

Optimization - Homework 1

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1 Problem Statement

Let

$$A = \begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 & 2 & 6 \\ 2 & 7 & 1 & 8 & 2 & 8 & 1 & 8 \\ 1 & 4 & 1 & 4 & 2 & 1 & 3 & 5 \end{bmatrix}, \vec{y} = \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix}$$

Find a 2-Space Vector $\vec{x} \in^8$ so that $\vec{y} = A\vec{x}$

At least one of the following vectors are a solution:

$$\vec{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a \\ b \\ 0 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ a \\ b \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a \\ b \end{bmatrix}$$

There are $\binom{8}{2} = 28$ 2-sparse vectors but only the four need to be considered.

2 Solution

2.1 Guiding Principles

$\vec{y} = A\vec{x}$ where \vec{x} contains free variables produces a system of equations which can be organized into a matrix R. Row reduction is applied to R to obtain discrete values a' , b' for the free variables. a' , b' are plugged into the remaining equation in R (figure out better terminology here). If the remaining equation is valid, then x_i is a valid solution. Otherwise, it is not.

2.2 \mathbf{x}_1

2.2.1 Organize $\vec{y} = Ax_1$ into a System of equations.

$$\vec{y} = Ax_1$$

$$\begin{aligned} \rightarrow \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix} &= \begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 & 2 & 6 \\ 2 & 7 & 1 & 8 & 2 & 8 & 1 & 8 \\ 1 & 4 & 1 & 4 & 2 & 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a \\ b \\ 0 \\ 0 \end{bmatrix} \\ \rightarrow \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix} &= \begin{bmatrix} 5a + 9b \\ 2a + 8b \\ 2a + b \end{bmatrix} \\ \rightarrow \begin{bmatrix} 5 & 9 & -7 \\ 2 & 8 & 20 \\ 2 & 1 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \end{aligned} \tag{1}$$

2.2.2 Apply Row Reduction to solve free variables.

$$\begin{bmatrix} 5 & 9 & -7 \\ 2 & 8 & 20 \\ 2 & 1 & 8 \end{bmatrix} \xrightarrow{-r_3+r_2} \begin{bmatrix} 5 & 9 & -7 \\ 0 & 7 & 12 \\ 2 & 1 & 8 \end{bmatrix} \xrightarrow{-9r_3+r_1} \begin{bmatrix} -13 & 0 & -79 \\ 0 & 7 & 12 \\ 2 & 1 & 8 \end{bmatrix} \tag{2}$$

$$-13a + 0 = -79 \tag{3a}$$

$$7b + 0 = 12 \tag{3b}$$

$$2a + b = 8 \tag{3c}$$

Solving for a, b using (3a) and (3b) yields $a = \frac{79}{13}$, $b = \frac{12}{7}$

2.2.3 Plug values into the remaining equation

Replacing a, b in (3c) yields $2(\frac{79}{13} + \frac{12}{7} \neq 8)$. Thus \vec{x}_1 is **not** a valid solution

2.3 x_2

2.3.1 Organize $\vec{y} = Ax_2$ into a System of equations.

$$\vec{y} = Ax_2$$

$$\begin{aligned} \rightarrow \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix} &= \begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 & 2 & 6 \\ 2 & 7 & 1 & 8 & 2 & 8 & 1 & 8 \\ 1 & 4 & 1 & 4 & 2 & 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \rightarrow \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix} &= \begin{bmatrix} 3a + b \\ 2a + 7b \\ a + 4b \end{bmatrix} \\ \rightarrow \begin{bmatrix} 3 & 1 & -7 \\ 2 & 7 & 20 \\ 1 & 4 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \end{aligned} \quad (4)$$

2.3.2 Apply Row Reduction to solve free variables.

$$\begin{bmatrix} 3 & 1 & -7 \\ 2 & 7 & 20 \\ 1 & 4 & 8 \end{bmatrix} \xrightarrow{-2r_3+r_2} \begin{bmatrix} 3 & 1 & -7 \\ 0 & -1 & 4 \\ 1 & 4 & 8 \end{bmatrix} \xrightarrow{-4r_2+r_3} \begin{bmatrix} 3 & 1 & -7 \\ 0 & -1 & 4 \\ 1 & 0 & 24 \end{bmatrix} \quad (5)$$

$$3a + b = -7 \quad (6a)$$

$$0 + -b = 4 \quad (6b)$$

$$a + 0 = 24 \quad (6c)$$

Solving for a , b using (6b) and (6c) yields $a = 24$, $b = -4$

2.3.3 Plug values into the remaining equation

Replacing a , b in (6a) yields $3(24) - 4 \neq 7$. Thus \vec{x}_2 is **not** a valid solution

