

# Course 02402 Introduction to Statistics

## Lecture 12: Two-way Analysis of Variance, ANOVA

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# Overview

- 1 Intro: Small example and TV-data from B&O
- 2 Model
- 3 Computation - decomposition and the ANOVA table
- 4 Hypothesis test (F-test)
- 5 Post hoc analysis
- 6 Model control / model validation
- 7 A complete example - from the book

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# TV set development at Bang & Olufsen

*Sound and image quality is measured by th human perceptual instrument:*



# Bang & Olufsen data in R

```
# Get the B&O data from the lmerTest-package
library(lmerTest)

## Error in library(lmerTest): there is no package called 'lmerTest'

data(TVbo)

## Warning in data(TVbo): data set 'TVbo' not found

# Each of 8 assessors scored each of 12 combinations 2 times.
# Take a look at the sharpness scores for one single picture
# and one of the two repetitions
TVbo_sub <- subset(TVbo, Picture == 1 & Repeat == 1)[, c(1, 2, 9)]

## Error in subset(TVbo, Picture == 1 & Repeat == 1): objekt 'TVbo' blev
ikke fundet

sharp <- matrix(TVbo_sub$Sharpness, nrow = 8, byrow = T)

## Error in matrix(TVbo_sub$Sharpness, nrow = 8, byrow = T): objekt
'TVbo_sub' blev ikke fundet

colnames(sharp) <- c("TV3", "TV2", "TV1")
```

## Bang & Olufsen data in R

```
## Error in library(lmerTest): there is no  
package called 'lmerTest'  
## Warning in data(TVbo): data set 'TVbo' not  
found  
## Error in subset(TVbo, Picture == 1 & Repeat  
== 1): objekt 'TVbo' blev ikke fundet  
## Error in matrix(TVbo_sub$Sharpness, nrow =  
8, byrow = T): objekt 'TVbo_sub' blev ikke  
fundet  
## Error in colnames(sharp) <- c("TV3", "TV2",  
"TV1"): objekt 'sharp' blev ikke fundet  
## Error in rownames(sharp) <- c("Person 1",  
"Person 2", "Person 3", "Person 4", : objekt
```

## Two-way ANOVA - example

- Same data as for one-way, but now we know that the experiment was split into blocks:

	Group A	Group B	Group C
Block 1	2.8	5.5	5.8
Block 2	3.6	6.3	8.3
Block 3	3.4	6.1	6.9
Block 4	2.3	5.7	6.1

- Hence three *groups* on four *blocks*,
- or three *treatments* on four *persons*,
- or three *varieties* on four *fields* (hence blocks),
- or something similar.

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- Hence three *groups* on four *blocks*,
  - or three *treatments* on four *persons*,
  - or three *varieties* on four *fields* (hence blocks),
  - or something similar.
- *One-way* vs. *two-way* ANOVA
- *Completely randomized design* vs. *Randomized block design*



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- Question: Is there a significant difference (in means) between the groups A, B and C?
- ANOVA can be used if the observations in each group are (approximately) normal distributed or if the  $n_i$ s are large enough (CLT).

# The toy data in R

```

# Observations
y <- c(2.8, 3.6, 3.4, 2.3,
      5.5, 6.3, 6.1, 5.7,
      5.8, 8.3, 6.9, 6.1)

# Treatments (groups, varieties)
treatm <- factor(c(1, 1, 1, 1,
                  2, 2, 2, 2,
                  3, 3, 3, 3))

# Blocks (persons, fields)
block <- factor(c(1, 2, 3, 4,
                 1, 2, 3, 4,
                 1, 2, 3, 4))

# No. of treatments and no. of blocks (for later formulas)
(k <- length(unique(treatm)))
(l <- length(unique(block)))

# Box plots by treatment
plot(treatm, y, xlab = "Treatment", ylab = "y")

# Box plots by block
plot(block, y, xlab = "Block", ylab="y")

```

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# Two-way ANOVA, model

- The model may be formulated as

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij},$$

where the errors are i.i.d. with

$$\varepsilon_{ij} \sim N(0, \sigma^2).$$

- $\mu$  is the overall mean
- $\alpha_i$  is the effect of treatment  $i$
- $\beta_j$  is the level for block  $j$
- There are  $k$  treatments and  $l$  blocks

# Estimates of parameters in the model

- We can compute the estimates of the parameters ( $\hat{\mu}$ ,  $\hat{\alpha}_i$ , and  $\hat{\beta}_j$ )

$$\hat{\mu} = \bar{y} = \frac{1}{k \cdot l} \sum_{i=1}^k \sum_{j=1}^l y_{ij}$$

$$\hat{\alpha}_i = \left( \frac{1}{k} \sum_{j=1}^l y_{ij} \right) - \hat{\mu}$$

$$\hat{\beta}_j = \left( \frac{1}{l} \sum_{i=1}^k y_{ij} \right) - \hat{\mu}$$

```
# Sample mean
(mu_hat <- mean(y))

# Sample mean deviation for each treatment
(alpha_hat <- tapply(y, treatm, mean) - mu_hat)

# Sample mean deviation for each block
(beta_hat <- tapply(y, block, mean) - mu_hat)
```

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# Two-way ANOVA, decomposition and the ANOVA table, Theorem 8.20

- With the model

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}, \quad \varepsilon_{ij} \stackrel{\text{i.i.d.}}{\sim} N(0, \sigma^2)$$

the total variation in the data can be decomposed:

$$SST = SS(Tr) + SS(BI) + SSE$$

- 'Two-way' refers to the fact that there are two factors (grouping variables) in the experiment.
- The method is called analysis of variance, because hypothesis testing is carried out by comparing certain variances.



