Class Notes

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Contents

1 Review & Introduction (2020/03/31)

1.1 Review

Orthogonal: Vectors are orthogonal when the dot product = 0.

1.1.1 Basis

$$
\begin{aligned}
\underset{(n\times 1)}{\vec{y}} &= \underset{(n\times p)}{A}\underset{(p\times 1)}{\vec{x}} \\
&= B\vec{c} \\
&= \Sigma c_i \vec{b_i} \text{ (most } c_i = 0)
\end{aligned} \tag{1}
$$

A: Basis Matrix

Properties of a Good Basis

- not all are orthogonal
- Allows for a sparse vector to be used ad the constant vector $\vec{c}$

Identity Matrices are the *worst* basis because most coefficients are non-zero.

2-Sparse Vector

$$
\vec{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ 4 \end{bmatrix} \tag{2}
$$

Very important!

> When dealing with Natural images and a good basis, there is a sparse vector.

1.1.2 Kernel

The kernel of a linear mapping is the set of vectors mapped to the 0 vector. The kernel is often referred to as the **null space**. Vectors should be linearly independent.

$$Ker(A) = \vec{x} \in \mathbb{R}^n \colon A\vec{x} = \vec{0} \tag{3}$$

A must be designed such that the Kernel of A does not contain any s-sparse vector other than $\vec{0}$

Main Idea: For (1), reduce $\vec{y}$ to a K-Sparse matrix to reduce the amount of non-zero numbers.

1.2 Linear Algebra Review

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \tag{4}$$

$$\begin{aligned} \underset{(1\times3)(3\times1)}{\vec{u}^T\vec{v}} &= \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 1 + 2 - 2 = 1 \\ &= \vec{u} \cdot \vec{v} \end{aligned} \tag{5}$$

$$\underset{(3\times1)(1\times3)}{\vec{u}\ \vec{v}^T} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ -1 & -1 & -2 \end{bmatrix} \tag{6}$$

$\vec{u}\ \vec{v}^T \neq \vec{u}^T\ \vec{v}$

1.2.1 Inner Product

$$\begin{aligned} < \vec{a}, \vec{b} > &= \vec{a} \cdot \vec{b} \\ &= \vec{a}^T \vec{b} \end{aligned} \tag{7}$$

1.2.2 Cauchy-Schwartz Inequality

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \tag{8}$$

$$\begin{aligned} | < \vec{a}, \vec{b} > | &\leq \sqrt{1^2 + 2^2 + (-1)^2} \times \sqrt{1^2 + 1^2 + 2^2} \\ | < \vec{a}, \vec{b} > | &\leq ||\vec{a}||_2\ ||\vec{b}||_2 \text{ (euclidean/l2-norm)} \end{aligned} \tag{9}$$

1.2.3 Norms

Why is the l1 norm preferred for ML opposed to the classic l2 norm?

Philosophically,

If we looked at a sphere in l2 norm, the shadow casted would be a circle regardless of the direction of the light.

Looking at a sphere in the l1 norm is shaped as a tetrahedron. The shadow cast by a tetrahedron is different for different angles so observing the shadow provides a lot more context about the sphere.

1. Euclidean/l2

   Sphere: $||\vec{x}||_2 = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = 5$

   (a) FOIL Given 2 fixed vectors x,y. Consider the l2-norm squared:

   $$f(t) = ||x + ty||_2^2$$

   $$\begin{aligned} f(t) =& ||x + ty||_2^2 \\ =& < x + ty, x + ty > \\ =& < x, x > + t < x, y > + t < y, x > + t^2 < y, y > \\ =& ||x||_2^2 + 2t < x, y > + t^2 ||y||_2^2 \end{aligned} \tag{10}$$

   Note: t<x,y> and t<y,x> can be combined because their dot-products are equivalent. $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$

   When using Machine Learning, don't use l2 norms. Use l1

   (b) Derivative

   $$\begin{aligned} \frac{d}{dt}(||x + ty||_2^2) =& 2 < x, y > + 2t||y||_2^2 \\ =& 2x^T y + 2ty^T y \end{aligned} \tag{11}$$

2. Simplex/l1

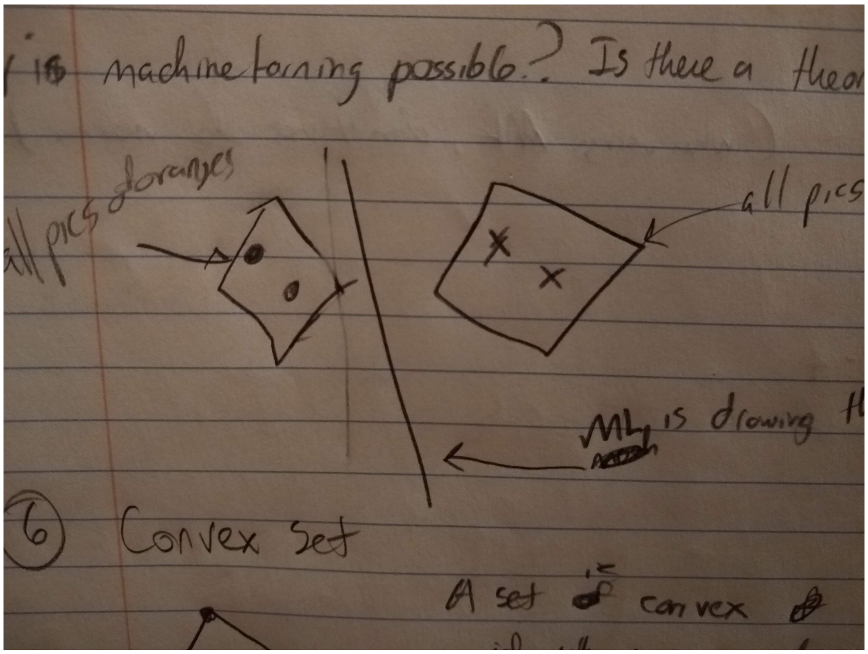
   Sphere: $||\vec{x}||_1 = |-4| + |3| = 7$

3. Infinity

   Sphere: $||\vec{x}||_\infty = Max|-4|, |3| = 4$

1.3 Optimization

Why is Machine Learning Possible? Is there a theoretical guarantee?



Imagine A is the set of all dogs and B is the set of all Cats

If the sets are convex and do not overlap, there exists a line between them which acts as a divider for determining whether a new pic belongs in A or B.

1.4 Convex Set

A set is convex if whenever X and Y are in the set, then for $0 \leq t \leq 1$ the points $(1-t)x + ty$ must also be in the set.

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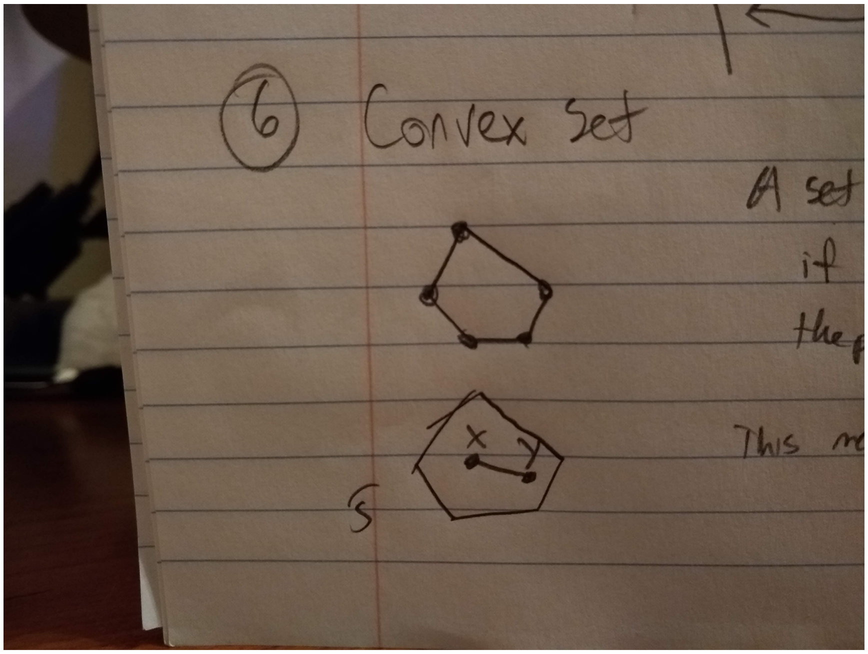


1.5 Separating Hyper-plane Theorem

Let C and D be 2 convex sets that do not intersect. i.e. the sets are **disjoint**.

