

ICP using SVD

January 2, 2023

Define points sets:

$$P = \{p_1, p_2, \dots, p_n\}, \quad P' = \{p'_1, p'_2, \dots, p'_n\} \quad (1)$$

Points transform:

$$\forall i, p_i = Rp'_i + t \quad (2)$$

Define error:

$$e_i = p_i - (Rp'_i + t) \quad (3)$$

$$E = \frac{1}{N} \sum_{i=1}^N \|p_i - (Rp'_i + t)\|^2 \quad (4)$$

Constructing least squares problems:

$$\arg \min_{R, t} \frac{1}{N} \sum_{i=1}^N \|p_i - (Rp'_i + t)\|^2 \quad (5)$$

Solve for the displacement matrix t:

$$\frac{dE}{dt} = \frac{2}{N} \sum_{i=1}^N (Rp'_i + t - p_i) = \frac{2}{N} \sum_{i=1}^N t + \frac{2}{N} \sum_{i=1}^N Rp'_i - \frac{2}{N} \sum_{i=1}^N p_i = 0 \quad (6)$$

$$t + \frac{1}{N} \sum_{i=1}^N Rp'_i - \frac{1}{N} \sum_{i=1}^N p_i = 0 \quad (7)$$

Define centroid:

$$p = \frac{1}{N} \sum_{i=1}^N p_i, \quad p' = \frac{1}{N} \sum_{i=1}^N p'_i \quad (8)$$

Thus:

$$t = p - Rp' \quad (9)$$

Substitute back to (4)

$$E = \frac{1}{N} \sum_{i=1}^N \|p_i - Rp'_i - p + Rp'\|^2 \quad (10)$$

$$= \frac{1}{N} \sum_{i=1}^N \|p_i - p - R(p'_i - p')\|^2 \quad (11)$$

Define decentroided point clouds:

$$q_i = p_i - p, \quad q'_i = p'_i - p' \quad (12)$$

Thus:

$$E = \frac{1}{N} \sum_{i=1}^N \|q_i - Rq'_i\|^2 \quad (13)$$

$$\arg \min_R \frac{1}{N} \sum_{i=1}^N \|q_i - Rq'_i\|^2 \quad (14)$$

Solve for the rotation matrix R :

$$\|q_i - Rq'_i\|^2 = (q_i - Rq'_i)^T (q_i - Rq'_i) = q_i^T q_i - q_i^T Rq'_i - q_i'^T R^T q_i + q_i'^T R^T Rq'_i \quad (15)$$

Since:

$$q_i^T Rq'_i : (1 \times 3)(3 \times 3)(3 \times 1) = (1 \times 1) \Rightarrow \text{scalar quantity} \quad (16)$$

$$q_i'^T R^T q_i : (1 \times 3)(3 \times 3)(3 \times 1) = (1 \times 1) \Rightarrow \text{scalar quantity} \quad (17)$$

Thus:

$$\|q_i - Rq'_i\|^2 = q_i^T q_i - 2q_i^T Rq'_i + q_i'^T q'_i \quad (18)$$

Substitute back to (13, 14)

$$E = \frac{1}{N} \sum_{i=1}^N (q_i^T q_i - 2q_i^T Rq'_i + q_i'^T q'_i) \quad (19)$$

$$\arg \min_R \frac{1}{N} \sum_{i=1}^N (q_i^T q_i - 2q_i^T Rq'_i + q_i'^T q'_i) \quad (20)$$

$$= \arg \min_R \frac{1}{N} \sum_{i=1}^N (-2q_i^T Rq'_i) \quad (21)$$

$$= \arg \max_R \frac{1}{N} \sum_{i=1}^N (q_i^T Rq'_i) \quad (22)$$

The trace of a constant is known to be equal to the constant itself, and $\text{tr}(AB) = \text{tr}(BA)$:

$$\frac{1}{N} \sum_{i=1}^N (q_i^T Rq'_i) = \text{tr} \left(\frac{1}{N} \sum_{i=1}^N (q_i^T Rq'_i) \right) \quad (23)$$

$$= \frac{1}{N} \text{tr} \left(R \sum_{i=1}^N (q'_i q_i^T) \right) \quad (24)$$

Define:

$$H = \sum_{i=1}^N (q_i q_i'^T) \quad \text{or} \quad H = \frac{1}{N} \sum_{i=1}^N (q_i q_i'^T) \quad (25)$$

Substitute back to (22)

$$\arg \max_R \text{tr}(RH^T) \quad (26)$$

SVD:

$$H = U \Sigma V^T \quad (27)$$

$$\arg \max_R \text{tr}(R(U \Sigma V^T)^T) = \arg \max_R \text{tr}(RV \Sigma U^T) \quad (28)$$

$$= \arg \max_R \text{tr}(\Sigma U^T R V) \quad (29)$$

Define:

$$M = U^T R V \quad (30)$$

Since R, U^T, V are orthogonal matrixes: M is a special orthogonal matrix:

$$\forall i, j, \quad |m_{ij}| < 1, \quad m \in M \quad (31)$$

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix}, \quad \sigma_1, \sigma_2, \sigma_3 > 0 \quad (32)$$

$$tr(\Sigma M) = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \quad (33)$$

$$= \sum_i \sigma_i m_{ii} \quad (34)$$

$$\leq \sum_i \sigma_i \quad (35)$$

Take the equal sign (maximum value) at $m_{ii} = 1$, since M is a identity matrix:

$$M = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = I \quad (36)$$

Thus:

$$M = U^T R V = I \quad (37)$$

$$R = U V^T \quad (38)$$

$$t = p - R p' = p - U V^T p' \quad (39)$$