Class Notes

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Contents

1 Review & Introduction (2020/04/02)

Monte Carlo Simulations: A family of computational algorithms using repeated sampling to get numerical results.

Applications

- get deterministic results
- Approximate High Dimension Integration

1.1 Review

Note that since every class has a review period for the first lecture, the notes documented here represent points that were either stressed or that I found particularly interesting

1.1.1 Distributions by Question

Binomial

- How *many* basketball free throws do you make out of a given number of attempts?
- How *many* people prefer the iPhone to other smart phones?

Poisson

- How *many* taxis pass by your corner in a given time?
- How *many* servers crash in a given time?

Gamma Distribution

- How *long* does it take for the next *several* taxis to pass by your corner?
- How *long* does it take for the next *several* servers to crash in a given time?

Gaussian

- How *far* does a stock price move in a given period of time?
- Describe averages

1. Gamma

   Time it takes for the next several taxis to pass by.

   $T\ Gamma(\alpha, \beta)$

   β: Average waiting between taxis

   α: number of taxis

   > I noted this because I haven't used the Gamma distribution too much and I thought this was an intuitive way to describe it.

1.1.2 Independence

If $Y_1 \sim N, Y_2 \sim N$, they are Bivariate Normal

If $\sigma_{12} = 0$, Y_1, Y_2 are independent since $\sigma_{12} = cov(Y_1, Y_2) = 0$.

1. Proof

   Any Bivariate Normal Random Var can be written as a linear function of two independent Normal R.V.s.

$$\begin{aligned} x_1 =& z_1 \\ x_2 =& \sigma_{12} z_1 \pm z_2 \sqrt{1 - \sigma_{12}^2} \end{aligned} \tag{1}$$

$$\begin{aligned} cov(x_1, x_2) &= cov(z_1, \sigma_{12} z_1 \pm z_2 \sqrt{1 - \sigma_{12}^2}) \\ &= cov(z_1, \sigma_{12} z_1 \pm z_2 \sqrt{1 - \sigma_{12}^2}) \\ &= cov(z_1, \sigma_{12} z_1) + cov(z_1, z_2 \sqrt{1 - \sigma_{12}^2}) \\ &= \sigma_{12} V(z_1) + 0 \\ &= \sigma_{12} \end{aligned} \tag{2}$$

   if $\sigma_{12} \neq 0$, x,y are **not** independent

   if $\sigma_{12} = 0$,

   - $x_1 = z_1$
   - $x_2 = z_2$

   This implies that x_1 and x_2 are independent.

1.2 Introduction

Metropolis Sampling - important method in Bayesian Statistics

Y represents some interesting quantity

- result of a game
- payoff of a derivative option
- daily profit

- time taken to travel by car to work

Compute the mean, $E(Y) = \mu$

- probability of winning
- fair price of a derivative option purchased today
- average daily profit

Or Y can be a percentile

The idea is to generate samples of Y with the *same* distribution to compute the sample mean, percentile as estimates of the true quantities.

1.3 Big Questions

- How do we generate the Y_i with a complicated distribution? Often $Y_i = f(X)$, where $X = (X_{i1}, ..., X_{id})$ is easy to generate and f is known.
- How do we generate X_i above?
- How large does n need to be?
- Can we reduce n(time, cost) by being clever? Yes, by choosing Y_i more carefully?

if $cov(Y_i, Y_j) = \rho$:

$$
\begin{aligned}
V(\bar{Y}) =& \frac{\sigma^2}{n} + \frac{2n(n-1)}{n^2} cov(Y_i, Y_j) \\
=& \frac{\sigma^2}{n} + \frac{n(n-1)}{n^2} \rho\sigma^2
\end{aligned} \tag{3}
$$

1.4 Example: Interest Rate

k = number of times per year interest is compounded r_k = interest rate per year compounded k times per year r_1 = annualized percentage rate (APR) r = interest rate per year compounded continuously

$$
\begin{aligned}
r_1 =& (1 + \frac{r_k}{k})^k - 1 = e^r - 1 \\
r =& k ln(1 + \frac{r_k}{k}) = ln(1 + r_1)
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\lim_{n\to\infty}(1+\frac{1}{n})^n &= e \\
\lim_{n\to\infty}(1+\frac{0.05}{n})^n &= \lim_{n\to\infty}(1+\frac{0.05}{n})^{\frac{n}{0.05}} = e^{0.05}
\end{aligned} \tag{5}
$$

1.5 Example: Estimating Pi

Assume the following:

- a piece of 1 x 1 square wood with a circle in it
- infinite darts

How to estimate the value of π?
Area of square: 1 $r = 0.5$ Area of a circle: $\pi r^2 = \frac{\pi}{4}$
$\hat{\pi} = 4 \times \frac{\text{\# of darts in circle}}{\text{\# of darts in square}}$

1.6 Example: Sandwich Shop Profit

$D_{ij} \sim U(5, ..., 35), i = 1, ..., n, j = 1, ..., d$
j: day i: random variable
profit: $P_{ij} = min(D_{ij}, O)R - OW$
average daily profit over d days: $\bar{P_i} = \frac{1}{d}(P_{i1} + ... + P_{id}), i = 1, ..., n$

$$
\begin{aligned}
\hat{\mu} &= \frac{1}{n}(\bar{P_1} + ... + \bar{P_n}) = \frac{1}{nd}\sum_{i,j=1}^{n,d} P_{ij} \\
\hat{\sigma}^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(\bar{P_i} - \hat{\mu})^2 \\
\hat{\mu} &\pm 2.58\frac{\hat{\sigma}}{\sqrt{n}}
\end{aligned} \tag{6}
$$

2.58 is p = 0.005 for a 99% C.I.

1.6.1 Questions

- What size order gives the maximum average daily profit? Why? Have you tried other order sizes?
- How accurately can you know the average daily profit from the simulation? How does this depend on the number of days for your simulation?

