

Class Notes

Dustin Leatherman

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1 Review & Introduction (2020/04/02)

Monte Carlo Simulations: A family of computational algorithms using repeated sampling to get numerical results.

Applications

- get deterministic results
- Approximate High Dimension Integration

1.1 Review

Note that since every class has a review period for the first lecture, the notes documented here represent points that were either stressed or that I found particularly interesting

1.1.1 Distributions by Question

1. Binomial

- How *many* basketball free throws do you make out of a given number of attempts?
- How *many* people prefer the iPhone to other smart phones?

2. Poisson

- How *many* taxis pass by your corner in a given time?
- How *many* servers crash in a given time?

3. Gamma Distribution

- How *long* does it take for the next *several* taxis to pass by your corner?
- How *long* does it take for the next *several* servers to crash in a given time?

4. Gaussian

- How *far* does a stock price move in a given period of time?
- Describe averages

5. Gamma

Time it takes for the next several taxis to pass by.

$$T \sim \text{Gamma}(\alpha, \beta)$$

β : Average waiting between taxis

α : number of taxis

I noted this because I haven't used the Gamma distribution too much and I thought this was an intuitive way to describe it.

1.1.2 Independence

If $Y_1 \sim N, Y_2 \sim N$, they are Bivariate Normal. If $\sigma_{12} = 0$, Y_1, Y_2 are independent since $\sigma_{12} = \text{cov}(Y_1, Y_2) = 0$.

1. Proof

Any Bivariate Normal Random Var can be written as a linear function of two independent Normal R.V.s.

$$\begin{aligned}x_1 &= z_1 \\x_2 &= \sigma_{12}z_1 \pm z_2\sqrt{1 - \sigma_{12}^2}\end{aligned}\tag{1}$$

$$\begin{aligned}\text{cov}(x_1, x_2) &= \text{cov}(z_1, \sigma_{12}z_1 \pm z_2\sqrt{1 - \sigma_{12}^2}) \\&= \text{cov}(z_1, \sigma_{12}z_1 \pm z_2\sqrt{1 - \sigma_{12}^2}) \\&= \text{cov}(z_1, \sigma_{12}z_1) + \text{cov}(z_1, z_2\sqrt{1 - \sigma_{12}^2}) \\&= \sigma_{12}V(z_1) + 0 \\&= \sigma_{12}\end{aligned}\tag{2}$$

if $\sigma_{12} \neq 0$, x, y are **not** independent

if $\sigma_{12} = 0$,

- $x_1 = z_1$
- $x_2 = z_2$

This implies that x_1 and x_2 are independent.

1.2 Introduction

Metropolis Sampling - important method in Bayesian Statistics

Y represents some interesting quantity

- result of a game
- payoff of a derivative option
- daily profit

- time taken to travel by car to work

Compute the mean, $E(Y) = \mu$

- probability of winning
- fair price of a derivative option purchased today
- average daily profit

Or Y can be a percentile

The idea is to generate samples of Y with the *same* distribution to compute the sample mean, percentile as estimates of the true quantities.

1.3 Big Questions

- How do we generate the Y_i with a complicated distribution? Often $Y_i = f(X)$, where $X = (X_{i1}, \dots, X_{id})$ is easy to generate and f is known.
- How do we generate X_i above?
- How large does n need to be?
- Can we reduce $n(\text{time, cost})$ by being clever? Yes, by choosing Y_i more carefully?

if $\text{cov}(Y_i, Y_j) = \rho$:

$$\begin{aligned} V(\bar{Y}) &= \frac{\sigma^2}{n} + \frac{2n(n-1)}{n^2} \text{cov}(Y_i, Y_j) \\ &= \frac{\sigma^2}{n} + \frac{n(n-1)}{n^2} \rho \sigma^2 \end{aligned} \tag{3}$$

1.4 Example: Interest Rate

k = number of times per year interest is compounded

r_k = interest rate per year compounded k times per year

r_1 = annualized percentage rate (APR)

r = interest rate per year compounded continuously

$$\begin{aligned} r_1 &= \left(1 + \frac{r_k}{k}\right)^k - 1 = e^r - 1 \\ r &= k \ln\left(1 + \frac{r_k}{k}\right) = \ln(1 + r_1) \end{aligned} \tag{4}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n &= e \\ \lim_{n \rightarrow \infty} \left(1 + \frac{0.05}{n}\right)^n &= \lim_{n \rightarrow \infty} \left(1 + \frac{0.05}{n}\right)^{\frac{n}{0.05}} = e^{0.05} \end{aligned} \quad (5)$$

1.5 Example: Estimating Pi

Assume the following:

- a piece of 1 x 1 square wood with a circle in it
- infinite darts

How to estimate the value of π ?

