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Math 135
Homework #4
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32, 41, 43, 44, 45, 3, 6

3.32

1. Show that R is symmetric iff $R^{-1} \subseteq R$.
2. Show that R is transitive iff $R \circ R \subseteq R$.

1. ($\Rightarrow$) If R is symmetric then: $xRy \implies yRx$. This means that $\langle x, y\rangle \in R$ and $\langle y, x\rangle \in R$, so if $\langle x, y\rangle \in R^{-1}$, then $\langle y, x\rangle \in R$ and $\langle x, y\rangle \in R$.
   ($\Leftarrow$) If $R^{-1} \subseteq R$, then if $\langle x, y\rangle \in R^{-1}$ then $\langle x, y\rangle \in R$. Additionally, because $R^{-1^{-1}} = R$, $\langle y, x\rangle \in R$, so xRy and yRx meaning R is symmetric.
2. ($\Rightarrow$) If R is transitive then $\forall x, y, z(xRy \,\&\, yRz \Rightarrow xRz)$ so, if $\langle x, z\rangle \in R \circ R$ then $\exists y(\langle x, y\rangle \in R \,\&\, \langle y, z\rangle \in R)$, or $xRy \,\&\, yRz$. By definition of transitivity, we have xRz, or $\langle x, z\rangle \in R$.
   ($\Leftarrow$) If $R \circ R \subseteq R$ then $(t \in R \circ R) \Rightarrow (t \in R)$. Take $\langle x, y\rangle \,\&\, \langle y, z\rangle \in R$. Then $\langle x, z\rangle \in R \circ R$ by composition which means $\langle x, z\rangle \in R$ which means $xRy \,\&\, yRz \Rightarrow xRz$.

$\blacksquare$

3.41 Let $\mathbb{R}$ be the set of real numbers and define the relation Q on $\mathbb{R} \times \mathbb{R}$ by $\langle u, v\rangle Q \langle x, y\rangle$ iff $u + y = x + v$.

1. Show that Q is an equivalence relation on $\mathbb{R} \times \mathbb{R}$.
2. Is there a function $G : \mathbb{R} \times \mathbb{R}/Q \to \mathbb{R} \times \mathbb{R}/Q$ satisfying the equation
$$G([\langle x, y\rangle]_Q) = [\langle x + 2y, y + 2x\rangle]_Q?$$

1. We must show three things:
   (a) Reflexive: $\langle u, v\rangle Q \langle u, v\rangle$. This is true if and only if $u + v = u + v$ which is true.
   (b) Symmetric: $\langle u, v\rangle Q \langle x, y\rangle \Rightarrow \langle x, y\rangle Q \langle u, v\rangle$. From the left side we have that $u + y = x + v$ which means that $x + v = u + y$ or $\langle x, y\rangle Q \langle u, v\rangle$.
   (c) Transitive: Assume $\langle u, v\rangle Q \langle x, y\rangle$ and $\langle x, y\rangle Q \langle n, m\rangle$. Then $u + y = x + v$ and $x + m = n + y$. Therefore we have that $u + m = n + v$ or: $\langle u, v\rangle Q \langle n, m\rangle$ which affirms transitivity.

2. By Theorem 3Q in the book, this function exists if and only if the function $F : \langle x, y\rangle \mapsto \langle x+2y, y+2x\rangle$ respects relation Q. If $\langle u, v\rangle Q\langle x, y\rangle$ then:

$$\begin{aligned} & u+y=x+v \\ \iff & 2u+v+2y+x=2x+y+u+2v \\ \iff & (u+2v)+(y+2x)=(x+2y)+(v+2u) \\ \iff & F(\langle u, v\rangle) Q F(\langle x, y\rangle) \end{aligned}$$

Therefore F respects Q so G exists.

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3.43 Assume that R is a linear ordering on a set A. Show that R^{-1} is also a linear ordering on A.

1. Transitive: If xRy and yRz then xRz. By the definition of the inverse we have that $zR^{-1}y$ and $yR^{-1}x$ and because xRz, we have that $zR^{-1}x$ so it is transitive.

2. Trichotomy: $\forall x, y \in A(\text{either } xRy, x=y, yRx)$. Given this, it follows that either $yR^{-1}x$, x=y, or $xR^{-1}y$. Therefore R^{-1} satisfies the trichotomy as well.

