Quiz 5

Dustin Leatherman

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1 1

Reggie Jackson hit Y = 10 home runs in N = 27 World Series games. Assume $Y|\lambda \sim Poisson(N\lambda)$ where λ is his World Series home run rate.

1.1 a

The conjugate prior is $\lambda \sim Gamma(a, b)$. Give the posterior under this prior.

Posterior of Poisson-Gamma: $\lambda|Y \sim Gamma(a + Y, b + N)$

$$\lambda|Y \sim Gamma(10 + a, 27 + b)$$

1.2 b

The Jeffreys prior is $\pi(\lambda) \propto \lambda^{-1/2}$. Give the posterior under this prior.

$$\begin{aligned}
\lambda|Y &\propto \lambda^{Y} e^{-\lambda} \cdot \lambda^{-1/2} \\
&\propto \lambda^{Y-1/2} e^{-\lambda} \\
&\propto \lambda^{(Y+1/2)-1} e^{-\lambda} \\
&\sim Gamma(Y + \frac{1}{2}, 1) \sim Gamma(\frac{11}{2}, 1)
\end{aligned} \tag{1}$$

1.3 c

> Give one situation where you would use the conjugate prior over the Jeffreys prior. Defend your answer.

The conjugate prior would be beneficial over the Jeffreys prior if the number of home runs over the regular season games were known. The parameters a and b typically represent the prior number of events and the prior observation time respectively so the conjugate model is more useful to use if this information is known.

1.4 d

> Give one situation where you would use the Jeffreys prior over the conjugate prior. Defend your answer.

The only parameter in the Posterior derived from Jeffreys Prior is Y which is known in this case. This is useful to use if there is no prior knowledge to incorporate into the model and the researcher wants to use an objective prior.

2 2

> Assume $Y_1, ..., Y_n$ are independent and $Y_1, ..., Y_n|\sigma^2 \sim N(0, \sigma^2)$. The prior is $\pi(\sigma^2) \propto 1$ for all $\sigma^2 > 0$.

2.1 a

> Assuming large n, derive the posterior distribution of σ^2.

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\begin{aligned}
f(\sigma^2|Y_1, ..., Y_n) \propto & f(Y_1|\sigma^2) \cdot ... \cdot f(Y_n|\sigma^2) f(\sigma^2) \\
\propto & (\sigma^2)^{-1/2} e^{-\frac{1}{2\sigma^2} y_1^2} \cdot ... \cdot (\sigma^2)^{-1/2} e^{-\frac{1}{2\sigma^2} y_n^2} \cdot 1 \\
\propto & (\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n Y_i^2} \\
\propto & (\sigma^2)^{-(\frac{n}{2}-1)-1} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n Y_i^2} \\
\sim & InvGamma(\frac{n}{2} - 1, \frac{1}{2} \sum_{i=1}^n Y_i^2)
\end{aligned} \tag{2}
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2.2 b

> Give a value of n so that for all n at least as large as this value, the posterior in (a) is guaranteed to be a proper distribution. Defend your answer.

To be a proper Gamma and Inverse Gamma distribution, the parameters a and b must both be positive. Since a is the only parameter using n, it is the only one that needs to be considered when determining a lower bound for n.

$$\begin{aligned}\frac{n}{2} - 1 &> 0 \\ \frac{n}{2} &> 1 \\ n &> 2\end{aligned} \tag{3}$$

For all n $>$ 2, the Posterior distribution will be valid.