Area of square: 1

$r = 0.5$

Area of a circle: $\pi r^2 = \frac{\pi}{4}$

$\hat{\pi} = 4 \times \frac{\# \text{ of darts in circle}}{\# \text{ of darts in square}}$

1.6 Example: Sandwich Shop Profit

$D_{ij} \sim U(5, \dots, 35), i = 1, \dots, n, j = 1, \dots, d$

j: day

i: random variable

profit: $P_{ij} = \min(D_{ij}, O)R - OW$

average daily profit over d days: $\bar{P}_i = \frac{1}{d}(P_{i1} + \dots + P_{id}), i = 1, \dots, n$

$$\begin{aligned} \hat{\mu} &= \frac{1}{n}(\bar{P}_1 + \dots + \bar{P}_n) = \frac{1}{nd} \sum_{i,j=1}^{n,d} P_{ij} \\ \hat{\sigma}^2 &= \frac{1}{n-1} \sum_{i=1}^n (\bar{P}_i - \hat{\mu})^2 \\ \hat{\mu} \pm 2.58 \frac{\hat{\sigma}}{\sqrt{n}} \end{aligned} \quad (6)$$

2.58 is $p = 0.005$ for a 99% C.I.

1.6.1 Questions

- What size order gives the maximum average daily profit? Why? Have you tried other order sizes?
- How accurately can you know the average daily profit from the simulation? How does this depend on the number of days for your simulation?
- How does the answer vary as you change your model assumptions?
- Plot daily profit and average daily profit with the number of days

2 Review & Estimating Integrals (2020/04/09)

2.1 Review

$$MSE(\hat{\mu}) = Var(\hat{\mu}) + [bias(\hat{\mu})]^2$$

$$bias(\hat{\mu}) = E(\hat{\mu}) - \mu$$

$$\text{Simple Monte Carlo Simulator: } \hat{\mu} = \bar{Y} = \frac{1}{n}(\sum Y_i)$$

2.1.1 Chebyshev Inequality

When working with an unknown distribute, the Chebyshev inequality can be used to construct Confidence Intervals (albeit wide).

$$P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

$$P(|Y - \mu| > k\sigma) \leq \frac{1}{k^2}$$

2.1.2 Determining N

$$\left| \frac{Z_{1-\alpha/2}\hat{\sigma}}{\sqrt{n}} \right| \leq \epsilon \rightarrow n \geq \left(\frac{Z_{1-\alpha/2}\hat{\sigma}}{\alpha} \right)^2 \quad (7)$$

$\hat{\sigma}$: Unbiased estimate of σ

α : Error tolerance.

In this class so far, $\alpha = 0.01$

1. Steps

- (a) Choose a small sample size ($n_0 = 1000$). Then generate n_0 random samples from an underlying probability distribution

- (b) Calculate $\hat{\sigma}$
- (c) Calculate n
- (d) Generate another sample of size n from the underlying probability distribution.
- (e) Compute $\hat{\mu}$ with error $\pm Z_{1-\alpha/2} \frac{\hat{\sigma}_n}{\sqrt{n}}$

2.2 Estimating Integrals

$$\mu = \int_{R^d} g(x) dx = ?$$

Let $f(x) = \frac{g(x)}{\rho(x)}$ where $\rho(x)$ is a probability density function (PDF) and $g(x)$ is the function of interest to be estimated.

