Homework 2

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1 Problem Statement

Find the point on the sphere $x^2 + y^2 + z^2 = 9$ that are closest to and farthest away from the point (2,3,4).

Hint: Construct and use a Jacobian Matrix. i.e gradient vector(s)

2 Solution

Let the function to be minimized be represented as

$$f(x, y, z) = (x-2)^2 + (y-3)^2 + (z-4)^2 = d^2$$

where d^2 is distance squared. $f(x, y, z)$ represents the distance squared between (x,y,z) and (2,3,4).

Let the constraint function be represented as

$$g(x, y, z) = x^2 + y^2 + z^2 = 9$$

The Jacobian Matrix is a matrix of partial derivatives with $N = 3$ columns and $m = 1$ rows. Let the Jacobian matrix for $f(x, y, z)$ be described as

$$\nabla f(x, y, z) = \begin{bmatrix} \frac{\partial f(x,y,z)}{\partial x} & \frac{\partial f(x,y,z)}{\partial y} & \frac{\partial f(x,y,z)}{\partial z} \end{bmatrix} = \begin{bmatrix} 2(x-2) & 2(y-3) & 2(z-4) \end{bmatrix}$$

and the Jacobian Matrix for $g(x,y,z)$ be

$$\nabla g(x,y,z) = \begin{bmatrix} \frac{\partial g(x,y,z)}{\partial x} & \frac{\partial g(x,y,z)}{\partial y} & \frac{\partial g(x,y,z)}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}$$

at the closest value for $g(x,y,z) = 9$.

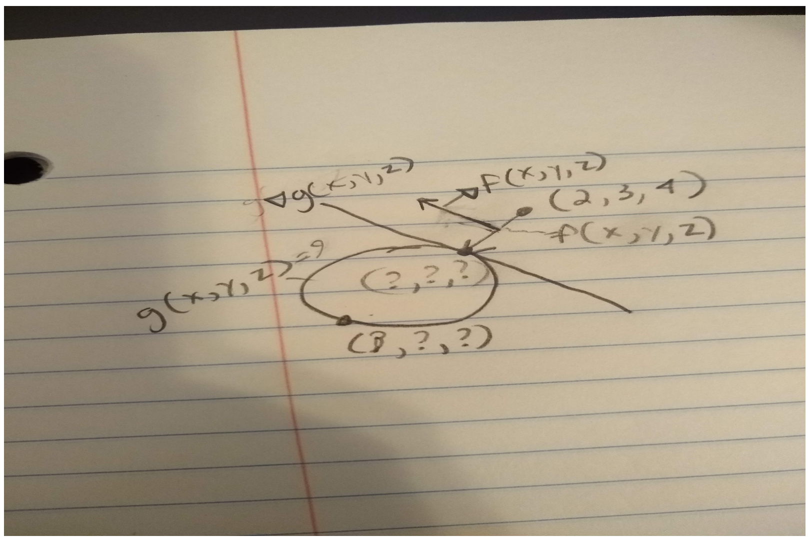

Figure 1: 2-D sketch of the problem

Since a function and its gradient vector are perpendicular, it can be shown that ∇f and ∇g are parallel. Thus it can be said, $\nabla f = \lambda \, \nabla g$ where λ is some constant.

This can be represented by the system of equations

$$\begin{aligned} 2(x-2) &= 2x\lambda && \text{(1a)} \\ 2(y-3) &= 2y\lambda && \text{(1b)} \\ 2(z-4) &= 2z\lambda && \text{(1c)} \\ x^2+y^2+z^2 &= 9 && \text{(1d)} \end{aligned}$$

Solving (1a) – (1c) in terms of λ will then be plugged into (1d) to determine the points on $g(x,y,z) = 9$ closest and furthest from (2,3,4).

1a

$$
\begin{aligned}
2(x-2) =& 2x\lambda \\
x-2 =& x\lambda \\
\frac{2}{1-\lambda} =& x
\end{aligned} \tag{2}
$$

1b

$$
\begin{aligned}
2(y-3) =& 2y\lambda \\
y-3 =& y\lambda \\
\frac{3}{1-\lambda} =& y
\end{aligned} \tag{3}
$$

1c

$$
\begin{aligned}
2(z-4) =& 2z\lambda \\
z-4 =& z\lambda \\
\frac{4}{1-\lambda} =& z
\end{aligned} \tag{4}
$$

1d

$$
\begin{aligned}
(\frac{2}{1-\lambda})^2 + (\frac{3}{1-\lambda})^2 + (\frac{4}{1-\lambda})^2 =& 9 \\
\frac{4}{(1-\lambda)^2} + \frac{9}{(1-\lambda)^2} + \frac{16}{(1-\lambda)^2} =& 9 \\
\frac{29}{(1-\lambda)^2} =& 9 \\
(1-\lambda)^2 =& \frac{29}{9} \\
1-\lambda = \frac{\sqrt{29}}{3} \text{ OR } 1-\lambda = -\frac{\sqrt{29}}{3} \\
1 \pm \frac{\sqrt{29}}{3} =& \lambda
\end{aligned} \tag{5}
$$

Using (1), let point P be a point on $g(x,y,z) = 9$ where
For (2,3,4) and $\lambda = 1 - \frac{\sqrt{29}}{3}$,

$$\begin{aligned}
&P(2x\lambda, 2y\lambda, 2z\lambda) \\
&= P(2(2)(1-\frac{\sqrt{29}}{3}), 2(3)(1-\frac{\sqrt{29}}{3}), 2(4)(1-\frac{\sqrt{29}}{3})) \\
&= P(4-\frac{4\sqrt{29}}{3}, 6-2\sqrt{29}, 8-\frac{8\sqrt{29}}{3})
\end{aligned} \tag{6}$$

For (2,3,4) and $\lambda = 1 + \frac{\sqrt{29}}{3}$,

$$\begin{aligned}
&Q(2x\lambda, 2y\lambda, 2z\lambda) \\
&= Q(2(2)(1+\frac{\sqrt{29}}{3}), 2(3)(1+\frac{\sqrt{29}}{3}), 2(4)(1+\frac{\sqrt{29}}{3})) \\
&= Q(4+\frac{4\sqrt{29}}{3}, 6+2\sqrt{29}, 8+\frac{8\sqrt{29}}{3})
\end{aligned} \tag{7}$$

Since P is smaller than Q, P is the closest point to (2,3,4) on the circle $x^2+y^2+z^2=9$ and Q is the furthest.