2.4 \mathbf{x}_3

2.4.1 Organize $\vec{y} = Ax_3$ into a System of equations.

$$\vec{y} = Ax_3$$

$$\rightarrow \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 & 2 & 6 \\ 2 & 7 & 1 & 8 & 2 & 8 & 1 & 8 \\ 1 & 4 & 1 & 4 & 2 & 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ a \\ b \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (7)$$

$$\rightarrow \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} a + 5b \\ 8a + 2b \\ 4a + 2b \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 5 & -7 \\ 8 & 2 & 20 \\ 4 & 2 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$$

2.4.2 Apply Row Reduction to solve free variables.

$$\begin{bmatrix} 1 & 5 & -7 \\ 8 & 2 & 20 \\ 4 & 2 & 8 \end{bmatrix} \xrightarrow{-2r_3+r_2} \begin{bmatrix} 1 & 5 & -7 \\ 0 & -2 & 4 \\ 4 & 2 & 8 \end{bmatrix} \xrightarrow{r_2+r_3} \begin{bmatrix} 1 & 5 & -7 \\ 0 & -2 & 4 \\ 4 & 0 & 12 \end{bmatrix} \xrightarrow{\frac{1}{4}r_3, -\frac{1}{2}r_2} \begin{bmatrix} 1 & 5 & -7 \\ 0 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix} \quad (8)$$

$$a + 5b = -7 \quad (9a)$$

$$0 + b = -2 \quad (9b)$$

$$a + 0 = 3 \quad (9c)$$

Solving for a , b using (9b) and (9c) yields $a = 3$, $b = -2$

2.4.3 Plug values into the remaining equation

Replacing a , b in (9a) yields $3 + 5(-2) = -7$. Thus \vec{x}_3 is a valid solution

2.5 \mathbf{x}_4

2.5.1 Organize $\vec{y} = Ax_4$ into a System of equations.

$$\vec{y} = Ax_4$$

$$\begin{aligned} \rightarrow \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix} &= \begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 & 2 & 6 \\ 2 & 7 & 1 & 8 & 2 & 8 & 1 & 8 \\ 1 & 4 & 1 & 4 & 2 & 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a \\ b \end{bmatrix} \\ \rightarrow \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix} &= \begin{bmatrix} 2a + 6b \\ a + 8b \\ 3a + 5b \end{bmatrix} \\ \rightarrow \begin{bmatrix} 2 & 6 & -7 \\ 1 & 8 & 20 \\ 3 & 5 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} \end{aligned} \quad (10)$$

2.5.2 Apply Row Reduction to solve free variables.

$$\begin{bmatrix} 2 & 6 & -7 \\ 1 & 8 & 20 \\ 3 & 5 & 8 \end{bmatrix} \xrightarrow{-3r_2+r_3} \begin{bmatrix} 2 & 6 & -7 \\ 1 & 8 & 20 \\ 0 & -19 & -52 \end{bmatrix} \xrightarrow{-\frac{8}{19}r_3+r_2} \begin{bmatrix} 2 & 6 & -7 \\ 1 & 0 & \frac{160}{19} \\ 0 & -19 & -52 \end{bmatrix} \quad (11)$$

$$2a + 6b = -7 \quad (12a)$$

$$a + 0 = \frac{160}{19} \quad (12b)$$

$$0 - 19b = -52 \quad (12c)$$

Solving for a , b using (12b) and (12c) yields $a = \frac{160}{19}$, $b = \frac{52}{19}$

2.5.3 Plug values into the remaining equation

Replacing a , b in (12a) yields $2(\frac{160}{19}) + 6(\frac{52}{19}) = -7$. Thus \vec{x}_4 is **not** a valid solution