Time Series Analysis Class Notes

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Contents

1 Characteristics of Time Series (2020/01/09)

- Must be correlation between data points which limits conventional statistical analysis.
- One variable, x_t, will be used in this course

Important Questions to Ask

- What patterns are visible over time?
- How can correlation between observations be used to help with the model?
- Can future state be predicted using this data?

Problem: We don't know how many previous time points should be used to predict the current value.

General Tips

- if non-constant variance, transform the predictors
- Find assumptions, then continue modeling
- Time is generally treated as discrete values instead of continuous

Stochastic Process: collection of random variables, x_t, indexed by t

- **Realization**: Realization of a stochastic process.

Time Series: collection of randome variables indexed and ordered by time

White Noise: $w_t \sim N(0, \sigma_w^2)$

One way to "smooth" a time series is to introduce a moving average.

MA(1): $x_t = \beta w_{t-1} + w_t$

AR(1): $x_t = \beta x_{t-1} + w_t$

$$\begin{aligned} E(x_t) =& E(\beta X_{t-1} + w_t) \\ =& \beta E(x_{t-1}) + E(w_t) \\ =& ... \\ =& 0 \end{aligned} \tag{1}$$

- $0 \leq \beta \leq 1$

$\gamma(s,t) = cov(x_s, x_t) = E[(x_s - \mu_s)(x_t - \mu_t)] \forall\ s, t$

if s == t, $cov(x_s, x_s) = var(x_s)$

$\gamma(s,t) = \begin{cases} \sigma_w^2 & s = t \\ 0 & s \neq t \end{cases}$

- given $w_t \sim ind\ N(0, \sigma_w^2)$

1.1 Moving Average

Let $m_t = \frac{w_t + w_{t-1} + w_{t-2}}{3}$

$$\begin{aligned} E[(m_s - \mu_s)(m_t - \mu_t)] =& E(m_s m_t) \\ =& \frac{1}{9} E[(w_s + w_{s-1} + w_{s-2})(w_t + w_{t-1} + w_{t-2})] \end{aligned} \tag{2}$$

<u>s = t</u>

$$\begin{aligned} E(m_t^2) =& var(m_t) + E(m_t)^2 \\ =& \frac{1}{9} var(w_t + w_{t-1} + w_{t-2}) + 0 \\ =& \frac{1}{9}(var(w_t) + var(w_{t-1} + var(w_{t-2}))) \\ =& \frac{1}{9}(1 + 1 + 1) \\ =& \frac{3}{9} \end{aligned} \tag{3}$$

s = t - 1: $E(m_{t-1}, m_t) = \frac{2}{9}$
s = t - 2: $E(m_{t-2}, m_t) = \frac{1}{9}$
s = t - 3: $E(m_{t-3}, m_t) = 0$

$$\gamma(s,t) = \begin{cases} \frac{3}{9} & s = t \\ \frac{2}{9} & |s-t| = 1 \\ \frac{1}{9} & |s-t| = 2 \\ 0 & |s-t| \geq 3 \end{cases}$$

1.2 Autocorrelation

$\rho_{xy} = \frac{cov(x,y)}{\sqrt{var(x)}\sqrt{var(y)}}$

AR: $\rho(s,t) = \begin{cases} 1 & s = t \\ 0 & s \neq t \end{cases}$

MA: $\rho(s,t) = \begin{cases} 1 & s = t \\ \frac{2}{3} & |s-t| = 1 \\ \frac{1}{3} & |s-t| = 2 \\ 0 & |s-t| \geq 3 \end{cases}$

positve linear dependence = smooth negative linear dependence = choppy

1.3 Stationarity

Strict stationary time series: the probabalistic behavior of $x_t, ..., x_{tk}$ os the exact same as the shifted set $x_{t+h}, ..., x_{tk+h}$ for any collection of time points $[t_1, t_k]$ for any $k = 1, 2,... $

$P(x_q \leq c_1, x_2 \leq c_2) = P(x_{10} \leq c_q, x_{11} \leq c_2)$

This is almost never used in practice because it is *too* strict.

Weakly Stationary Time Series: The first two moments (mean, covariance) of the time series are invariant to time shifts

$E(x_t) = \mu \forall t$

$\gamma(t, t+h) = \gamma(0, h) \forall t$

- μ and $\gamma(0, h)$ are *not* functions of t
- Assumption of **Equal Variance**
- $\gamma(h) = \gamma(-h)$ if weakly stationary

$$\begin{aligned}\rho(t,t+h) &= \frac{\gamma(t,t+h)}{\sqrt{\gamma(t,t)}\sqrt{\gamma(t+h,t+h)}} \\ &= \frac{\gamma(h)}{\sqrt{\gamma(0)}\sqrt{\gamma(0)}} \\ &= \frac{\gamma(h)}{\gamma(0)}\end{aligned} \tag{4}$$

Is there a correlation between lags? $H_0: \ \rho(h) = 0$ $H_A: \ \rho(h) \neq 0$
Sample Mean: $\bar{x} = \frac{1}{n}\Sigma x_t$
Sample Covariance: $\gamma(\hat{h}) = \frac{1}{n}\sum_{t=1}^{n-h}(x_{t+h} - \bar{x})(x_t - \bar{x})$