Then there exists a vector $\vec{a} \neq 0$ and a number b such that.

$$a^T x \leq b \forall x \in C$$

and

$$a^T x \geq b \forall x \in D$$

The Separating Hyper-plane is defined as $x\colon a^T x = b$ for sets C, D.
This is the theoretical guarantee for ML

vector a is perpendicular to the plane b.

2 Why Separating Hyperplane Theorem & Subspace Segmentation Example (2020/04/07)

2.1 Why is Separating Hyper-plane Theorem true?

2.1.1 Math Background

Let $x = d - c,\ y = u - d$

1. Square of the \$l$_2$\$-norm is the inner product

$$\|x\|_2^2 = \langle x, x \rangle = x^T x$$

$$(d-c)^T(d-c) = \|d-c\|_2^2$$

2. Expansion of Vectors

$$\begin{aligned} &\|x+ty\|_2^2 \\ =&\langle x+ty, x+ty \rangle \\ =&\|x\|_2^2 + 2t\langle x, y \rangle + t^2\|y\|_2^2 \end{aligned} \tag{12}$$

3. Derivative of vector products

$$\frac{d}{dt}(\|x+ty\|_2^2) = 2x^T y + 2ty^T y$$

$$\frac{d}{dt}(\|x+ty\|_2^2)|_{t=0} = 2x^T y$$

$$\frac{d}{dt}(\|d+t(u-d)-c\|_2^2)|_{t=0} = 2(d-c)^T(u-d)$$

2.1.2 Separating Hyper-plane Theorem

C, D are convex disjoint sets. Thus there exists a vecto $\vec{a} \neq 0$ and a number b such that

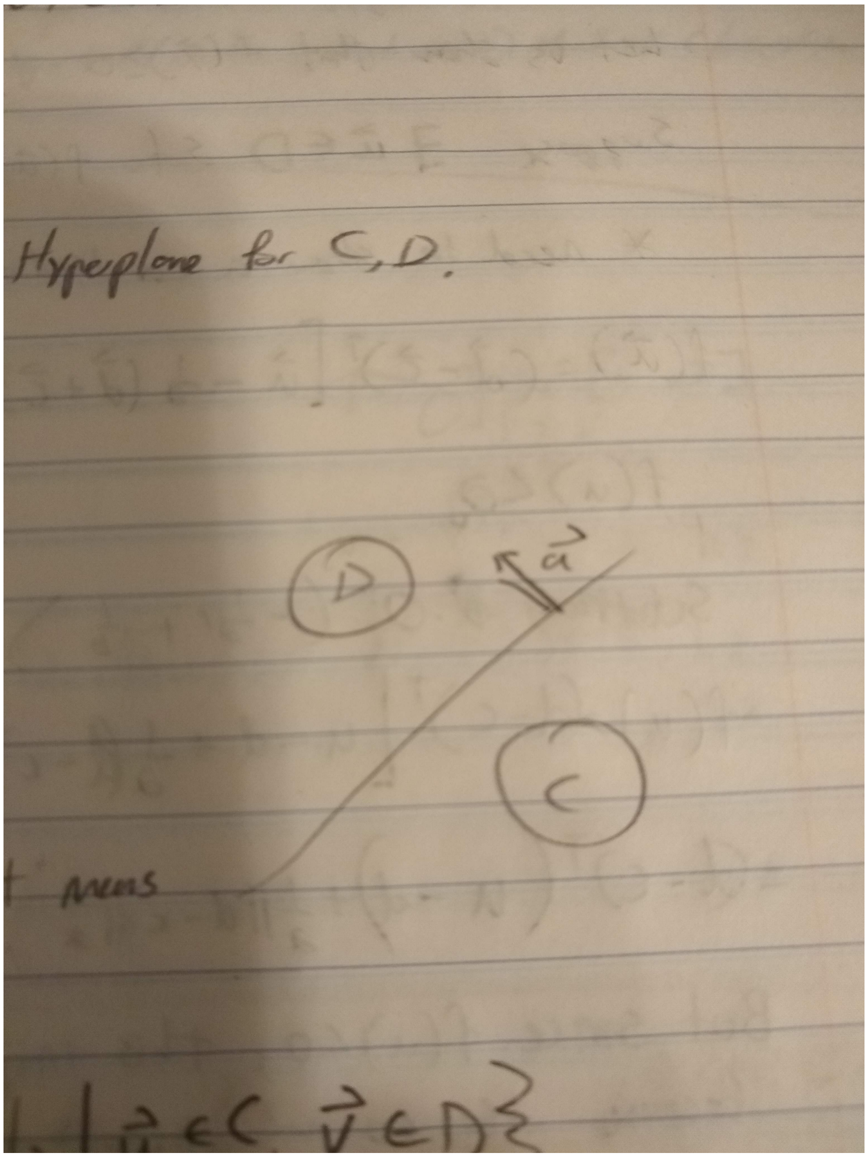
$$a^T x \leq b, \forall x \in C$$

and

$$a^T x \geq b, \forall x \in D$$

$x : a^T x = b$ is the separating hyper-plane for C,D.
When $b = 0$, then inconclusive answer.

2.1.3 Why is it true?



$$\begin{aligned} \vec{a}^T\vec{x} &\leq b \text{ on side C} \\ \vec{a^T}\vec{x} &\geq \text{ on side D} \end{aligned} \tag{13}$$

Goal: Prove $\vec{a}$ exists as that means a separating hyperplane exists.

$$dist(C, D) = min\|\vec{u} - \vec{v}\|_2 | \vec{u} \in C, \vec{v} \in D = \|\vec{c} - \vec{d}\|_2$$

where $\|\vec{u} - \vec{v}\|_2$ is the euclidean distance.
Let $\vec{a} = \vec{d} - \vec{c},\ b = \frac{1}{2}(\|\vec{d}\|_2^2 - \|\vec{c}\|_2^2)$
We will show that

$$f(\vec{x}) = a^T x - b$$

has the property that

$$f(\vec{x}) \leq 0,\ \forall \vec{x} \in C$$

and

$$f(\vec{x}) \geq 0,\ \forall \vec{x} \in D$$

Note: $(\vec{d} - \vec{c})^T \frac{1}{2}(\vec{d} + \vec{c}) = \frac{1}{2}(\|\vec{d}\|_2^2 - \|\vec{c}\|_2^2)$
What does showing something mean?
Let us show that $F(\vec{x}) \geq 0,\ \forall \vec{x} \in D$ (Argue by Contradiction)
Suppose $\exists \vec{u} \in D$ such that $f(\vec{x}) < 0$

$$f(\vec{u}) = (\vec{d} - \vec{c})^T[\vec{u} - \frac{1}{2}(\vec{d} + \vec{c})] = (\vec{d} - \vec{c})^T \vec{u} - \frac{1}{2}(\|\vec{d}\|_2^2 - \|\vec{c}\|_2^2)$$

Subtract 0

$$f(u) = (d - c)^T[u - d + \frac{1}{2}\|d - c\|]$$

$u - \frac{1}{2}d + \frac{1}{2}c$
$u - d + \frac{1}{2}d - \frac{1}{2}c$

$$f(u) = (d - c)^T(u - d) + \frac{1}{2}\|d - c\|_2^2$$

Now we observe that

$$\frac{d}{dt}(\|d + t(u - d) - c\|_2^2)|_{t=0} = 2(d - c)^T(u - d) < 0$$

and so for some small $t > 0$,

$$\|d + t(u - d) - c\|_2^2 < \|d - c\|_2^2$$

$g'(t) < 0$ means decreasing. Thus $g(t) < g(0)$.
Let's call point $p = d + t(u - d)$
Then

