ECE 450 - Exam #2

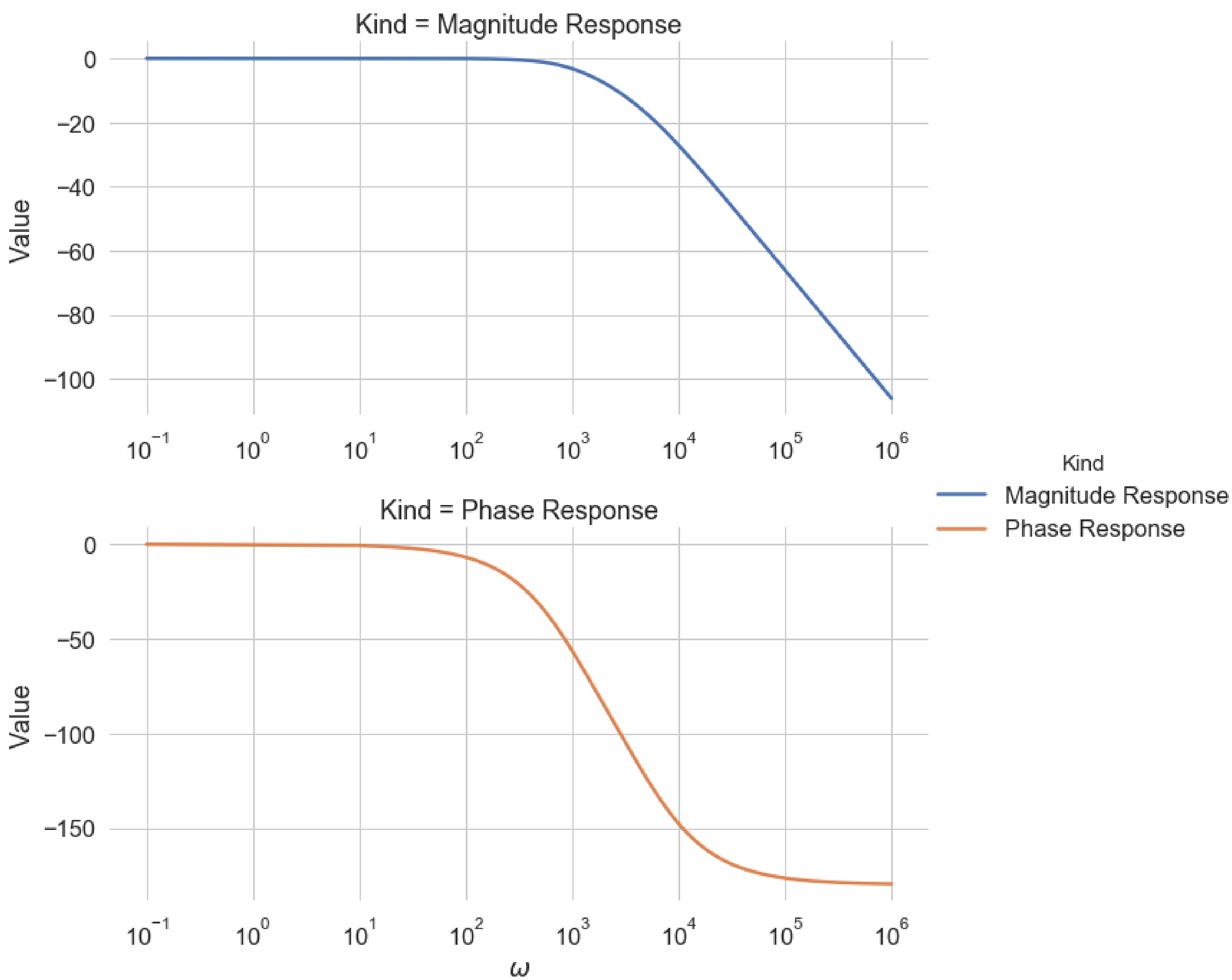
Collin Heist

October 13, 2019

1 Solution

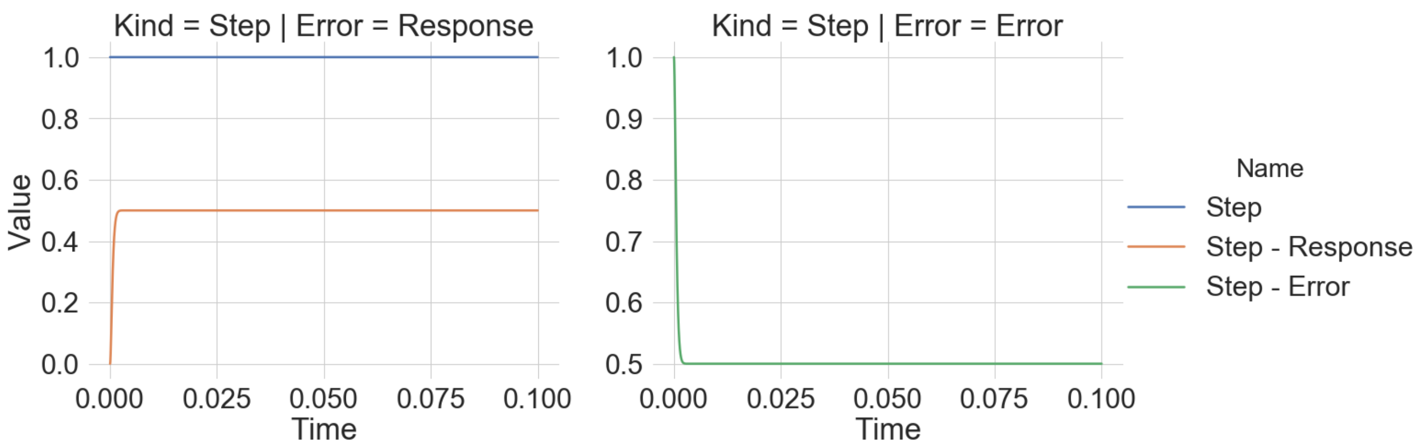
1.1 Problem A

1.1.1 Part i



1.1.2 Part ii

For starters, lets' take a look at the step response of this system to the system with no compensation network.



First, I'll test a compensation network with just an increased gain. The steady state error of the unity response is determined by:

$$e_{ss}^{step} = \frac{1}{1 + lim_{s\to 0} H_0(s)}$$

$$e_{ss}^{step} = \frac{1}{1 + lim_{s\to 0}(\frac{5\cdot 10^6}{s^2+6\cdot 10^3 s+5\cdot 10^6})}$$

$$e_{ss}^{step} = \frac{1}{1+1} = 0.5$$

In order to get this below one percent (0.01), I'll choose a gain according to the following:

