \geq 0.9$$

$$\omega_s \leq 200, |H_s| \leq 0.1$$

In order to implement the Butterworth design procedure, the filter must be a low-pass with a cutoff frequency of $\omega_c = 1$. I will start by selecting a center frequency of 300.

Thus, the pass and stopband frequencies need to be moved as follows:

$$\omega_p = \frac{1}{\frac{400}{300}} = 0.75, \omega_s = \frac{1}{\frac{200}{300}} = 1.5$$

Now, let's look at the order requirements for each of these two criteria:

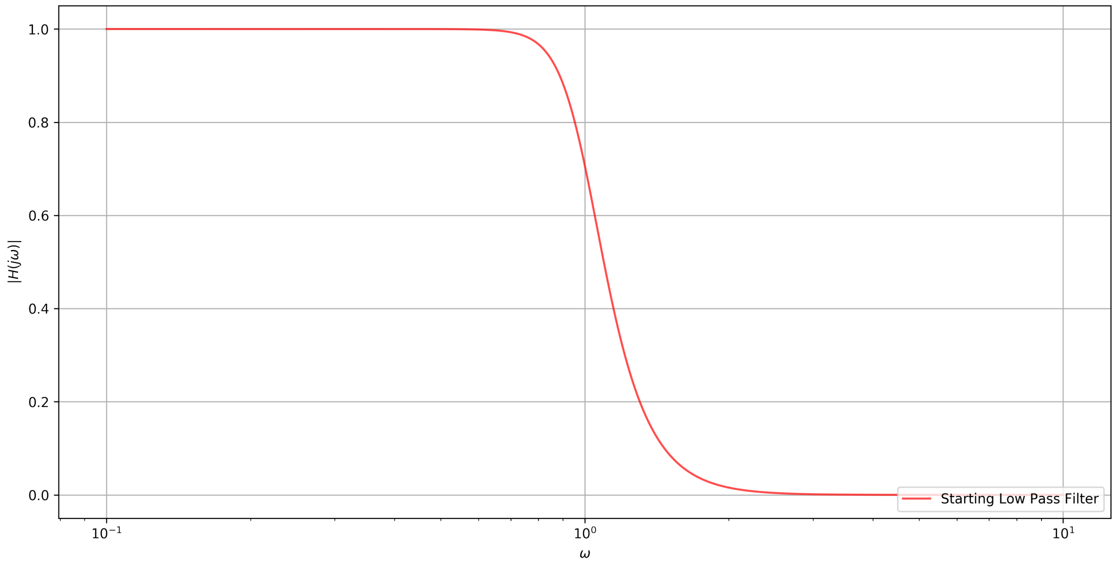
$$n_p = \frac{log_{10}(\frac{1}{H_p^2} - 1)}{2log_{10}(\omega_p)} = \frac{log_{10}(\frac{1}{0.9^2} - 1)}{2log_{10}(0.75)} = 2.75$$

$$n_s = \frac{log_{10}(\frac{1}{H_s^2} - 1)}{2log_{10}(\omega_p)} = \frac{log_{10}(\frac{1}{0.1^2} - 1)}{2log_{10}(1.5)} = 5.67$$

Clearly, I'd need a sixth order filter in order to meet these requirements.

```
In [15]: n = 6

         pole_list = []
         for m in range(1, n + 1):
             phi = np.pi / 2 + np.pi / 2 * (2 * m - 1) / n
             pole_list.append(complex(np.cos(phi), np.sin(phi)))

         num = [1]
         den = convolve_all([[1, -pole] for pole in pole_list])
         print ("Num: ", num, "\nDen: ", np.real(den))

         system = sig.lti(num, den)
         w, h_mag, h_phase = sig.bode(system, np.arange(0.1, 10, 0.001))
         create_plot([w], [(10 ** (0.05 * h_mag), )], ["$\omega$"], ["$|H(j\omega)|$"],
                     [("Starting Low Pass Filter", )], size=(14, 7), logx=True)
```

```
Num:  [1]
Den:  [1.         3.86370331 7.46410162 9.14162017 7.46410162 3.86370331
 1.        ]
```