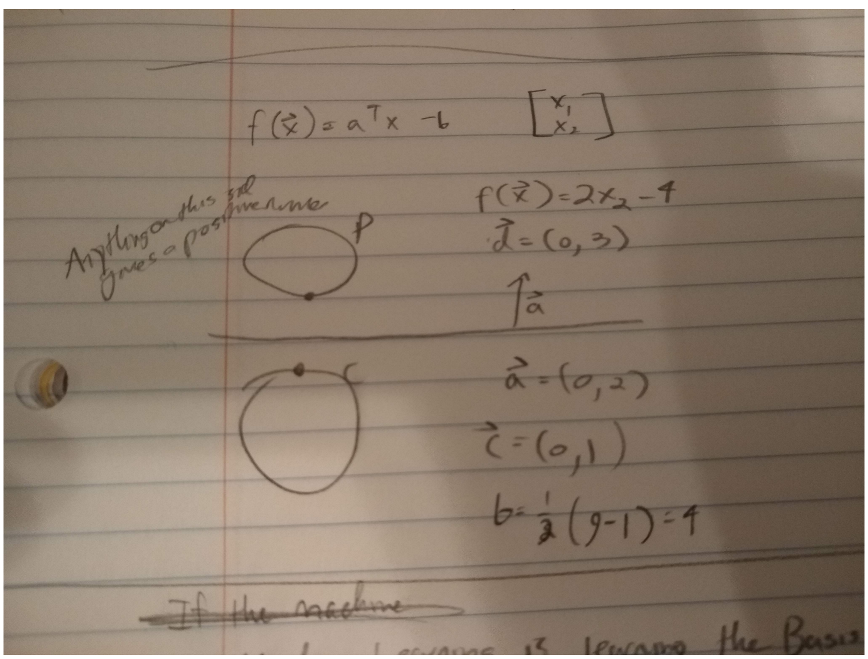
$$\|p - c\|_2^2 < \|d - c\|_2^2$$

This is a contradiction. Both d and u are in set D. Thus by the definition of convexity, $p = (1 - t)d + tu$

D is a convex set so p must also be in D. This situation is impossible since d is the point in D that is closest to c.

2.1.4 Example

Let $f(\vec{x}) = a^T x - b$



2.2 Subspace Segmentation Example

Machine Learning is learning the Basis A. If we can deduce that a vector $\vec{x}$ is a linear combination of A, then a vector is a subspace of Basis A and we know that it belongs to A.

$$V_1 = (x, y, z) \in R^3 : z = 0$$
$$V_2 = (x, y, z) \in R^3 : x = 0, y = 0$$

V_i is the affine variety (it is also a Ring, Module)
Apply a Veronase map with degree 2 to lift up from 3 to 6 dimensions.

$$\nu_n \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \end{bmatrix}, \nu_n : R^3 \to R^6$$

$$\begin{aligned} z_1 = (3, 4, 0), z_2 = (4, 3, 0), \\ z_3 = (2, 1, 0), z_4 = (1, 2, 0), \\ z_5 = (0, 0, 1), z_6 = (0, 0, 3), z_7 = (0, 0, 4) \end{aligned} \tag{14}$$

Plug the sample points into the Veronase map to produce a matrix L

$$L = \begin{bmatrix} 9 & 16 & 4 & 1 & 0 & 0 & 0 \\ 16 & 9 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 9 & 6 \\ 12 & 12 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in R^{6\times 7}$$

solve for $\vec{c}$, where $\vec{c}^T L = \vec{0}$

$$\vec{c}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Rank(L) = 4 (since there are 4 linearly independent rows)

$$\begin{aligned} q_1(X) =& \vec{c}^T \nu_n(X) \\ =& xz \\ q_2(X) =& \vec{c}_2^T \nu_n(X) \\ =& yz \end{aligned} \tag{15}$$

We have:

$$
\begin{aligned}
q_1(X) &= xz \quad V_1 = (z = 0) \\
q_2(X) &= yz \quad V_2 = (x = 0, y = 0)
\end{aligned} \tag{16}
$$

Observe:
$V_1 \cup V_2 = ((x, y, z) \in R^3 : q_1(X) = 0, q_2(X) = 0)$
Construct the Jacobian matrix
J(Q)(X) $= \begin{bmatrix} \frac{\partial q_1}{\partial x} & \frac{\partial q_1}{\partial y} & \frac{\partial q_1}{\partial z} \\ \frac{\partial q_2}{\partial x} & \frac{\partial q_2}{\partial y} & \frac{\partial q_2}{\partial z} \end{bmatrix} = \begin{bmatrix} z & 0 & x \\ 0 & z & y \end{bmatrix}$

1. When $z = z_1 = (3, 4, 0), J(Q)(z_1) = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix}$

   When $z = z_3 = (2, 1, 0),\ J(Q)(z_3) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

   The right null space of $J(Q)(z_1)$ has basis $\vec{b_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{b_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

2. When $z = z_5 = (0, 0, 1),\ J(Q)(z_5) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

   When $z = z_7 = (0, 0, 4),\ J(Q)(z_7) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}$ The right null space of $J(Q)(z_5)$ has basis $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$C = [\vec{c}_1 | \vec{c}_2]$

3 Sparse Representation & Problem P0 . P1 (2020/04/14)

3.1 Big Idea

Your Data is a vector $x \in R^N$ where all vectors are column vectors. Each x is s-sparse i.e. each vector has at **most** s non-zero entries. Let s = 5000. We don't know where the non-zero entries are located.

Let $\underset{(m \times N)}{A},\ m < N$
$N = 100,000,\ m = 20,000$
Short + Wide Matrix

This is the opposite of the kinds of matrices seen in Linear Regression which are tall and skinny.

What if we can design a matrix $A \in R^{m \times N}$ so that for each s-sparse $\vec{x} \in R^N$, you can store $\vec{y}$ instead? $(A\vec{x} = \vec{y})$

Q: Is there a way to get back $\vec{x}$ from $\vec{y}$? We observe $\vec{y}$.

A: Yes!

Properties of A

- A cannot be the 0 matrix.
- if $\vec{x}_1$ is s-sparse and $\vec{x} \neq 0$, what if $\vec{x}_1$ is in $ker(A)$? No! that would return $\vec{0}$ which means we cannot reconstruct the original matrix since there are multiple vectors in Ker(A).

Using Techniques from 1955

1. Is $\vec{x}$ the inverse of $\vec{y}$ or psuedo-inverse, or Moore-Penrose inverse, or. . . ?

$$\begin{aligned} \vec{y} =& A\vec{x} \\ A^{\#}\vec{y} =& A^{\#}A\vec{x} \text{ where } \mathrm{A}^{\#}A = I \end{aligned} \tag{17}$$

Doesn't work! This is because there is no way to guarantee that $\vec{x}$ is a s-sparse vector.

1. Can we use gradient descent to solve for $\vec{x}$ to minimize $\|\vec{y} - A\vec{x}\|_2$

   No! Why?

   pick any vector $\vec{v} \in Ker(A)$. $\vec{y} = A(\vec{x} + \vec{v})$ however, $(\vec{x} + \vec{v})$ may not be sparse.

New math was needed to solve this problem so it was created in 2005 by Donoho, Candes, and Tao using the \$l$_1$\$-norm instead of the euclidean norm (l_2).