Then,

$$\mu = \int_{R^d} f(x) \rho(x) dx = E(Y)$$

where $Y = f(X)$

2.2.1 Example - Normal Probability

$$\mu = \int_0^1 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = \Phi(1) - \Phi(0)$$

$$RMSE(\hat{\mu}) = \sqrt{\text{Var}(\hat{\mu}) + [\text{bias}(\hat{\mu})]^2}$$

Summary

Estimator($\hat{\mu}$)	bias($\hat{\mu}$)	Var($\hat{\mu}$)	RMSE($\hat{\mu}$)
$\hat{\mu}_{MC1}$	0	$0.0023345 \ n^{-1}$	$0.048420 \ n^{-\frac{1}{2}}$
$\hat{\mu}_{MC2}$	0	$0.22483 \ n^{-1}$	$0.47416 \ n^{-\frac{1}{2}}$
$\hat{\mu}_{MC3}$	$O(n^{-1})$	0	$O(n^{-1})$
$\hat{\mu}_{MC4}$	0	$O(n^{-3})$	$O(n^{-\frac{3}{2}})$

1. First Estimator - Simple Monte Carlo Estimator

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$X_i \sim U[0, 1]$$

$$Y = f(X)$$

$$\hat{\mu}_{MC1} = E(Y) = \frac{1}{n} \sum f(X_i) = \frac{1}{n} \sum \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{X_i^2}{2}\right)$$

$$\begin{aligned}
MSE_{MC1} &= Var(\hat{\mu}_{MC1}) + 0 \\
&= \frac{Var(Y)}{N} \\
&= n^{-1}Var(Y) \propto n^{-1} \\
&= O(n^{-1})
\end{aligned} \tag{8}$$

2. Second Estimator - Standard Normal R.V.

$$f(x) = 1_{[0,1]}(x) = \begin{cases} 1, & x \in [0, 1] \\ 0, & else \end{cases}$$

$$\mu = E(Y), Y = f(X), X_i \sim N(0, 1)$$

$$\hat{\mu}_{MC2} = \frac{1}{n} \sum Y_i = \frac{1}{n} f(X_i) = \frac{1}{n} 1_{[0,1]}(X_i)$$

In this case, $Y \sim Bernoulli(p)$. Thus $E(Y) = p$ and $\bar{Y} = \hat{p}$

$$\begin{aligned}
Var(Y) &= p(1-p) \\
&= (\Phi(1) - \Phi(0))(1 - (\Phi(1) - \Phi(0))) \\
&= 0.2248
\end{aligned} \tag{9}$$

$$MSE_{MC2} = Var(\bar{Y}) = n^{-1}Var(Y) = 0.2248n^{-1}$$

3. Third Estimator - Left Rectangle Rule

$$\text{Let } x_i = \frac{i-1}{n}$$

$$\hat{\mu}_{Rect} = \frac{1}{n} \sum \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x_i^2}{2}\right)$$

Deterministic, thus $Var(\hat{\mu}) = 0$. Not a R.V.

$$MSE_{MC3} = (\hat{\mu} - \mu)^2 + 0$$

$$\text{Let error } \epsilon = \left| \int_0^1 \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) - \hat{\mu} \right|$$

$$\text{Let } k = \max |f(x)| \text{ for } x \in [0, 1]$$

$$\epsilon \leq \frac{k(1-0)}{2n}$$

$$\begin{aligned}\mu - \hat{\mu} &\leq \frac{k}{2n} = O(n^{-1}) \\ MSE_{\hat{\mu}} &= O(n^{-2})\end{aligned}\tag{10}$$

4. Fourth Estimator - Stratified Sampling Estimator

Simulates a random sample for each stratum.

Let $x_i = \frac{(i-1+U_i)}{n}$, U_i iid $U[0, 1]$

$$\hat{\mu}_{MC4} = \frac{1}{n} \sum \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$$

$$MSE = Bias^2 + Var(\hat{\mu}) = 0 + O(n^{-3})$$

3 European Call/Put Options and Brownian Motion (2020/04/16)

3.1 Options

Call Option: Contract that gives the buyer of the option the right to buy an asset at a specific price at a specific time.

Put Option: Contract that gives the buyer the right to sell an asset at a specific price at a specific time.

European Option: This is a type of option that allows execution time to be at the expiration/maturity date.