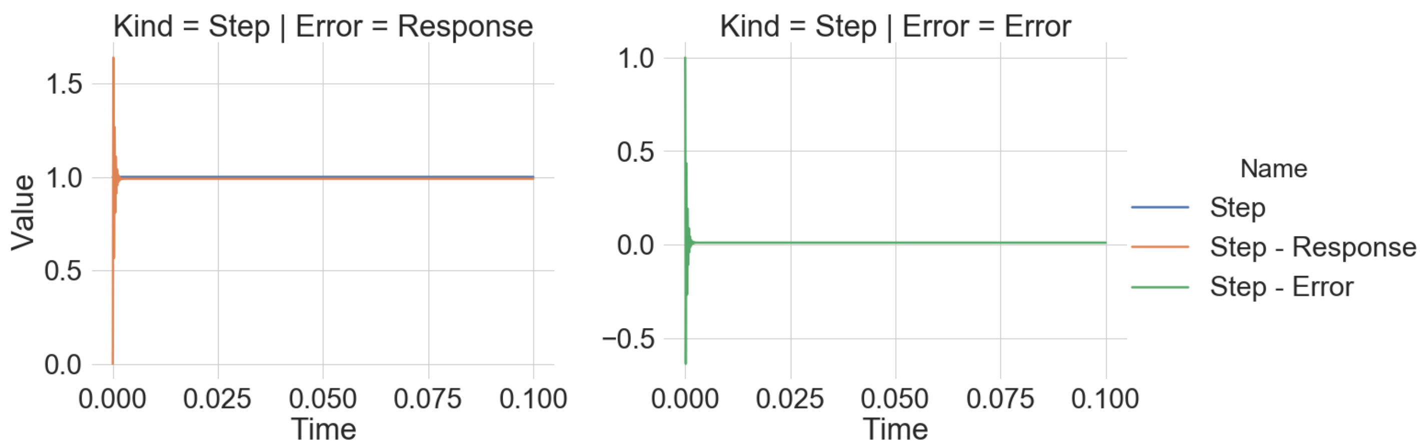
$$e_{ss}^{step} = \frac{1}{1 + lim_{s\to 0}(K\cdot\frac{5\cdot 10^6}{s^2+6\cdot 10^3 s+5\cdot 10^6})} = 0.01$$

$$e_{ss}^{step} = \frac{1}{1+K} = 0.01$$

$$1 = 0.01 + 0.01\cdot K \rightarrow 0.99 = 0.01\cdot K$$

$$K = 99$$

Therefore a gain of 100 would produce the desired reduction in error (+ some wiggle room).



This seems to satisfy the problem requirements for the steady-state error, but let's look at the values at 5 milliseconds, just to be sure.

```
        Time     Value  Kind          Name  Error
20500  0.005  0.009901  Step  Step - Error  Error
```

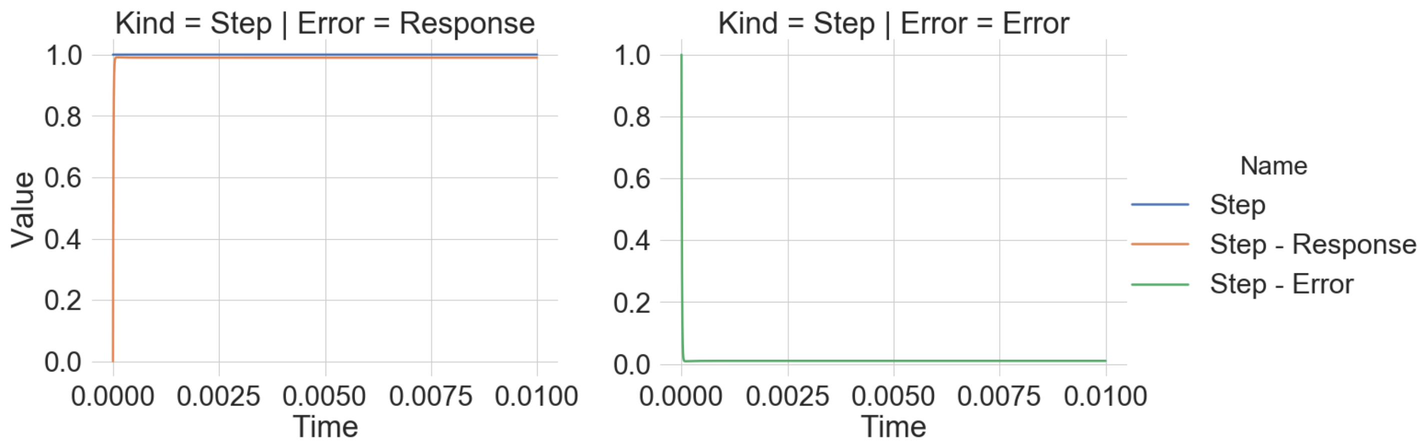
As we can see, at 5 milliseconds, the step error for this new transfer function is below 1%. Now, the overshoot needs to be dealt with.

Let's take a look at the phase margin for this new system, to identify how much phase offset we need to add. From my code, the crossover is at 20603.534 (rad/s), and the phase-margin is 16.266 degrees.

This phase margin of only 16.266 degrees, is clearly the reason we have such an overshoot when it comes to the step response. To deal with this, let's add a phase lead compensation. I chose a desired phase of 100^o, since we want **no overshoot whatsoever**. My code reveals that the zero and pole frequencies are 5228.13 (rad/s) and 1744826.86 (rad/s) respectively. After computing the calculated zero and pole of the compensation network, the final transfer function is:

$$G_c(s) = 100 \cdot \frac{1744826.86}{5228.13} \cdot \frac{s + 5228.13}{s + 1744826.86}$$

Now, I'll take a look at the final step response of this system - with the compensation network added.



This appears to do the job. There is no overshoot, and the response to the unit step function definitely reaches equilibrium by 5 milliseconds. However, just to verify, here is the error at this time:

```
Out[16]:         Time     Value  Kind          Name  Error
         25000  0.005  0.009901  Step  Step - Error  Error
```

1.2 Problem B

1.2.1 Part i

There is clearly a low pass filter with two poles. The reason we know this is because the phase settles out to -180 degrees, and does not appear to increase or have any zeros of the plotted range.

My first estimation is that the dominant pole exists at 10^0, or 1. I am making this estimation because the magnitude response begins to decrease at this point, and the phase reaches -45^o at this frequency. The next pole clearly exists at 10^1, or 10. Once again, we can see the magnitude transition from a $-20\frac{dB}{dec}$ slope to $-40\frac{dB}{dec}$ (suggesting a pole) - **and** and frequency is at -135^o at this point.