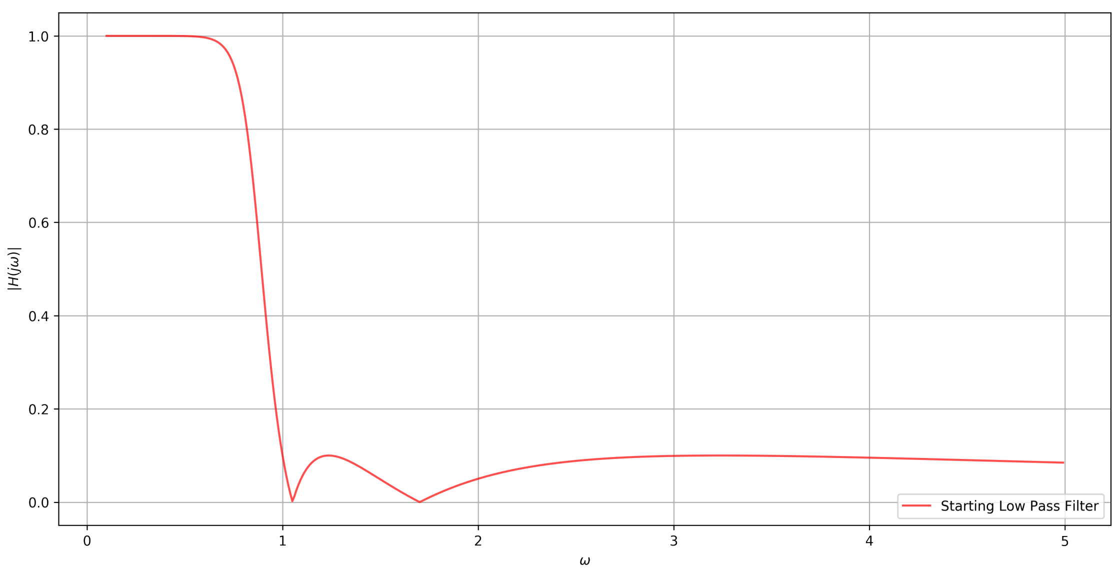


Let's change this to a Chebychev II filter.

```
In [7]: h_s = 0.1
        h_p = 0.9
        w_s = 1.5
        w_p = 0.75

        epsilon = np.sqrt(h_s ** 2 / (1 - h_s ** 2))
        n = int(np.ceil(np.arccosh(np.sqrt(1 / (epsilon ** 2 * (1 - h_p ** 2)))) / np.arccosh(1
        alpha = 1 / epsilon + np.sqrt(1 + 1 / (epsilon ** 2))
        a = 0.5 * (alpha ** (1 / n) - alpha ** (-1 / n))
        b = 0.5 * (alpha ** (1 / n) + alpha ** (-1 / n))
        print ("Epsilon: {:.3f}\nn: {:.3f}\nAlpha: {:.3f}\na: {:.3f}\nb: {:.3f}\n".format(epsilo

        # Develop the poles (denomenator) for the ChevyChase Type II
        pole_list = []
        for m in range(1, n + 1):
            phi = np.pi / 2 + np.pi / 2 * (2 * m - 1) / n
            pole_list.append(1 / complex(a * np.cos(phi), b * np.sin(phi)))
        den_chev2 = convolve_all([[1, -pole] for pole in pole_list])

        # Develop the zeros (numerator) for the Chevy-chase Type II
        zero_list = [1 / np.cos(k * np.pi / (2 * n)) for k in range(1, 2 * n, 2)]
        zero_list = [zero for zero in zero_list if zero < 10 ** 10]
```

