# Course 02402 Introduction to Statistics

## Lecture 2: Random variables and discrete distributions

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## Agenda

1. Random variables and density functions
2. Distribution functions
3. Specific (discrete) distributions I: The binomial
   - Example 1
4. Specific distributions II: The hypergeometric
   - Example 2
5. Specific distributions III: The Poisson
   - Example 3
6. Distributions in R
7. Mean and variance of discrete distributions

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## Random variables

A random variable represents a value of an outcome *before* the corresponding *experiment* is carried out.

- A throw of a dice.
- The number of six'es in ten dice throws.
- Fuel consumption of a car.
- Measurement of glucose level in blood sample.
- ...

## Discrete and continuous random variables

- We distinguish between *discrete* and *continuous* random variables.
- Discrete:
  - Number of people in this room who wear glasses.
  - Number of planes departing from CPH within the next hour.
- Continuous:
  - Wind speed measurement.
  - Transport time to DTU.
- Today: Discrete. Next week: Continuous.

## Random variable

Before the experiment is carried out, we have a random variable

$$X \text{ (or } X_1, \ldots, X_n\text{)}$$

indicated with capital letters.

After the experiment is carried out, we have a *realization* or *observation*

$$x \text{ (or } x_1, \ldots, x_n\text{)}$$

indicated with lowercase letters.

## Simulate rolling a dice in R

```r
# One random draw from (1,2,3,4,5,6)
# with equal probability for each outcome
sample(1:6, size = 1)
```

```
[1] 1
```

## Discrete distributions

- Random variables are used to describe an experiment before it is carried out.
- How to do this without yet knowing the outcome?
- Solution: Use a *density function*.

## Density function, discrete random variable: Definition 2.6

The *density function* (probability density function, pdf) of a discrete random variable:

### Definition

$$f(x) = P(X = x)$$

Describes the probability that $X$ takes the value $x$ when the experiment is carried out.

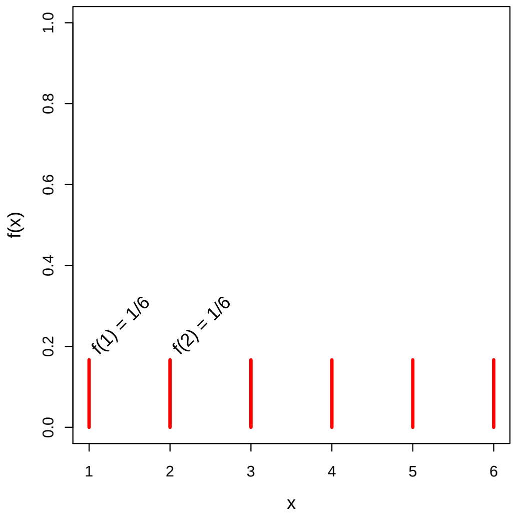
## Density function, discrete random variable, Definition 2.6

The density function of a discrete random variable satisfies two properties:

### Definition

$$f(x) \geq 0 \text{ for all } x \quad \text{and} \quad \sum_{\text{all } x} f(x) = 1$$

## Density function for a fair dice



## Sample

If we only have a single observation, can we see the distribution? No!

But if we have $n$ observations, then we have a *sample*

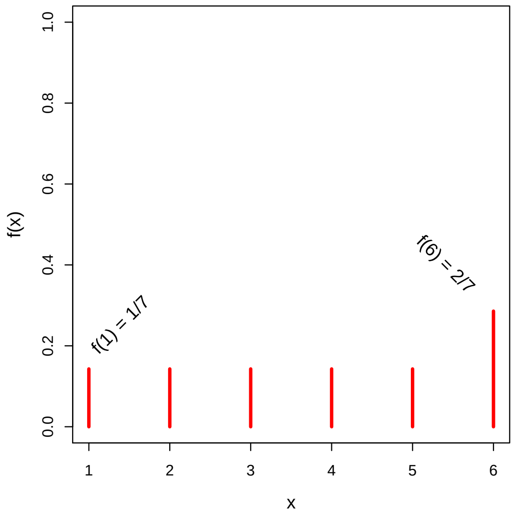
$$\{x_1, x_2, \dots, x_n\}$$

and we can begin to get an idea of the distribution.