Strike Price: The predetermined price that the holder can buy or sell.

Premium: Expected value of the return at maturity.

3.1.1 Examples

1. Call Option

Premium: \$4 Strike price: \$50 Expiration: 3 Months

Three month

(a) Stock Market Price = \$100 pay \$4, then can buy for \$50 when its 100

\|—————\| 50 100

The buyer executes. The return is $100 - 50 - 4 = \$46$ dollars

(b) Stock market price is \$20

\|—————\| 50 20

The buyer does **not** execute. Buyer loses \$4.

2. Put Option

(a) Stock Price is \$100

\|—————\| 50 100

The buyer does **not** execute because selling for \$50 is a loss. Loses \$4.

(b) Stock price is \$20

\|—————\| 50 20

The buyer executes. The return is $50 - 20 - 4 = \$26$

3.1.2 European Options

t = time in years

$S(t)$ = the price of the asset at time t

T = time to expiry (maturity) of the contract

K = strike price (the price decided at $t = 0$)

r = risk-neutral interest rate

Discounted Euro Call payoff: $\max(S(T) - K, 0)e^{-rT}$

Discounted Euro Put payoff: $\max(K - S(T), 0)e^{-rT}$

We only need to model $S(T)$ not $S(\cdot)$. The fair call/put option prices are $\mu = E(Y)$, where Y is the discounted call/put payoff.

3.2 Geometric Brownian Motion

A simple model for asset prices

$$S(t) = S(0)\exp((r - \sigma^2/2)t + \sigma B(t)), \quad t \geq 0$$

$B_t \sim N(0, t)$: Brownian Motion. This produces wave-like noise that fans wider as t increases.

σ : volatility. Measure the spread of an asset. Determined by no arbitrary principle. i.e. the return cannot be greater than the interest rate if there is no risk.

3.2.1 Properties

- $B(0) = 0$ with probability one

- $B(\tau)$ and $B(t) - B(\tau)$ independent for $0 \leq \tau \leq t$
- $B(t) - B(\tau) \sim N(0, t - \tau) \forall 0 \leq \tau \leq t$
- $cov(B(t), B(\tau)) = \min(t, \tau)$ for $0 \leq t, \tau$

3.3 Black-Sholes Formula for Option Prices

Fair European Call Price:

$$S(0)\Phi\left(\frac{\ln(S(0)/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\ln(S(0)/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right)$$

$$Ke^{-rT}\Phi\left(\frac{\ln(K/S(0)) - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) - S(0)\Phi\left(\frac{\ln(K/S(0)) - (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right)$$

3.3.1 Assumptions

1. The stock underlying call/put options provides no dividends during the call/put lifetime.
2. There are no transaction costs for the sale/purchase of stock.
3. Risk free interest rate (r) is constant during the option time

Put-call parity: Fair European call price - fair European put price = $S(0) - K \exp(-rT)$

3.4 Monte Carlo Computation of European Put

1. Generate X_1, \dots, X_n by a normal pseudo-random number generator.
2. Compute the sample ending stock prices: $S_i(T) = S(0) \exp((r - \sigma^2/2)T + \sigma\sqrt{T}X_i)$
3. Compute sample discounted payoffs, $Y_i = \max(K - S_i(T), 0)e^{-rT}$
4. Average the discounted payoffs,

Fair European Put Price

$$\mu = E(Y) \approx \frac{1}{n} \sum_{i=1}^n \max(K - S_i(T), 0)e^{-rT}$$

Estimated error = $\pm \frac{2.58\hat{\sigma}}{\sqrt{n}}$ where $\hat{\sigma}$ is the sample standard deviation of the discounted payoffs.

4 Linear Congruential Generators and Inverse Distributions (2020/04/23)

4.1 Linear Congruential Generators

Random numbers aren't truly random.