3.2 Background

\$l$_1$\$-norm: $\|x\|_1 = |x_1| + |x_2| + |x_3|$

\$l$_2$\$-norm: $\|x\| = \sqrt{|x_1|^2 + |x_2|^2 + |x_3|^2}$

For $\vec{x} \in R^n$, $\vec{y} \in R^N$, then

$$\|\vec{x}+\vec{y}\| \leq \|x\|_1 + \|y\|_1$$

For a norm to be valid, it must uphold the **Triangle Inequality**.
$\vec{a}$ is one side of a triangle, $\vec{b}$ is a second side, third side, ...

$$\begin{aligned} |\vec{a}+\vec{b}| &\leq |\vec{a}| + |\vec{b}| \\ \|\vec{x}+\vec{y}\|_1 &\leq \|\vec{x}\|_1 + \|\vec{y}\|_1 \\ \|\vec{x}+\vec{y}\|_2 &\leq \|\vec{x}\|_2 + \|\vec{y}\|_2 \\ \|\vec{x}+\vec{y}\|_2 &\leq \|\vec{x}\|_\infty + \|\vec{y}\|_\infty \end{aligned} \tag{18}$$

It also must be distributive:
If $\vec{x}_1 + \vec{x}_2 = \vec{y}$, then $(\vec{x}_1 + \vec{x}_2) \cdot \vec{a} = \vec{y} \cdot \vec{a}$ for any $\vec{a}$

$$\langle \vec{x}_1 + \vec{x}_2, \vec{a} \rangle = \langle \vec{y}, \vec{a} \rangle \rightarrow \langle \vec{x}_1, \vec{a} \rangle + \langle \vec{x}_2, \vec{a} \rangle = \langle \vec{y}, \vec{a} \rangle$$

3.3 Warm-up

$A = [\vec{a}_1|...|\vec{a}_N]$
$\|\vec{a}_j\|_2 = 1 = \langle \vec{a}_j, \vec{a}_j \rangle$
Let $\vec{v} \in Ker(A),\ \vec{v} \neq \vec{0},\ \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_N \end{bmatrix}$
Assume $\vec{a}_j$ are unit vectors.
Pick $i = 3$ observations.

1. Multiply by 1. Be Sneaky.
   $v_i = v_i \langle \vec{a}_i, \vec{a}_i \rangle$
2. $\vec{v} \in Ker(A)$

$$\begin{aligned} &v_1 a_1 + v_2 a_2 + ... + v_n a_n = \vec{0} \\ \rightarrow &\langle v_1 a_1 + ... + v_N a_N, a_i \rangle = \langle \vec{0}, a_i \rangle \\ \rightarrow &\langle v_1 a_1, a_i \rangle + ... + \langle v_N a_N, a_i \rangle = \langle \vec{0}, a_i \rangle \end{aligned} \tag{19}$$

Keep $v_3 \langle a_3, a_i \rangle$ on the left side. Move everything to the other side. Thus,

$$v_i = \langle v_i a_i, a_i \rangle = - \sum_{j=1, j \neq i} v_j \langle a_j, a_i \rangle$$

Since $i = 3$, $v_3 \langle a_3, a_i \rangle = v_i$

$$|v_i| \leq \sum_{j=1, ji} |v_j| \cdot |\langle a_j, a_i \rangle|$$

What is the absolute value of a single number in $Ker(A)$? There is a relation between v_i and the rest of the entries in $\vec{v}$.

Why "=" becomes $\leq$

For example, if -2 = 3 + (- 5), then

3.4 Getting Ready to Formulate the Problem

3.4.1 Problem P0

Find the s-sparse $\vec{x} \in R^N$ such that $\vec{y} = A\vec{x}$.

Ex. Problem 1 HW 1.

Find a 2-sparse vector $\vec{x} \in R^8$ such that $\vec{y} = A\vec{x}$.

There are $\binom{8}{2}$ 2-sparse vectors. (28).

Imagine N = 100,000 and s = 5000. Not feasible to try all sparse-vectors.

3.4.2 Problem P1 (Convex Optimization)

Given $A \in R^{m \times N}$ and measurement $\vec{y} = R^m$, solve the optimization problem,

$$\min_{x \in R^N} \|x\|_1$$

subject to constraint $y = A\vec{x}$

Find a condition on matrix A, so that solving P1 will recover the s-sparse vector $x \in R^N$

3.5 Null Space Property of Order s

3.5.1 Setting up Notation

Let $\vec{v} \in Ker(A)$, $\vec{v} \neq \vec{0}$

Let the set of indices , where $\vec{v}[j] \neq 0$ to be S.

e.g. $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \\ 3 \\ 0 \\ 4 \end{bmatrix}$

$S = \{3, 5, 7\}$ (non-zero indices. Also called the support vector of $\vec{v}$).
$|S| = s$ (number of elements. i.e. sparsity)
$\bar{S} = \{1, 2, 4, 6\}$ (complement. i.e zero indices)

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ -2 \\ 2 \end{bmatrix}, \vec{v}_S = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \ \vec{v}_{\bar{S}} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

$\vec{v} = \vec{v}_S + \vec{v}_{\bar{S}}$

3.5.2 Definition

Let A be a $m \times N$ matrix.

Let S be a subset or $\{1, 2, 3, ..., N\}$. Suppose $N = 50$, and $S = \{3, 5, 7\}$

1. We say that a matrix A satisfies the null space property with respect to a set S if
$$\|\vec{v}_S\|_1 < \|\bar{S}\|, |\forall \vec{v} \,\vec{\in}\, Ker(A)$$
2. If it satisfies the null space property with respect to any set S of size s where S is a subset of $\{1, 2, 3, ..., N\}$. $s < N$

If a matrix satisfies this property, what does it buy us?

If a matrix A satisfies the Null Space property of order s, then solving problem P1 will solve P0. i.e. you can recover any s-sparse vector $\vec{x}$ from the measurement y where $\vec{y} = A\vec{x}$

> If A has a small coherence, then it satisfies the Null Space Property of order s.

Let $A = [\vec{a}_1|...|\vec{a}_N]$

$$\mu_1 = \max_{j \neq k} |\langle \vec{a}_j, \vec{a}_k \rangle|$$

Assume $\vec{a}_j$ has \$l$_2$\$-norm equal to 1.

3.5.3 Theorem

Same assumptions as above.

Suppose $\mu_1 \cdot s + \mu_1 \cdot (s-1) < 1$

The matrix satisfies the Null Space property of order s.

Remarks

1. $\mu_1(2s-1) < 1$ if true, then A satisfies NSP of order s. It is not a necessary condition. It is a sufficient condition.

2. From the warm up, if we fix an index i, then for $\vec{v} \in Ker(A)$,

$$|v_i| \leq \sum_{j=1, j \neq i} |v_j| \cdot |\langle \vec{a}_j, \vec{a}_i \rangle| \tag{20}$$

1. Note that $|v_i|$ is just one term in $\|v\|_1$ because

$$\|v\|_1 = |v_1| + |v_2| + ...$$

3.5.4 Proof

Given A is an $m \times N$ matrix. $A = [\vec{a}_1|...|\vec{a}_N]$.

Suppose $\|\vec{a}_j\| = 1,\ \mu_1 \cdot s + \mu_1 \cdot (s-1) < 1$

Show that NSP of order s holds.

i.e.

$$\|\vec{v}_S\| < \|\vec{v}_{\bar{S}}\|, \forall \vec{v} \in ker(A)|\{\vec{0}\}$$

and for every set

$$S \subset \{1, 2, 3, ..., N\}\text{with}|S| = s$$

Let $\vec{v} = Ker(A)$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix}$

$A\vec{v} = v_1\vec{a}_1 + ... + v_N\vec{a}_N = \vec{0}$
Let $S \subset \{1, 2, ..., N\},\ |S| = s$. Pick any $\vec{a}_i, i \in S$
Then $v_i = v_i\langle\vec{a}_i, \vec{a}_i\rangle$. Also, $v_1\langle\vec{a}_i, \vec{a}_i\rangle + ... + v_N\langle\vec{a}_N, \vec{a}_i\rangle = 0$

$$\begin{aligned} \rightarrow v_i = v_i\langle\vec{a}_i, \vec{a}_i\rangle &= -\sum_{j=1, j\neq i} v_i\langle\vec{a}_j, \vec{a}_i\rangle \\ \rightarrow v_i = -\sum_{l\in S} v_l\langle\vec{a}_l, \vec{a}_i\rangle &- \sum_{j\in S, j\neq i} v_j\langle\vec{a}_j, \vec{a}_i\rangle \\ \rightarrow |v_i| \leq \sum_{l\in S} |v_l||\langle\vec{a}_l, \vec{a}_i\rangle| &+ \sum_{j\in S, j\neq i} |v_j||\langle\vec{a}_j, \vec{a}_i\rangle| \end{aligned} \tag{21}$$

sum over all $i \in S$ to get
$\|\vec{v}_S\|_1 = \sum_{i\in S}|v_i|$

This adds up all the inequalities for one inequality to rule them all.

