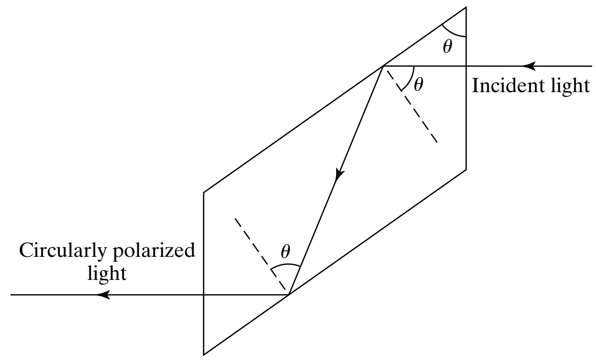

23 *Fresnel Equations*

INTRODUCTION

The basic laws of reflection and refraction in geometrical optics were derived earlier on the basis of either Huygens' or Fermat's principles. In this chapter we regard light as an electromagnetic wave and show that the laws of reflection and refraction can also be deduced from this point of view. More importantly, this approach also leads to the *Fresnel equations*, which describe the fraction of incident energy transmitted or reflected at a plane surface. These quantities will be seen to depend not only on the change in refractive index and the angle of incidence at the surface but also on the polarization of the incident light. Finally, the important differences between internal and external reflection are clarified.

1 THE FRESNEL EQUATIONS

Consider Figure 1, which shows a ray of light incident at point P on a plane interface—the xy boundary plane—and the resulting reflected and refracted rays. The plane of incidence is the xz-plane. Let us assume the incident light consists of plane harmonic waves, expressed by

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \mathbf{e}^{i(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t)} \tag{1}$$

where the origin of coordinates is taken to be point O. The wave vector $\vec{\mathbf{E}}$ of the incident wave is chosen in the $+y$-direction, so that the wave is linearly

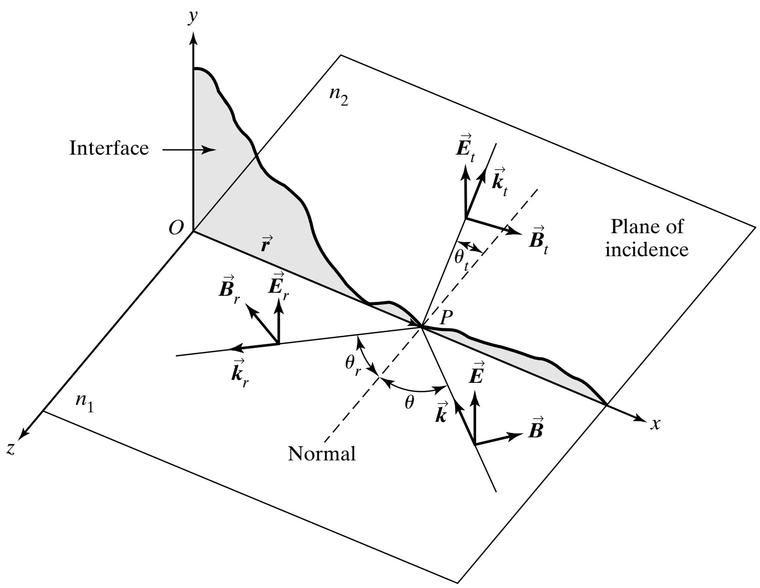

Figure 1 Defining diagram for incident, reflected, and transmitted rays at an *xy*-plane interface when the electric field is perpendicular to the plane of incidence, the TE mode.

polarized. The direction of the corresponding magnetic field vector $\vec{\mathbf{B}}$ is then determined to ensure that the direction of $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$ is the direction of wave propagation $\vec{\mathbf{k}}$. This mode of polarization, in which the $\vec{\mathbf{E}}$-field is perpendicular to the plane of incidence and the $\vec{\mathbf{B}}$-field lies in the plane of incidence, is called the *transverse electric* (TE) mode. If instead $\vec{\mathbf{B}}$ is transverse to the plane of incidence, a case to be considered later, the mode is a *transverse magnetic* (TM) mode. An arbitrary polarization direction represents some linear combination of these two special cases. The reflected and transmitted waves in Figure 1 can be expressed, respectively, in forms like that of the incident wave of Eq. (1):

$$\vec{\mathbf{E}}_r = \vec{\mathbf{E}}_{0r} e^{i(\vec{\mathbf{k}}_r \cdot \vec{\mathbf{r}} - \omega_r t)} \tag{2}$$

$$\vec{\mathbf{E}}_t = \vec{\mathbf{E}}_{0t} e^{i(\vec{\mathbf{k}}_t \cdot \vec{\mathbf{r}} - \omega_t t)} \tag{3}$$

In the boundary plane *xy*, where all three waves exit simultaneously, there must be a fixed relationship between the three wave amplitudes (and thus their irradiances) that has yet to be determined. Since such a relationship cannot depend on the arbitrary choice of a boundary point $\vec{\mathbf{r}}$ nor a time t, it follows that the phases of the three waves, which depend on $\vec{\mathbf{r}}$ and t, must themselves be equal:

$$(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega \mathrm{t}) = (\vec{\mathbf{k}}_r \cdot \vec{\mathbf{r}} - \omega_r \mathrm{t}) = (\vec{\mathbf{k}}_t \cdot \vec{\mathbf{r}} - \omega_t \mathrm{t}) \tag{4}$$

In particular, at the boundary point $\vec{\mathbf{r}} = 0$ of Figure 1,

$$-\omega t = -\omega_r t = -\omega_t t$$

or

$$\omega = \omega_r = \omega_t \tag{5}$$

so that all frequencies are equal. On the other hand, at $t = 0$ *within the* *boundary plane*, Eq. (4) yields:

$$\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} = \vec{\mathbf{k}}_r \cdot \vec{\mathbf{r}} = \vec{\mathbf{k}}_t \cdot \vec{\mathbf{r}} \tag{6}$$

Several conclusions can be drawn from the relations of Eq. (6). First notice that by subtracting any two members, these relations are equivalent to

$$(\vec{\mathbf{k}} - \vec{\mathbf{k}}_r) \cdot \vec{\mathbf{r}} = (\vec{\mathbf{k}} - \vec{\mathbf{k}}_t) \cdot \vec{\mathbf{r}} = (\vec{\mathbf{k}}_r - \vec{\mathbf{k}}_t) \cdot \vec{\mathbf{r}} = 0 \tag{7}$$

Equation (7) requires that the vectors $\vec{\mathbf{k}}_r$ and $\vec{\mathbf{k}}_t$ lie in the plane determined by the vectors $\vec{\mathbf{k}}$ and $\vec{\mathbf{r}}$. Thus all three propagation vectors are coplanar in the xz-plane, and we conclude that the reflected and refracted waves lie in the plane of incidence. Next, consider the first two members of Eq. (6), which govern the relationship between the incident and reflected waves. In terms of the angles designated in Figure 1, they are equivalent to

$$kr \sin\theta = k_r r \sin\theta_r$$

Since both waves travel in the same medium, their wavelengths are identical and so $k = k_r$. Therefore, we have the

$$\text{law of reflection:} \quad \theta = \theta_r \tag{8}$$

Finally, the last two members of Eq. (6) are equivalent to

$$k_r r \sin\theta_r = k_t r \sin\theta_t \tag{9}$$

Writing $k_r = \omega/v_r = n_r\omega/c$ and $k_t = n_t\omega/c$, Eq. (9) becomes Snell's

$$\text{law of refraction:} \quad n_r \sin\theta_r = n_t \sin\theta_t \tag{10}$$

We continue now to specify further the situation at the boundary with the help of boundary conditions arising out of Maxwell's equations and treated in texts on electricity and magnetism. We employ them here without proof. These boundary conditions require that the components of both the electric and magnetic fields parallel to the boundary plane be continuous as the boundary is crossed.