“Anyone who considers arithmetic methods of producing random digits is, of course, in a state of sin.” - John Neumann

M : A large Integer

a : large primitive root of M ($a \bmod M \neq 0$)

$i = 1, \dots, M - 1$

m_0 : integer seed

$$m_i = a \cdot m_{i-1} \bmod M, \quad x_i = \frac{m_i}{M}, \quad i = 1, 2, \dots$$

$x_i \neq x_j$ for $j = i + 1, \dots, i + M - 2$

$M - 1$: Period

$$\begin{aligned} m_i &= a \cdot m_{i-1} \bmod M \\ &= m_0 \cdot a^i \bmod M \end{aligned} \tag{11}$$

a is a primary root of M if $a^i \bmod M > 0$ for $i = 1, \dots, M - 1$

$$\begin{aligned} m_i &= a[m_0 a^{i-1} \bmod M] \bmod M \\ &= a[m_0 a^{i-1} - \left(\frac{m_0 a^{i-1}}{M}\right) \cdot M] \bmod M \\ &= (m_0 a^i - a \left[\frac{m_0 a^{i-1}}{M}\right] M) \\ &= m_0 a^i \bmod M \end{aligned} \tag{12}$$

4.1.1 Example 1

$M - 1 = 16$

$a = 5$

$m_n = 5 \cdot m_{n-1} \bmod 16$

$$\begin{aligned}
m_0 &= 5 \\
m_1 &= 10 \\
m_2 &= 3 \\
&\dots \\
m_5 &= 6 \\
m_6 &= 15 \\
&\dots
\end{aligned} \tag{13}$$

$$0 \leq \frac{m_i}{16} \leq 1$$

at $m_1 6$, it starts over again

period length: any linear congruential generator will eventually repeat itself.

reproducibility: Using the same seed can produce the same *random*

4.2 Tests for Pseudo random numbers

A given M may have primary roots, a, but not all may produce good sequences of random numbers.

The numbers should fill the d-dim hypercube.

Spectral Tests

Quantitative measure of how well the points

$(x_i, x_{i+1}, \dots, x_{i+d-1})$ fill $[0, 1]^d$.

This test, $l(0, M, d)$ is the largest possible distance between planes covering the points.

4.2.1 Collision Test

Y_1, \dots, Y_n iid R.V. with the common cumulative prob distr. function F so $x_i = F(Y_i) \sim iid U[0, 1]$

$$Z_i = (X_{(i-1)d+1}, \dots, X_{id}), \quad i = 1, \dots, k = \frac{n}{d}$$

$$Z_i \sim iid[0, 1]^d$$

$W = \#$ of Bins with more than one point. (collisions)

Break the cube $[0, 1]^d$ into l non overlapping Bins

Check if the points are uniformly random.

$$\lim_{n \rightarrow \infty} W \sim Poisson$$

$$\lambda = \frac{k^2}{l}$$

If W is much smaller than λ or much larger than λ , then it is not pseudo random.

4.3 Inverse Distribution

Y with CDF $F(Y)$

$$0 \leq F(Y) \leq 1$$

Define a new R.V. X : $X = F(Y) \sim Unif(0, 1)$

$$\begin{aligned} P(X < x) &= P(F(Y) < X) \\ &= P(Y < F^{-1}(x)) \\ &= F(F^{-1}(x)) \\ &= x \end{aligned} \tag{14}$$

5 GBM Explanation (2020/04/30)

Geometric + Random term to model that the price is always increasing.

GBM model used to simulate stock prices at a given time.

$$S(t) = S(0) \exp((-r - \sigma^2/2)t + \sigma B(t)) \tag{15}$$

Random log Return between t_1 and t_2

$$R(t_1, t_2) = \ln\left(\frac{S(t_2)}{S(t_1)}\right) = (r - \sigma^2/2)(t_2 - t_1) + \sigma[B(t_2) - B(t_1)] \tag{16}$$

$$B(t_2) - B(t_1) \sim N(0, t_2 - t_1)$$

Risk Free investment (no volatility or money in the bank) ($\sigma = 0$)

$$E(S(t)) = S(0) \exp(rt)$$

$\exp(-\sigma^2 t/2)$: comes from no arbitrage principle. The mean return is the return on a risk-free investment.