$$\begin{aligned} &\leq \sum_{i\in S}\sum_{l\in\bar{S}} |v_l| \cdot |\langle\vec{a}_l, \vec{a}_i\rangle| + \sum_{i\in S}\sum_{j\in S, j\neq i} |v_j| \cdot |\langle\vec{a}_j, \vec{a}_i\rangle| \\ &= \sum_{l\in\bar{S}} |v_l| \sum_{i\in S} |\langle\vec{a}_l, \vec{a}_i\rangle| + \sum_{j\in S} |v_j| \sum_{i\in S, i\neq j} |\langle\vec{a}_j, \vec{a}_i\rangle| \\ &\leq \sum_{l\in S} |v_l|\mu_1 \cdot s + \sum_{j\in S} |v_j|\mu_1(s-1) \\ \|\vec{v}_S\|_1 &\leq \mu_1 \cdot s\|\vec{v}_{\bar{S}}\| + \mu_1(s-1)\|\vec{v}_{\S}\| \end{aligned} \tag{22}$$

$$(1 - \mu_1(s-1))\|\vec{v}_{\bar{S}}| \ \leq \mu_1 \cdot s\|\vec{v}_S\|$$

Since $\mu_1(s-1) + \mu_1(s) < 1$ by hypothesis, so $1 - \mu_1(s-1) \geq \mu_1(s)$ and hence $\|\vec{v}_S\|_1 < \|\vec{v}_{\bar{S}}\|_1$

3.6 Ways to Solve P1

There are 8 algos to solve P1. The worst performing one is Linear programming.

This is one of the Algos

3.6.1 Algos

$A = \begin{bmatrix} 1 & 1 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$a_{11} = a_{12} = 1$

$Q = \begin{bmatrix} \frac{1}{w_1} & 1 \\ 0 & \frac{1}{w_2} \end{bmatrix}$

1. Minimize $\|\vec{x}_1\|$ subject to $\vec{y} = A\vec{x}$

$$\begin{aligned} \vec{y} =& (AA^T)(AA^T)^{-1}\vec{y} \\ \vec{y} =& A(A^T(AA^T)^{-1}\vec{y}) \end{aligned} \tag{23}$$

Why not let $\vec{x} = (A^T(AA^T)^{-1}\vec{y})$
maybe we can do better.
$\vec{y} = AQA^T(AQA^T)\vec{y}$
Why not let $\vec{x} = (QA^T(AQA^T)^{-1}\vec{y})$
How to choose Q?

1. $min \sum_{i=1}^{N} W_i x_i^2$ subject to $\vec{y} = A\vec{x}$

   This is not the \$l$_1$\$-norm but it would be if $w_i = \frac{1}{|x_i|}$.

   solve 2. then substitute w_i

2. min: $w_1 x_1^2 + w_2 + x_2^2$ subject to $y = a_{11}x_1 + a_{12}x_2$

   $f(x_1) = w_1 x_1^2 + w_2(y - x_1)^2$

   $f'(x_1) = 0$ solve for x$_1$

   $2w_1 x_1 + 2(y - x_1)(-1)w_2 = 0$

   $x_1 = \frac{w_2}{w_1+w_2} y,\ x_2 = \frac{w_1}{w_1+w_2} v$

$$\begin{aligned} AQA^T =& \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{w_1} & 0 \\ 0 & \frac{1}{w_2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ =& \frac{w_1 + w_2}{w_1 w_2} \end{aligned} \tag{24}$$

$$QA^T(AQA^T)^{-1}y = \begin{bmatrix} \frac{1}{w_1} \\ \frac{1}{w_2} \end{bmatrix} \frac{w_1 w_2}{w_1 + w_2} y \tag{25}$$

4 Sparse Representation pt 2 (2020/04/21)

4.1 Historical Perspective

Why is the visual system so powerful? Hypothesis is our brain uses sparse representation of Visual Data.

Let a picture $\vec{y} = c_1\vec{b}_1 + ... + c_n\vec{b}_n$
so that most c_j are zero.
Sparse representation used to be called Sparse Coding.
Robust Facial Recognition uses Sparse Subspace Clustering.
Given 19 x 19 images, let $Y = [\vec{Y}_1|...|\vec{Y}_{45}],\ \vec{y}_j \in R^{361}$
19 * 19 = 361
Given Y, solve for matrix C

$$Y = YC,\ diag(C) = \vec{0}$$

Since we don't want $Y_i = Y_i$, that is why the constraint $diag(C) = \vec{0}$ is introduced. It ensures that a group of vectors can be a linear combination of others.

Each column of C is sparse since we want all column vectors to be a linear combination of a smaller set of columns.

4.2 Example - Handwritten Digit Recognition

Given 28 x 28 images, Let $B = [\vec{y}_1|...|\vec{y}_{4000}]$ where each $\vec{y}_j \in R^{784}$

- 800 images of 0, 1-800
- 800 images of 1, 801-1600
- 800 images of 2, 1601-2400
- 800 images of 3, 2401-3200
- 800 images of 8, 3201-4000

Let $\vec{f}$ be a new image of 2. Solve for X such that $\vec{f} = B\vec{x}$
Assume $\vec{x}$ is 20-sparse.
We would like to see the only **non-zero** entries at position 1601-2400.
Columns outside the range may be non-zero as well. There is a 95% probability that a digit will be 2, 5% it will be another digit.

4.2.1 Qualitative Theorem

Given $A^{m \times N}$ with $m << N$. If A is a Gaussian random matrix, then with overwhelming high probability, it satisfies some Exact Recovery Condition for s-sparse Vectors.

For most large undetermined systems of linear equations, the minimal \$$l_1$\$-norm solution is also the sparsest solution.

Topics of Research:

- Theory of Random Matrices
- Banach Spaces

4.3 Solving P1 solves P0. Why?

PO

Find the s-sparse $\vec{x} \in R^N$ such that $\vec{y} = A\vec{x}$.

P1

$A \in R^{m \times N}$ and measurement $\vec{y} \in R^m$. Solve optimization problem,

$$\min_{x \in R^N} \|x\|_1$$

subject to the constraint $y = A\vec{x}$

Suppose $\vec{y} = A\vec{x}$ and $\vec{y} = A\vec{z}$. Suppose $\vec{x}$ is a sparse vector and $\vec{z}$ is **not**.

We want to show that $\|\vec{x}\|_1 < \|\vec{x}\|_1$ - Null Space property of order S

$\|\vec{x}\|_1 = \|\vec{x} - \vec{z}_S + \vec{z}_S\|_1$ - $\vec{z}$ restricted to some Set S. (Subtract 0 so we can use triangle inequality).

Let $\vec{v} = \vec{x} - \vec{z},\ \vec{v} \in Ker(A)$

$A(\vec{x} + \vec{z}) = A\vec{v} = \vec{0}$

$$\begin{aligned}
\|\vec{x}\|_1 \leq & \|\vec{x} - \vec{z}_S\|_1 + \|z_S\|_1 && \text{(26a)}\\
= & \|\vec{v}_S\|_1 + \|\vec{z}_S\|_1 && \text{(26b)}\\
< & \|\vec{z}_S\|_1 + \|\vec{v}_{\bar{S}}\|_1 \quad \text{via Null Space Property} && \text{(26c)}\\
= & \| - \vec{z}_{\bar{S}}\|_1 + \|z_S\|_1 \quad \|x_{\bar{s}}\|_1 = 0 \text{ since x is sparse} && \text{(26d)}\\
= & \|\vec{z}\|_1 && \text{(26e)}
\end{aligned}$$

4.4 Adjoint

Let $T\colon V \to W$. For example, T can be a matrix from R^3 to R^2. In this case, V is R^3 and W is R^2

We write T^* for the adjoint of T.

$$\forall x \in V,\ \forall y \in W,\ \langle Tx, y\rangle = \langle x, T^* y\rangle$$

Horrible way to think of it, when T is a matrix, the adjoint is the same as the transpose.