- How does the answer vary as you change your model assumptions?
- Plot daily profit and average daily profit with the number of days

2 Review & Estimating Integrals (2020/04/09)

2.1 Review

$MSE(\hat{\mu}) = Var(\hat{\mu}) + [bias(\hat{\mu})]^2$

$bias(\hat{\mu}) = E(\hat{\mu}) - \mu$

Simple Monte Carlo Simulator: $\hat{\mu} = \bar{Y} = \frac{1}{n}(\Sigma Y_i)$

2.1.1 Chebyshev Inequality

When working with an unknown distribute, the Chebyshev inequality can be used to construct Confidence Intervals (albeit wide).

$$P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

$$P(|Y - \mu| > k\sigma) \leq \frac{1}{k^2}$$

2.1.2 Determining N

$$|\frac{Z_{1-\alpha/2}\hat{\sigma}}{\sqrt{n}}| \leq \epsilon \rightarrow n \geq (\frac{Z_{1-\alpha/2}\hat{\sigma}}{\alpha})^2 \tag{7}$$

$\hat{\sigma}$: Unbiased estimate of σ

α: Error tolerance.

In this class so far, $\alpha = 0.01$

1. Steps

   (a) Choose a small sample size ($n_0 = 1000$). Then generate n_0 random samples from an underlying probability distribution

   (b) Calculate $\hat{\sigma}$

   (c) Calculate n

   (d) Generate another sample of size n from the underlying probability distribution.

   (e) Compute $\hat{\mu}$ with error $\pm Z_{1-\alpha/2}\frac{\hat{\sigma}_n}{\sqrt{n}}$

2.2 Estimating Integrals

$$\mu = \int_{R^d} g(x)dx = ?$$

Let $f(x) = \frac{g(x)}{\rho(x)}$ where $\rho(x)$ is a probability density function (PDF) and $g(x)$ is the function of interest to be estimated.

Then,

$$\mu = \int_{R^d} f(x)\rho(x)dx = E(Y)$$

where $Y = f(X)$

2.2.1 Example - Normal Probability

$$\mu = \int_0^1 \frac{1}{\sqrt{2\pi}} exp(\frac{-x^2}{2})dx = \Phi(1) - \Phi(0)$$

$$RMSE(\hat{\mu}) = \sqrt{Var(\hat{\mu}) + [bias(\hat{\mu})]^2}$$

Summary

Estimator($\hat{\mu}$)	bias($\hat{\mu}$)	Var($\hat{\mu}$)	RMSE($\hat{\mu}$)
$\hat{\mu}_{MC1}$	0	0.0023345 n^{-1}	0.048420 $n^{-\frac{1}{2}}$
$\hat{\mu}_{MC2}$	0	0.22483 n^{-1}	0.47416 $n^{\frac{-1}{2}}$
$\hat{\mu}_{MC3}$	O(n^{-1})	0	O(n^{-1})
$\hat{\mu}_{MC4}$	0	O(n^{-3})	O($n^{\frac{-3}{2}}$)

1. First Estimator - Simple Monte Carlo Estimator
   $f(x) = \frac{1}{\sqrt{2\pi}} exp(\frac{-x^2}{2})$
   $X_i \sim U[0,1]$
   $Y = f(X)$

$$\hat{\mu}_{MC1} = E(Y) = \frac{1}{n}\Sigma f(X_i) = \frac{1}{n}\Sigma \frac{1}{\sqrt{2\pi}} exp(\frac{-X_i^2}{2})$$

$$\begin{aligned} MSE_{MC1} =& Var(\hat{\mu}_{MC1}) + 0 \\ =& \frac{Var(Y)}{N} \\ =& n^1 Var(Y) \propto n^{-1} \\ =& O(n^{-1}) \end{aligned} \tag{8}$$

2. Second Estimator - Standard Normal R.V.

$$f(x) = 1_{[0,1]}(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & else \end{cases}$$

$\mu = E(Y), Y = f(X), X_i \sim N(0,1)$

$$\hat{\mu}_{MC2} = \frac{1}{n}\Sigma Y_i = \frac{1}{n}f(X_i) = \frac{1}{n}1_{[0,1]}(X_i)$$

In this case, $Y \sim Bernoulli(p)$. Thus $E(Y) = p$ and $\bar{Y} = \hat{p}$

$$\begin{aligned} Var(Y) =& p(1-p) \\ =& (\Phi(1) - \Phi(0))(1 - (\Phi(1) - \Phi(0))) \\ =& 0.2248 \end{aligned} \tag{9}$$

$MSE_{MC2} = Var(\bar{Y}) = n^{-1}Var(Y) = 0.2248n^{-1}$

3. Third Estimator - Left Rectangle Rule

Let $x_i = \frac{i-1}{n}$

$$\hat{\mu}_{Rect} = \frac{1}{n}\Sigma\frac{1}{\sqrt{2\pi}}exp(\frac{-x_i^2}{2})$$

Deterministic, thus $Var(\hat{\mu}) = 0$. Not a R.V.

$MSE_{MC3} = (\hat{\mu} - \mu)^2 + 0$

Let error $\epsilon = |\int_0^1 \frac{1}{\sqrt{2\pi}}exp(\frac{-x^2}{2}) - \hat{\mu}|$

Let $k = max|f(x)|$ for $x \in [0,1]$

$\epsilon \leq \frac{k(1-0)}{2n}$

$$\begin{aligned} \mu - \hat{\mu} \leq \frac{k}{2n} =& O(n^{-1}) \\ MSE_{\hat{\mu}} =& O(n^{-2}) \end{aligned} \tag{10}$$

4. Fourth Estimator - Stratified Sampling Estimator

   Simulates a random sample for each stratum.

   Let $x_i = \frac{(i-1+U_i)}{n}$, U_i iid $U[0,1]$

   $$\hat{\mu}_{MC4} = \frac{1}{n}\Sigma\frac{1}{\sqrt{2\pi}}exp(\frac{-x^2}{2})$$

   $MSE = Bias^2 + Var(\hat{\mu}) = 0 + O(n^{-3})$

3 European Call/Put Options (2020/04/16)

3.1 Brownian Motion

3.2 Options

Call Option: Contract that gives the buyer of the option the right to buy an asset at a specific price at a specific time.

Put Option: Contract that gives the buyer the right to sell an asset at a specific price at a specific time.

European Option: This is a type of option that allows execution time to be at the expiration/maturity date.

Strike Price: The predetermined price that the holder can buy or sell.

Premium: Expected value of the return at maturity.