Return is a Gaussian random variable. May be positive or negative

$$\ln\left(\frac{S(t + \Delta)}{S(t)}\right) = (r - \sigma^2/2)\Delta + \sigma[B(t + \Delta) - B(t)] \tag{17}$$

$$B(t + \Delta) - B(t) \sim N(0, \Delta)$$

$$\text{GBM } S(t) = S(0)\exp((r - \sigma^2/2)t + \sigma B(t))$$

$$\begin{aligned} E(S(t)) &= E[S(0)\exp((r - \sigma^2/2)t + \sigma B(t))] \\ &= S(0)\exp((r - \sigma^2/2)t)E(\exp(\sigma B(t))) \end{aligned} \quad (18)$$

The moment generating function for $N(0, \sigma^2)$ is
 $M_x(t) = \exp(\frac{\sigma^2 t^2}{2})$

$$E[\exp(\sigma B(t))] = \exp(\sigma^2 t/2) \quad (19)$$

Cancels out the term in $E[S(t)]$

5.1 Fair European Put Price

$E[\text{discounted payoff at time } T]$

$$\begin{aligned} & E[\max(K - S(T, X), 0)e^{-rT}] \\ &= \int_{-\infty}^{\infty} \max(K - S(T, X), 0)e^{-rT} dx \\ &= \int_{-\infty}^{\infty} \max(K - S(0)\exp((r - \sigma^2/2)T + \sigma\sqrt{T}X), 0)e^{-rT} f(x) dx \\ &= \int_{-\infty}^{X_{hi}} K - S(0)\exp((r - \sigma^2/2)T + \sigma\sqrt{T}X)e^{-rT} f(x) dx + \int_{X_{hi}}^{\infty} 0 \\ &= \int_{-\infty}^{X_{hi}} Ke^{-rT} f(x) dx - \int_{-\infty}^{X_{hi}} S(0)\exp((r - \sigma^2/2)T + \sigma\sqrt{T}X - rT) f(x) dx \\ &= ke^{-rT}\Phi(X_{hi}) - S(0) \int_{-\infty}^{X_{hi}} \exp(r - \sigma^2/2)T + \sigma\sqrt{T}X - \frac{1}{\sqrt{2\pi}}\exp(-x^2/2) \\ &= ke^{-rT}\Phi(X_{hi}) - S(0) \int_{-\infty}^{X_{hi}} \frac{1}{\sqrt{2\pi}}\exp(-\sigma^2 T/2 + \sigma\sqrt{T}X - X^2/2) \\ &= ke^{-rT}\Phi(X_{hi}) - S(0)\Phi(X_{hi} - \sigma\sqrt{T}) \end{aligned} \quad (20)$$

Need to find out when this becomes 0. $K - S(0)\exp((r - \sigma^2/2)T + \sigma\sqrt{T}X) \leq 0$
 $X_{hi} \geq \frac{\ln(K/S(0)) - (r - \sigma^2/2)T}{\sigma\sqrt{T}}$

6 Random Number Generation pt 2 (2020/05/07)

6.1 Acceptance-Rejection Method

$Y_i, W_i \sim iid R.V$

$Y_i \sim$ common PDF, f_Y

$W_i \sim U[0, 1]$

Let $c \leq 1$ s.t. $\frac{cf_Z(z)}{f_Y(z)}$

f_Z : PDF of the Random variable we want to generate.

We want C to be as close to 1 as possible. It is generally found by calculation

$$\frac{1}{c} = \sup_z \frac{f_Z(z)}{f_Y(z)} == \frac{f_Y(y)}{c} = \sup f_Z(y)$$

$1/c$ is the largest value of the ratio between $f_Z(y)$, $f_Y(y)$

sup = supremum = maximum of a given set

$k = 0, \dots, m$ for $i = 1, 2, \dots$

if $W_i \leq \frac{cf_Z(Y_i)}{f_Y(Y_i)}$, $k = k + 1$, $Z_k = Y_i$

Z_1, \dots, Z_n iid with PDF f_Z

Goal: Want to simulate a random sample from Z with known PDF $f_Z(y)$

What we know

We can simulate random samples Y_1, \dots, Y_n from $f_Y(y)$ since $f_Y(y)$ is known and has the same support as Z . The support being the domain of a R.V where the pdf is non-zero. For example, values using the Beta distr. PDF is between 0 and 1, so is $U[0, 1]$

We can simulate random samples from a Uniform Distribution:

$W_1, \dots, W_n \sim U[0, 1]$

6.1.1 Proof

How do we know whether the accepted samples are sufficient for a random sample?