Q: When A is an orthogonal matrix, what is A^*A? I

Hint: each column has l_2-norm 1, distinct cols are perpendicular.

Q: When A is an orthogonal matrix, why is $\|Ax\|_2 = \|x\|_2$ for every vector x? (This is known as an isometry)

$$\|Ax\|_2^2 = \langle Ax, Ax\rangle = \langle x, A^*Ax\rangle = \langle x, x\rangle = \|x\|_2^2$$

This shows that $\|Ax\|_2^2$ is not too different than $\|x\|_2^2$

4.5 Restricted Isometry Property (RIP)

$A \in R^{m\times N}$ satisfies the restricted isometry property of order s and level δ_s $(0 < \delta_s \leq 1)$

$$(1-\delta_s)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta_s)\|x\|_2^2,\ \forall \text{ s-sparse } x \in R^N$$

Any s columns of the matrix A are **nearly** orthogonal to each other.

Q: What can we say about $|\langle (I - A^*A)x, x\rangle|$ when vector is s-sparse?
This is a small number.
Let $u,\ v \in R^N$ and $S \in \{1, 2, 3, ..., N\},\ |S| = s$
What can we say about the following?

$$|\langle u, (I - A * A)v\rangle|$$

We would like to be able to say
$|\langle u, (I - A^*A)v\rangle| \leq \delta_t \|u\|_2\|v\|_2$

4.5.1 How to think about RIP?

Suppose A satisfies the restricted isometry property of order s.

Intuition: **Hopefully**, the matrix A^*A behaves like the Identity Matrix. $(I - A^*A)$ is small.

If you take some s-sparse vector $\vec{x}$ and multiply it by $I - A^*A$, hopefully, the resulting vector will also be small.

4.5.2 Algorithm

Consider the following vectors,

$$\vec{x}_1 = \begin{bmatrix} 10 \\ -20 \\ 3 \\ -4 \\ 5 \\ -6 \\ -7 \\ 8 \\ 4 \end{bmatrix}, \ \vec{x}_2 = \begin{bmatrix} 10 \\ -20 \\ 0 \\ 0 \\ 0 \\ 0 \\ -7 \\ 8 \\ 0 \end{bmatrix}$$

Hard Threshold

$\tau_s(\vec{x})$ is the vector that keeps the s entries that are the largest in Absolute Value.

Example: When $s = 4,\ \tau_s(\vec{x}_1) = \vec{x}_2$

$\tau_s(\cdot)$ is an operator that takes a vector and will output a sparse vector.

$$\begin{aligned} \vec{u}_n =& \vec{x}_n + A^*(\vec{y} - A\vec{x}_n), \text{ where } \vec{y} = A\vec{x} && \text{(27a)} \\ =& \vec{x}_n + (A^*A\vec{x} - A^*A\vec{x}_n) && \text{(27b)} \\ =& (I - A^*A)\vec{x}_n + A^*A\vec{x} && \text{(27c)} \end{aligned}$$

- expect $\vec{u}_n$ close to $\vec{x}$
- however, $\vec{u}_n$ may not be sparse. Thus use $\tau_s(\cdot)$

  Iterative Hard Thresholding

$$\vec{x}_{n+1} = \tau_x(\vec{x}_n + A^*(\vec{y} - A\vec{x}_n))$$

4.6 Operator Norm

$\|A\| = \underset{x\neq 0}{max} \frac{\|Ax\|_2}{\|x\|_2}$

How much influence does A have on a vector x? Shrink, stretch, compress?

Describes how big a matrix is. If A is 2 x 3, then take $\vec{x} \in R^3,\ x \neq 0$

What is

$$\|A\| = max\{\|Ax\|_2 \colon \|x\|_2 = 1\}$$

4.6.1 Inner Product

Let A be a matrix . The inner product of two vectors Ax and y has this property,

$$|\langle Ax, y\rangle| \leq \|A\| \cdot \|x\|_2 \|y\|_2$$

Where $\|A\|$ is the operator norm of A.

By Cauchy-Schwartz Inequality,

$$\|\langle Ax, y\rangle\| \leq \|Ax\|_2 \cdot \|y\|$$

By def,

$$\|Ax\| \leq \|A\| \cdot \|x\|_2$$

Thus,

$$\|\langle Ax, y\rangle\| \leq \|A\| \cdot \|x\|_2 \cdot \|y\|_2$$

5 Sparse Representation Pt 3 (2020/04/28)

5.1 Expanding on RIP

Expanding upon RIP

Any S columns of the matrix A are nearly orthogonal to each other.

5.2 Expanding on IHT

Expanding upon the IHT Algorithm,

$\tau_x(\cdot)$ is an non-linear operator that outputs a sparse matrix. The operator is non-linear because it does not *change* the dimensions on the vector. i.e. $R^n \to R^n$. You will not be able to find a matrix that will return the same output as this operator.

$\tau_s(\vec{x}_1) = x_2$

Which means both $\vec{x}_1$ and $\vec{x}_2$ have an inner product.

The IHT algorithm is described below:

$$\vec{u}_n = \vec{x}_n + A^*(\vec{y} - A\vec{x}_n), \text{ where } \vec{y} = A\vec{x} \tag{28a}$$

$$= \vec{x}_n + (A^*A\vec{x} - A^*A\vec{x}_n) \tag{28b}$$

$$= (I - A^*A)\vec{x}_n + A^*A\vec{x} \tag{28c}$$

We expect $\vec{u}_n$ is close to $\vec{x}$.

What does it mean for a matrix A to be small? matrix A is small when $A\vec{x}$ is small.

5.3 IHT Proof

Suppose A satisfies RIP of order 3s with

$$\delta_{3s} < \frac{1}{2}$$

δ_{3s}: relaxation.

$3s$: every 3s columns need to be orthogonal

$\frac{1}{2}$: how far from orthogonality the difference can be.

Then the sequence $\{\vec{x}_n\}$ defined by

$$\vec{x}_{n+1} = \tau_S(\vec{x}_n + A^*(\vec{y} - A\vec{x}_n))$$

will converge to $\vec{x}$

Note: 3s-sparse vectors and s-sparse vectors are **not** the same.

5.3.1 How to think about this?

u and v are 2s-sparse.

Let S_1 be the support of u. Meaning $S_1 = \{j : u(j) \neq 0\}$

Let S_2 be the support of v.