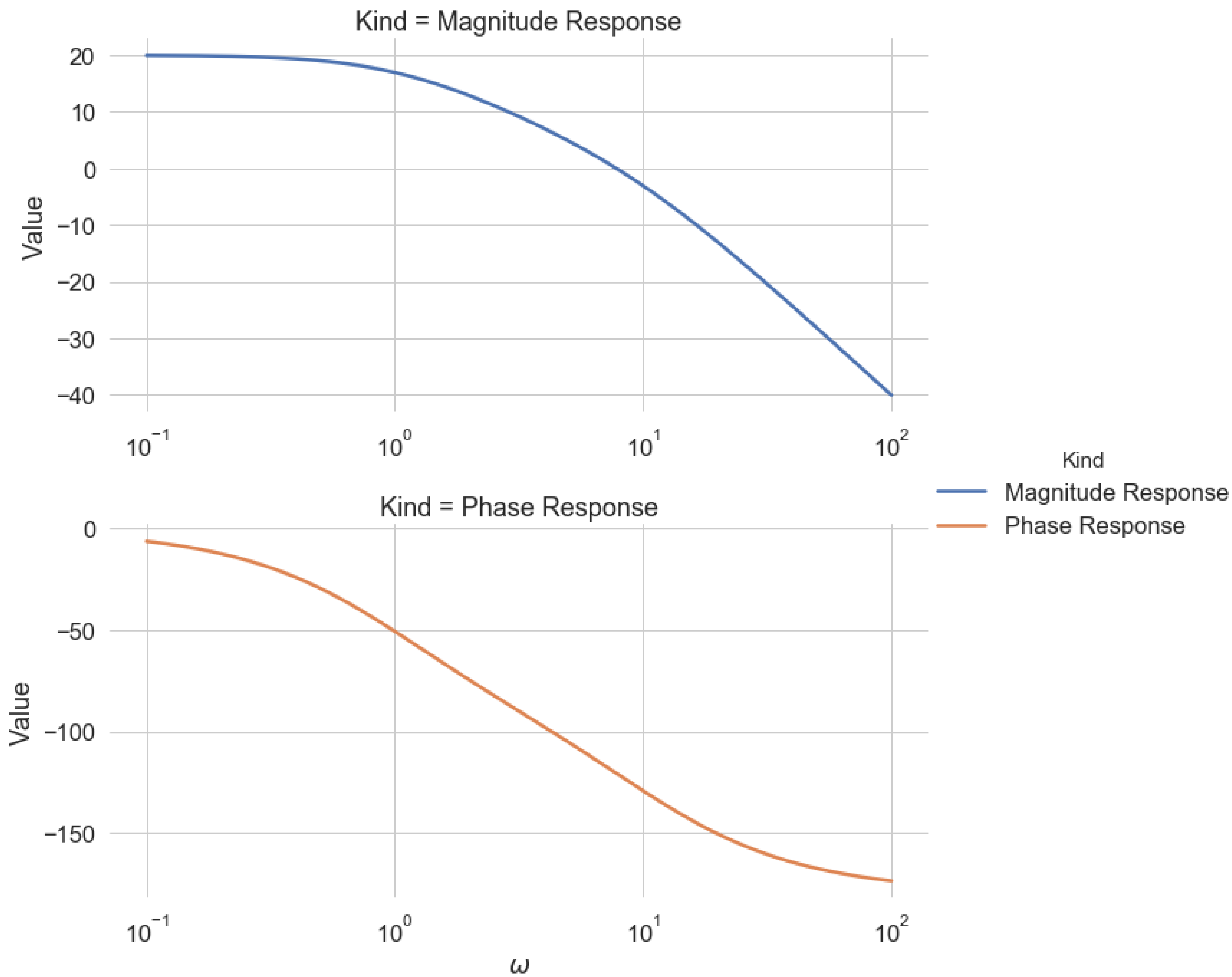
That results in the following approximated transfer function:

$$H(s) = K \cdot \frac{1}{(s+1)(s+10)}$$

However, we see that at near zero frequencies ($w \approx 0$), the magnitude of the response is 20dB. In order to achieve this gain, K needs to be approximately 100 - we know this because each gain of 10 corresponds to 20dB, and the starting gain of this equation is -20dB.

Thus, the following is the final transfer function:

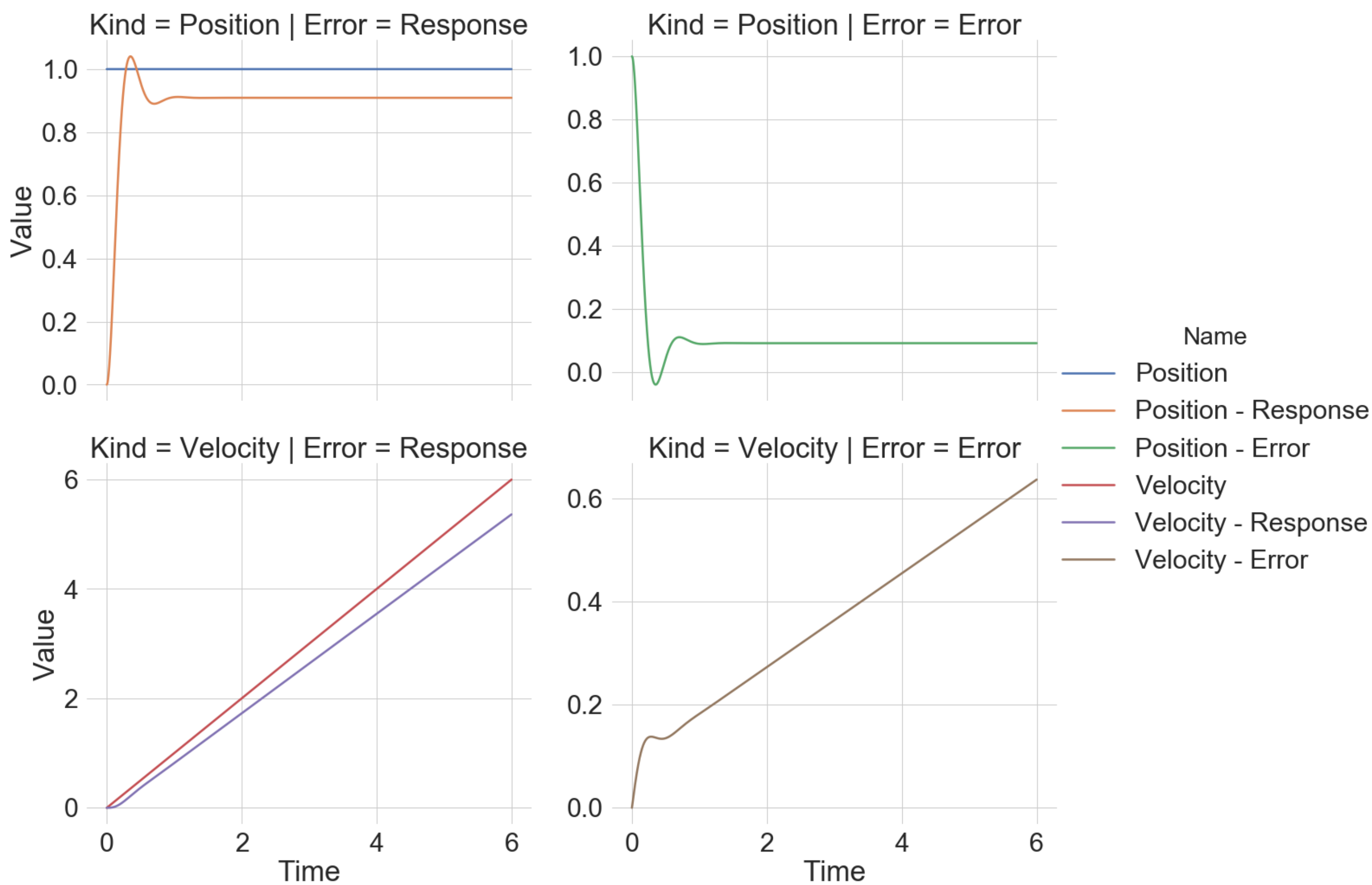
$$H(s) = \frac{100}{(s+1)(s+10)}$$



Which seems to match the given Bode plot.

1.2.2 Part ii

Let's start by looking at the position and velocity response.