3.2.1 Examples

1. Call Option

   Premium: \$4 Strike price: \$50 Expiration: 3 Months

   Three month

   (a) Stock Market Price = \$100 pay \$4, then can buy for \$50 when its 100
   \|————————————————\| 50 100
   The buyer executes. The return is 100 - 50 - 4 = \$46 dollars

   (b) Stock market price is \$20
   \|————————————————\| 50 20
   The buyer does **not** execute. Buyer loses \$4.

2. Put Option

(a) Stock Price is \$100
\|——————————————\| 50 100
The buyer does **not** execute because selling for \$50 is a loss. Loses \$4.

(b) Stock price is \$20
\|——————————————\| 50 20
The buyer executes. The return is 50 - 20 - 4 = \$26

3.2.2 European Options

t = time in years S(t) = the price of the asset at time t T = time to expiry (maturity) of the contract K = strike price (the price decided at t = 0) r = risk-neutral interest rate

Discounted Euro Call payoff: $max(S(T) - K, 0)e^{-rT}$

Discounted Euro Put payoff: $max(K - S(T), 0)e^{-rT}$

We only need to model $S(T)$ not $S(.)$. The fair call/put option prices are $\mu = E(Y)$, where Y is the discounted call/put payoff.

3.3 Geometric Brownian Motion

A simple model for asset prices

$S(t) = S(0)exp((r - \sigma^2/2)t + \sigma B(t)),\ t \geq 0$

$B_t \sim N(0, t)$: Brownian Motion. This produces wave-like noise that fans wider as t increases.

σ: volatility. Measure the spread of an asset. Determined by no arbitrary principle. i.e. the return cannot be greater than the interest rate if there is no risk.

3.3.1 Properties

- $B(0) = 0$ with probability one
- $B(\tau)$ and $B(t) - B(\tau)$ independent for $0 \leq \tau \leq t$
- $B(t) - B(\tau) \sim N(0, t - \tau) \forall 0 \leq \tau \leq t$
- $cov(B(t), B(\tau)) = min(t, \tau)$ for $0 \leq t, \tau$

3.4 Black-Sholes Formula for Option Prices

Fair European Call Price:

$$S(0)\Phi(\frac{ln(S(0)/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}) - Ke^{-rT}\Phi(\frac{ln(S(0)/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}})$$

$$Ke^{-rT}\Phi(\frac{ln(K/S(0))-(r-\sigma^2/2)T}{\sigma\sqrt{T}}) - S(0)\Phi(\frac{ln(K/S(0))-(r+\sigma^2/2)T}{\sigma\sqrt{T}})$$

3.4.1 Assumptions

1. The stock underlying call/put options provides no dividends during the call/put lifetime.
2. There are no transaction costs for the sale/purchase of stock.
3. Risk free interest rate (r) is constant during the option time

Put-call parity: Fair European call price - fair European put price = $S(0) - k\ exp(-rT)$

3.5 Monte Carlo Computation of European Put

1. Generate $X_1, ..., X_n$ by a normal pseudo-random number generator.
2. Compute the sample ending stock prices: $S_i(T) = S(0)\ exp((r - \sigma^2/2)T + \sigma\sqrt{T}X_i)$
3. Compute sample discounted payoffs, $Y_i = max(K - S_i(T), 0)e^{-rT}$
4. Average the discounted payoffs,

   Fair European Put Price

   $$\mu = E(Y) \approx \frac{1}{n}\sum_{i=1}^{n} max(K - S_i(T), 0)e^{-rT}$$

   Estimated error $= \pm\frac{2.58\hat{\sigma}}{\sqrt{n}}$ where $\hat{\sigma}$ is the sample standard deviation of the discounted payoffs.

4 Linear Congruential Generators (2020/04/23)

4.1 and Inverse Distributions

4.2 Linear Congruential Generators

Random numbers aren't truly random.

> "Anyone who considers arithmetic methods of producing random digits is, of course, in a state of sin." - John Neumann

M: A large Integer
a: large primitive root of M ($amodM \neq 0$)
$i = 1, .., M - 1$
m_0: integer seed

$$m_i = a\ m_{i-1} modM,\ x_i = \frac{m_i}{M},\ i = 1, 2, ...$$

$x_i \neq x_j$ for $j = i + 1, ..., i + M - 2$
$M - 1$: Period

$$\begin{aligned} m_i =& a\ m_{i-1}\ mod\ M \\ =& m_0\ a^i\ mod\ M \end{aligned} \tag{11}$$

a is a primary root of M if a^i mod M > 0 for $i = 1, ..., M - 1$

$$\begin{aligned} m_i =& a[m_o a^{i-1}\ mod\ M]\ modM \\ =& a[m_0 a^{i-1} - (\frac{m_o a^{i-1}}{m}) \cdot M]\ mod\ M \\ =& (m_o a^i - a[\frac{m_o a^{i-1}}{m}]\ M) \\ =& m_o a^i\ mod\ M \end{aligned} \tag{12}$$

4.2.1 Example 1

$M - 1 = 16$
$a = 5$
$m_n = 5\ m_{n-1} mod 16$

$$\begin{aligned} m_0 =& 5 \\ m_1 =& 10 \\ m_2 =& 3 \\ &... \\ m_5 =& 6 \\ m_6 =& 15 \\ &... \end{aligned} \tag{13}$$

$0 \leq \frac{m_i}{16} \leq 1$
at $m_1 6$, it starts over again

period length: any linear congruential generator will eventually repeat itself.

reproducability: Using the same seed can produce the same *random*

4.3 Tests for Pseudo random numbers

A given M may have primary roots, a, but not all may produce good sequences of random numbers.

The numbers should fill the d-dim hypercube.

Spectral Tests

Quantitative measure of how well the points
$(x_i, x_{i+1}, ..., x_{i+d-1})$ fill $[0,1]^d$.

This test, $l(0, M, d)$ is the largest possible distance between planes covering the points.

4.3.1 Collision Test

$Y_1, ..., Y_n$ iid R.V. with the common cumulative prob distr. function F so $x_i = F(Y_i) \sim iidU[0,1]$

$Z_i = (X_{(i-1)d+1}, ..., X_{id}),\ i = 1, ..., k = \frac{n}{d}$

$Z_i \sim iid[0,1]^d$

W = # of Bins with more than one point. (collisions)

Break the cube $[0,1]^d$ into **l** non overlapping Bins

Check if the points are uniformly random.