Let S be the union of S_1 and S_2. Assume $|S| = 3s$
If A satisfies RIP of order 3s. Then
$|\langle u, (I - A^*A)v\rangle| \le \delta_{3s}\|u\|_2 \cdot \|v\|_2$

$$\begin{aligned} \|\langle u, (I - A^*A)\rangle\| &\le \|u\|_2 \; \|v(I - A^*A)\|_2 && (29a) \\ &\le \|u\|_2 \; \|v\delta_{3s}\|_2 && (29b) \\ &\le \delta_{3s} \; \|u\|_2 \; \|v\|_2 && (29c) \end{aligned}$$

5.3.2 Explanation: Why is the theorem true?

We want to find a constant $\lambda,\ 0 \le \lambda < 1$ s.t.

$$\|x_{n+1} - x\|_2 \le \lambda\|x_n - x\|_2,\ \forall\ n = 1, 2, 3, ...$$

Why?

$$\begin{aligned} \|x_4 - x\|_2 \le& \lambda\|x_3 - x\|_2 \\ \|x_3 - x\|_2 \le& \lambda\|x_2 - x\|_2 \\ \|x_2 - x\|_2 \le& \lambda\|x_1 - x\|_2 \end{aligned} \qquad (30)$$

Therefore,

$$\|x_4 - x\|_2 \le \lambda^{n-1} \; \|x_1 - x\|_2 \qquad (31)$$

In general,

$$\|x_{n+1} - x\|_2 \le \lambda^{n-1} \; \|x_1 - x\|_2 \qquad (32)$$

as $n \to \infty,\ \lambda \to 0$ (because $\lambda < 1$)

$$\vec{x}_{n+1} = \tau_S(\vec{x}_n + A^*(\vec{y} - A\vec{x}_n))$$

and

$$x_{n+1} = \tau_S(u_n)$$

$x_{n+1},\ x$ are s-sparse.
Key Observation: Which one (x_{n+1} or x) is a better approximation to u_n?

x_{n+1}
Thus,

$$\|u_n - x_{n+1}\|_2^2 \leq \|u_n - x\|_2^2 \tag{33}$$

What is $u_n - x$?

$$\begin{aligned} u_n - x =& x_n + A^*A(x - x_n) - x && \text{(34a)} \\ =& (I - A^*A)x_n + (A^*A - I)x && \text{(34b)} \\ =& (I - A^*A)(x_n - x) && \text{(34c)} \end{aligned}$$

What is $u_n - x_{n+1}$?

$$\begin{aligned} \|u_n - x_{n+1}\|_2^2 =& \|u_- x_{n+1} - (x - x)\|_2^2, && \text{subtract 0} \quad \text{(35a)} \\ =& \|(u_n - x) - (x_{n+1} - x)\|_2^2, && \text{square of l2 norm os inner product} \quad \text{(35b)} \\ =& \langle (u_n - x) - (x_{n+1} - x), (u_n - x) - (x_{n+1} - x)\rangle && \text{(35c)} \\ =& \|u_n - x\|_2^2 - 2\langle u_n - x, x_{n+1} - x\rangle + \|x_{n+1} - x\|_2^2 && \text{(35d)} \end{aligned}$$

From the above two formulas, we getattr

$$-2\langle u_n - x, x_{n+1} - x\rangle + \|x_{n+1} - x\|_2^2 \leq 0 \tag{36}$$

This is the same as

$$\|x_{n+1} - x\|_2^2 \leq 2\langle u_n - x, x_{n+1} - x\rangle$$

What is $u_n - x$?

$$u_n - x = (I - A^*A)(x_n - x)$$

$$\langle u_n - x, x_{n+1} - x\rangle = \langle (I - A^*A)(x_n - x), x_{n+1} - x\rangle$$

Thus,

$$u = x_{n-x}, \;\; v = x_{n+1} - x$$

Why? $x_n - x$ is 2s-sparse and $x_{n+1} - x$ is also 2s-sparse.

We have shown that

$$\langle u_n - x, x_{n+1} - x \rangle \leq \delta_{3s} \|x_n - x\|_2 \cdot \|x_{n+1} - x\|_2$$

$$\begin{aligned} \|x_{n+1} - x\|_2^2 &\leq 2\delta_{3s} \|x_n - x\|_2 \cdot \|x_{n+1} - x\|_2 \\ \|x_{n+1} - x\|_2 &\leq 2\delta_{3s} \cdot \|x_n - x\|_2 \end{aligned} \tag{37}$$

The hypothesis is $\delta_{3s} < \frac{1}{2}$ and so $0 \leq \lambda < 1$

$$\|x_{n+1} - x\|_2 \leq \lambda \|x_n - x\|_2 \tag{38}$$

Explanation succeeded

5.4 Convex Functions

Pick any norm, $\| \cdot \|_1$, $\| \cdot \|_2$

We have the triangle inequality

$$\|x + y\| \leq \|x\| + \|y\| \tag{39}$$

Suppose we define $f(x) = \|x\|$ for any $x \in R^d$ and $0 \leq \theta \leq 1$.

$$\begin{aligned} f(\theta x = (1 - \theta)y) =& \| + (1 - \theta)y\| \leq \|\theta x\| + \|(1 - \theta)y\| \\ =& \theta \|x\| + (1 - \theta)\|y\| \end{aligned} \tag{40}$$

Hence, $f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$ so $f(x)$ is a convex function.

5.5 Convex Optimization

Suppose you have a convex function defined over a convex set C, and you want to find the minimum of the function over the set C.

What do you have? A convex optimization problem!

Let $f(x)$ be a convex function over R^d. Minimize $f(x)$ subject to $Ax = b$.

The domain D is the set of $x \in R^d$ such that $Ax = b$.

If $Ax = b$, and $Ay = b$, then $A(tx + (1 - t)y) = b$. Thus D is a convex set.

If x and y are both in D, then the line segment joining x and y is entirely in D.

5.6 Why is convex optimization important?

Fundamental property of Convex optimization:

Any local minimum of a convex function f over a convex set C **must** also be a global minimum of f over C.

6 Gradient Descent (2020/05/05)

6.1 Method of Steepest Descent

Let $x \in R^3,\ y \in R^3$. these are column vectors in R^3

$$\begin{aligned} f(x) =& f(x_1, x_2, x_3) \\ f(y) =& f(y_1, y_2, y_3) \\ G(y) =& G(y_1, y_2, y_3) \end{aligned} \tag{41}$$

$\nabla f(x)$ is a gradient vector. The convention is that the gradient is a **row** vector.

$G(y) = f(y) - \nabla f(x)y$

$$\begin{aligned} \nabla f(x) \equiv& (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}) \\ \nabla f(x)y =& \frac{\partial f}{\partial x_1}y_1 + \frac{\partial f}{\partial x_2}y_2 + \frac{\partial f}{\partial x_3}y_3 \\ =& \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \end{aligned} \tag{42}$$

6.1.1 Warm Up

$\nabla G(y) = \nabla[f(y) - \nabla f(x)y] = \nabla f(y) - \nabla f(x)$

We assume

$$f(x) - f(y) - \nabla f(y)(x - y) \leq \frac{b}{2}\|x - y\|_2^2$$

> This assumption drives from Taylor's Theorem where the Hessian Matrix (Matrix of 2ND Derivatives) is bounded by the largest Eigenvalue.

For any given x, consider the function

$$G(y) = f(y) - \nabla f(x) y$$

G is convex.
$G(y) \equiv G_x(y)$ because G depends on x.
Suppose x is the minimizer of $G(y)$

$$G(x) \leq G(y - \frac{1}{b}\nabla G(y))$$

and

$$\nabla G(y) = \nabla[f(y) - \nabla f(x) y] = \nabla f(y) - \nabla f(x)$$

We assume $f(x)$ is C^1 and satisfies the condition:

$$\forall x,\ y,\ f(x) - f(y) \leq \nabla f(y)(x-y) + \frac{b}{2}\|x-y\|_2^2$$

C^1: continuously differentiable.