# Formulas for sums of squares

- Total sum of squares (or “the total variance”, same as for one-way)

$$SST = \sum_{i=1}^k \sum_{j=1}^l (y_{ij} - \hat{\mu})^2$$

# Formulas for sums of squares

- Total sum of squares (or “the total variance”, same as for one-way)

$$SST = \sum_{i=1}^k \sum_{j=1}^l (y_{ij} - \hat{\mu})^2$$

- Treatment sum of squares (or “variance explained by the treatment part of the model”)

$$SS(Tr) = l \cdot \sum_{i=1}^k \hat{\alpha}_i^2$$

## Formulas for sums of squares

- Sum of squares for blocks/persons (“variance explained by the block part of the model”)

$$SS(Bl) = k \cdot \sum_{j=1}^l \hat{\beta}_j^2$$

- Sum of squares for the residuals (“residual variance after model fit”)

$$SSE = \sum_{i=1}^k \sum_{j=1}^l (y_{ij} - \hat{\alpha}_i - \hat{\beta}_j - \hat{\mu})^2$$

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## Two-way ANOVA: Hypothesis of no effect of treatment, Theorem 8.22

- We want to compare (more than 2) means  $\mu + \alpha_i$  in the model

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}, \quad \varepsilon_{ij} \stackrel{\text{i.i.d.}}{\sim} N(0, \sigma^2)$$

- The hypothesis of no difference between treatment means may be formulated as

$$H_{0,Tr} : \quad \alpha_i = 0 \quad \text{for all } i$$

$$H_{1,Tr} : \quad \alpha_i \neq 0 \quad \text{for at least one } i$$

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$$H_{1,Tr} : \quad \alpha_i \neq 0 \quad \text{for at least one } i$$

- Under  $H_{0,Tr}$  the following is true:

$$F_{Tr} = \frac{SS(Tr)/(k-1)}{SSE/((k-1)(l-1))}$$

is  $F$ -distributed with  $k-1$  and  $(k-1)(l-1)$  degrees of freedom.

## Two-way ANOVA: hypothesis of no effect of blocks/persons, Theorem 8.22

- We want to compare (more than 2) means  $\mu + \beta_j$  in the model

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}, \quad \varepsilon_{ij} \stackrel{\text{i.i.d.}}{\sim} N(0, \sigma^2)$$

- The hypothesis of no difference between block means may be formulated as

$$H_{0,BI} : \quad \beta_j = 0 \quad \text{for all } j$$

$$H_{1,BI} : \quad \beta_j \neq 0 \quad \text{for at least one } j$$

## Two-way ANOVA: hypothesis of no effect of blocks/persons, Theorem 8.22

- We want to compare (more than 2) means  $\mu + \beta_j$  in the model

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}, \quad \varepsilon_{ij} \stackrel{\text{i.i.d.}}{\sim} N(0, \sigma^2)$$

- The hypothesis of no difference between block means may be formulated as

$$H_{0,BI} : \beta_j = 0 \quad \text{for all } j$$

$$H_{1,BI} : \beta_j \neq 0 \quad \text{for at least one } j$$

- Under  $H_{0,BI}$  the following is true:

$$F_{BI} = \frac{SS(BI)/(l-1)}{SSE/((k-1)(l-1))}$$

follows an  $F$ -distribution with  $l-1$  and  $(k-1)(l-1)$  degrees of freedom.



# F-distribution and treatment hypothesis

```

# Plot density of relevant F-distribution. Remember that this is "under H0"
# (computed as if H0 were true)
xseq <- seq(0, 10, by = 0.1)
plot(xseq, df(xseq, df1 = k-1, df2 = (k-1)*(l-1)), type = "l")

# Show critical value (5% signif. level) for test of treatment hypothesis
critical_value <- qf(0.95, df1 = k-1, df2 = (k-1)*(l-1))
abline(v = critical_value, col = "red")

# Compute value of the test statistic
(FTr <- (SSTr/(k-1)) / (SSE/((k-1)*(l-1))))

# Compute p-value for the test
1 - pf(FTr, df1 = k-1, df2 = (k-1)*(l-1))

```

# F-distribution and block hypothesis

```

# Plot density of relevant F-distribution. Remember that this is "under H0"
# (computed as if H0 were true)
xseq <- seq(0, 10, by = 0.1)
plot(xseq, df(xseq, df1 = l-1, df2 = (k-1)*(l-1)), type = "l")

# Show critical value (5% signif. level) for test of treatment hypothesis
critical_value <- qf(0.95, df1 = l-1, df2 = (k-1)*(l-1))
abline(v = critical_value, col = "red")

# Compute value of the test statistic
(FB1 <- (SSB1/(l-1)) / (SSE/((k-1)*(l-1))))

# Compute p-value for the test
1 - pf(FB1, df1 = l-1, df2 = (k-1)*(l-1))