Boundary Conditions for TE Waves

As mentioned earlier, TE waves have electric fields that are perpendicular to the plane of incidence and therefore are parallel to the boundary plane separating the two media. In terms of the choices made for the direction of the electric fields in Figure 1, the vector amplitudes of the complex fields of Eqs. (1)–(3) can be written as

$$\vec{\mathbf{E}}_0 = E\hat{\mathbf{y}} \quad \vec{\mathbf{E}}_{0r} = E_r\hat{\mathbf{y}} \quad \vec{\mathbf{E}}_{0t} = E_t\hat{\mathbf{y}} \tag{11}$$

Here, E, E_r and E_t are the complex field amplitudes associated, respectively, with the incident, reflected, and transmitted waves. The requirement that the component of the electric field parallel to the boundary plane be continuous at the boundary then gives

$$E + E_r = E_t \tag{12}$$

The magnetic fields associated with the electric fields of Figure 1 have the form,

$$\vec{\mathbf{B}} = (B\cos\theta\hat{\mathbf{x}} - B\sin\theta\hat{\mathbf{z}})e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)}$$

$$\vec{\mathbf{B}}_r = (-B_r\cos\theta_r\hat{\mathbf{x}} - B_r\sin\theta_r\hat{\mathbf{z}})e^{i(\vec{\mathbf{k}}_r\cdot\vec{\mathbf{r}}-\omega t)} \tag{13}$$

$$\vec{\mathbf{B}}_t = (B_t\cos\theta_t\hat{\mathbf{x}} - B_t\sin\theta_t\hat{\mathbf{z}})e^{i(\vec{\mathbf{k}}_t\cdot\vec{\mathbf{r}}-\omega t)}$$

Continuity of the parallel components of the magnetic field requires that the field amplitudes be related by

$$B\cos\theta - B_r\cos\theta = B_t\cos\theta_t \tag{14}$$

where we have made use of Eq. (8). Equations (12) and (14) are correct for the $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ vectors as chosen in Figure 1. If a different choice is made, for example, by reversing the $\vec{\mathbf{E}}$ vector of the incident wave (and also $\vec{\mathbf{B}}$ to keep the direction of wave propagation the same), Eqs. (12) and (14) appear with a change of signs. However, the physical import of these equations is the same when they are interpreted in terms of their original figures.

Boundary Conditions for TM Waves

Before pursuing the significance of Eqs. (12) and (14) for the TE mode, we parallel their development for the TM mode pictured in Figure 2. The electric and magnetic fields in this figure can be written as

$$\vec{\mathbf{E}} = (E\cos\theta\hat{\mathbf{x}} - E\sin\theta\hat{\mathbf{z}})e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)}$$

$$\vec{\mathbf{E}}_r = (E_r\cos\theta_r\hat{\mathbf{x}} + E_r\sin\theta_r\hat{\mathbf{z}})e^{i(\vec{\mathbf{k}}_r\cdot\vec{\mathbf{r}}-\omega t)} \tag{15}$$

$$\vec{\mathbf{E}}_t = (E_t\cos\theta_t\hat{\mathbf{x}} - E_t\sin\theta_t\hat{\mathbf{z}})e^{i(\vec{\mathbf{k}}_t\cdot\vec{\mathbf{r}}-\omega t)}$$

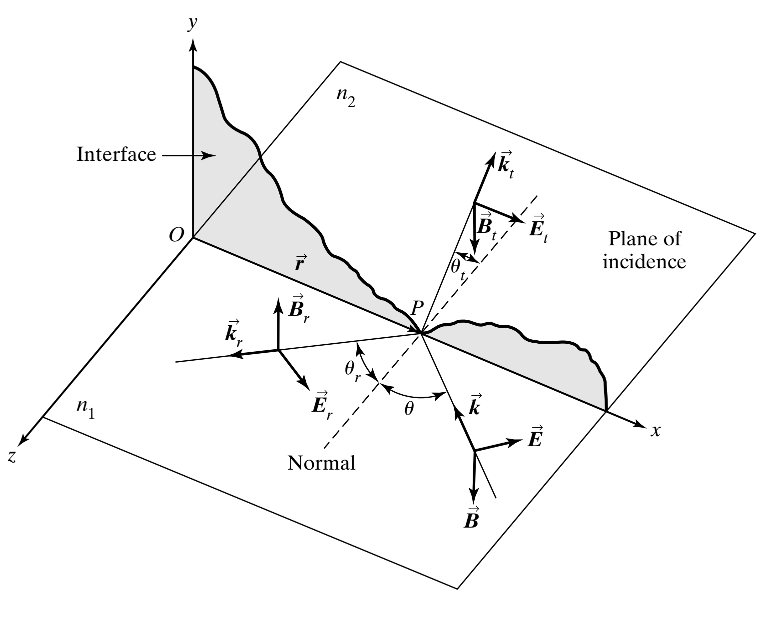

Figure 2 Defining diagram for incident, reflected, and transmitted rays at an *xy*-plane interface when the magnetic field is perpendicular to the plane of incidence, the TM mode.

and

$$\vec{\mathbf{B}} = -B\hat{\mathbf{y}}e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)}$$
$$\vec{\mathbf{B}}_r = B_r\hat{\mathbf{y}}e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)} \tag{16}$$
$$\vec{\mathbf{B}}_t = -B_t\hat{\mathbf{y}}e^{i(\vec{\mathbf{k}}_t\cdot\vec{\mathbf{r}}-\omega t)}$$

Requiring continuity of the components of the electric and magnetic fields that are parallel to the boundary gives, in this case,

$$-B + B_r = -B_t \tag{17}$$

$$E\cos\theta + E_r\cos\theta = E_t\cos\theta_t \tag{18}$$

Reflection and Transmission Coefficients

The magnetic field amplitudes of Eqs. (14) and (17) can be expressed in terms of the corresponding electric field amplitudes through the generic relation

$$E = vB = \left(\frac{c}{n}\right)B \tag{19}$$

Writing the index of refraction for incident and refracting media as n_1 and n_2, respectively, Eqs. (12), (14), (17), and (18) can be recast as follows:

$$\text{TE:}\begin{cases} E + E_r = E_t & (20) \\ n_1 E\cos\theta - n_1 E_r\cos\theta = n_2 E_t\cos\theta_t & (21) \end{cases}$$

$$\text{TM:}\begin{cases} -n_1 E + n_1 E_r = -n_2 E_t & (22) \\ E\cos\theta + E_r\cos\theta = E_t\cos\theta_t & (23) \end{cases}$$

Next, eliminating E_t from each pair of equations and solving for the *reflection coefficient* $r = E_r/E$,

$$r_{TE} = \frac{E_r}{E} = \frac{\cos\theta - n\cos\theta_t}{\cos\theta + n\cos\theta_t} \tag{24}$$

$$r_{TM} = \frac{E_r}{E} = \frac{-n\cos\theta + \cos\theta_t}{n\cos\theta + \cos\theta_t} \tag{25}$$

where we have introduced a *relative refractive index* $n \equiv n_2/n_1$. Note that we use subscripts to distinguish between the TE and TM cases. Finally, since n and θ_t are related to θ through Snell's law, $\sin\theta = n\sin\theta_t$, θ_t may be eliminated using

$$n\cos\theta_t = n\sqrt{1 - \sin^2\theta_t} = \sqrt{n^2 - \sin^2\theta} \tag{26}$$