# Simulation of $n$ rolls with a fair dice

```
# Number of simulated realizations (sample size)
n <- 30

# n independent random draws from the set (1,2,3,4,5,6)
# with equal probability of each outcome
xFair <- sample(1:6, size = n, replace = TRUE)
xFair

# Count number of each outcome using the 'table' function
table(xFair)

# Plot the empirical pdf
plot(table(xFair)/n, lwd = 10, ylim = c(0,1), xlab = "x",
     ylab = "Density f(x)")
# Add the true pdf to the plot
lines(rep(1/6,6), lwd = 4, type = "h", col = 2)
# Add a legend to the plot
legend("topright", c("Empirical pdf","True pdf"), lty = 1, col = c(1,2),
       lwd = c(5, 2), cex = 0.8)
```

# Density function for an unfair dice



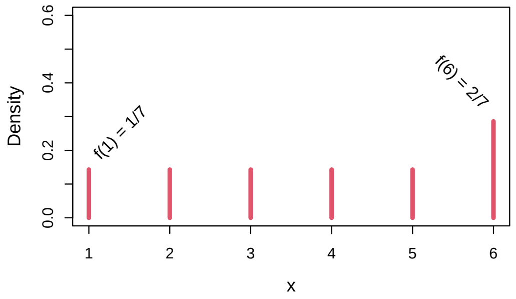
# Simulation of $n$ rolls with an unfair dice

```
# Number of simulated realizations (sample size)
n <- 30

# n independent random draws from the set (1,2,3,4,5,6)
# with higher probability of getting a six
xUnfair <- sample(1:6, size = n, replace = TRUE, prob = c(rep(1/7,5),2/7))
xUnfair

# Plot the empirical pdf
plot(table(xUnfair)/n, lwd = 10, ylim = c(0,1), xlab = "x",
     ylab = "Density f(x)")
# Add the true pdf to the plot
lines(c(rep(1/7,5),2/7), lwd = 4, type = "h", col = 2)
# Add a legend to the plot
legend("topright", c("Empirical pdf","True pdf"), lty = 1, col = c(1,2),
       lwd = c(5, 2), cex = 0.8)
```

# Some questions

Let $X$ describe one throw with the *unfair* dice. What is:

- The probability of getting a 4?
- The probability of getting a 5 or a 6?
- The probability of getting less than 3?



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## Distribution function, discrete random variable: Definition 2.9

The *distribution function* (cumulative distribution function, cdf) of a discrete random variable:

**Definition**

$$F(x) = P(X \leq x) = \sum_{j \text{ where } x_j \leq x} f(x_j)$$

## Fair dice example

Let $X$ represent one throw with a fair dice.

Find the probability of throwing less than 3:

$$\begin{aligned} P(X<3) &= P(X \leq 2) \\ &= F(2) \quad \textit{the distribution function} \\ &= P(X=1)+P(X=2) \\ &= f(1)+f(2) \quad \textit{the density function} \\ &= \frac{1}{6}+\frac{1}{6}=\frac{1}{3} \end{aligned}$$

## Fair dice example

Find the probability of throwing greater than or equal to 3:

$$\begin{aligned} P(X \geq 3) &= 1-P(X \leq 2) \\ &= 1-F(2) \quad \textit{the distribution function} \\ &= 1-\frac{1}{3}=\frac{2}{3} \end{aligned}$$

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## Specific discrete distributions

- A number of different statistical distributions exist, which may be used to describe and analyse different types of problems.
- Today, we consider only discrete distributions:
  - The binomial distribution
  - The hypergeometric distribution
  - The Poisson distribution

## The Binomial distribution

- An experiment with two outcomes, "success" or "failure", is repeated (independent repetitions).
- $X$ is the number of successes after $n$ repetitions.
- Then $X$ follows a binomial distribution:

$$X \sim B(n,p)$$

  - $n$: number of repetitions
  - $p$: probability of success in each repetition

## The density function of the binomial distribution

The probability of $x$ successes:

$$f(x;n,p) \;=\; P(X=x) \;=\; \binom{n}{x} p^x (1-p)^{n-x}$$

where

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

## Example: Binomial distribution

Suppose that $X \sim B(4,p)$, i.e. $n = 4$. Find the probability of 3 successes.