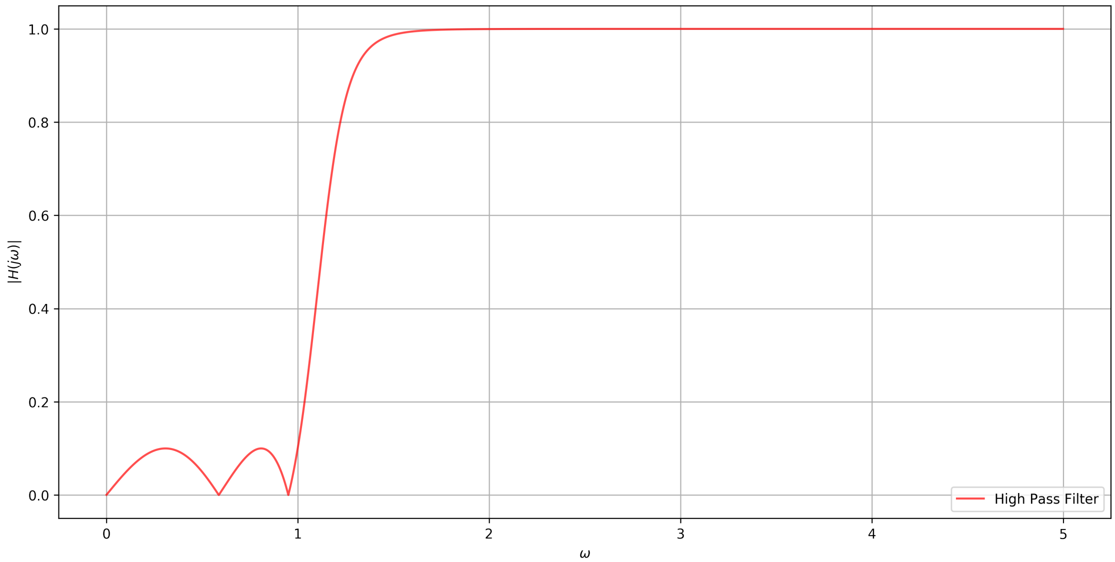
```
num_chev2 = convolve_all([[1, complex(0, -zero)] for zero in zero_list])
K = den_chev2[-1] / num_chev2[-1]
num_chev2 = np.multiply(K, num_chev2)

print ("Num: ", np.real(num_chev2), "\nDen: ", np.real(den_chev2))

system = sig.lti(num_chev2, den_chev2)
w, h_mag, h_phase = sig.bode(system, np.arange(0.1, 5, 0.01))
create_plot([w], [(10 ** (0.05 * h_mag), )], ["$\omega$"], ["$|H(j\omega)|$"],
            [("Starting Low Pass Filter", )], size=(14, 7))
```

```
Epsilon: 0.101
n: 5.000
Alpha: 19.950
a: 0.635
b: 1.185

Num:  [ 5.02518908e-01 -1.27111376e-31  2.01007563e+00 -1.69481835e-31
  1.60806050e+00]
Den:  [1.         3.24707087 5.14547198 5.40570185 3.30465502 1.6080605 ]
```



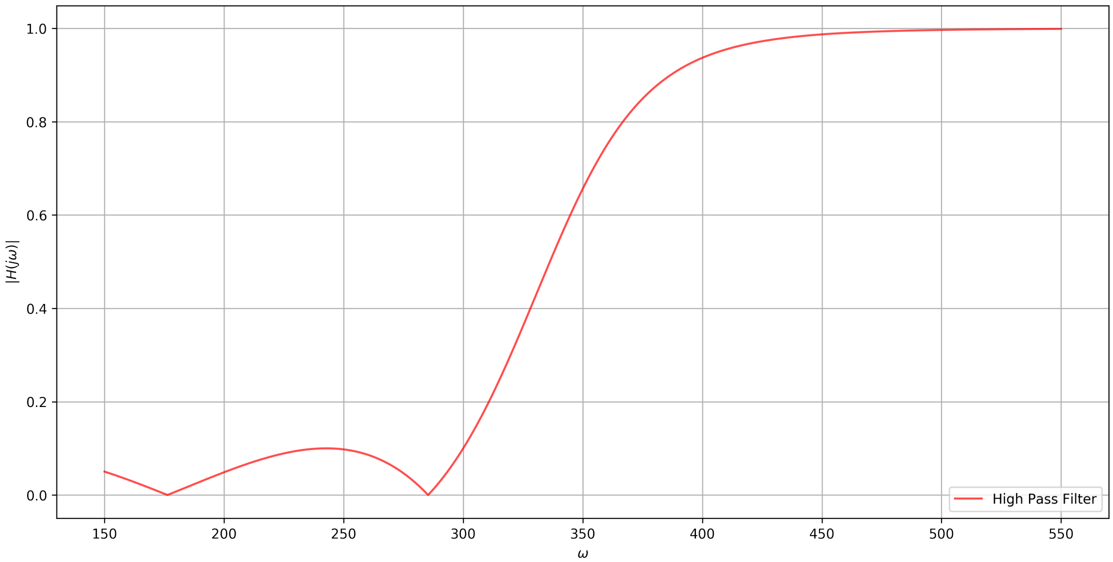

```
In [8]: num_chev2, den_chev2 = low_to_high(num_chev2, den_chev2)
        print ("Num: ", np.real(num_chev2), "\nDen: ", np.real(den_chev2))

        system = sig.lti(num_chev2, den_chev2)
        w, h_mag, h_phase = sig.bode(system, np.arange(0.001, 5, 0.001))
        create_plot([w], [(10 ** (0.05 * h_mag), )], ["$\omega$"], ["$|H(j\omega)|$"],
                    [("High Pass Filter", )], size=(14, 7))
```

```
Num:  [ 1.60806050e+00 -1.69481835e-31  2.01007563e+00 -1.27111376e-31
  5.02518908e-01  0.00000000e+00]
Den:  [1.6080605  3.30465502 5.40570185 5.14547198 3.24707087 1.        ]
```