```

# The two-way ANOVA table

Source of variation	Deg. of freedom	Sums of squares	Mean sum of squares	Test-statistic $F$	$p$ -value
<i>Treatment</i>	$k - 1$	$SS(Tr)$	$MS(Tr) = \frac{SS(Tr)}{k-1}$	$F_{Tr} = \frac{MS(Tr)}{MSE}$	$P(F > F_{Tr})$
<i>Block</i>	$l - 1$	$SS(Bl)$	$MS(Bl) = \frac{SS(Bl)}{l-1}$	$F_{Bl} = \frac{MS(Bl)}{MSE}$	$P(F > F_{Bl})$
<i>Residual</i>	$(k - 1)(l - 1)$	$SSE$	$MSE = \frac{SSE}{(k-1)(l-1)}$		
<i>Total</i>	$n - 1$	$SST$			

```
anova(lm(y ~ treatm + block))
```

```
## Analysis of Variance Table
```

```
##
```

```
## Response: y
```

```
##          Df Sum Sq Mean Sq F value    Pr(>F)
```

```
## treatm    2  30.79   15.40   74.40 5.8e-05 ***
```

```
## block     3   3.95    1.32    6.37  0.027 *
```

```
## Residuals 6   1.24    0.21
```

```
## ---
```

```
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

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## Post hoc confidence interval

- Like for one-way ANOVA (use methods 8.9 and 8.10) but substitute  $n - k$  degrees of freedom with  $(k - 1)(l - 1)$  (and use MSE from the two-way ANOVA).
- Can be done with either treatments or blocks.

## Post hoc confidence interval

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- Can be done with either treatments or blocks.
- A single pre-planned CI for the difference between treatment  $i$  and  $j$ :

$$\bar{y}_i - \bar{y}_j \pm t_{1-\alpha/2} \sqrt{\frac{SSE}{(k-1)(l-1)} \left( \frac{1}{n_i} + \frac{1}{n_j} \right)} \quad (1)$$

where  $t_{1-\alpha/2}$  is based on the t-distribution with  $(k - 1)(l - 1)$  degrees of freedom.

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where  $t_{1-\alpha/2}$  is based on the t-distribution with  $(k - 1)(l - 1)$  degrees of freedom.

- If all  $M = k(k - 1)/2$  combinations of pairwise confidence intervals are found use the formula  $M$  times but each time with  $\alpha_{\text{Bonferroni}} = \alpha/M$ .

# Post hoc pairwise hypothesis test

- A single pre-planned level  $\alpha$  hypothesis tests:

$$H_0 : \mu_i = \mu_j, \quad H_1 : \mu_i \neq \mu_j$$

is carried out as:

$$t_{\text{obs}} = \frac{\bar{y}_i - \bar{y}_j}{\sqrt{MSE \left( \frac{1}{n_i} + \frac{1}{n_j} \right)}} \quad (2)$$

and:

$$p\text{-value} = 2P(t > |t_{\text{obs}}|)$$

where the  $t$ -distribution with  $(k-1)(l-1)$  degrees of freedom is used.



# Post hoc pairwise hypothesis test

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# Model validation: Variance homogeneity

Make box plots of the *residuals* to check whether the variability seems different across the groups.

```
# Save the fitted model
fit <- lm(y ~ treatm + block)

# Make box plots of residuals
par(mfrow = c(1,2))
plot(treatm, fit$residuals, xlab = "Treatment")
plot(block, fit$residuals, xlab = "Block")
```

# Model validation: Normality

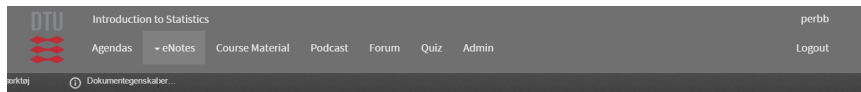
Make a normal QQ-plot to check whether the distribution of the residuals seems normal.

```
# Normal QQ-plot of the residuals  
qqnorm(fit$residuals)  
qqline(fit$residuals)
```

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# A complete example - from the book



## 8.8.5 A complete worked through example: Car tires

### |||| Example 8.26 Car tires

In a study of 3 different types of tires ("treatment") effect on the fuel economy, drives of 1000 km in 4 different cars ("blocks") were carried out. The results are listed in the following table in km/l.

	Car 1	Car 2	Car 3	Car 4	Mean
Tire 1	22.5	24.3	24.9	22.4	22.525
Tire 2	21.5	21.3	23.9	18.4	21.275
Tire 3	22.2	21.9	21.7	17.9	20.925
Mean	21.400	22.167	23.167	19.567	21.575

Let us analyse these data with a two-way ANOVA model, but first some explorative plotting:

```
## Collecting the data in a data frame
```

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