■

3.44 Assumer that $<$ is a linear ordering on a set A. Assume that $f : A \to A$ and that f has the property that whenever $x<y$, then $f(x)<f(y)$. Show that f is one-to-one and that whenever $f(x)<f(y)$, then $x<y$.

1. One-to-one: Assume that $f(x)=f(y)$. Then we have that $f(x) \nless f(y)$ and $f(y) \nless f(x)$ which means that neither $x<y$ or $y<x$. Because of the trichotomy of linear orderings we have that $x=y$ so f is one-to-one.

2. If we have that $f(x)<f(y)$ then either $x<y$, which is what we want. $x=y$ which is impossible becuase then $f(x)=f(y)$ which contradicts the hypothesis. Finally we could have that $y<x$ but this would imply that $f(y)<f(x)$ which also contradicts the hypothesis.

■

3.45 Assume that $<_{A}$ and $<_{B}$ are linear ordering on A and B respectively. Define the binary relation $<_{L}$ on the Cartesian product $A \times B$ by:

$$\langle a_1, b_1\rangle <_{L} \langle a_2, b_2\rangle \text{iff either } a_1 <_{A} a_2 \text{ or } (a_1=a_2 \ \& \ b_1 <_{B} b_2)$$

Show that $<_{L}$ is a linear ordering on $A \times B$.

1. Transitive: Assume that $\langle a_1, b_1\rangle <_{L} \langle a_2, b_2\rangle$ and $\langle a_2, b_2\rangle <_{L} \langle a_3, b_3\rangle$. Then if $a_1 = a_2 \ \& \ a_2 <_{A} a_3$ or $a_1 <_{A} a_2 \ \& \ a_2 = a_3$. In any of these, we have that $a_1 <_{A} a_3$ which confirms transitivity. By the assumptions the only other option for the a variables is that $a_1=a_2=a_3$. If this is the case then we have that $b_1 <_{B} b_2 <_{B} b_3$ in which case $b_1 <_{B} b_3$ which also confirms transitivity.

2. Trichotomy: If $t = \langle a_1, b_1 \rangle$ & $u = \langle a_2, b_2 \rangle \in A \times B$ then we have trichotomy of the a's under $<_{\text{A}}$. If $a_1 <_{\text{A}} a_2$ then $t <_{\text{L}} u$. If $a_2 <_{\text{A}} a_1$ then $u <_{\text{L}} t$. If $a_1 = a_2$ then we have the trichotomy of the b's under $<_{\text{B}}$. In this case if $b_1 <_{\text{B}} b_2$ then $t <_{\text{L}} u$. If $b_2 <_{\text{B}} b_1$ then $u <_{\text{L}} t$. Finally if $b_1 = b_2$ then $t = u$. And out trichotomy of $<_{\text{L}}$ is complete.

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4.3

1. Show that if a is a transitive set, then $\mathscr{P}a$ is also a transitive set.
2. Show that if $\mathscr{P}a$ is a transitive set, then a is also a transitive set.

1. If a is a transitive set then $x \in t \in a \Rightarrow x \in a$. Suppose we have a $y \in u \in \mathscr{P}a$. Then $y \in u \subseteq a$. Since $u \subseteq a$, $y \in a$. Becuase a is transitive, it follows that $y \subseteq a$ which implies that $y \in \mathscr{P}a$. Hence, $\mathscr{P}a$ is transitive.
2. If $\mathscr{P}a$ is transitive then $x \in t \in \mathscr{P}a \Rightarrow x \in \mathscr{P}a$. Since $t \in \mathscr{P}a$, $t \subseteq a$ so x is in a. But since x is also in $\mathscr{P}a$, it follows that a is transitive. ($x \in a \Rightarrow x \subseteq a$).

$\blacksquare$

4.6 Prove that converse to Theorem 4E: If $\bigcup(a^+) = a$, then a is a transitive set.

$\bigcup(a^+) = \bigcup(a \cup \{a\}) = \bigcup a \cup \bigcup\{a\} = \bigcup a \cup a = a$. By the definition of binary union, If $x \in \bigcup a$ then $x \in \bigcup a \cup a = a$ so $x \in a$ which implies $\bigcup a \subseteq a$ so a is transitive. $\blacksquare$