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Math 135
Homework #2
09/17/2014

Elements of Set Theory 3.1, 3.3, 3.8, 3.11, 3.15, 3.19, 3.25, 3.30

3.1 Suppose that we attempted to generalize the Kuratowski definitions of ordered pairs to ordered triples by defining

$$\langle x, y, z\rangle^* = \{\{x\}, \{x, y\}, \{x, y, z\}\}$$

Show that this definition is unsuccessful by giving examples of objects u, v, w, x, y, z with $\langle x, y, z\rangle^* = \langle u, v, w\rangle^*$ but with either $y \neq v$ or $z \neq w$ (or both).

One such ordered pair would be

$$\begin{aligned}
\langle a, b, b\rangle^* &= \langle a, a, b\rangle^* \\
\text{Expanded:}\{\{a\}, \{a, b\}, \{a, b, b\}\} &= \{\{a\}, \{a, a\}, \{a, a, b\}\} \\
\{\{a\}, \{a, b\}, \{a, b\}\} &= \{\{a\}, \{a\}, \{a, b\}\} \\
\{\{a\}, \{a, b\}\} &= \{\{a\}, \{a, b\}\}
\end{aligned}$$

■

3.3 Show that $A \times \bigcup \mathscr{B} = \bigcup\{A \times X \mid X \in \mathscr{B}\}$.

($\rightarrow$) We want to show that any $t = \langle x, y\rangle \in A \times \bigcup \mathscr{B}$ that $t \in \bigcup\{A \times X \mid X \in \mathscr{B}\}$

$$\begin{aligned}
\langle x, y\rangle \in A \times \bigcup \mathscr{B} &\implies x \in A \;\&\; y \in \bigcup \mathscr{B} \\
y \in \bigcup \mathscr{B} &\implies \exists N \in \mathscr{B} \text{ s.t. } (t \in N) \\
\text{So } \exists N \in \mathscr{B} &\text{ s.t. } (t \in A \times N)
\end{aligned}$$

Taking $X = N$, $\exists M \in \{A \times X \mid X \in \mathscr{B}\}$ s.t. $t \in M$ which means that $t \in \bigcup\{A \times X \mid X \in \mathscr{B}\}$

($\leftarrow$) In this direction we start with $t \in \bigcup\{A \times X \mid X \in \mathscr{B}\}$ and show that $t \in A \times \bigcup \mathscr{B}$

$$\begin{aligned}
t \in \bigcup\{A \times X \mid X \in \mathscr{B}\} &\implies \exists N \in \{A \times X \mid X \in \mathscr{B}\} \text{ s.t. } t \in N \\
\exists X \in \mathscr{B} &\text{ s.t. } t \in A \times X \\
t \in A \times X &\implies x \in A \;\&\; y \in X \\
y \in X &\implies y \in \bigcup \mathscr{B}
\end{aligned}$$

Therefore $\langle x, y\rangle \in A \times \bigcup \mathscr{B}$. ■

3.8 Show that for any set $\mathscr{A}$:

$$\operatorname{dom} \bigcup \mathscr{A} = \bigcup \{\operatorname{dom} R \mid R \in \mathscr{A}\},$$
$$\operatorname{ran} \bigcup \mathscr{A} = \bigcup \{\operatorname{ran} R \mid R \in \mathscr{A}\},$$

Let $\langle x, y \rangle$ be in $\bigcup \mathscr{A}$ so that x is in dom $\bigcup \mathscr{A}$. Then $\exists N \in \mathscr{A}$ such that $\langle x, y \rangle \in N$. Taking this $N = R$, $x \in$ dom R so $R \in \bigcup \{\operatorname{dom} R \mid R \in \mathscr{A}\}$. Conversely if x starts in $\bigcup \{\operatorname{dom} R \mid R \in \mathscr{A}\}$, then there is some R such that $x \in$dom R or $\langle x, y \rangle \in R \in \mathscr{A}$. Therefore $\langle x, y \rangle \in \bigcup \mathscr{A}$, and x is in it's domain.
For the second part one can make an identical argument using y being in the ran $\langle x, y \rangle$. ∎

3.11 Prove the following: Assume that F and G are functions, dom F = dom G, and $F(x) = G(x)$ for all x in the common domain. Then $F = G$.

Let $\langle x, y \rangle \in F$. Then becuase the domains are equal, there exists an $\langle x, t \rangle$ in G. Since $F(x) = G(x)$, there exists a t in $G(x)$ such that $y = t$. Therefore $\langle x, y \rangle \in G$ and $F \subseteq G$. By making the identical argument swapping F for G, one can see that $G \subseteq F$ so $F = G$. ∎

3.15 Let $\mathscr{A}$ be a set of functions such that for any f and g in $\mathscr{A}$, either $f \subseteq g$ or $g \subseteq f$. Show that $\bigcup \mathscr{A}$ is a function.

