COLLINS KIBET

HW 6 : Ch 4: 54, 55, 60; Ch 5: 3, 4, 6

Theoretical 7

Chapter 4

54. $E[X] = 0.2 = np$

→ distribution of number of errors is Binomial.

Poisson approximation can enable us to find $\lambda = np = 0.2$.

So X - num of errors has Pois(0.2) distribution.

a) $P(X=0) = e^{-\lambda} = 0.82$

b) $P(X \geq 2) = 1 - P(X=0) - P(X=1)$

$= 1 - e^{-\lambda} - \lambda e^{-\lambda}$

$= 1 - 0.82 - 0.82(0.2) = 0.0175$

55) $X \to$ plane crashes $= 3.5$

a) $P(X \geq 2)$

- Let n be no. of flights in a month.
- n is obviously very large.
- So, define X_i, a Bernoulli r.v. where $P(X_i = 1) = p$ is probability of a plane crashing.
- We know,

$S = X_1 + X_2 + X_3 + \dots + X_n \sim B(n, p) = S$

- Access of trials is independent. (assumed)

$P(S \geq 2) = 1 - P(S = 0) - P(S = 1)$

$= 1 - e^{-3.5} - 3.5\, e^{-3.5}$

$= 0.86$

b) $P(S \leq 1) = P(S = 0) + P(S = 1)$

$= e^{-3.5} + 3.5\, e^{-3.5}$

$= 0.136$

60) a) $P(X \geq 3) = 1 - \sum_{i=0}^{2} \frac{3^i}{i!\, e^3}$

$= 1 - \left(\frac{3^0}{0!\, e^3} + \frac{3^1}{1!\, e^3} + \frac{3^3}{2!\, e^3} \right)$

$= 0.577$

b) $P\{(X \geq 3) \mid (X \geq 1)\}$:
(Assumption: at least one accident occurs.

$= \frac{P\{(X \geq 3) \cap (X \geq 1)\}}{P\{X \geq 1\}}$

$X \geq 1 \subset X \geq 3$ implies that

$\frac{P\{(X \geq 3) \cap (X \geq 1)\}}{P\{X \geq 1\}} = \frac{P\{X \geq 3\}}{P\{X \geq 1\}}$

$= \frac{0.577 \text{ (from a)}}{1 - P\{X = 0\}}$

$= \frac{0.577}{1 - e^{-3}}$

$= \underline{\underline{0.607}}$

Chapter 5: 3, 4, 6

3.) Case 1

$$f(x) = \begin{cases} c(2x - x^3) & 0 < x < 5/2 \\ 0 & \text{otherwise} \end{cases}$$

Check if:

(1) $f(x) \geq 0$ for $-\infty < x < \infty$

(2.) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

- We don't consider $f(x) = 0$ because it meets property 1.
- Consider $0 < x < 5/2$:

- Assume $f(x) = 0 = c(2x - x^3)$ and $c = 1$

$2x - x^3 = 0$

$2x = x^3$

$2 = x^2$

$x = \sqrt{2}$

- If $x > \sqrt{2}$ and $x < 5/2$, $f(x) < 0$.
- Thus fails property 1 at interval $\sqrt{2} < x < 5/2$

Case 2

- Follows from explanation in case I above. But here, if $X > 2$ and $x < 5/2$, $f(x)$ will be negative.

- In both cases, f cannot be a probability density function.

4)
$$f(x) = \begin{cases} 10/x^2 & x > 10 \\ 0 & 0 \le 10 \end{cases}$$

a) $P(X > 20) = 1 - \int_{10}^{20} 10/x^2 \, dx = 1 - 10 \int_{10}^{20} \frac{1}{x^2}$

$$= 1 + 10 \cdot \frac{1}{x} \Big|_{10}^{20}$$

$$= 1 - 10 \cdot 1/20 = 1/2$$

b) Cumulative distribut function

$$F_X(x) = P(X \le x)$$

$$= \int_{10}^{x} 10/a^2 \, da = -10 \cdot 1/a \Big|_{10}^{x}$$

$$= -10 \cdot \left(\frac{1}{x} - \frac{1}{10}\right) = 1 - 10/x$$

(where $x > 10$).

A = no. of devices that work

(c) $P(A \geq 3) = 1 - P(A \leq 2) = 1 - P(A=0) - P(A=1) - P(A=2)$.

$$P(A \geq 3) = 1 - \frac{1}{3^6} - \frac{12}{3^6} - \frac{60}{3^6}$$

$$= \frac{656}{729}$$

6) a) $f(x) = \begin{cases} \frac{1}{4} x e^{-x/2} & x > 0 \\ 0. & \text{otherwise} \end{cases}$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{0} 0\, dx + \int_{0}^{\infty} x \frac{1}{4} x e^{-x/2} dx$$

(separation by parts)

$$= \int_0^{\infty} \frac{x^2}{4} e^{-x^2/2} dx$$

$$= \frac{1}{4}\left[x^2 \int e^{-x/2} dx - \int (x^2)' \left(\int e^{-x/2} dx\right) dx\right]_0^{\infty}$$

$$= \frac{1}{4}\left[x^2 \frac{e^{-x/2}}{-1/2} - \int (2x)\frac{e^{-x/2}}{-1/2}\,dx\right]_0^\infty$$

$$= \frac{1}{4}\left[-2x^2 e^{-x/2} - \int 4x e^{-x/2}\,dx\right]_0^\infty$$

$$= \frac{1}{4}\left[-2x^2 e^{-x/2} - \frac{4xe^{-x/2}}{-1/2} + 8\int e^{-x/2}\,dx\right]_0^\infty$$

$$= \frac{1}{4}\left[-2x^2 e^{-x/2} - 8xe^{-x/2}\ 16e^{-x/2}\right]_0^\infty$$

$$= \frac{1}{4}\left[-0+16\right] \cdot \left(\text{for } e^{-\infty} = \frac{1}{e^{\infty}} \approx 0\right)$$

$$= 4$$

$E[X] = 4$

b) $f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} x\,c(1-x^2)\,dx + \int_{1}^{\infty} dx$$

$$= c\int_{-1}^{1} (x - x^3)\,dx$$

$$= c\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{-1}^{1}$$

$$= c\left[\left(\frac{1}{2} - \frac{1}{4}\right) - \frac{1}{2} + \frac{1}{4}\right]$$

$$= 0$$

$E[x] = 0$

c) $f(x) = \begin{cases} 5/x^2 & x \geq 5 \\ 0 & x < 5 \end{cases}$

$$E[x] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{5} 0\,dx + \int_{0}^{\infty} x \frac{5}{x^2}\,dx$$

$$= \int_{5}^{\infty} \frac{5}{x}\,dx = [\ln x]_5^{\infty} = \ln\infty - \ln 5$$

$$= \infty$$

Theoretical Exercises

7.) $SD(X) = \sqrt{Var(X)}$

Find $SD(aX+b)$ if X has variance σ^2 $\quad Var(X) = \sigma^2$

$$SD(aX+b) = \sqrt{Var(aX+b)}$$

$$= \sqrt{a^2 Var(X)}$$

$$= |a| \sqrt{Var(X)}$$

Since $Var(X) = \sigma^2 \Rightarrow \sqrt{Var(X)} = \sigma$

$$\therefore SD(aX+b) = |a|\sigma$$