- Probability of 3 successes: $P(X = 3)$.
- Three successes can be obtained in four "ways": SSSF, SSFS, SFSS, FSSS.
- Thus,

$$\binom{n}{x} = \binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 1} = 4,$$

and

$$P(X = 3) = 4p^3(1-p).$$

## Simulation from a binomial distribution

```r
## Probability of success
p <- 0.1

## Number of repetitions
nRepeat <- 30

## Simulate Bernoulli experiment 'nRepeat' times
tmp <- sample(c(0,1), size = nRepeat, prob = c(1-p,p), replace = TRUE)

# Compute 'x'
sum(tmp)

## Or: Use the binomial distribution simulation function
rbinom(1, size = 30, prob = p)
```

## Example: Fair dice

```r
# Number of simulated realizations (sample size)
n <- 30

# n independent random draws from the set (1,2,3,4,5,6)
# with equal probability for each outcome
xFair <- sample(1:6, size = n, replace = TRUE)

# Count the number of six'es
sum(xFair == 6)

## Do the same using 'rbinom()' instead
rbinom(n = 1, size = 30, prob = 1/6)
```

## Example 1

In the call center of a phone company, customer satisfaction is an issue. It is especially important that when errors/faults occur, they are corrected within the same day.

Assume that six errors occur, and that the probability of any error being corrected within the same day is $70\%$. **What is the probability that all six errors are corrected within the same day that they occurred?**

- Step 1) What should be described by a random variable $X$?

  The number of corrected errors.
- Step 2) What is the distribution of X?

  A binomial distribution with $n = 6$ and $p = 0.7$.

## Example 1

In the call center of a phone company, customer satisfaction is an issue. It is especially important that when errors/faults occur, they are corrected within the same day.

Assume that six errors occur, and that the probability of any error being corrected within the same day is 70%. **What is the probability that all six errors are corrected within the same day that they occurred?**

- Step 3) Which probability should be computed?

  $\underline{P(X=6)=f(6;6,0.7)}$

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## The hypergeometric distribution

- Again, $X$ is the the number of successes, but now *without* replacement when repeating.
- $X$ follows the hypergeometric distribution

$$X \sim H(n,a,N)$$

  - $n$ is the number of draws (repetitions)
  - $a$ is the number of successes in the population
  - $N$ is the number of elements in the (entire) population

## The hypergeometric distribution

- The probability of getting $x$ successes is

$$f(x;n,a,N) = P(X=x) = \frac{\binom{a}{x}\binom{N-a}{n-x}}{\binom{N}{n}}$$

  - $n$ is the number of draws (repetitions)
  - $a$ is the number of successes in the population
  - $N$ is the number of elements in the (entire) population

## Example 2

In a shipment of 10 harddisks, 2 of them have small scratches.

A random sample of 3 harddisks is taken. **What is the probability that at least 1 of them has scratches?**

- Step 1) What should be described by a random variable $X$?
  Number of harddisks with scratches in the random sample.
- Step 2) What is the distribution of $X$?
  A hypergeometric distribution with $n = 3$, $a = 2$, $N = 10$.
- Step 3) Which probability should be computed?
  $P(X \geq 1) = 1 - P(X = 0) = 1 - f(0; 3, 2, 10)$

## Binomial vs. hypergeometric

- The binomial distribution is used to analyse samples with replacement.
- The hypergeometric distribution is used to analyse samples without replacement.

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## The Poisson distribution

- The Poisson distribution is often used as a distribution (model) for counts, which do not have a natural upper bound.
- The Poisson distribution is often characterized by its *intensity*, which is on the form "number/unit", and often denoted $\lambda$.

## The Poisson distribution

$$X \sim Po(\lambda)$$

The density function:

$$f(x) = P(X = x) = \frac{\lambda^x}{x!} e^{-\lambda}$$

The distribution function:

$$F(x) = P(X \leq x)$$

## Example 3

Assume that, on average, 0.3 patients per day are hospitalized in Copenhagen due to air pollution.