The results are then

$$r_{TE} = \frac{E_r}{E} = \frac{\cos\theta - \sqrt{n^2 - \sin^2\theta}}{\cos\theta + \sqrt{n^2 - \sin^2\theta}} \tag{27}$$

$$r_{TM} = \frac{E_r}{E} = \frac{-n^2 \cos\theta + \sqrt{n^2 - \sin^2\theta}}{n^2 \cos\theta + \sqrt{n^2 - \sin^2\theta}} \tag{28}$$

Returning to Eqs. (20) through (23), if E_r is eliminated instead of E_t, similar steps lead to the following equations describing the *transmission coefficient* $t = E_t/E$:

$$t_{TE} = \frac{E_t}{E} = \frac{2\cos\theta}{\cos\theta + \sqrt{n^2 - \sin^2\theta}} \tag{29}$$

$$t_{TM} = \frac{E_t}{E} = \frac{2n\cos\theta}{n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}} \tag{30}$$

Eqs. (29) and (30) can also be found more quickly by using Eqs. (20) and (22) written in the form

$$t_{TE} = 1 + r_{TE}$$

$$nt_{TM} = 1 - r_{TM}$$

into which the results expressed by Eqs. (27) and (28) can be conveniently substituted. Equations (27) through (30) are the *Fresnel equations*, giving reflection and transmission coefficients, the ratio of both reflected and transmitted $\vec{\mathbf{E}}$-field amplitudes to the incident $\vec{\mathbf{E}}$-field amplitude. Note that, for normal incidence, the reflection and transmission coefficients for the TE case are identical to those for the TM case. This is sensible since for normal incidence there is no distinction between the two cases.[1] In practice, measured reflection and transmission coefficients also depend on scattering losses from a nonplanar surface.

Example 1

Calculate the reflection and transmission coefficients for both TE and TM modes of light incident from air at 30° onto glass of index 1.60.

Solution

Using Eqs. (27) and (28),

$$r_{TE} = \frac{\cos(30°) - \sqrt{1.6^2 - \sin^2(30°)}}{\cos(30°) + \sqrt{1.6^2 - \sin^2(30°)}} = -0.2740$$

$$r_{TM} = \frac{-1.6^2\cos(30°) + \sqrt{1.6^2 - \sin^2(30°)}}{1.6^2\cos(30°) + \sqrt{1.6^2 - \sin^2(30°)}} = -0.1866$$

Using the relations below Eq. (30),

$$t_{TE} = 1 + r_{TE} = 1 - 0.2740 = 0.7260$$

$$t_{TM} = \frac{1 - r_{TM}}{n} = \frac{1 + 0.1866}{1.60} = 0.7416$$

[1] Some texts use a different convention, in which the positive direction of the reflected electric field for the TM case is opposite to that shown in Figure 2, leading to an expression for the reflection coefficient for the TM case that differs from ours by a factor of -1. Of course, both conventions lead to the same physical result since the extra factor of -1 simply reverses the direction of the reflected electric field.

2 EXTERNAL AND INTERNAL REFLECTIONS

When interpreting these equations, it is useful to distinguish between two physically different situations:

$$\text{external reflection:} \quad n_1 < n_2 \quad \text{or} \quad n = \frac{n_2}{n_1} > 1$$

$$\text{internal reflection:} \quad n_1 > n_2 \quad \text{or} \quad n = \frac{n_2}{n_1} < 1$$

Figure 3 is a plot of Eqs. (27) through (30) for the case of external reflection with $n = 1.50$. Notice that at both normal and grazing incidence—angles of 0° and 90°, respectively—TE and TM modes have reflection coefficients of the same magnitude and transmission coefficients of the same magnitude. Negative values of r for both the TE and TM modes indicate a phase change of the $\vec{\mathbf{E}}$- or $\vec{\mathbf{B}}$-field vectors on reflection and will be discussed presently. The fraction of power P in the incident wave that is reflected or transmitted, called the *reflectance* and the *transmittance*, respectively, depends on the ratio of the squares of the amplitudes.

$$\text{reflectance} = R = \frac{P_r}{P_i} = r^2 = \left(\frac{E_r}{E}\right)^2 \tag{31}$$

$$\text{transmittance} = T = \frac{P_t}{P_i} = n\left(\frac{\cos\theta_t}{\cos\theta}\right)t^2 \tag{32}$$

These expressions are justified later in this chapter.

In Figure 4, reflectance is plotted as a function of the angle of incidence θ. The curve for the case of external reflection, TM mode, indicates that no wave energy is reflected when the angle of incidence is near 60°. The angle θ_p at which $R_{TM} = 0$ is known as *Brewster's angle* or the *polarizing* angle and takes the value,

$$\theta_p = \tan^{-1}(n) = \tan^{-1}(n_2/n_1)$$

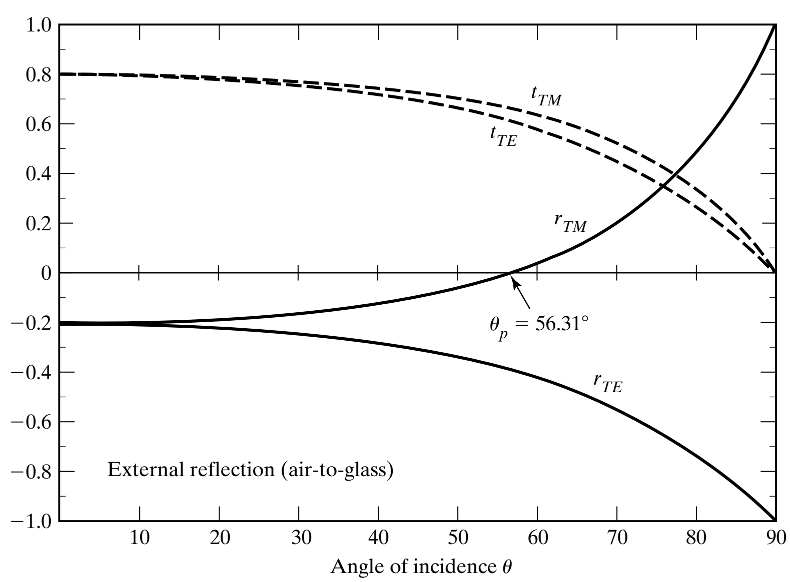

Figure 3 Reflection and transmission coefficients for the case of external reflection, with $n = n_2/n_1 = 1.50$.

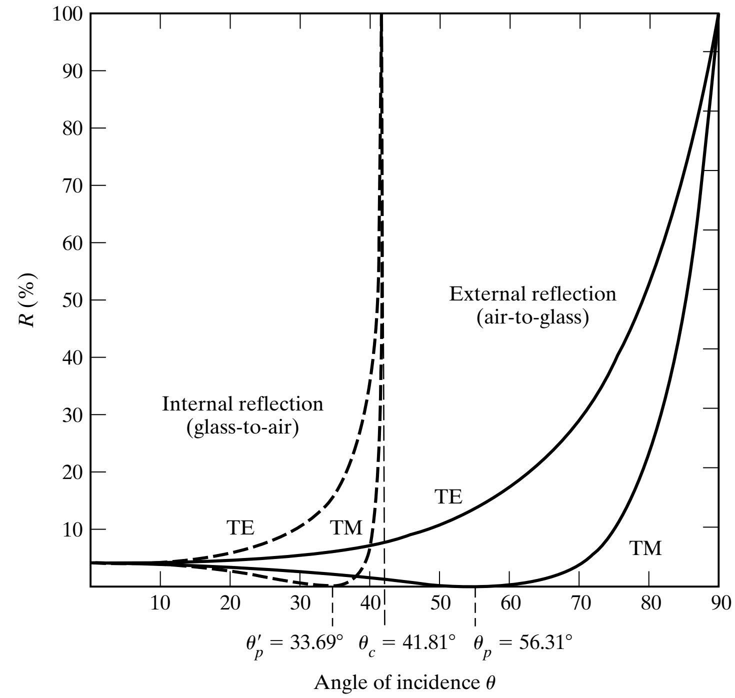

Figure 4 Reflectance for both external and internal reflection when $n_1 = 1$ and $n_2 = 1.50$.