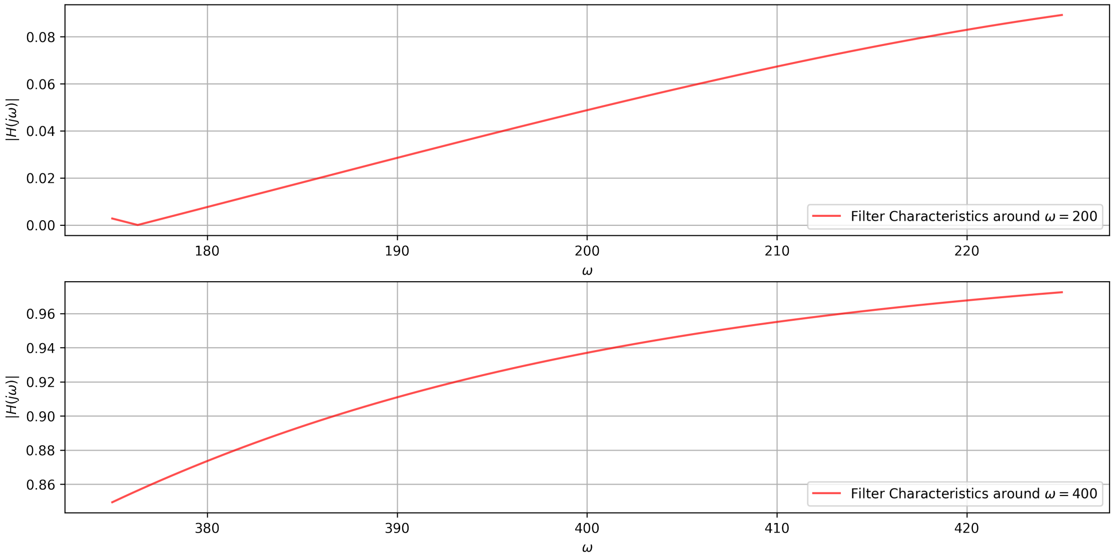


In [9]:
```
final_num, final_den = shift_filter(num_chev2, den_chev2, 300)
print ("Num: ", np.real(final_num), "\nDen: ", np.real(final_den))

system = sig.lti(final_num, final_den)
w, h_mag, h_phase = sig.bode(system, np.arange(150, 550, 0.01))
create_plot([w], [(10 ** (0.05 * h_mag), )], ["$\omega$"], ["$|H(j\omega)|$"],
            [("High Pass Filter", )], size=(14, 7))
```

```
Num:  [ 1.60806050e+00 -5.08445505e-29  1.80906807e+05 -3.43200716e-24
  4.07040315e+09  0.00000000e+00]
Den:  [1.60806050e+00 9.91396507e+02 4.86513166e+05 1.38927744e+08
 2.63012740e+10 2.43000000e+12]
```




```
In [10]: w1, h_mag1, _ = sig.bode(system, np.arange(175, 225, 0.001))
         w2, h_mag2, _ = sig.bode(system, np.arange(375, 425, 0.001))
         create_plot([w1, w2], [(10 ** (0.05 * h_mag1), ), (10 ** (0.05 * h_mag2), )],
                     ["$\omega$", "$\omega$"], ["$|H(j\omega)|$", "$|H(j\omega)|$"],
                     [("Filter Characteristics around $\omega=200$", ),
                      ("Filter Characteristics around $\omega=400$", )], size=(14, 7), num_rows=
```




```
In [12]: dt = 0.00001
         NN = 10000
```

```python
TT = np.arange(0, NN * dt, dt)
response_200 = np.zeros(NN)
response_400 = np.zeros(NN)
force_200 = np.zeros(NN)
force_400 = np.zeros(NN)

a, b, c, d = sig.tf2ss(final_num, final_den)
a, b, c, d = np.real(a), np.real(b), np.real(c), np.real(d)

for n in range(NN):
    force_200[n] = np.sin(200 * n * dt)
    force_400[n] = np.sin(400 * n * dt)

x_200 = np.zeros(np.shape(b))
x_400 = np.zeros(np.shape(b))
for m in range(NN):
    x_200 += dt * a.dot(x_200) + dt * b * force_200[m]
    x_400 += dt * a.dot(x_400) + dt * b * force_400[m]

    response_200[m] = c.dot(x_200) + d * force_200[m]
    response_400[m] = c.dot(x_400) + d * force_400[m]

create_plot([TT, TT],
            [(force_200, response_200), (force_400, response_400)],
            ["$t$", "$t$"], ["$y(t)$", "$y(t)$"],
            [("$sin(200t)$", "System's Response"),
             ("$sin(400t)$", "System's Response")], 2, size=(14, 7))
```