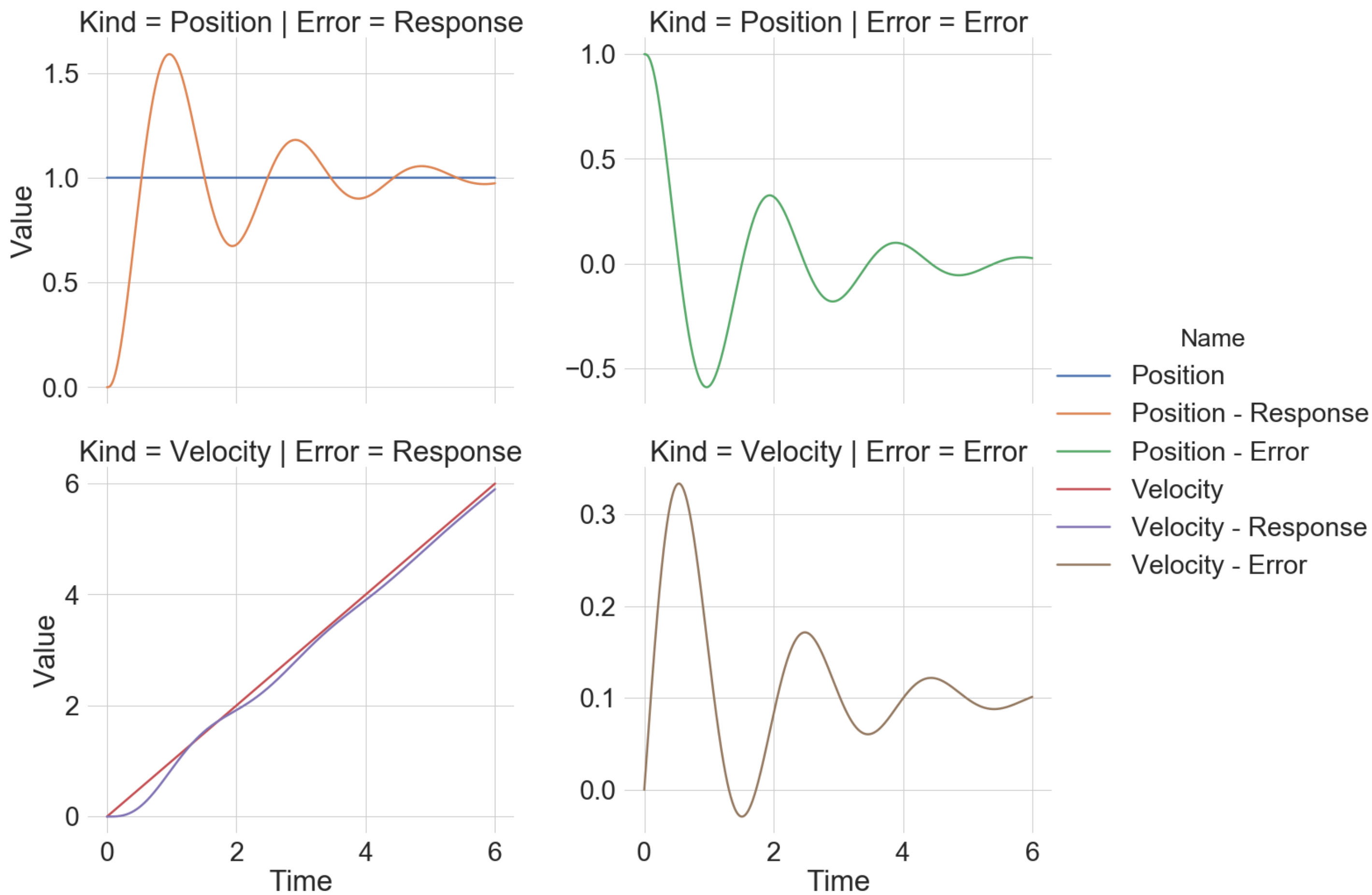


I can see that the position error seems to stabilize to about 10%, but the velocity error is unbounded. In order to address this, I need to institute a compensation network that reduces the steady-state velocity constant to a number less than 0.1.

To accomplish this, I will add a phase-lead offset, with the addition of another pole at $\omega = 0$, in order to change the velocity error to a bounded value. The general form of this will be:

$$G_c(s) = \frac{\omega_p}{\omega_z} \cdot \frac{s + \omega_z}{s \cdot (s + \omega_p)}$$

After calling my helper-function, the zero and pole frequencies are 6.19 (rad/s) and 25.71 (rad/s) accordingly.



This seems to have done the trick. Just to verify, let's look at the error of the response function at $t = 5$ seconds.

Out[21]:

	Time	Value	Kind	Name	Error
17000	5.0	-0.050009	Position	Position - Error	Error
35000	5.0	0.098880	Velocity	Velocity - Error	Error

As requested, the error of both the position and velocty have been reduced to below 10% after 5 seconds.

2 Code Appendix

2.1 Package Imports

```
In [1]: import numpy as np
        import seaborn as sns
        import pandas as pd
        import matplotlib.pyplot as plt
        from scipy import signal as sig
        from control import margin, tf
```

2.2 Generic functions for the step, ramp, and parabolic inputs

```
In [2]: step = lambda times:     [0 if t < 0 else 1 for t in times]
        ramp = lambda times:     [0 if t < 0 else t for t in times]
        parabola = lambda times: [0 if t < 0 else t * t for t in times]
```

2.3 Generic function to convolve any number of equations

```
In [3]: def convolve_all(values):
            temp_conv = values[0]
            if len(values) > 1:
                for next_val in values[1:]:
                    temp_conv = np.convolve(temp_conv, next_val)

            return temp_conv
```

2.4 Generic function to generate the magnitude and phase of $H(j\omega)$ values

```
In [4]: def magnitude_phase_response(num, den, omega_range, omega_step=10, gain_num=None, gain_d
            if isinstance(gain_num, (np.ndarray, list)) and isinstance(gain_den, (np.ndarray, li
                num = convolve_all([num, gain_num])
                den = convolve_all([den, gain_den])

            system = sig.lti(num, den)
            w, h_mag, h_phase = sig.bode(system, np.arange(omega_range[0], omega_range[1], omega
            _, phase_margin, _, crossover_w = margin(h_mag, h_phase, w)

            df_list = []
            df = pd.DataFrame(list(zip(w, h_mag)), columns=["$\omega$", "Value"])
            df["Kind"] = "Magnitude Response"
            df_list.append(df)

            df = pd.DataFrame(list(zip(w, h_phase)), columns=["$\omega$", "Value"])
            df["Kind"] = "Phase Response"
            df_list.append(df)

            return pd.concat(df_list, ignore_index=True, axis=0), phase_margin, crossover_w
```

2.5 Generic function to obtain response of a system to inputs

```
In [5]: def response_to_inputs(num, den, input_funcs, input_names, time, gain_num=None, gain_den
            df_list = []

            # If a gain equation was given, adjust the system num / dun
            if isinstance(gain_num, (np.ndarray, list)) and isinstance(gain_den, (np.ndarray, li
                num = convolve_all([num, gain_num])
                den = convolve_all([den, gain_den])

            num = np.pad(num, (len(den) - len(num), 0), "constant") # Make arrays same length
            den = np.add(den, num)
            for in_name, in_f in zip(input_names, input_funcs):
                df = pd.DataFrame(list(zip(time, in_f(time))), columns=["Time", "Value"])
                df["Kind"] = df["Name"] = in_name
                df["Error"] = "Response"
                df_list.append(df)

                _, response, _ = sig.lsim((num, den), in_f(time), time)
                df = pd.DataFrame(list(zip(time, response)), columns=["Time", "Value"])
                df["Kind"] = in_name
                df["Name"] = in_name + " - Response"
                df["Error"] = "Response"
                df_list.append(df)

                response_err =  np.subtract(in_f(t), response)
                df = pd.DataFrame(list(zip(time, response_err)), columns=["Time", "Value"])
                df["Kind"] = in_name
                df["Name"] = in_name + " - Error"
                df["Error"] = "Error"
                df_list.append(df)

            return pd.concat(df_list, ignore_index=True, axis=0)
```