**What is the probability that at most two patients are hospitalized in Copenhagen due to air pollution on any given day?**

- Step 1) What should be described by a random variable $X$?

  The number of patients on a given day.
- Step 2) What is the distribution of $X$?

  A Poisson distribution with $\lambda = 0.3$.
- Step 3) Which probability should be computed?

  $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

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| R | Name |
|---|---|
| `binom` | Binomial |
| `hyper` | Hypergeometric |
| `pois` | Poisson |

`d` $f(x)$, probability density function

`p` $F(x)$, cumulative distribution function

`r` random numbers from the distribution

`q` quantiles of the distribution (”inverse” of $F(x)$)

**Example:** The binomial distribution, $P(X \leq 5) = F(5; 10, 0.6)$

```
pbinom(q = 5, size = 10, prob = 0.6)
```

```
[1] 0.37
```

```
# Get help with:
?pbinom
```

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## Mean (expectation, expected value)

Mean of a discrete random variable, Definition 2.13:

### Definition

$$\mu = E(X) = \sum_{\text{all } x} x f(x)$$

- The *"true mean"* of X (as opposed to the sample mean).
- Expresses the "center" of the distribution of $X$.

## Example: Mean of a throw with a fair dice

$$\begin{aligned}\mu &= E(X)\\ &= 1\cdot\frac{1}{6}+2\cdot\frac{1}{6}+3\cdot\frac{1}{6}+4\cdot\frac{1}{6}+5\cdot\frac{1}{6}+6\cdot\frac{1}{6}\\ &= 3.5\end{aligned}$$

## Link to sample mean - learning from simulations

```r
# Number of simulated realizations (sample size)
n <- 30

# Sample independently from the set (1,2,3,4,5,6)
# with equal probability of outcomes
xFair <- sample(1:6, size = n, replace = TRUE)

# Compute the sample mean
mean(xFair)
```

```
[1] 3.3
```

## Asymptotics, increasing the sample size

The more observations (the larger the sample size), the closer you get to the true mean:

$$\lim_{n\to\infty} \hat{\mu} = \mu$$

- Try increasing $n$ in the simulations in R.

## Variance

Variance of a discrete random variable, Definition 2.16:

**Definition**

$$\sigma^2 = \mathrm{Var}(X) = \sum_{\text{all } x} (x-\mu)^2 f(x)$$

- Measures average dispersion/spread.
- The “true variance“ of $X$ (as opposed to the sample variance).

## Example: Variance of a throw with a fair dice

$$\begin{aligned}
\sigma^2 &= E[(X-\mu)^2] \\
&= (1-3.5)^2 \cdot \frac{1}{6} + (2-3.5)^2 \cdot \frac{1}{6} + (3-3.5)^2 \cdot \frac{1}{6} \\
&\quad + (4-3.5)^2 \cdot \frac{1}{6} + (5-3.5)^2 \cdot \frac{1}{6} + (6-3.5)^2 \cdot \frac{1}{6} \\
&\approx 2.92
\end{aligned}$$

## Link to sample variance - learning from simulations

```
# Number of simulated realizations (sample size)
n <- 30

# Sample independently from the set (1,2,3,4,5,6)
# with equal probability of outcomes
xFair <- sample(1:6, size = n, replace = TRUE)

# Compute the sample variance
var(xFair)
```

```
[1] 2.4
```

## Mean and variance of specific discrete distributions

The binomial distribution

- Mean:
  $\mu = n \cdot p$
- Variance:
  $\sigma^2 = n \cdot p \cdot (1 - p)$

## Mean and variance of specific discrete distributions

The hypergeometric distribution

- Mean:
  $\mu = n \cdot \frac{a}{N}$
- Variance:
  $\sigma^2 = \frac{n \cdot a \cdot (N-a) \cdot (N-n)}{N^2 \cdot (N-1)}$

## Mean and variance of specific discrete distributions

The poisson distribution

- Mean:
  $\mu = \lambda$
- Variance:
  $\sigma^2 = \lambda$

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