$\lim_{n \to \infty} W \sim Poisson$

$$\lambda = \frac{k^2}{l}$$

If W is much smaller than λ or much larger than λ, then it is not pseudo random.

4.4 Inverse Distribution

Y with CDF F(Y)

$$0 \leq F(Y) \leq 1$$

Define a new R.V. X: $X = F(Y) \sim Unif(0,1)$

$$\begin{aligned} P(X < x) =& P(F(Y) < X) \\ =& P(Y < F^{-1}(x)) \\ =& F(F^{-1}(x)) \\ =& x \end{aligned} \tag{14}$$

5 GBM Explanation (2020/04/30)

Geometric + Random term to model that the price is always increasing.

GBM model used to simulate stock prices at a given time.

$$S(t) = S(0)exp((-r - \sigma^2/2)t + \sigma B(t)) \tag{15}$$

Random log Return between t1 and t2

$$R(t_1, t_2) = ln(\frac{S(t_2)}{S(t_1)}) = (r - \sigma^2/2)(t_2 - t_1) + \sigma[B(t_2) - B(t_1)] \tag{16}$$

$B(t_2) - B(t_1) \sim N(0, t_2 - t_1)$

Risk Free investment (no volatility or money in the bank) ($\sigma = 0$)

$E(S(t)) = S(0)exp(rt)$

$exp(-\sigma^2 t/2)$: comes from no arbitrage principle. The mean return is the return on a risk-free investment.

Return is a Gaussian random variable. May be positive or negative

$$ln(\frac{S(t+\Delta)}{S(t)}) = (r - \sigma^2/2)\Delta + \sigma[B(t+\Delta) - B(t)] \tag{17}$$

$B(t+\Delta) - B(t) \sim N(0, \Delta)$

GBM $S(t) = S(0)exp((r - \sigma^2/2)t + \sigma B(t))$

$$\begin{aligned} E(S(t)) =& E[S(0)exp((r - \sigma^2/2)t + \sigma B(t))] \\ =& S(0)exp((r - \sigma^2/2)t)E(exp(\sigma B(t))) \end{aligned} \tag{18}$$

The moment generating function for $N(0, \sigma^2)$ is
$M_x(t) = exp(\frac{\sigma^2 t^2}{2})$

$$E[exp(\sigma B(t))] = \sigma^2/2 \tag{19}$$

Cancels out the term in $E[S(t)]$

5.1 Fair European Put Price

E[discounted payoff at time T]

$$\begin{aligned} & E[max(K - S(T,X), 0)e^{-rT}] \\ =& \int_{-\infty}^{\infty} max(K - S(T,X), 0)e^{-rT} dx \\ =& \int_{-\infty}^{\infty} max(K - S(T,X), 0)e^{-rT} f(x)dx \\ =& \int_{-\infty}^{\infty} max(K - S(0)exp((r - \sigma^2/2)T + \sigma\sqrt{T}X), 0)e^{-rT} f(x)dx \\ =& \int_{-\infty}^{X_{hi}} K - S(0)exp((r - \sigma^2/2)T + \sigma\sqrt{T}X)e^{-rT} f(x)dx + \int_{X_{hi}}^{\infty} 0 \\ =& \int_{-\infty}^{X_{hi}} Ke^{-rT} f(x)dx - \int_{-\infty}^{X_{hi}} S(0)exp((r - \sigma^2/2)T + \sigma\sqrt{T}X - rT)f(x)dx \\ =& kexp(-rT)\Phi(X_{hi}) - S(0)\int_{-\infty}^{X_{hi}} exp(r - \sigma^2/2)T + \sigma\sqrt{T}X - \frac{1}{\sqrt{2\pi}} exp(-x^2/2) \\ =& ke^{-rT}\Phi(X_{hi}) - S(0)\int_{-\infty}^{X_{hi}} \frac{1}{\sqrt{2\pi}} exp(-\sigma^2 T/2 + \sigma\sqrt{T}X - X^2/2) \\ =& ke^{-rT}\Phi(X_{hi}) - S(0)\Phi(X_{hi} - \sigma\sqrt{T}) \end{aligned} \tag{20}$$

Need to find out when this becomes 0. $K - S(0)exp((r - \sigma^2/2)T + \sigma\sqrt{T}X) \leq 0$

X_{hi} $X \geq \frac{ln(k/S(0)) - (r - \sigma^2/2)T}{\sigma\sqrt{T}}$

6 Random Number Generation pt 2 (2020/05/07)

6.1 Acceptance-Rejection Method

$Y_i,\ W_i \sim iidR.V$

$Y_i \sim$ common PDF, f_Y

$W_i \sim U[0,1]$

Let $c \leq 1\ s.t.\ \frac{cf_Z(z)}{f_Y(z)}$

f_Z: PDF of the Random variable we want to generate.

We want C to be as close to 1 as possible. It is generally found by calculation

$\frac{1}{c} = sup_z \frac{f_Z(z)}{f_Y(z)} == \frac{f_Y(y)}{c} = supf_Z(y)$

1/c is the largest value of the ratio between $f_Z(y),\ f_Y(y)$

sup = supremum = maximum of a given set

$k = 0, ..., m$ for \$i = 1,2,. . . \$

if $W_i \leq \frac{cf_Z(Y_i)}{f_Y(Y_i)},\quad k = k+1,\quad Z_k = Y_i$

$Z_1, ..., Z_n$ iid with PDF f_Z

Goal: Want to simulate a random sample from Z with known PDF $f_Z(y)$

What we know

We can simulate random samples $Y_1, ..., Y_n$ from $f_Y(y)$ since $f_Y(y)$ is known and has the same support as Z. The support being the domain of a R.V where the pdf is non-zero. For example, values using the Beta distr. PDF is between 0 and 1, so is $U[0,1]$

We can simulate random samples from a Uniform Distribution:

$W_1, ..., W_n \sim U[0,1]$

6.1.1 Proof

How do we know whether the accepted samples are sufficient for a random sample?