2.6 Generic function to plot the responses of a system

```
In [6]: def create_plots(df, error=False):
            if error:
                sns.set(style="whitegrid", font_scale=2.75)
                g = sns.FacetGrid(df, hue="Name", row="Kind", col="Error", height=7.5, aspect=1.
                g.map(sns.lineplot, "Time", "Value", **dict(linewidth=2.5)).add_legend().despine
            else:
                sns.set(style="whitegrid", font_scale=1.5)
                g = sns.FacetGrid(df, hue="Kind", row="Kind", height=5.5, aspect=1.75,
                                  sharey=False, gridspec_kws={"hspace":0.3})
                g.map(sns.lineplot, "$\omega$", "Value", **dict(linewidth=2.5)).add_legend().des

            return g
```

```
In [7]: def compute_phase_lead(df, desired_pm):
            alpha = (1 + np.sin(desired_pm * np.pi / 180)) / (1 - np.sin(desired_pm * np.pi / 18
            alpha_db = -10 * np.log10(alpha)
            omega_m = df["$\omega$"].iloc[(df["Value"] - alpha_db).abs().argsort()[:1].values[0]
            omega_p = np.sqrt(alpha) * omega_m
            omega_z = omega_m / np.sqrt(alpha)

            return omega_z, omega_p
```

2.7 Problem A

2.7.1 Part i

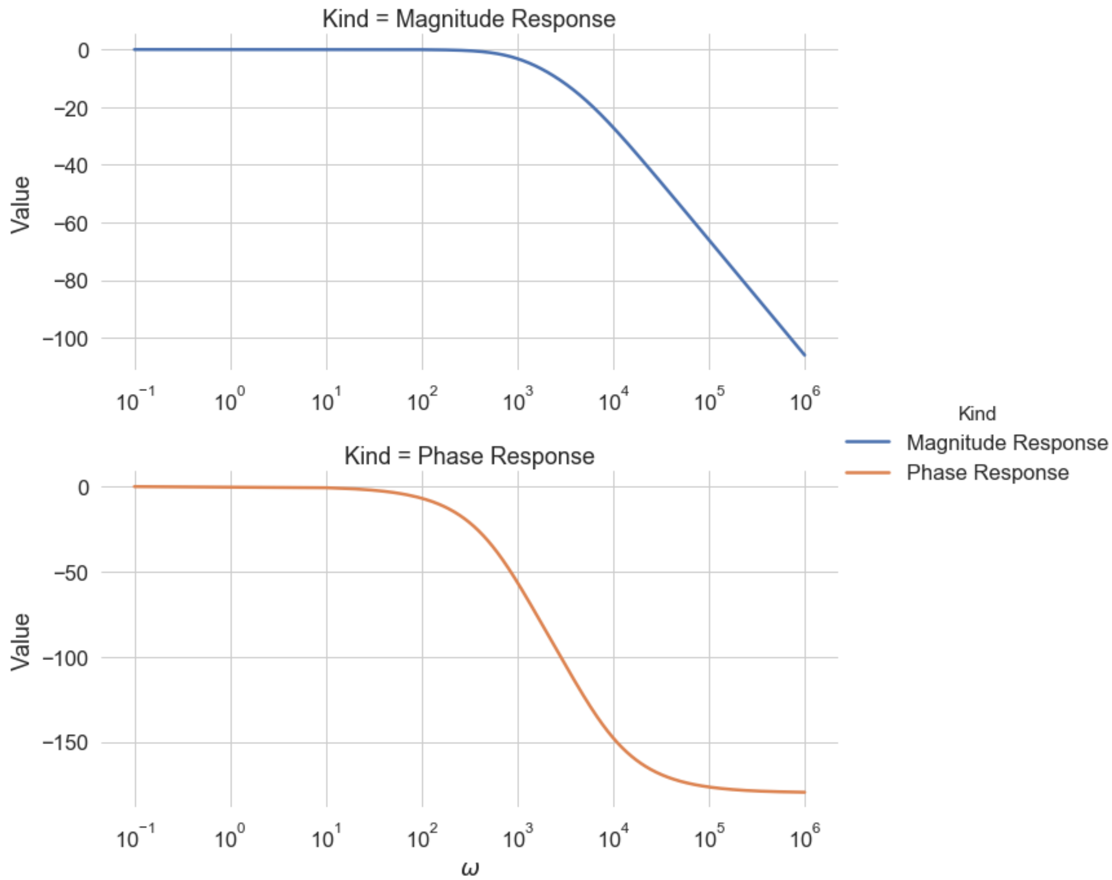
A system is described by:

$$H_0(s) = \frac{5 \cdot 10^6}{s^2 + 6 \cdot 10^3 s + 5 \cdot 10^6}$$

```
In [8]: num = [5 * 10 ** 6]
        den = convolve_all([[1, 6 * 10 ** 3, 5 * 10 ** 6]])

        df, m, w = magnitude_phase_response(num, den, [0.1, 10 ** 6], 10)
        g = create_plots(df)
        for ax in g.axes.flatten():
            ax.tick_params(labelbottom=True)
```

```
/usr/local/lib/python3.7/site-packages/matplotlib/figure.py:2359: UserWarning: This figure inclu
  warnings.warn("This figure includes Axes that are not compatible "
```

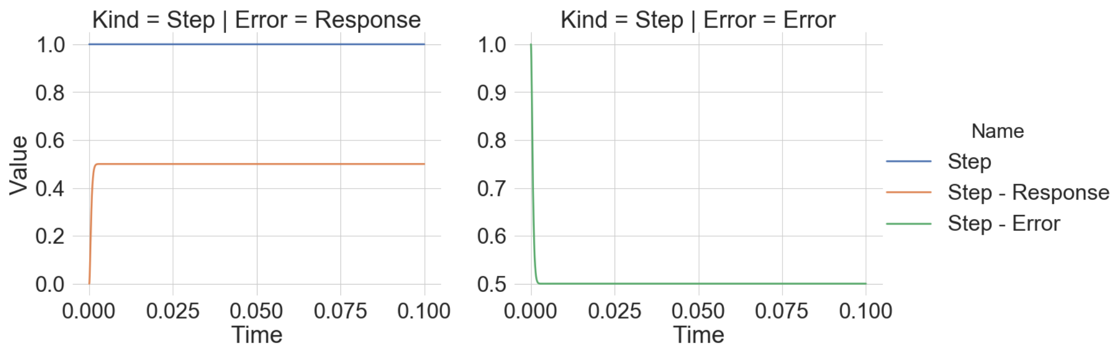


2.7.2 Part ii

For starters, lets' take a look at the step response of this system to the system with no compensation network.

```
In [9]: t = np.arange(0, 0.1, 0.00001)
        df = response_to_inputs(num, den, [step], ["Step"], t)
        create_plots(df, True);
```

```
/usr/local/lib/python3.7/site-packages/scipy/signal/filter_design.py:1551: BadCoefficients: Badl
  "results may be meaningless", BadCoefficients)
```