This condition is also evident in the vanishing of r_{TM} in Figure 3 and the vanishing of the numerator of Eq. (28). See problem 1. For the case $n = 1.50$ used in Figures 3 and 4, $\theta_p = 56.31°$. R_{TE} does not go to zero under this condition, so reflected light contains only the TE mode and is linearly polarized, with $R_{TE} = 15\%$. At normal incidence $(\theta = 0°)$, for both TE and TM modes, Eqs. (24) and (25) simplify to give

$$R = r^2 = \left(\frac{1 - n}{1 + n}\right)^2 \tag{33}$$

Equation (33) gives a reflectance of 4% from an air/glass interface with $n = 1.5$. Keep in mind, however, that n is a function of wavelength. As the angle of incidence increases to grazing incidence $(\theta = 90°)$, both R_{TE} and R_{TM} become unity, although R_{TM} remains quite small until Brewster's angle has been exceeded.

The reflection coefficient for the case of internal reflection is shown in Figure 5 with $n = 1/1.50$, as when light encounters a glass/air interface from the glass side. Evidence of phase changes and of a polarizing, or Brewster's, angle may also be seen here. For the case of internal reflection we give Brewster's angle the symbol θ'_p. Examination of Figures 4 and 5 shows that, for the case of internal reflection, both $R_{TE} = r_{TE}^2$ and $R_{TM} = r_{TM}^2$ reach values of unity before the angle of incidence θ reaches 90°. This is the phenomenon of *total internal reflection*, which occurs at the critical angle $\theta_c = \sin^{-1}(n) = \sin^{-1}(n_2/n_1)$. For the example of glass $(n = 1/1.5)$ used in Figure 5, $\theta'_p = 33.7°$ and $\theta_c = 41.8°$. When $\sin\theta_c > n$, the radical $\sqrt{n^2 - \sin^2\theta}$ is negative and both r_{TE} and r_{TM} are complex. Their magnitudes, however, are easily shown to be unity in this range, giving total reflection for $\theta > \theta_c$.

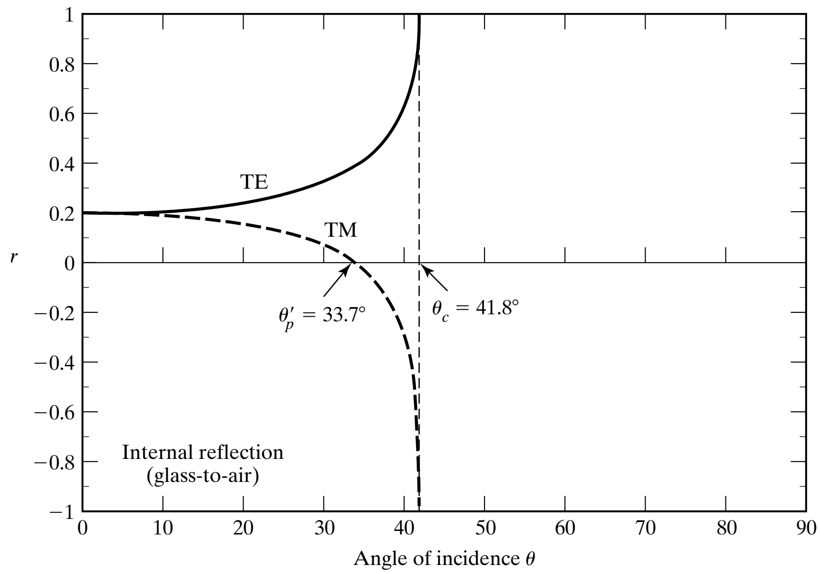

Figure 5 Reflection coefficient for the case of internal reflection with $n = n_1/n_2 = 1/1.50$.

3 PHASE CHANGES ON REFLECTION

The negative values of the reflection coefficient in Figures 3 and 5 indicate that $E_r = -|r|E$ in certain situations. Evidently, the electric field vector may reverse direction on reflection. Equivalently, in such cases there is a π-phase shift of E on reflection, as the following mathematical argument demonstrates:

$$E_r = -|r|E = e^{i\pi}|r|E_0 e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)} = |r|E_0 e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t+\pi)}$$

Thus in the case of external reflection, Figure 3, a π-phase shift of E occurs at any angle of incidence for the TE mode and for $\theta < \theta_p$ for the TM mode. When reflection is internal, Figure 5, we conclude that a π-phase shift occurs for the TM mode for $\theta'_p < \theta < \theta_c$. However, the situation in the region $\theta > \theta_c$, where r is complex, requires further investigation. When $\theta > \theta_c = \sin^{-1}(n)$, the radical in Eqs. (27) and (28) becomes imaginary, and the equations may be written in the form

$$r_{TE} = \frac{\cos\theta - i\sqrt{\sin^2\theta - n^2}}{\cos\theta + i\sqrt{\sin^2\theta - n^2}} \tag{34}$$

$$r_{TM} = \frac{-n^2\cos\theta + i\sqrt{\sin^2\theta - n^2}}{n^2\cos\theta + i\sqrt{\sin^2\theta - n^2}} \tag{35}$$

The reflection coefficients can be written in polar form as $r = |r|e^{i\phi}$ and we shall refer to ϕ as the *phase shift on reflection.* In Eq. (34), the reflection coefficient takes the form $r_{TE} = (a - ib)/(a + ib)$. Since the real and imaginary parts of the numerator and denominator are the same, except for a sign, the magnitudes of the numerator and denominator are equal, and r_{TE} has unit amplitude. The phase of r_{TE} may be investigated by expressing Eq. (34) in complex polar form, as

$$r_{TE} = \frac{e^{-i\alpha}}{e^{i\alpha}} = e^{-i(2\alpha)}$$

where $\tan\alpha = \sqrt{\sin^2\theta - n^2}/\cos\theta$. So, for the TE case, the phase shift on reflection is $\phi_{TE} = -2\alpha$. A similar analysis (see problem 6) can be used to show that r_{TM}, like r_{TE}, has unit magnitude when the angle of incidence exceeds the critical angle and enables one to find the phase shift on total internal reflection ϕ_{TM} for the TM case. The phase shifts on *total internal reflection* for the two cases have the form,

$$\tan\left(\frac{\phi_{TE}}{2}\right) = -\frac{\sqrt{\sin^2\theta - n^2}}{\cos\theta} \tag{36}$$

$$\tan\left(\frac{\phi_{TM} - \pi}{2}\right) = -\frac{\sqrt{\sin^2\theta - n^2}}{n^2\cos\theta} \tag{37}$$

Clearly, the phase shift on reflection, for total internal reflection, may take on values other than 0 and π, depending on the angle of incidence. The phase shift ϕ, as determined from Eqs. (36) and (37), is plotted in Figure 6.