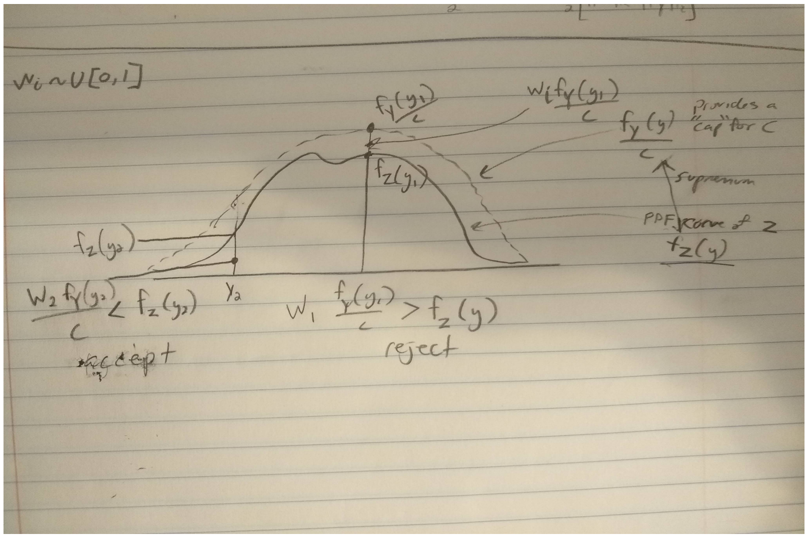

Figure 1: Acceptance-Rejection method using two distributions

$$\lim_{\Delta \to 0} \frac{P(Y \in [y, y+\Delta] | \text{ Y accepted to be Z})}{\Delta} = f_Z(y)$$

$$\lim_{\Delta \to 0} \frac{P(Y \in [y, y+\Delta] \cap \text{ Y accepted to be Z})}{\Delta \cdot P(\text{Y accepted to be Z})}$$

$$\lim_{\Delta \to 0} \frac{f_Y(y) \times P(W \leq c f_Z(y)/f_Y(y))}{\Delta \cdot c}$$

Using definition for P(Y is Z)

$$\lim_{\Delta \to 0} \frac{f_Y(y) \times c f_Z(y)/f_Y(y)}{c} = f_Z(y) \tag{21}$$

$$
\begin{aligned}
P(\text{Y accepted to be Z}) &= \lim_{n\to\infty}\sum_{i=1}^{n} P(w \le \frac{cf_Z(y_i)}{f_Y(y_i)})f_Y(y_i)\cdot\Delta \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} P(\text{accept } y_i|y_i)\cdot P(y_i) \\
&= \int_\Omega \frac{cf_Z(y_i)}{f_Y(y_i)}\cdot f_Y(y_i)dy \\
&= c\int_\Omega f_Z(y)dy \\
&= c\cdot 1 = c
\end{aligned} \tag{22}
$$

Ω: Support of Y and Z

- You **must** know the PDF function $f_Y,\ f_Z$ explicitly
- Generating \$Y$_1$, Y$_2$,...\$ with PDF f_Y may be done using the inverse transformation method.

6.1.2 Application

Generate a Sequence of 1000 random numbers.

$$f_Z(z) = 20z(1-z)^3,\ 0<z<1,\ z\sim Beta(\alpha=2,\beta=4)$$

$f_Y \sim U[0,1]$

1. The candidate distribution $f_Y(y)=1,\ 0<y<1$
2. What is the value of C?

$$\frac{1}{c} = sup\ \frac{f_Z(y)}{f_Y(y)}$$

$Q = \frac{f_Z(y)}{f_Y(y)} = \frac{20z(1-z)^3}{1}$. Need to find max of Q

$$\frac{dQ}{dz} = 20z(1-z)^3 - 60z(1-z)^2 \tag{23a}$$

$$= (1-z)^2(20(1-z)-60z) \tag{23b}$$

$$= (1-z)^2(20-80z) = 0 \tag{23c}$$

$$z = 1, \frac{1}{4} \tag{23d}$$

Since $\frac{1}{4}$ is the smallest,

$$\frac{20(0.25)(1-0.25)^3}{1} = \frac{135}{64} \to \frac{1}{c} \to c = \frac{64}{135}$$

3. How many random samples are required?

   Let N be the number of iterations required.

   $E(N) = \frac{1000}{c} = \frac{1000 \cdot 135}{64} = \frac{135000}{64}$

   $N = 1.1 \cdot E(N) = 2321$ random samples

4. How to simulate

   (a) Sim N random samples from $U[0,1]$ Y

   (b) Sim N random samples from $U[0,1]$ W

   (c) Make decision. if $W_i < \frac{cf_Z(y)}{f_Y(y)}$ then reject

6.1.3 Normal Distribution

$f_Z(z) = \frac{1}{\sqrt{2\pi}} exp(-z^2/2), \quad -\infty \leq z \leq \infty$

Want to simulate a Normal Z but we need to find a candidate distribution with the same support. No other distributions have the same support as the Normal but many have the support $0 \leq y < \infty$.

$|z| < \infty$

$$\begin{aligned} P(|z| \leq z) =& P(-z \leq Z \leq z) \\ =& \Phi(z) - \Phi(-z) \\ =& \Phi(z) - (1 - \Phi(z)) \\ =& 2\Phi(z) - 1 \end{aligned} \tag{24}$$

$f_{|z|}(z) = \frac{2}{\sqrt{2\pi}} exp(-z^2/2)$

$P(z < 0) = P(z \geq 0) = 0.5$

We will need to simulate random samples from $|z|$.

1. What is the candidate distr. for Y?

$$Y \sim exp(1), \quad f_Y(z) = exp(-z), \ 0 \leq z < \infty$$

2. What is the value of c?

$$Q = \frac{1}{c} = \frac{f_Z(z)}{f_Y(z)} = \frac{\frac{2}{\sqrt{2\pi}}exp(-z^2/2)}{exp(-z)} = \frac{2}{\sqrt{2\pi}}exp(-z^2/2 + z)$$

Maximize Q which means maximize $-z^2/2 + z$ (z = 1)

$Q = \frac{2}{\sqrt{2\pi}}exp(0.5) = \sqrt{\frac{2e}{\pi}} = \frac{1}{6}$

$c = \sqrt{\frac{\pi}{2e}}$

$Q = \frac{1}{c} = \frac{cf_Z(y)}{f_Y(y)} = sqrt\frac{\pi}{2e} \cdot \sqrt{\frac{2}{\pi}}exp(-z^2/2 + z) = exp(-z^2/2 + z - 0.5)$

$c\frac{f_Z(y)}{f_Y(y)} = exp(-\frac{y^2}{2} + y - 0.5) = exp(-\frac{1}{2}(y-1)^2)$

3. How many random samples?

$E(N) = \frac{1000}{c} \approx 1316$

$N = 1.1 \cdot 1316 = 1448$ random samples

4. Simulate

   (a) Simulate R.S $U[0, 1]$ W

   (b) Simulate \$.S from $Y \sim exp(1)$ Inverse transformation method yields $Y_i = -log(x_i)$

$$\begin{aligned} F_Y(y) =& 1 - e^{-1} \\ F_Y^{-1}(x) =& -log(x_i) OR \; -log(1 - x_i) \end{aligned} \tag{25}$$

   (c) If $W_i \leq exp(-0.5(y-1)^2)$, accept.

