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Theorem 7.3.10:
Let X be a set and assume that $\sim$ is an equivalence relation on X . Then the following are true:
1. For every x $\in$ X, x $\in$ [x].
2. For all x,y $\in$ X,we have x $\sim$ y if and only if [x] = [y].
3. For all x,y $\in$ X, if [x] $\cap$ [y] $\neq \emptyset$,then [x] = [y].
We prove each of the three required properties.

Proof. **Proof of(1):** Let x $\in$ X. From the reflexive property of equivalence relations, we know that for all x $\in$ X, x $\sim$ x.

The definition of the equivalence class of x in a set X is defined to be [x] = { y $\in$ X : x $\sim$ y}.

Since x $\sim$ x from the the reflexive property, it follows from the definition of the equivalence class of. x that x $\in$ [x]. □

Proof. **Proof of(2):** We must prove both directions. Let x,y $\in$ X be arbitrary.

Proof of $\Rightarrow$ Assume x $\sim$ y. We must show that [x] = [y].

First, we will show that [x] $\subset$ [y]. Let a $\in$ [x] be arbitrary. We will show that a $\in$ [y].

By definition of an equivalence class, we find that a $\sim$ x. Using our assumption that x $\sim$ y, it follows from the transitive property of equivalence relations that since a $\sim$ x and x $\sim$ y, a $\sim$ y.

By definition of an equivalence class, since a $\sim$ y, it follows that a $\in$ [y]. Thus, [x] $\subset$ [y] as desired.

Next, we will show that [y] $\subset$ [x]. Let m $\in$ [y] be arbitrary. We will show that m $\in$ [x].

By definition of an equivalence class, we find that m $\sim$ y. Also, the symmetry property of equivalence relations tells us that if x $\sim$ y, then y $\sim$ x. Using this property in our assumption that x $\sim$ y, it follows that y $\sim$ x. The transitive property of equivalence relations then shows that since m $\sim$ y and y $\sim$ x, m $\sim$ x.

By definition of an equivalence class, since m $\sim$ x, it follows that m $\in$ [x] $\dot{\text{T}}$hus, [y] $\subset$ [x]as desired.

Hence, since [x] $\subset$ [y] and [y] $\subset$ [c], [x] = [y].

Proof of $\Leftarrow$ Assume [x] = [y]. We must show that x $\sim$ y.

Given that [x] = [y], we know that [x] $\subset$ [y] and [y] $\subset$ [x]. Let us use the fact that [x] $\subset$ [y]. Let n $\in$ [x] be an arbitrary element. The definition of equivalence classes shows us that n $\sim$ x. The symmetry property of equivalence relations shows us that x $\sim$ n.

Since [x] $\subset$ [y], n must also be in [y]. The logic described above for n $\in$ [x] therefore follows for n $\in$ [y] and we end up concluding that n $\sim$ y.

Since x $\sim$ n and n $\sim$ y, the transitive property of equivalence relations shows us that x $\sim$ y as desired.

Thus, since we've proven both conditionals to be true, the biconditional statement must also be true. □

Proof. **Proof of(3):** Suppose [x] $\cap$ [y] $\neq$. We will show that [x] = [y].

Let z $\in$ [x] $\cap$ [y]. From the definition of intersection of sets, z $\in$ [x] and z $\in$ [y]. By the definition of equivalence classes, z $\sim$ x and z $\sim$ y. From the reflexive property of equivalence relations, we know that if z $\sim$ x, then x $\sim$ z. Using the transitive property of equivalence relations, it follows that since x $\sim$ z and z $\sim$ y, x $\sim$ y. The proof described in part 2 shows that x $\sim$ y if and only if [x] = [y] as desired. □