ICP using SVD

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Define points sets:

$$P = \{p_1, p_2, \dots, p_n\}, \quad P' = \{p'_1, p'_2, \dots, p'_n\} \tag{1}$$

Points transform:

$$\forall i, p_i = Rp'_i + t \tag{2}$$

Define error:

$$e_i = p_i - (Rp'_i + t) \tag{3}$$

$$E = \frac{1}{N}\sum_{i=1}^{N} ||p_i - (Rp'_i + t)||^2 \tag{4}$$

Constructing least squares problems:

$$\arg\min_{R,t} \frac{1}{N}\sum_{i=1}^{N} ||p_i - (Rp'_i + t)||^2 \tag{5}$$

Solve for the displacement matrix t:

$$\frac{dE}{dt} = \frac{2}{N}\sum_{i=1}^{N}(Rp'_i + t - p_i) = \frac{2}{N}\sum_{i=1}^{N} t + \frac{2}{N}\sum_{i=1}^{N} Rp'_i - \frac{2}{N}\sum_{i=1}^{N} p_i = 0 \tag{6}$$

$$t + \frac{1}{N}\sum_{i=1}^{N} Rp'_i - \frac{1}{N}\sum_{i=1}^{N} p_i = 0 \tag{7}$$

Define centroid:

$$p = \frac{1}{N}\sum_{i=1}^{N} p_i, \quad p' = \frac{1}{N}\sum_{i=1}^{N} p'_i \tag{8}$$

Thus:

$$t = p - Rp' \tag{9}$$

Substitute back to (4)

$$E = \frac{1}{N}\sum_{i=1}^{N} ||p_i - Rp'_i - p + Rp'||^2 \tag{10}$$

$$= \frac{1}{N}\sum_{i=1}^{N} ||p_i - p - R(p'_i - p')||^2 \tag{11}$$

Define decentroided point clouds:

$$q_i = p_i - p, \quad q'_i = p'_i - p' \tag{12}$$

Thus:

$$E = \frac{1}{N}\sum_{i=1}^{N} ||q_i - Rq'_i||^2 \tag{13}$$

$$\arg\min_{R} \frac{1}{N}\sum_{i=1}^{N} ||q_i - Rq'_i||^2 \tag{14}$$

Solve for the rotation matrix R:

$$||q_i - Rq_i'||^2 = (q_i - Rq_i')^T(q_i - Rq_i') = q_i^T q_i - q_i^T Rq_i' - q_i'^T R^T q_i + q_i'^T R^T Rq_i' \tag{15}$$

Since:

$$q_i^T Rq_i' : (1 \times 3)(3 \times 3)(3 \times 1) = (1 \times 1) \Rightarrow scalar \quad quantity \tag{16}$$
$$q_i'^T R^T q_i : (1 \times 3)(3 \times 3)(3 \times 1) = (1 \times 1) \Rightarrow scalar \quad quantity \tag{17}$$

Thus:

$$||q_i - Rq_i'||^2 = q_i^T q_i - 2q_i^T Rq_i' + q_i'^T q_i' \tag{18}$$

Substitute back to (13, 14)

$$E = \frac{1}{N}\sum_{i=1}^{N}(q_i^T q_i - 2q_i^T Rq_i' + q_i'^T q_i') \tag{19}$$

$$\arg\min_R \frac{1}{N}\sum_{i=1}^{N}(q_i^T q_i - 2q_i^T Rq_i' + q_i'^T q_i')) \tag{20}$$
$$= \arg\min_R \frac{1}{N}\sum_{i=1}^{N}(-2q_i^T Rq_i') \tag{21}$$
$$= \arg\max_R \frac{1}{N}\sum_{i=1}^{N}(q_i^T Rq_i') \tag{22}$$

The trace of a constant is known to be equal to the constant itself, and tr(AB) = tr(BA):

$$\frac{1}{N}\sum_{i=1}^{N}(q_i^T Rq_i') = tr(\frac{1}{N}\sum_{i=1}^{N}(q_i^T Rq_i')) \tag{23}$$
$$= \frac{1}{N} tr(R\sum_{i=1}^{N}(q_i' q_i^T)) \tag{24}$$

Define:

$$H = \sum_{i=1}^{N}(q_i q_i'^T) \quad or \quad H = \frac{1}{N}\sum_{i=1}^{N}(q_i q_i'^T) \tag{25}$$

Substitute back to (22)

$$\arg\max_R tr(RH^T) \tag{26}$$

SVD:

$$H = U\Sigma V^T \tag{27}$$

$$\arg\max_R tr(R(U\Sigma V^T)^T) = \arg\max_R tr(RV\Sigma U^T) \tag{28}$$
$$= \arg\max_R tr(\Sigma U^T RV) \tag{29}$$

Define:

$$M = U^T RV \tag{30}$$

Since R, U^T, V are orthogonal matrixs: M is a special orthogonal matrix:

$$\forall i, j, \quad |m_{ij}| < 1, \quad m \in M \tag{31}$$

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix}, \quad \sigma_1, \sigma_2, \sigma_3 > 0 \tag{32}$$

$$
\begin{aligned}
tr(\Sigma M) &= \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} && (33) \\
&= \sum_i \sigma_i m_{ii} && (34) \\
&\leq \sum_i \sigma_i && (35)
\end{aligned}
$$

Take the equal sign (maximum value) at $m_{ii} = 1$, since M is a orthogonal matrix:

$$
M = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = I \tag{36}
$$

Thus:

$$
\begin{aligned}
M = U^T R V = I && (37) \\
R = U V^T && (38) \\
t = p - Rp' = p - UV^T p' && (39)
\end{aligned}
$$