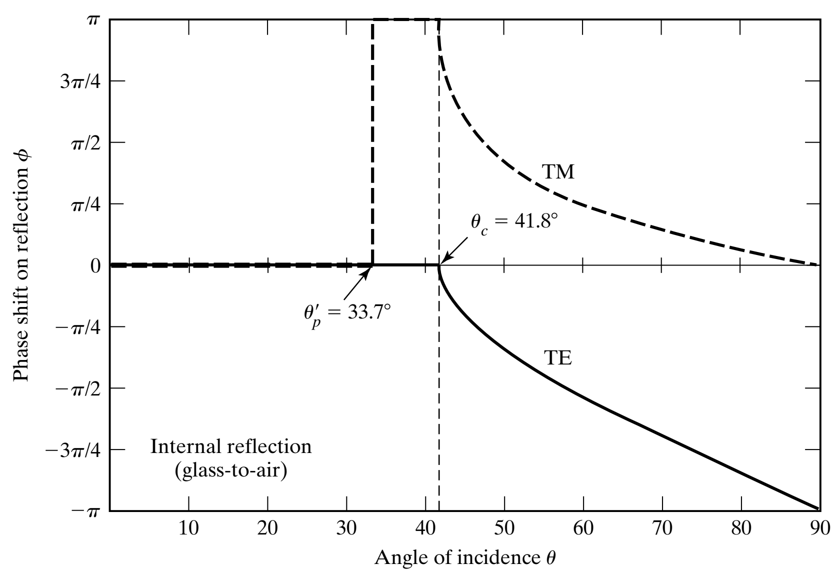

Figure 6 Phase shift ϕ on reflection of the electric field for internally reflected rays, with $n = n_1/n_2 = 1/1.5$.

It happens that the relative phase shift $\phi_{TE} - \phi_{TM}$ is about $-3\pi/4$ at an angle of incidence near 53°. Two consecutive internal reflections thus produce a relative phase shift of $2(-3\pi/4) = -3\pi/2$ (equivalently, $+\pi/2$) between the perpendicular components of the $\vec{\mathbf{E}}$-field. Recall that circularly polarized light consists of equal amplitude components with phases that differ by $\pm\pi/2$. Thus linearly polarized incident light with equal TM and TE components, after two internal reflections at 53°, will be transformed into circularly polarized light. This technique is utilized in the *Fresnel rhomb* (Figure 7).

Summarizing these results for the case of internal reflection,

$$\phi_{TM} = \begin{cases} 0, & \theta < \theta'_p \\ \pi, & \theta'_p < \theta < \theta_c \\ -2\arctan\left(\dfrac{\sqrt{\sin^2\theta - n^2}}{n^2\cos\theta}\right) + \pi, & \theta > \theta_c \end{cases} \tag{38}$$

$$\phi_{TE} = \begin{cases} 0, & \theta < \theta_c \\ -2\arctan\left(\dfrac{\sqrt{\sin^2\theta - n^2}}{\cos\theta}\right), & \theta > \theta_c \end{cases} \tag{39}$$

Phase shifts for both TM and TE modes and for both internal and external reflection are summarized in Figure 8.

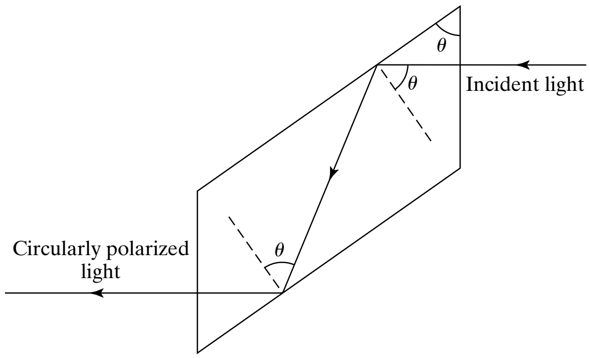

Figure 7 The Fresnel rhomb. With the incident light polarized at 45° to the plane of incidence, two internal reflections produce equal-amplitude TE and TM amplitudes with a relative phase of $\pi/2$, or circularly polarized light. For $n = 1.50$, the angle should be $\theta = 53°$. The device is effective over a wide range of wavelengths.

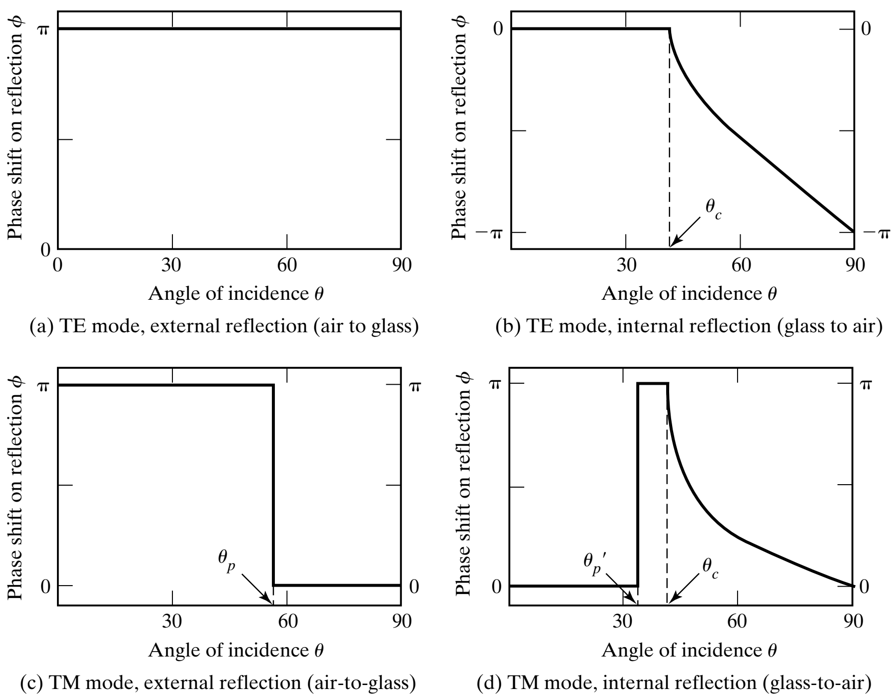

Figure 8 Phase changes on reflection ϕ between incident and reflected rays versus angle of incidence. Discontinuities occur at $\theta_c = 41.8°$, $\theta_p = 56.3°$, and $\theta'_p = 33.7°$ for refractive indices of $n_1 = 1$ and $n_2 = 1.50$.

Example 2

What is the phase shift of the TM and TE rays reflected both externally and internally for the situation discussed in Example 1?

Solution

For this interface,

$$\theta_c = \sin^{-1}\left(\frac{1}{1.6}\right) = 38.7°$$

$$\theta_p = \tan^{-1}(1.6) = 58.0°$$

$$\theta'_p = \tan^{-1}\left(\frac{1}{1.6}\right) = 32.0°$$

Since the angle of incidence of 30° is less than either θ'_p or θ_c, Eqs. (38) and (39) or Figure 8 require that for internal reflection, $\phi_{TM} = 0$ and $\phi_{TE} = 0$, while Figure 8 shows that for external reflection, $\phi_{TM} = \pi$ and $\phi_{TE} = \pi$.

A general conclusion can be drawn from the phase changes for the TE and TM modes under internal and external reflection: Near normal incidence, for both TE and TM modes, the phase shift for an internally reflected beam differs from that of an externally reflected beam by π. For a thin film in air, we are interested in the *relative* phase shift between rays reflected from the first surface (external) and the second surface (internal). Inspection of Figure 8 shows that a *relative* phase shift of π occurs in the TE mode for *internal* angles of incidence less than θ_c and in the TM mode for *internal* angles of incidence less than θ'_p. The corresponding ranges of *external* angles of incidence at the first surface are 0° to 90° (TE mode) and 0° to θ_p (TM mode). Thus in the TE mode a relative phase shift of π occurs for *all* external angles of incidence, but in the TM mode this is true only for external angles less than θ_p.