$G(y-a) - G(y)$
Let $x = y - a, a = \frac{1}{b}\nabla G(y)$

When making an assumption, make an assumption that allows you to learn something interesting.

$$\begin{aligned}
&\leq \nabla G(y)(x-y) + \frac{b}{2}\|x-y\|_2^2 \\
&= \nabla G(y)(-a) + \frac{b}{2}\|x-y\|_2^2 \\
&= \nabla G(y)(-\frac{1}{b}\nabla G(y)^T) + \frac{b}{2}\ \frac{1}{b^2}\|\nabla G(y)\|_2^2
\end{aligned} \tag{43}$$

We just demonstrated

$$\begin{aligned}
&G(y - \frac{1}{b}\nabla G(y)) - G(y) \\
&\leq \nabla G(y)(-\frac{1}{b}\nabla G(y)^T) + \frac{b}{2}\ \frac{1}{b^2}\|\nabla G(y)\|_2^2
\end{aligned} \tag{44}$$

6.1.2 Proving Gradient Descent

$$\nabla G(y) = \nabla[f(y) - \nabla f(x)y] = \nabla f(y) - \nabla f(x)$$

$$\begin{aligned}
\rightarrow & f(x) - f(y) - \nabla f(x)(x-y) && (45a)\\
= & f(x) - \nabla f(x)x - (f(y) - \nabla f(x)y) && (45b)\\
= & G(x) - G(y) && (45c)\\
= & G(y - \frac{1}{b}\nabla G(y)) - G(y) && (45d)\\
\leq & \nabla G(y)(-\frac{1}{b}\nabla G(y)^T) + \frac{b}{2}\ \frac{1}{b^2}\|\nabla G(y)\|_2^2 && (45e)\\
= & -\frac{1}{2b}\|\nabla G(y)\|_2^2 && (45f)\\
= & -\frac{1}{2b}\|\nabla f(x) - \nabla f(y)\|_2^2 && (45g)
\end{aligned}$$

[g] says

$$f(x) - f(y) - \nabla f(x)(x-y) \leq -\frac{1}{2b}\|\nabla f(x) - \nabla f(y)\|_2^2$$

We define a sequence of vectors

$$x_{k+1} = x_k - \frac{1}{b}g_k$$

$x_{k+1} = x_k - \frac{1}{b}\nabla f(x_k)$
Using $1\overline{}$ $\frac{}{bis\mathbf{Bold}.Theoldstyleupdatedthestepateachiterationwhichresultsinlessiterationsbutmorecompute.}$
$h = \frac{1}{b}$
Let us write

$$\begin{aligned}
d_k = & x_k - x^* \\
\text{How far the current estimate is from the minimum} & \\
\delta_k = & f(x_k) - f(x^*) \\
\text{Actual Error} &
\end{aligned} \quad (46)$$

Thus,

$$d_{k+1} = x_{k+1} - x^*$$

Apply [g] with $x = x_k,\ y = x^*$

$$
\begin{aligned}
&f(x_k) - f(x^*) - g_k^T(x_k - x^*) \leq -\frac{1}{2b}\|\nabla f(x_k) - \nabla f(x^*)\|_2^2 \\
&\to \delta_k \leq g_k^T d_k - \frac{1}{2b}\|g_k\|_2^2
\end{aligned} \tag{47}
$$

because $g_k = \nabla f(x_k)$ and $d_k = x - x^*$
G: scalar everything else: vector
Look Closer!

$$x_{k+1} - x_k = -\frac{1}{b}g_k$$

<= Using $x_{k+1} - \frac{1}{b}g_k$
$g_k = -b(x_{k+1} - x_k)$

$$
\begin{aligned}
\delta_k \leq & g_k^T d_k - \frac{1}{2b}\|g_k\|_2^2 && \text{(48a)}\\
= & -b(x_{k+1} - x_k)^T d_k - \frac{b}{2}\|x_{k+1} - x_k\|_2^2 && \text{(48b)}\\
= & -\frac{b}{2}(\|x_{k+1} - x_k\|_2^2 + 2(x_{k+1} - x_k)^T d_k) && \text{(48c)}\\
= & -\frac{b}{2}(\|d_{k+1} - d_k\|_2^2 + 2(d_{k+1} - d_k)^T d_k) && \text{(48d)}\\
= & \frac{b}{2}(\|d_k\|_2^2 + \|d_{k+1}\|_2^2) && \text{(48e)}\\
= & \|d_{k+1} - d_k\|_2^2 + 2(d_{k+1} - d_k)^T d_k && \text{(48f)}\\
= & (\langle d_{k+1}, d_{k+1}\rangle - 2\langle d_{k+1}, d_k\rangle + \langle d_k, d_k\rangle) + (2d_{k+1}^T d_k - d_k^T d_k) && \text{(48g)}
\end{aligned}
$$

why [f]?

To summarize,

$$\delta_k \leq \frac{b}{2}(\|d_k\|_2^2 - \|d_{k+1}\|_2^2)$$

$$\sum_{i=1}^{n} \delta_i \leq \frac{b}{2}(\|d_0\|_2^2 - \|d_n\|_2^2 \leq \frac{b}{2}\|d_0\|_2^2$$

What do we know about convergent series?
If $\sum_{k=1}^{\infty} \delta_k$ is convergent, then $\delta_k \to 0$ as $k \to \infty$

6.2 Global Convergence

Start with any x_0. We define the sequence of vectors

$$x_{k+1} = x_k - \frac{1}{b} g_k$$

$x_{k+1} = x_k - \frac{1}{b}\nabla f(x_k)$
Then, $f(x_k) - f(x^*) \to 0$ as $k \to \infty$
We can pick N as large as we want,

$$\sum_{k=0}^{N} \delta_k \leq \frac{b}{2}\|d_0\|_2^2$$

Recall that $g_k \equiv \nabla f(x_k)$ and $g_{k+1} \equiv \nabla f(x_{k+1})$
We can also show that $\|g_{k+1}\| \leq \|g_k\|$
The length of the gradient vectors are monotone decreasing.
We've shown that

$$f(x) - f(y) - \nabla f(x)(x-y) \leq -\frac{1}{2b}\|\nabla f(x) - \nabla f(y)\|_2^2$$

Similarly,

$$f(y) - f(x) - \nabla f(y)(y-x) \leq -\frac{1}{2b}\|\nabla f(x) - \nabla f(y)\|_2^2$$

Summing the above inequalities yields

$$-\nabla f(x)(x-y) - \nabla f(y)(y-x) \leq -\frac{1}{b}\|\nabla f(x) - \nabla f(y)\|_2^2$$

which means,

$$(\nabla f(x) - \nabla f(y))(x-y) \geq \frac{1}{b}\|\nabla f(x) - \nabla f(y)\|_2^2 \quad **$$

Let $x = x_{k+1},\ y = x_k$. Then, from (**),

$$(x_{k+1} - x_k)^T(g_{k+1} - g_k) \geq \frac{1}{b}\|g_{k+1} - g_k\|_2^2$$

But $x_{k+1} = x_k - \frac{1}{b}g_k$ so that

$$-\frac{1}{b}(g_k)^T(g_{k+1} - g_k) \geq \frac{1}{b}\|g_{k+1} - g_k\|_2^2$$

$$\begin{aligned} \|g_{k+1}\|_2^2 \leq & g_{k+1}^T g_k && (49a) \\ \leq & \|g_{k+1}\|\|g_k\| && \text{By Cauchy-Schwartz} \quad (49b) \end{aligned}$$

Why is a true?

That means, $\|g_{k+1}\| \leq \|g_k\|$, which is the desired conclusion

6.3 About Gradient Descent

Gradient Descent is *not* a single method. It is a large collection of methods.

1. Steepest Descent with a constant step size

$$x_{k+1} = x_k - h\nabla f(x_k)$$

2. Use a different step size at each iteration

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k)$$

6.3.1 Example

Select α_k to minimize $f(x_k - d_k g_k)$, where $g_k = \nabla f(x_k)$. Lots of algorithms to choose α_k

We assume $f(x)$ is C^1 and satisfies

$$f(x) - f(y) \leq \nabla f(y)(x-y) + \frac{b}{2}\|x-y\|_2^2$$

If we assume f is convex, differentiable, and its gradient vector satisfies the Lipshitz Condition

$$\|\nabla f(x) - \nabla f(y)\| \leq b\|x-y\|$$

for any two points $x,\ y$, then the condition (*) is true.

6.4 Challenge

We have already demonstrated

$$\sum_{i=1}^{100} \delta_i \leq \frac{b}{2}\|d_0\|_2^2$$

and $\|g_{k + 1}\| \leq \|g_k\|$. Our notation is $\delta_k = f(x_k) - f(x^*)$
You can show that the rate of convergence is given by

$$\delta_k \leq (\frac{1}{k+1})\frac{b}{2}\|d_0\|_2^2$$

TODO: Prove this out.