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Math 135
Homework #2
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Elements of Set Theory 3.1, 3.3, 3.8, 3.11, 3.15, 3.19, 3.25, 3.30

3.1 Suppose that we attempted to generalize the Kuratowski definitions of ordered pairs to ordered triples by defining

$$\langle x, y, z\rangle^* = \{\{x\}, \{x, y\}, \{x, y, z\}\}$$

Show that this definition is unsuccessful by giving examples of objects u, v, w, x, y, z with $\langle x, y, z\rangle^* = \langle u, v, w\rangle^*$ but with either $y \neq v$ or $z \neq w$ (or both).

One such ordered pair would be

$$\begin{aligned} \langle a, b, b\rangle^* &= \langle a, a, b\rangle^* \\ \text{Expanded:} \{\{a\}, \{a, b\}, \{a, b, b\}\} &= \{\{a\}, \{a, a\}, \{a, a, b\}\} \\ \{\{a\}, \{a, b\}, \{a, b\}\} &= \{\{a\}, \{a\}, \{a, b\}\} \\ \{\{a\}, \{a, b\}\} &= \{\{a\}, \{a, b\}\} \end{aligned}$$

■

3.3 Show that $A \times \bigcup \mathscr{B} = \bigcup\{A \times X \mid X \in \mathscr{B}\}$.

Taking fifth roots of the equation yields

$$z + 1 = ze^{ik\frac{2\pi}{5}},$$

where $k \in \mathbb{Z}$. We note that $k = 0$ (and all other multiples of 5) yields $z + 1 = z$, which reduces to $1 = 0$, an inconsistent equation. Isolating z, we therefore have the solutions

$$\boxed{z = \frac{1}{e^{ik\frac{2\pi}{5}} - 1},}$$

with four unique solutions obtained using $k = 1, 2, 3, 4$. We expect 4 unique solutions because $(z + 1)^5 - z^5$ is a fourth-degree polynomial. ■

SS 1.7.5ad Describe the projections on the Riemann sphere of the following sets in the complex plane:

(a) the right half-plane $\{z \mid \text{Re}\, z > 0\}$,

(d) the set $\{z \mid |z| > 3\}$.

(a) The right half-plane corresponds to the right open hemisphere $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 > 0, x_1^2 + x_2^2 + x_3^2 = 1\}$. ■

(d) Since $|z| > 3$, $|z|^2 + 1 > 10$, so

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1} = 1 - \frac{2}{|z|^2 + 1} > 1 - \frac{2}{10} = \frac{4}{5}.$$

Thus, the set $\{z \mid |z| > 3\}$ corresponds to the dome of the Riemann sphere above the plane $x_3 = 4/5$. Hand sketches of these projections are shown below:

(a) (d)

(to be drawn in later)

■

DF 1.1.25 Prove that if $x^2 = 1$ for all $x \in G$ then G is abelian.

Suppose $x^2 = 1$ for all $x \in G$. Then $x = x^2x^{-1} = 1x^{-1} = x^{-1}$ for all $x \in G$. For two arbitrary elements a and b of G,

$$ab = (ab)^{-1} = b^{-1}a^{-1} = ba,$$

so a and b commute. Since a and b are arbitrary, $ab = ba$ for all a and $b \in G$, and G is abelian. ■

Logan 1.8.6 This exercise illustrates an important numerical procedure for solving Laplace's equation on a reactangle. Consider Laplace's equation on the rectangle $D : 0 < x < 4, 0 < y < 3$ with boundary conditions given on the bottom and top by $u(x, 0) = 0$, $u(x, 3) = 0$ for $0 \le x \le 4$ and on the sides by $u(0, y) = 2y(3 - y)$, $u(4, y) = 0$ for $0 \le y \le 3$. Apply the average value property (1.45) with $h = 1$ at each of the six lattice points $(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)$ inside D to obtain a system of six equations for the six unknown temperatures on these lattice points. Solve the system to approximate the steady temperature distribution and plot the approximate surface using a software package.

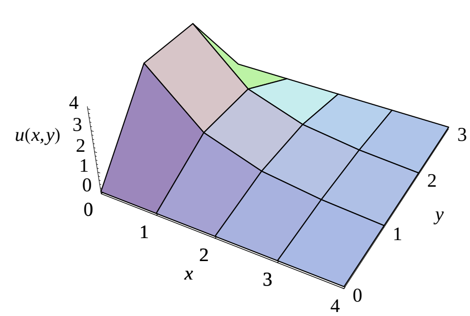
Applying this average value property with $h = 1$ yields the following linear system:

$$\begin{aligned}
u(1,1) &= \frac{1}{4}(u(1,0) + u(1,2) + u(0,1) + u(2,1)) = \frac{1}{4}(4 + u(1,2) + u(2,1)),\\
u(1,2) &= \frac{1}{4}(u(1,1) + u(1,3) + u(0,2) + u(2,2)) = \frac{1}{4}(4 + u(1,1) + u(2,2)),\\
u(2,1) &= \frac{1}{4}(u(2,0) + u(2,2) + u(1,1) + u(3,1)) = \frac{1}{4}(u(1,1) + u(2,2) + u(3,1)),\\
u(2,2) &= \frac{1}{4}(u(2,1) + u(2,3) + u(1,2) + u(3,2)) = \frac{1}{4}(u(2,1) + u(2,3) + u(3,2)),\\
u(3,1) &= \frac{1}{4}(u(3,0) + u(3,2) + u(2,1) + u(4,1)) = \frac{1}{4}(u(3,2) + u(2,1)),\\
u(3,2) &= \frac{1}{4}(u(3,1) + u(3,3) + u(2,2) + u(4,2)) = \frac{1}{4}(u(3,1) + u(2,2)),
\end{aligned}$$

which we solve in *Mathematica* 5.0 to obtain

$$\begin{pmatrix} u(1,1) \\ u(1,2) \\ u(2,1) \\ u(2,2) \\ u(3,1) \\ u(3,2) \end{pmatrix} = \begin{pmatrix} \frac{32}{21} \\ \frac{32}{21} \\ \frac{4}{7} \\ \frac{4}{7} \\ \frac{4}{21} \\ \frac{4}{21} \end{pmatrix}.$$

Plotting these lattice point values yields the following approximate temperature surface:



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