First, I'll test a compensation network with just an increased gain. The steady state error of the unity response is determined by:

$$e_{ss}^{step} = \frac{1}{1 + lim_{s \to 0} H_0(s)}$$

$$e_{ss}^{step} = \frac{1}{1 + lim_{s \to 0}(\frac{5 \cdot 10^6}{s^2 + 6 \cdot 10^3 s + 5 \cdot 10^6})}$$

$$e_{ss}^{step} = \frac{1}{1+1} = 0.5$$

In order to get this below one percent (0.01), I'll choose a gain according to the following:

$$e_{ss}^{step} = \frac{1}{1 + lim_{s \to 0}(K \cdot \frac{5 \cdot 10^6}{s^2 + 6 \cdot 10^3 s + 5 \cdot 10^6})} = 0.01$$

$$e_{ss}^{step} = \frac{1}{1+K} = 0.01$$

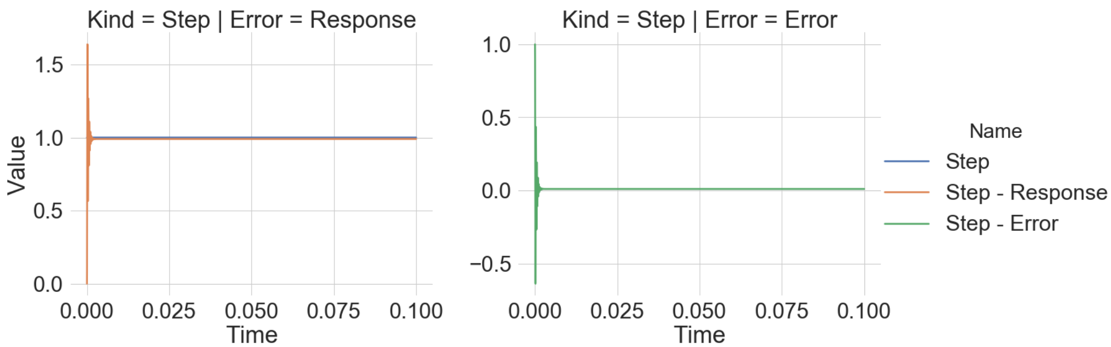
$$1 = 0.01 + 0.01 \cdot K \to 0.99 = 0.01 \cdot K$$

$$K = 99$$

Therefore a gain of 100 would produce the desired reduction in error (+ some wiggle room).

```
In [10]: comp_network_num = [100]
         comp_network_den = [1]

         t = np.arange(0, 0.1, 0.00001)
         df = response_to_inputs(num, den, [step], ["Step"], t, comp_network_num, comp_network_d
         create_plots(df, True);
```



This seems to satisfy the problem requirements for the steady-state error, but let's look at the values at 5 milliseconds, just to be sure.

```
In [11]: display(df[(df["Time"] == 5 / 1000) & (df["Error"] == "Error")])
```

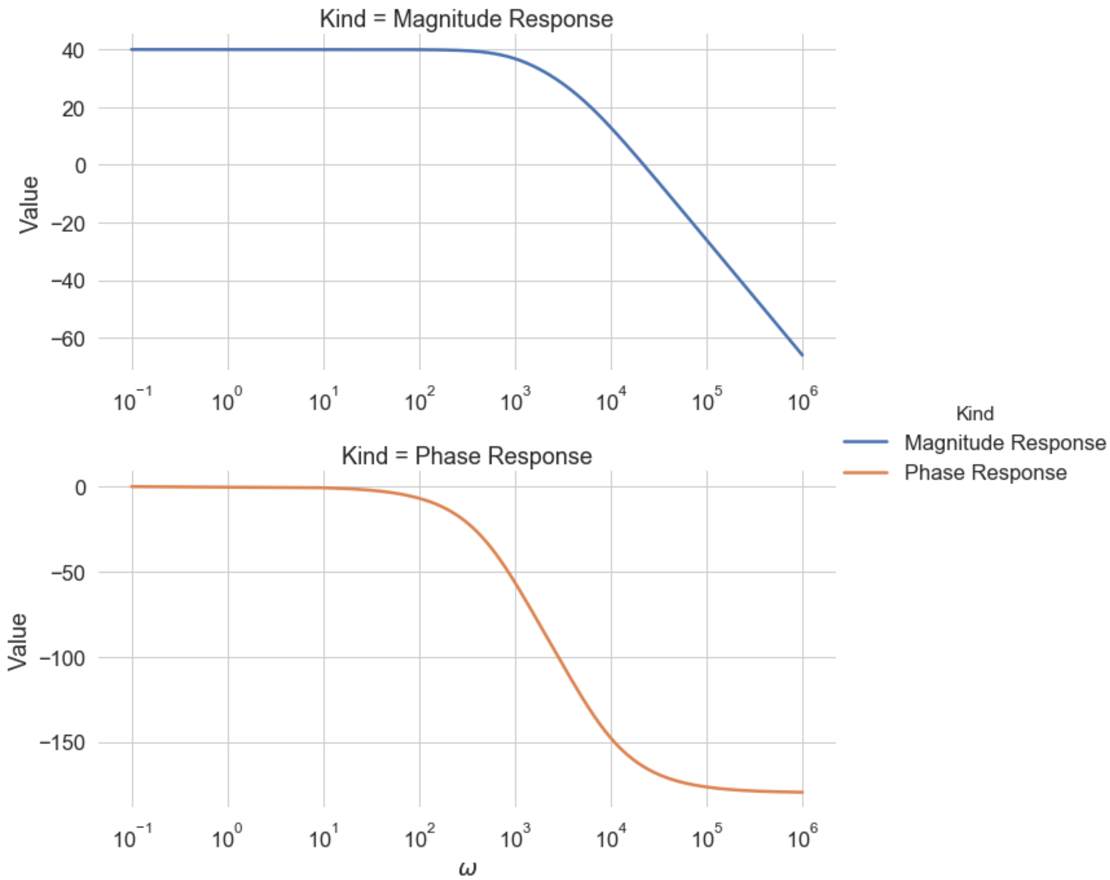
```
        Time     Value  Kind          Name  Error
20500  0.005  0.009901  Step  Step - Error  Error
```

As we can see, at 5 milliseconds, the step error for this new transfer function is below 1%. Now, the overshoot needs to be dealt with.