To show that $\bigcup \mathscr{A}$ is a function it must only contain 2-pairs and $(\langle x, y \rangle \in \bigcup \mathscr{A})$ & $(\langle x, t \rangle \in \bigcup \mathscr{A}) \implies y = t$. If $\langle x, y \rangle$ & $\langle x, t \rangle \in \bigcup \mathscr{A}$ then let $\langle x, y \rangle \in f \in \mathscr{A}$ and $\langle x, t \rangle \in g \in \mathscr{A}$. If $f = g$ then since f is a function $y = t$. If $f \neq g$ then since $f \subseteq g$ or $g \subseteq f$, either $\langle x, y \rangle \in g$ or $\langle x, t \rangle \in f$. In either case, becuase f and g are function, one of them will have both $\langle x, y \rangle$ & $\langle x, t \rangle$ in them which means $x = t$. ∎

3.19 Let
$$A = \{\langle \varnothing . \{\varnothing, \{\varnothing\}\}\rangle, \langle \{\varnothing\}, \varnothing\rangle\}.$$
Evaluate each of the following: $A(\varnothing)$, $A[\varnothing]$, $A[\{\varnothing\}]$, $A[\{\varnothing, \{\varnothing\}\}]$, A^{-1}, $A \circ A$, $A \upharpoonright \varnothing$, $A \upharpoonright \{\varnothing\}$, $A \upharpoonright \{\varnothing, \{\varnothing\}\}$, $\bigcup\bigcup A$.

$$\begin{aligned}
A(\varnothing) &= \{\varnothing, \{\varnothing\}\} \\
A[\varnothing] &= \varnothing \\
A[\{\varnothing\}] &= \{\varnothing, \{\varnothing\}\} \\
A[\{\varnothing, \{\varnothing\}\}] &= \{\varnothing, \{\varnothing, \{\varnothing\}\}\} \\
A^{-1} &= \{\langle \{\varnothing, \{\varnothing\}\}, \varnothing\rangle,\ \langle \varnothing, \{\varnothing\}\rangle\} \\
A \circ A &= \{\langle \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\rangle\} \\
A \upharpoonright \varnothing &= \varnothing \\
A \upharpoonright \{\varnothing\} &= \{\langle \varnothing, \{\varnothing, \{\varnothing\}\}\rangle\} \\
A \upharpoonright \{\varnothing, \{\varnothing\}\} &= \varnothing \\
\bigcup\bigcup A &= \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}
\end{aligned}$$

■

3.25

1. Assume that G is a one-to-one function. Show that $G \circ G^{-1}$ is $I_{\text{ran } G}$, the identity function on ran G.

2. Show that the result of part (a) holds for any function G, not necessarily one-to-one.

To show that they are equal, assume $\langle x, y\rangle \in G \circ G^{-1}$. This means that: $\exists t(\langle x, t\rangle \in G^{-1} \,\&\, \langle t, y\rangle \in G)$. Since $\langle x, t\rangle \in G^{-1}$, it must be that $\langle t, x\rangle \in G$. But since both $\langle t, x\rangle$ and $\langle t, y\rangle$ are in G, a function, $x = y$. Therefore the only elements in $G \circ G^{-1}$ are of the form $\langle x, x\rangle$ where x is in ran G which is exatly $I_{\text{ran } G}$ ■

3.30 Assume that $F : \mathscr{P}A \to \mathscr{P}A$ and that F has the monotonicity property:
$$X \subseteq Y \subseteq A \quad \Longrightarrow \quad F(X) \subseteq F(Y).$$
Define
$$B = \bigcap\{X \subseteq A \mid F(X) \subseteq X\} \quad \text{and} \quad C = \bigcup\{X \subseteq A \mid X \subseteq F(X)\}.$$

1. Show that $F(B) = B$ and $F(C) = C$.

2. Show that if $F(X) = X$, then $B \subseteq X \subseteq C$.

1. Pick any $t \in F(B)$. Then its preimage x is in B. Since $x \in B$, every set that contains x is a superset of itself under F. Therefore, B is a superset of $F(B)$ and $t \in B$. On the other hand assume $t \in B$ but $t \notin F(B)$. This is a contradiction becuase B must be a super set of $F(B)$. Therefore $F(B) = B$).
   For set C, assume $t \in C$, then $\exists X(t \in X)$ and $X \subseteq F(X)$, which implies $t \in F(X)$ and $t \in F(C)$. Assuming $t \in F(C)$, then since $F(C) \subseteq C$, $F(F(C)) \subseteq F(C)$ so $F(C)$ belongs to the union and anything in C is in $F(C)$.

2. If $F(X) = X$ then $F(X) \subseteq X$ so B includes it in it's dijoint which means $B \subseteq X$. On the other hand, $X \subseteq F(X)$ and since C is the union of X and other sets, $X \subseteq C$.

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