4 CONSERVATION OF ENERGY

To conserve energy, at a given boundary, it must be true that the power incident on the boundary be equal to the sum of the power reflected at that boundary and the power transmitted through the boundary. That is,

$$P_i = P_r + P_t \tag{40}$$

If we represent the reflectance R as the ratio of reflected to incident power and the transmittance T as the ratio of transmitted to incident power,

$$R = \frac{P_r}{P_i} \quad \text{and} \quad T = \frac{P_t}{P_i} \tag{41}$$

then Eq. (40) takes the form

$$1 = R + T \tag{42}$$

The irradiance I is the power density (W/m^2), so that we may write, in place of Eq. (40),

$$I_i A_i = I_r A_r + I_t A_t \tag{43}$$

The cross-sectional areas of the three beams (see Figure 9) that appear in Eq. (43) are all related to the area A intercepted by the beams in the boundary plane through the cosines of the angles of incidence, reflection, and refraction. We may then write

$$I_i(A\cos\theta) = I_r(A\cos\theta_r) + I_t(A\cos\theta_t)$$

Of course, $\theta = \theta_r$ by the law of reflection. Also using the relation between irradiance and electric field amplitude,

$$I = \left(\frac{\varepsilon v}{2}\right)E_0^2$$

and the facts that $v_i = v_r$, and $\varepsilon_i = \varepsilon_r$, since they correspond to the same medium, we arrive at the equation

$$E_0^2 = E_{0r}^2 + \left(\frac{v_t\varepsilon_t}{v_i\varepsilon_i}\right)\left(\frac{\cos\theta_t}{\cos\theta}\right)E_{0t}^2 \tag{44}$$

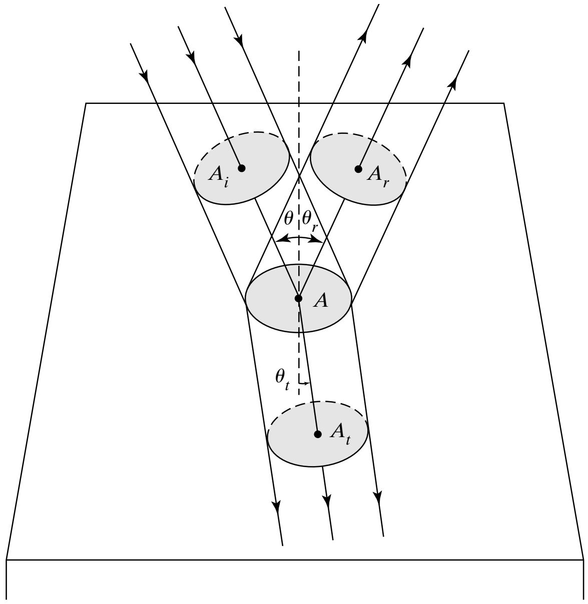

Figure 9 Comparison of cross sections of incident, reflected, and transmitted beams.

The quantity $(v_t\varepsilon_t/v_i\varepsilon_i)$ is just a complicated way of expressing the relative refractive index n, which we can show as follows:

$$\frac{v_t\varepsilon_t}{v_i\varepsilon_i} = \frac{v_t}{v_i}\frac{v_i^2\,\mu_i}{v_t^2\,\mu_t} = \frac{v_i}{v_t} = n \tag{45}$$

In arriving at this result we have used

$$\mu_i = \mu_t = \mu_0$$

for nonmagnetic materials and the relation

$$v^2 = \frac{1}{\mu\varepsilon}$$

for the velocity of a plane electromagnetic wave. Incorporating Eq. (45) in Eq. (44),

$$E_{0i}^2 = E_{0r}^2 + n\left(\frac{\cos\theta_t}{\cos\theta}\right)E_{0t}^2 \tag{46}$$

Dividing the equation by the left member, it becomes

$$1 = r^2 + n\left(\frac{\cos\theta_t}{\cos\theta}\right)t^2 \tag{47}$$

where the reflection and transmission coefficients r and t have been introduced. Now the quantity r^2 is just the reflectance R:

$$R = \frac{P_r}{P_i} = \frac{I_r}{I_i} = \left(\frac{E_{0r}}{E_{0i}}\right)^2 = r^2$$

Comparing Eq. (47) with Eq. (42), it follows that the transmittance T is expressed by the relation

$$T = n\left(\frac{\cos\theta_t}{\cos\theta}\right)t^2 \tag{48}$$

Notice that T is not simply t^2 since it must take into account a different speed and direction in a new medium. The change in speed modifies the rate of energy propagation and thus the power of the beam; the change in direction modifies the cross section and thus the power density of the beam. However, for normal incidence, Eq. (48) reduces to $T = nt^2$ and Eq. (47) becomes

$$\text{normal incidence: } 1 = r^2 + nt^2 \tag{49}$$

Note that throughout this section we have assumed that the reflection and transmission coefficients are real, as they will be for all external reflections and all internal reflections with angles of incidence less than the critical angle.

Example 3

Calculate the reflectance R and transmittance T for both TE and TM modes of light incident at 30° on glass of index 1.60.

Solution

The reflection and transmission coefficients for this situation are given in the solution to Example 1. Using these, the reflectance and transmittance are found to be

$$R_{TE} = r_{TE}^2 = (-0.2740)^2 = 0.075 \quad \text{and} \quad T_{TE} = 1 - R_{TE} = 0.925$$
$$R_{TM} = r_{TM}^2 = (-0.1866)^2 = 0.035 \quad \text{and} \quad T_{TM} = 1 - R_{TM} = 0.965$$

5 EVANESCENT WAVES

In discussing the propagation of a light wave by total internal reflection (TIR) through an optical fiber, we mentioned the phenomenon of *cross talk*, the coupling of wave energy into another medium when it is brought close enough to the reflecting wave. This loss of energy is described as *frustrated total internal reflection*. The theory presented in this chapter allows us to describe this phenomenon quantitatively.

The transmitted wave at a refraction can be represented as

$$E_t = E_{0t}e^{i(\vec{\mathbf{k}}_t \cdot \vec{\mathbf{r}} - \omega t)}$$

where, according to the coordinates chosen in Figure 1,

$$\vec{\mathbf{k}}_t \cdot \vec{\mathbf{r}} = k_t\,(-\sin\theta_t\,\hat{\mathbf{x}} - \cos\theta_t\hat{\mathbf{z}}) \cdot (x\,\hat{\mathbf{x}} + z\,\hat{\mathbf{z}})$$
$$\vec{\mathbf{k}}_t \cdot \vec{\mathbf{r}} = k_t\,(-x\sin\theta_t - z\cos\theta_t)$$

We can express $\cos\theta_t$ as

$$\cos\theta_t \equiv \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - \frac{\sin^2\theta}{n^2}}$$

where we have used Snell's law, $n\sin\theta_t = \sin\theta$ in writing the last equality. At the critical angle, $\sin\theta = n$ and $\cos\theta_t = \cos(90°) = 0$. For angles such that $\sin\theta > n$, when TIR occurs, $\cos\theta_t$ becomes purely imaginary and we can write

$$\cos\theta_t = i\sqrt{\frac{\sin^2\theta}{n^2} - 1}$$

Thus the exponential factor

$$\vec{\mathbf{k}}_t \cdot \vec{\mathbf{r}} = -k_t x \frac{\sin\theta}{n} - ik_t z\sqrt{\frac{\sin^2\theta}{n^2} - 1} = -k_t x \frac{\sin\theta}{n} + ik_t|z|\sqrt{\frac{\sin^2\theta}{n^2} - 1}$$

In writing the last equality we have noted that, for the situation depicted in Figure 1, the transmitted wave exists in the region for which $z < 0$, and so in this region $z = -|z|$. With the definition of the real, positive number,

$$\alpha \equiv k_t\sqrt{\frac{\sin^2\theta}{n^2} - 1}$$

the transmitted wave may be expressed as

$$E_t = E_{0t}e^{i(\vec{\mathbf{k}}_t \cdot \vec{\mathbf{r}} - \omega t)} = E_{0t}e^{-i\omega t}e^{-ixk_t \sin\theta/n}e^{-\alpha|z|}$$

The last factor on the right-hand side of this relation describes an exponential decrease in the amplitude of the wave as it enters the medium of lesser refractive index along the negative z-direction. When the wave penetrates into the medium of lesser refractive index by an amount

$$|z| = \frac{1}{\alpha} = \frac{\lambda}{2\pi\sqrt{\dfrac{\sin^2\theta}{n^2} - 1}} \tag{50}$$

the amplitude is decreased by a factor of $1/e$. The energy of this *evanescent wave* returns to its original medium unless a second medium is introduced into its region of penetration. Although detrimental in the case of cross talk in closely bound fibers lacking sufficient thickness of protective cladding, the *frustration* of the total internal reflection is put to good use in devices such as variable output couplers, made of two right-angle prisms whose separation along their diagonal faces can be carefully adjusted to vary the amount of evanescent wave coupled from one prism into the other. Another application involves a prism face brought near to the surface of an optical waveguide so that the evanescent wave emerging from the prism can be coupled into the waveguide at a given angle (mode) of propagation.