Let's take a look at the phase margin for this new system, to identify how much phase offset we need to add.

```
In [13]: df, phase_margin, crossover_w = magnitude_phase_response(num, den, [0.1, 10 ** 6], 10,
                                                                  comp_network_num, comp_network
         print ("Crossover at {:.3f} (rad/s) has a phase-margin of {:.3f} degrees".format(crosso
```

```
Crossover at 20803.534 (rad/s) has a phase-margin of 16.266 degrees
```



This phase margin, of only 16.266 degrees, is clearly the reason we have such an overshoot when it comes to the step response. To deal with this, let's add a phase lead compensation. I chose a desired phase of 100^o, since we want **no overshoot whatsoever**.

```
In [14]: z, p = compute_phase_lead(df, 100 - phase_margin)
         print ("Zero: {:.2f} (rad/s)\nPole: {:.2f} (rad/s)".format(z, p))
```

```
Zero: 5228.13 (rad/s)
Pole: 1744826.86 (rad/s)
```

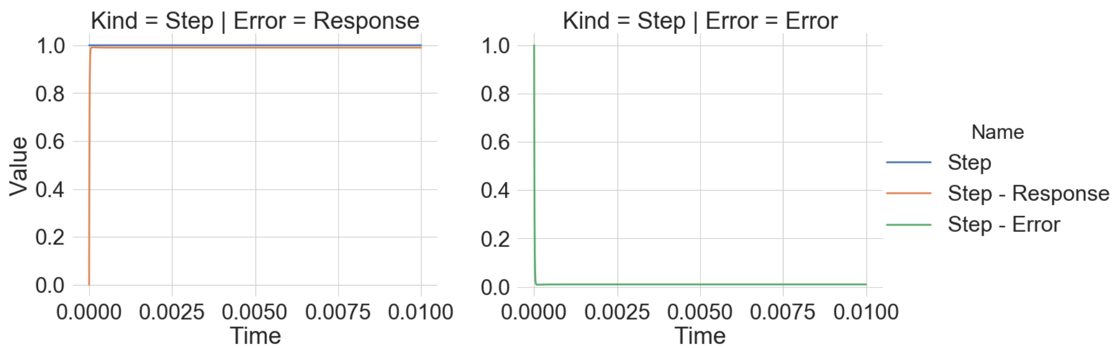
After computing the calculated zero and pole of the compensation network, the final transfer function is:

$$G_c(s) = 100 \cdot \frac{1744826.86}{5228.13} \cdot \frac{s + 5228.13}{s + 1744826.86}$$

Now, I'll take a look at the final step response of this system - with the compensation network added.

```
In [15]: comp_network_num = np.multiply(100 * p / z, [1, z])
         comp_network_den = [1, p]
```

```
t = np.arange(0, 0.01, 0.000001)
df = response_to_inputs(num, den, [step], ["Step"], t, comp_network_num, comp_network_d
create_plots(df, True);
```



This appears to do the job. There is no overshoot, and the response to the unit step function definitely reaches equilibrium by 5 milliseconds. However, just to verify, here is the error at this time:

```
In [16]: df[(df["Error"] == "Error") & (df["Time"] == 5 / 1000)]

Out[16]:          Time     Value  Kind          Name  Error
         25000   0.005  0.009901  Step  Step - Error  Error
```

2.8 Problem B

2.8.1 Part i

There is clearly a low pass filter with two poles. The reason we know this is because the phase settles out to -180 degrees, and does not appear to increase or have any zeros of the plotted range.

My first estimation is that the dominant pole exists at 10^0, or 1. I am making this estimation because the magnitude response begins to decrease at this point, and the phase reaches -45^o at this frequency. The next pole clearly exists at 10^1, or 10. Once again, we can see the magnitude transition from a $-20\frac{dB}{dec}$ slope to $-40\frac{dB}{dec}$ (suggesting a pole) - **and** and frequency is at -135^o at this point.

That results in the following approximated transfer function:

$$H(s) = K \cdot \frac{1}{(s+1)(s+10)}$$

However, we see that at near zero frequencies ($w \approx 0$), the magnitude of the response is 20dB. In order to achieve this gain, K needs to be approximately 100 - we know this because each gain of 10 corresponds to 20dB, and the starting gain of this equation is -20dB.

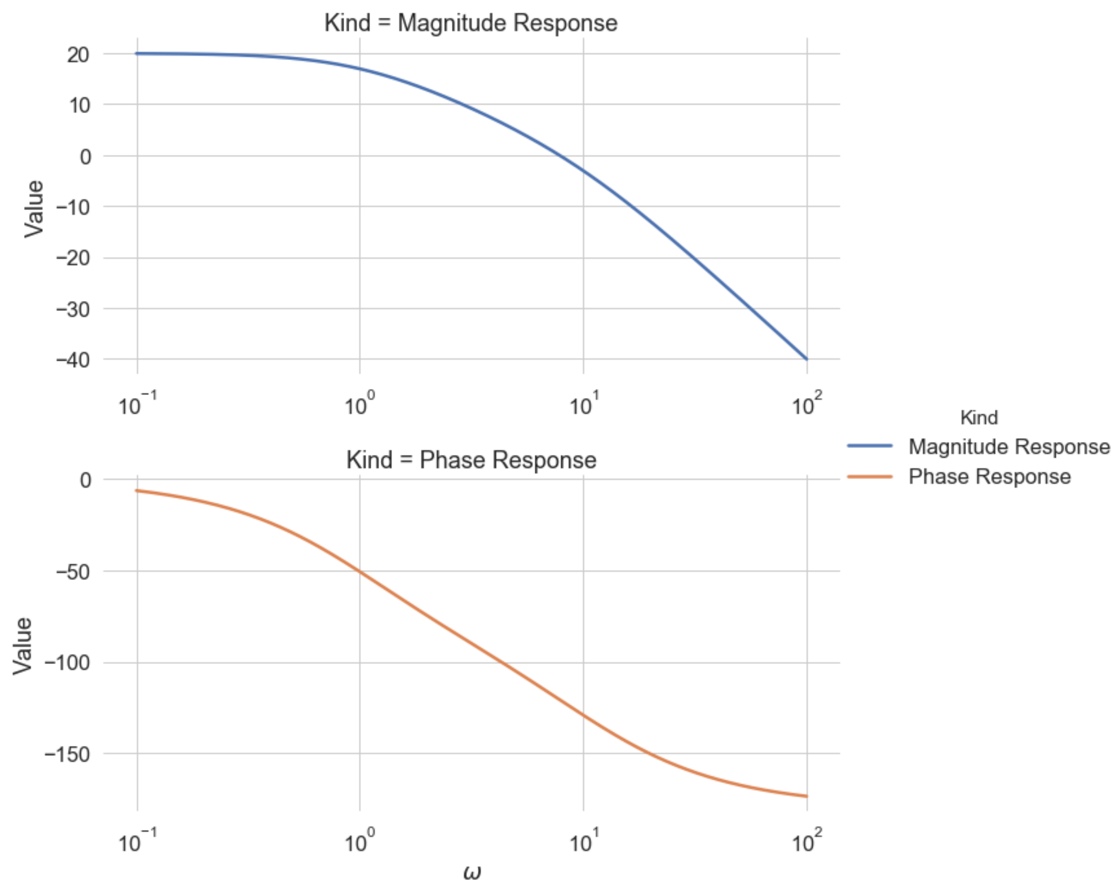
Thus, the following is the final transfer function:

$$H(s) = \frac{100}{(s+1)(s+10)}$$

```
In [17]: num = [100]
         den = convolve_all([[1, 1], [1, 10]])

         df, m, w = magnitude_phase_response(num, den, [10 ** -1, 10 ** 2], .01)
         print ("Crossover at {:.3f} (rad/s) has a phase-margin of {:.3f} degrees".format(w, m))
         g = create_plots(df)
         for ax in g.axes.flatten():
             ax.tick_params(labelbottom=True)
```

```
Crossover at 7.173 (rad/s) has a phase-margin of 62.286 degrees
```