   (d) Simulate R.S $V_i \sim U[0, 1]$
   $Z_k = sign(V_i - 0.5)Y_i$

6.2 Brownian Motion Time Differencing

Sometimes instead of a scalar, we want to generate random functions, B.

6.2.1 Properties

- $B(0) = 0$
- $B(\tau)$ and $B(t) - B(\tau)$ are indep for all $0 \leq \tau \leq t$
- $B(t) - B(\tau) \sim N(0, t - \tau)$
- $B(t)$ and $B(\tau)$ are <u>not</u> independent. $cov(t, \tau) = min(t, \tau) = \tau$
- May be generated at discrete times, $0 = t_0 < t_1 < ... < t_\alpha = T$
  $B(0) = 0, B(t_k) = B(t_{k-1}) + X_k\sqrt{t_k - t_{k-1}}, k = 1, ..., d$
  $X_1, ..., X_d$ are iid.

6.2.2 Linear Interpolation

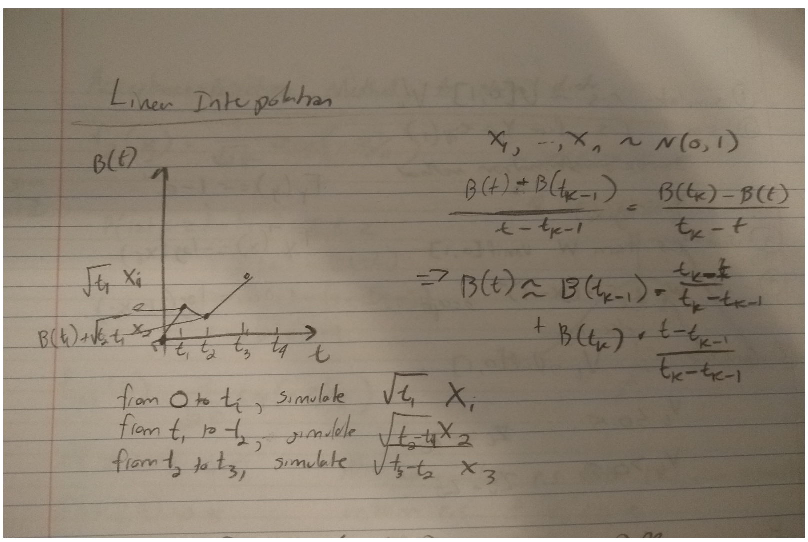

Figure 2: Linear Interpolation

6.2.3 Generating Brownian Sample Paths

d: number of time nodes

$$0, \frac{T}{d}, \frac{2T}{d}, ..., \frac{(d-1)T}{d}, \frac{dT}{d} = T$$

to generate sample paths of brownian motion.

1. Generate d standard normal random numbers $X_1, ..., X_d$
2. Brownian motion at time $\frac{kT}{d}, k = 1, 2, ..., d$

Relationship between Brownian Motion and Geometric Brownian Motion

They are the same thing.

Brownian Motion

$$S(T) = S(0)exp(rT - T\frac{\sigma^2}{2} + \sigma B(T))$$

Geometric Brownian Motion

$$S(T) = S(0)exp(T(r - \frac{\sigma^2}{2}) + \sigma B(T))$$

7 Types of Options (2020/05/14)

7.1 Vanilla European Call/Put Option

Asset path not important for European options

Review

$$S(t) = S(0)exp(r - \sigma^2/2)t + \sigma B(t) = S(0)exp(r - \sigma^2/2)t + \sigma\sqrt{t}X$$

$\sqrt{T}X \approx B(t)$
where $X \sim N(0, 1)$
$E(S(t)) = S(0)exp(rt)$

$$\begin{aligned}
S(\frac{t}{d}) =& S(0)exp(r - \sigma^2/2)\frac{t}{d} + \sigma\sqrt{\frac{t}{d}}X_1 \\
S(\frac{2t}{d}) =& S(0)exp(r - \sigma^2/2)\frac{2t}{d} + \sigma\sqrt{\frac{t}{d}}(X_1 + X_2) \\
S(\frac{3t}{d}) =& S(0)exp(r - \sigma^2/2)\frac{3t}{d} + \sigma\sqrt{\frac{t}{d}}(X_1 + X_2 + x_3) \\
&... \\
S(t) =& S(0)exp(r - \sigma^2/2)t + \sigma B(t) = S(0)exp(r - \sigma^2/2)t + \sigma\sqrt{t}X
\end{aligned} \tag{26}$$

More generically,

$$S(\frac{kt}{d}) = S(0)exp(r - \sigma^2/2)\frac{kt}{d} + \sigma\sqrt{\frac{t}{d}}\sum_1^k X_i$$

where $\sum_i^k X_i \sim N(0, \frac{kt}{d})$

d: Number of increments. It can be years, days, hours, etc. In previous discussions, d was *years*. In this case, it is *days*.

Simulation Steps

1. Simulate independent Std. Norm R.V.s $n \times d$
2. Generate Brownian Motion Path for each sample $\sqrt{\frac{t}{d}} \cdot cumsum(x)$
3. Generate Geometric Brownian Motion using formula above (asset price path) payoff $= max(K - S(t), 0)$ discounted payoff $= max(K - S(t), 0)exp(-rt)$

7.2 Asian Call/Put Options

Uses the average price of the asset from purchase to maturity instead of the asset price at maturity.

European Option = Arithmetic or Geometric Asian Option where d = 1.