Example 4

Calculate the penetration depth of an evanescent wave undergoing TIR at a glass- ($n = 1.50$) to-air interface, such that the amplitude is attenuated to $1/e$ of its original value. Assume light of wavelength 500 nm is incident on the interface at an angle of 60°.

Solution

Since $\theta_c = \sin^{-1}(1/1.5) = 41.8°$, TIR occurs at 60°. The penetration depth is given by Eq. (50):

$$|z| = \frac{0.500\ \mu\text{m}}{2\pi\sqrt{\dfrac{\sin^2 60}{(1/1.5)^2} - 1}} = 0.096\ \mu\text{m}$$

6 COMPLEX REFRACTIVE INDEX

We wish now to show that when the reflecting surface is metallic, the Fresnel equations we have derived continue to be valid, with one important modification: The index of refraction becomes a complex number, including an imaginary part that is a measure of the absorption of the wave.

When the reflecting surface is that of a homogeneous dielectric—the case we have been discussing in this chapter—the *conductivity* σ of the material is zero. The conductivity is the proportionality constant in *Ohm's law*,

$$\vec{\mathbf{j}} = \sigma\vec{\mathbf{E}}$$

where $\vec{\mathbf{j}}$ is the *current density* $(\mathrm{A/m^2})$ produced by the field $\vec{\mathbf{E}}$. In such cases, both the $\vec{\mathbf{E}}$- and $\vec{\mathbf{B}}$-fields satisfy a differential wave equation of the form

$$\nabla^2 E = \left(\frac{1}{c^2}\right)\frac{\partial^2 E}{\partial t^2} \tag{51}$$

We have written harmonic waves satisfying Eq. (51) in the form

$$E = E_0 e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t)} \tag{52}$$

Now if the material is metallic or has an appreciable conductivity, the fundamental Maxwell equations of electricity and magnetism lead to a modification of Eqs. (51) and (52). The differential wave equation to be satisfied by the $\vec{\mathbf{E}}$-field is then

$$\nabla^2 E = \left(\frac{1}{c^2}\right)\frac{\partial^2 E}{\partial t^2} + \left(\frac{\sigma}{\varepsilon_0 c^2}\right)\frac{\partial E}{\partial t} \tag{53}$$

Note that, compared with Eq. (51), the new wave equation, given as Eq. (53), includes an additional term involving the conductivity and the first time derivative of E. As a result, when a harmonic wave in the form of Eq. (52) is substituted into Eq. (53), we find that the propagation vector $\vec{\mathbf{k}}$ must have the complex magnitude

$$\tilde{k} = \frac{\omega}{c}\left[1 + i\left(\frac{\sigma}{\varepsilon_0 \omega}\right)\right]^{1/2} \tag{54}$$

Since the refractive index n is related to k by $n = (c/\omega)k$, the refractive index is now the complex number

$$\tilde{n} = \left[1 + i\left(\frac{\sigma}{\varepsilon_0 \omega}\right)\right]^{1/2} \tag{55}$$

or we write, in general,

$$\tilde{n} = n_R + i n_I \tag{56}$$

where Re $(\tilde{n}) = n_R$ and Im $(\tilde{n}) = n_I$. Combing Eqs. (55) and (56) and equating their real and imaginary parts, the *optical constants* n_R and n_I can be found in terms of the conductivity by the equations

$$\begin{aligned} n_R^2 - n_I^2 &= 1 \\ 2n_R n_I &= \frac{\sigma}{\varepsilon_0 \omega} \end{aligned} \tag{57}$$

Furthermore, if the complex character of k in the form

$$\tilde{k} = \left(\frac{\omega}{c}\right)\tilde{n} = \left(\frac{\omega}{c}\right)[n_R + in_I] \tag{58}$$

is introduced into the harmonic wave, Eq. (52), the result is

$$E = E_0 e^{-(\omega n_I s/c)} e^{i\omega(n_R s/c - t)} \tag{59}$$

where s is the directed distance along the propagation direction. We conclude from Eq. (59) that the wave propagates in the material at a wave speed c/n_R and is absorbed such that the amplitude decreases at a rate governed by the exponential factor $e^{-(\omega n_I s/c)}$. Thus, Re $(\tilde{n}) = n_R$ must behave as the ordinary refractive index, and Im $(\tilde{n}) = n_I$, called the *extinction coefficient*, determines the rate of absorption of the wave in the conductive medium. This absorption, due to the energy contributed to the production of conduction current j in the material, is usually described by the decrease in power density I with distance, given by

$$I = I_0 e^{-\alpha s} \tag{60}$$

By comparison with the power density as determined from Eq. (59), where $I \propto |E|^2$,

$$I = I_0 e^{-2\omega n_I s/c} \tag{61}$$

Thus, the *absorption coefficient* α is related to the *extinction coefficient* n_I by

$$\alpha = \frac{2\omega n_I}{c} = \frac{4\pi n_I}{\lambda} \tag{62}$$

7 REFLECTION FROM METALS

Replacing n by $\tilde{n}$ in the Fresnel equations, Eqs. (27) and (28), we have for metals,

$$\text{TE:} \quad \frac{E_r}{E} = \frac{\cos\theta - \sqrt{\tilde{n}^2 - \sin^2\theta}}{\cos\theta + \sqrt{\tilde{n}^2 - \sin^2\theta}} \tag{63}$$

$$\text{TM:} \quad \frac{E_r}{E} = \frac{-\tilde{n}^2\cos\theta + \sqrt{\tilde{n}^2 - \sin^2\theta}}{\tilde{n}^2\cos\theta + \sqrt{\tilde{n}^2 - \sin^2\theta}} \tag{64}$$

Introducing $\tilde{n}$ as $n_R + in_I$ into Eqs. (63) and (64) gives

$$\text{TE:} \quad \frac{E_r}{E} = \frac{\cos\theta - \sqrt{(n_R^2 - n_I^2 - \sin^2\theta) + i(2n_R n_I)}}{\cos\theta + \sqrt{(n_R^2 - n_I^2 - \sin^2\theta) + i(2n_R n_I)}} \tag{65}$$

$$\text{TM:}\quad \frac{E_r}{E} = \frac{-[n_R^2 - n_I^2 + i(2n_R n_I)]\cos\theta + \sqrt{(n_R^2 - n_I^2 - \sin^2\theta) + i(2n_R n_I)}}{[n_R^2 - n_I^2 + i(2n_R n_I)]\cos\theta + \sqrt{(n_R^2 - n_I^2 - \sin^2\theta) + i(2n_R n_I)}} \tag{66}$$

In calculating the reflectance $R = |E_r/E|^2$, the complex quantity E_r/E can first be reduced to a ratio of complex numbers in the form $(a + ib)/(c + id)$, so that

$$R = (a^2 + b^2)/(c^2 + d^2)$$

In the process, we must take the square root of a complex number, which is done by first putting it into polar form. For example, if $z = A + iB$, then, in polar form,

$$z = (A^2 + B^2)^{1/2} e^{i[\tan^{-1}(B/A)]}$$

and the square root becomes

$$z^{1/2} = (A^2 + B^2)^{1/4} e^{i[(1/2)\tan^{-1}(B/A)]} \tag{67}$$

The complex expression in Eq. (67) can then be returned to the general complex form $C + iD$ using Euler's equation. These mathematical steps are easily performed with a programmable calculator or a computer. In Figure 10, the results of such calculations are shown for two metal surfaces, solid sodium and single-crystal gallium. High reflectance in the visible spectrum is characteristic of metallic surfaces, as shown by the curves for solid sodium at a wavelength of 589.3 nm. Strong discrimination between the TE and TM modes in the incident radiation is exhibited by the curves for single-crystal gallium surfaces.