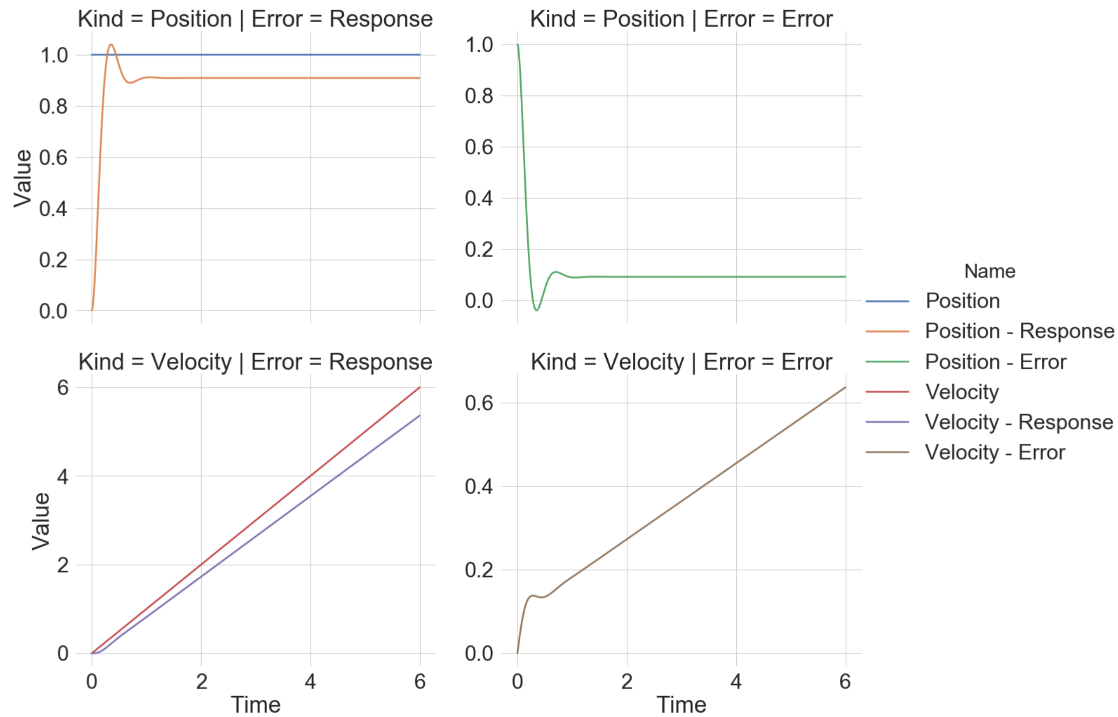


Which seems to match the given Bode plot.

2.8.2 Part ii

Let's start by looking at the position and velocity response.

```
In [18]: t = np.arange(0, 6, 0.001)
         df2 = response_to_inputs(num, den, [step, ramp], ["Position", "Velocity"], t)
         create_plots(df2, True);
```



I can see that the position error seems to stabilize to about 10%, but the velocity error is unbounded. In order to address this, I need to institute a compensation network that reduces the steady-state velocity constant to a number less than 0.1.

To accomplish this, I will add a phase-lead offset, with the addition of another pole at $\omega = 0$, in order to change the velocity error to a bounded value. The general form of this will be:

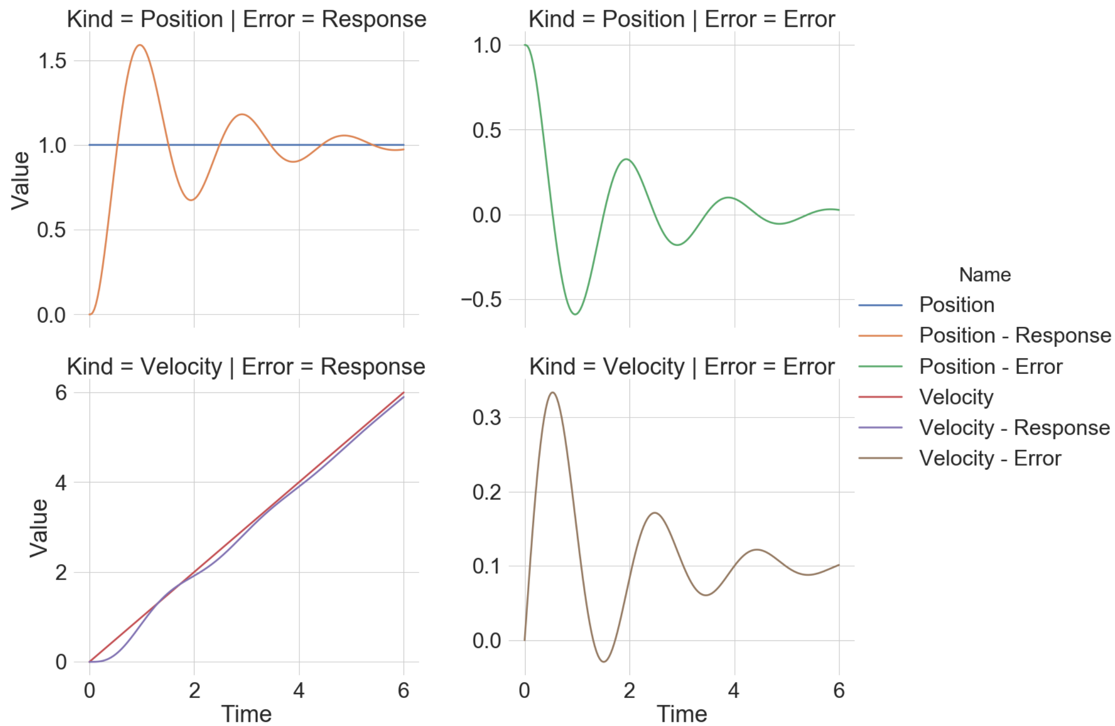
$$G_c(s) = \frac{\omega_p}{\omega_z} \cdot \frac{s + \omega_z}{s \cdot (s + \omega_p)}$$

```
In [19]: z, p = compute_phase_lead(df, 100-m)
         print ("The new frequency of the zero is {:.2f} (rad/s)\
                \nThe new frequency of the pole is {:.2f} (rad/s)".format(z, p))
```

```
The new frequency of the zero is 6.19 (rad/s)
The new frequency of the pole is 25.71 (rad/s)
```

```
In [20]: comp_num = np.multiply(p / z, [1, z]) # Was 7.5
         comp_den = convolve_all([[1, 0], [1, p]])

         t = np.arange(0, 6, 0.001)
         df = response_to_inputs(num, den, [step, ramp], ["Position", "Velocity"], t, comp_num,
         create_plots(df, True);
```



This seems to have done the trick. Just to verify, let's look at the error of the response function at $t = 5$ seconds.

```
In [21]: df[(df["Time"] == 5) & (df["Error"] == "Error")]
```

```
Out[21]:        Time     Value      Kind              Name  Error
         17000   5.0 -0.050009  Position  Position - Error  Error
         35000   5.0  0.098880  Velocity  Velocity - Error  Error
```

As requested, the error of both the position and velocty have been reduced to below 10% after 5 seconds.