$\bar{S}_{geo} \leq \bar{S}_{ari}$

7.2.1 Arithmetic Mean

call payoff $= max(\frac{1}{d}\sum_{j=1}^d S(\frac{jT}{d}) - K, 0)$

put payoff $= max(K - \frac{1}{d}\sum_{j=1}^d S(\frac{jT}{d}), 0)$

European options will be a higher payout because of higher volatility.

7.2.2 Geometric Mean

$\sqrt{ab}$

$$\bar{S}_{geo} = [\Pi_1^d S(\frac{jT}{d})]^{1/d}$$

$$
\begin{aligned}
S(\frac{T}{d}) =& S(0)exp((r-\sigma^2/2)\frac{T}{d}+\sigma\sqrt{\frac{T}{d}}X_1) \\
S(\frac{2T}{d}) =& S(0)exp((r-\sigma^2/2)\frac{2T}{d}+\sigma\sqrt{\frac{T}{d}}(X_1+X_2)) \\
&... \\
S(T) =& S(0)exp((r-\sigma^2/2)T+\sigma\sqrt{\frac{T}{d}}\sum_1^d X_i)
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
[\Pi_1^d S(\frac{jT}{d})]^{1/d} =& [S(0)exp(d(r-\sigma^2/2)\frac{(1+2+...+d)T}{d}+\sigma\sqrt{\frac{T}{d}}[X_1+(d-1)X_2+(d-2)X_3+...+X_d \\
=& S(0)[exp((r-\sigma^2/2)\frac{d(d+1)}{d}\cdot\frac{T}{d}+\sigma\sqrt{\frac{T}{d}}W)]
\end{aligned} \tag{28}
$$

$W = dX_1 + (d-1)X_2 + ... + x_0 d \sim N(0, \frac{d(d+1)(d+2)}{6})$
$V(W) = d^2 + (d-1)^2 + ... + 1^2 = \frac{d(d+1)(d+2)}{6}$
We replace W with $\sqrt{\frac{d(d+1)(d+2)}{6}}X$
$T = (\frac{d+1}{2d})T = (\frac{1}{2}+\frac{1}{d})T \leq T$
$\bar{\sigma}^2 = \frac{\sigma^2(2+\frac{1}{d})}{3} = (\frac{2}{3}+\frac{1}{3d})\sigma^2 \leq \sigma^2$
$r = r + (\bar{\sigma}^2 - \sigma^2)/2 \leq r$

7.3 Barrier Options

There may be a barrier price, b, which one must cross to trigger the option (one way or another).

7.3.1 Knock-in Barrier Option

> Knock-out is the opposite of Knock-in. I.e. option no longer valid if the barrier is crossed.

1. Up-and-in If barrier is met during the time to maturity, the option is activated.
2. Down-and-out

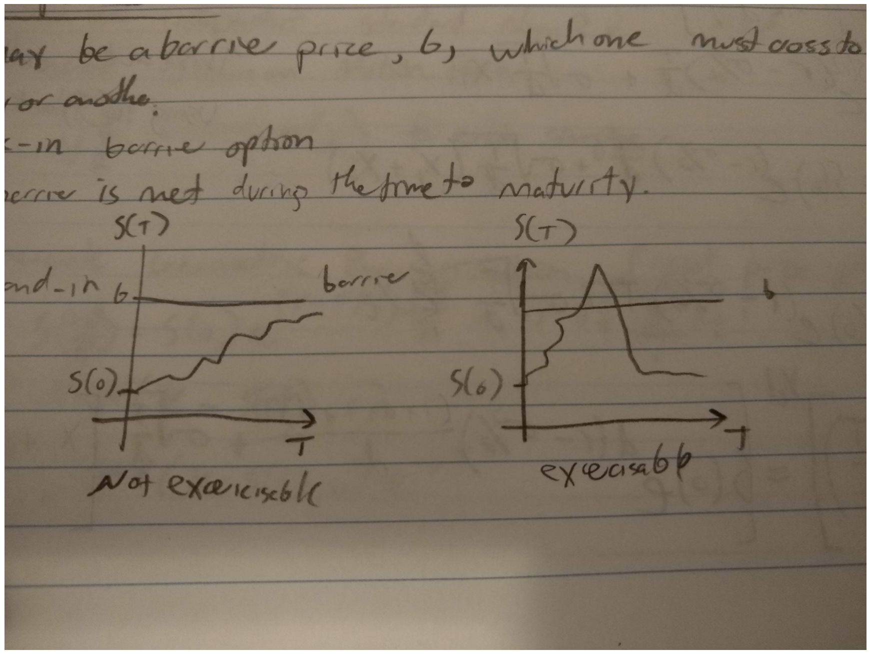

Figure 3: Barrier Options: Up and In

7.4 Lookback Options

Minimum or maximum value of the asset price before expiry acts as a strike price.

Call Payoff

$$S(T) - \min_{j=1,\dots,d} S(jT/d) \text{ or } S(T) - \min_{0<t\leq T} S(t)$$

Put Payoff

$$\max_{j=1,\dots,d} S(jT/d) - S(T) \text{ or } \max_{0<t\leq T} S(t) - S(T)$$

8 Control-Variate Method for Efficiency (2020/05/21)

Suppose we want to estimate estimate $\mu = E(Y)$.