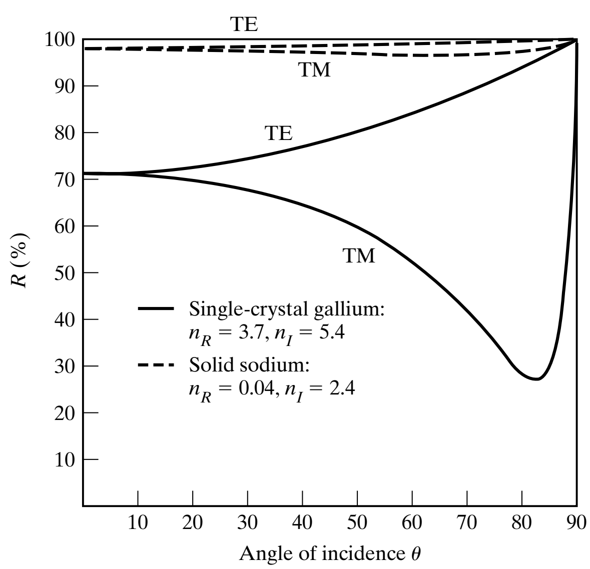

Figure 10 Reflectance from metal surfaces by using Fresnel's equations. The values of n_R and n_I are given for sodium light of $\lambda = 589.3$ nm.

PROBLEMS

1 Show that the vanishing of the reflection coefficient in the TM mode, Eq. (28), occurs at *Brewster's angle*, $\theta_p = \tan^{-1}(n)$.

2 The critical angle for a certain oil is found to be 33°33′. What are its Brewster's angles for both external and internal reflections?

3 Determine the critical angle and polarizing angles for (a) external and (b) internal reflections from dense flint glass of index $n = 1.84$.

4 For what refractive index are the critical angle and (external) Brewster angle equal when the first medium is air?

5 Show that the Fresnel equations, Eqs. (27) to (30), may also be expressed by

$$\text{TE:} \quad r = -\frac{\sin(\theta - \theta_t)}{\sin(\theta + \theta_t)} \qquad t = \frac{2\cos\theta\sin\theta_t}{\sin(\theta + \theta_t)}$$

$$\text{TM:} \quad r = -\frac{\tan(\theta - \theta_t)}{\tan(\theta + \theta_t)} \qquad t = \frac{2\cos\theta\sin\theta_t}{\sin(\theta + \theta_t)\cos(\theta - \theta_t)}$$

6 Show that Eq. (37) follows from Eq. (35).

7 Using Eqs. (27) through (30) and a computer program or a computer algebra system, reproduce Figures 3 and 5. Also, change the value of n to produce graphs for the case of external and internal reflection from diamond ($n = 2.42$).

8 Use a computer to calculate and plot the reflectance curves of Figure 4. Also plot the corresponding transmittance.

9 Use a computer to calculate and plot the phase shifts on reflection as a function of angle of incidence for $\theta > \theta_c$. Take $n = 1/1.5$ to reproduce Figure 6 and then make similar plots for $n = 1/1.3$ and $n = 1/2.42$.

10 A film of magnesium fluoride is deposited onto a glass substrate with optical thickness equal to one-fourth the wavelength of the light to be reflected from it. Refractive indices for the film and substrate are 1.38 and 1.52, respectively. Assume that the film is nonabsorbing. For monochromatic light incident normally on the film, determine (a) reflectance from the air–film surface; (b) reflectance from the film–glass surface; (c) reflectance from an air–glass surface without the film; (d) net reflectance from the combination.

11 Calculate the reflectance of water ($n = 1.33$) for both (a) TE and (b) TM polarizations when the angles of incidence are 0°, 10°, 45°, and 90°.

12 Light is incident upon an air–diamond interface. If the index of diamond is 2.42, calculate the Brewster and critical angles for both (a) external and (b) internal reflections. In each case distinguish between polarization modes.

13 Calculate the percent reflectance and transmittance for both (a) TE and (b) TM modes of light incident at 50° on a glass surface of index 1.60.

14 Derive Eqs. (29) and (30) for the transmission coefficients both by (a) eliminating E_r from Eqs. (20) to (23) and by (b) using the corresponding equations for the reflection coefficients, together with the relationships between reflection and transmission coefficients implied by Eqs. (20) and (22).

15 Unpolarized light is reflected from a plane surface of fused silica glass of index 1.458.

- **a.** Determine the critical and polarizing angles.
- **b.** Determine the reflectance and transmittance for the TE mode at normal incidence and at 45°.
- **c.** Repeat (b) for the TM mode.
- **d.** Calculate the phase difference between TM and TE modes for internally reflected rays at angles of incidence of 0°, 20°, 40°, 50°, 70°, and 90°.

16 A Fresnel rhomb is constructed of transparent material of index 1.65.

- **a.** What should be the apex angle θ, as in Figure 7?
- **b.** What is the phase difference between the TE and TM modes after both reflections, when the angle is 5% below and above the correct value?

17 Determine the reflectance for metallic reflection of sodium light (589.3 nm) from steel, for which $n_R = 2.485$ and $n_I = 1.381$. Calculate reflectance for (a) TE and (b) TM modes at angles of incidence of 0°, 30°, 50°, 70°, and 90°.

18 Determine the reflectance from tin at angles of incidence of 0°, 30°, and 60°. Do this for the (a) TE and (b) TM modes of polarization. Real and imaginary parts of the complex refractive index are 1.5 and 5.3, respectively, for light of 589.3 nm.

19
- **a.** What is the absorption coefficient for tin, with an imaginary part of the refractive index equal to 5.3 for 589.3-nm light?
- **b.** At what depth is 99% of normally incident sodium light absorbed in tin?

20
- **a.** From the power conservation requirement, as expressed by Eq. (47), show that for an external reflection the transmission coefficient t must be less than 1, but for an internal reflection t' may be greater than 1.
- **b.** Show further, using the Fresnel Eqs. (29) and (30), that as the angle of incidence approaches the critical angle, t' must approach a value of 2 in the TE mode and $2/n$ in the TM mode.
- **c.** Plot the transmission coefficient t' for an interface between glass ($n = 1.5$) and air.

21 A narrow beam of light ($\lambda = 546$ nm) is rotated through 90° by TIR from the hypotenuse face of a 45°–90°–45° prism made of glass with $n = 1.60$.

- **a.** What is the penetration depth at which the amplitude of the evanescent wave is reduced to $1/e$ of its value at the surface?
- **b.** What is the ratio of irradiance of the evanescent wave at 1 μm beyond the surface to that at the surface?