Possible estimators

- $\bar{Y}$

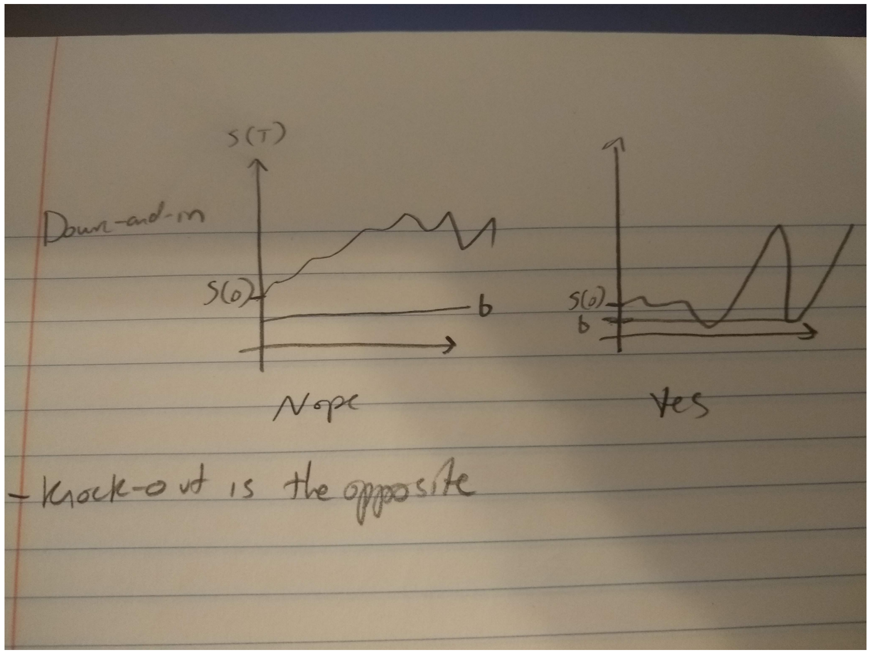

Figure 4: Barrier Options: Down and Out

- Y_1
- $Y_1 + \frac{Y_2+Y_3}{2}$

$\hat{\mu}$ is a point estimator of μ.
How do we evaluate a point estimator? MSE

$$\begin{aligned} MSE =& E((\hat{\mu} - \mu)^2) \\ =& Var(\hat{\mu}) + Bias^2(\hat{\mu}) \end{aligned} \tag{29}$$

Let Y be a R.V. whose μ you want to estimate. Let X be a R.V. whose μ_x you already know.

Example

A student takes two tests. We would like to estimate the score of the second test. We know the distribution and score of the first test. The second test is *correlated* with the score of the first. For example, if they do well on the first exam, it is likely that they will do well on the second exam.

$X \sim N(85, 5^2),\ W \sim N(0, 3^2)$
$Y = X + W$

X in this case is the **variate**. The **Control Variate** means Control X.

Let $(X_i, Y_i),\ i = 1, ..., n$ be iid draws of (X, Y) which are not independent of each other.

$$\hat{Y} = \bar{Y} + \beta(\mu_X - \bar{X})E(\hat{Y}) = \mu$$

Without X, $\bar{Y}$ is an unbiased estimator for μ.
$V(\bar{Y}) = \frac{\sigma^2}{n}$
$V(\hat{Y}) = V(\bar{Y}) + \beta(\mu_X - \bar{X})$
We would like to choose β such that $Var(\hat{Y})$ is minimized.

$$\begin{aligned} \hat{Y} =& \frac{Y_1 + ... + Y_n}{n} + \frac{\beta(\mu_X - (x_1 + ... + x_n))}{n} \\ =& \frac{1}{n}\sum_1^n Y_i + \beta(\mu_X - X_i) \end{aligned} \tag{30}$$

Adjust $\mu_X - X_i$ to be closer to the mean. This reduces variance of the point estimator.

$$\begin{aligned} V(\hat{Y}) =& V(\bar{Y} + \beta(\mu_X - \bar{X})) \\ =& V(\bar{Y}) + V(\beta(\mu_X + \bar{X})) + 2cov(\hat{Y}, \beta(\mu_X - \bar{X})) \\ =& \frac{\sigma^2}{n} + \beta^2\frac{\sigma_X^2}{n} - 2\beta cov(\bar{Y}, \bar{X} - \mu_X) \end{aligned} \tag{31}$$

$$\begin{aligned} cov(\bar{Y}, \bar{X}) =& cov(\frac{Y_1 + ... + Y_n}{n}, \frac{X_1 + ... + X_n}{n}) \\ =& \frac{n}{n^2}cov(X, Y) \\ =& \frac{cov(X, Y)}{n} = \frac{corr(X, Y)\sigma_x\sigma_y}{n} \end{aligned} \tag{32}$$

$$\frac{\partial V(\hat{Y})}{\partial\beta} = \frac{2\beta\sigma_X^2}{n} - \frac{2corr(X, Y)\sigma_x\sigma_Y}{n} = 0$$

$\beta = \frac{corr(X,Y)\sigma_X\sigma_Y}{\sigma_X} = \frac{cov(X,Y)}{\sigma_X^2} = \frac{\sigma_{xy}^2}{\sigma_x^2}$

$$
\begin{aligned}
V(\hat{Y}) =& \frac{\sigma_Y^2}{n} + (\frac{\sigma_{xy}^2}{\sigma_X}^{2})^2 \frac{\sigma_X^2}{n} - \frac{2\sigma_{xy}^2}{\sigma_X^2} - \frac{\sigma_{xy}^2}{n} \\
=& \frac{\sigma_Y^2}{n} + \frac{\sigma_{xy}^4}{\sigma_X}^{4} \cdot \frac{\sigma_X^2}{n} - \frac{2\sigma_{xy}^4}{n\sigma_X^2} \\
=& \frac{\sigma_Y^2}{n} + \frac{\sigma_{xy}^4}{n\sigma_X^2} \\
=& \frac{\sigma_Y^2}{n}[1 - \frac{\sigma_{xy}^4}{\sigma_X^2\sigma_Y^2}] = \frac{\sigma_Y^2}{n}[1 - corr(X, Y)]
\end{aligned}
\tag{33}
$$

The larger the correlation, the **better**.
$V(\bar{Y}) = \frac{\sigma_Y^2}{n}$

$$
RMSE(\hat{Y}) = \sqrt{Var(\hat{Y})} = \frac{\sigma_Y}{\sqrt{n}}[1 - corr^2(X, Y)]^{1/2}
$$

Choosing the Control Variate, X, to mark Corr(X, Y) as close to ± 1 as possible.

$$
\begin{aligned}
\hat{Y} =& \bar{Y} + \hat{\beta}(\mu_X - \bar{X}) = \bar{Y} + \frac{(\mu_X - \bar{X}) \sum_1^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_1^n (X_i \bar{X})^2} \\
=& \sum_1^n W_i Y_i \text{ where } w_i = \frac{1}{n} + \frac{(\mu_X - \bar{X})(X_i - \bar{X})}{\sum_1^n (X_i - \bar{X})^2}
\end{aligned}
\tag{34}
$$

If $X_i < \bar{X}$ and $\mu_X > \bar{X}$, then opposite side of $\bar{X}$ has a negative weight. Otherwise, its positive.

Remarks

- X,Y must be highly correlated
- All pairs of X,Y must be independent
- Reduces MSE but doesn't necessarily lead to a more accurate estimator.