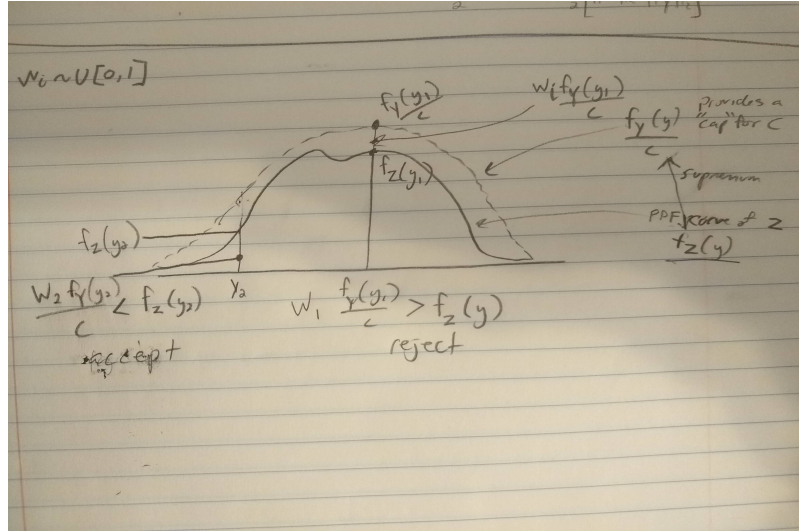


Figure 1: Acceptance-Rejection method using two distributions

$$\lim_{\Delta \rightarrow 0} \frac{P(Y \in [y, y + \Delta] | Y \text{ accepted to be } Z)}{\Delta} = f_Z(y)$$

$$\lim_{\Delta \rightarrow 0} \frac{P(Y \in [y, y + \Delta] \cap Y \text{ accepted to be } Z)}{\Delta \cdot P(Y \text{ accepted to be } Z)}$$

$$\lim_{\Delta \rightarrow 0} \frac{f_Y(y) \times P(W \leq cf_Z(y)/f_Y(y))}{\Delta \cdot c}$$

Using definition for $P(Y \text{ is } Z)$

$$\lim_{\Delta \rightarrow 0} \frac{f_Y(y) \times cf_Z(y)/f_Y(y)}{c} = f_Z(y) \quad (21)$$

$$\begin{aligned}
P(\text{Y accepted to be Z}) &= \lim_{n \rightarrow \infty} \sum_{i=1}^n P(w \leq \frac{cf_Z(y_i)}{f_Y(y_i)}) f_Y(y_i) \cdot \Delta \\
&= \lim_{n \rightarrow \infty} \sum_{i=1}^n P(\text{accept } y_i | y_i) \cdot P(y_i) \\
&= \int_{\Omega} \frac{cf_Z(y)}{f_Y(y)} \cdot f_Y(y) dy \\
&= c \int_{\Omega} f_Z(y) dy \\
&= c \cdot 1 = c
\end{aligned} \tag{22}$$

Ω : Support of Y and Z

- You **must** know the PDF function f_Y , f_Z explicitly
- Generating Y_1, Y_2, \dots with PDF f_Y may be done using the inverse transformation method.

6.1.2 Application

Generate a Sequence of 1000 random numbers.

$$f_Z(z) = 20z(1-z)^3, \quad 0 < z < 1, \quad z \sim \text{Beta}(\alpha = 2, \beta = 4)$$

$$f_Y \sim U[0, 1]$$

1. The candidate distribution $f_Y(y) = 1, \quad 0 < y < 1$
2. What is the value of C?

$$\frac{1}{c} = \sup \frac{f_Z(y)}{f_Y(y)}$$

$$Q = \frac{f_Z(y)}{f_Y(y)} = \frac{20z(1-z)^3}{1}. \text{ Need to find max of } Q$$

$$\frac{dQ}{dz} = 20z(1-z)^3 - 60z(1-z)^2 \tag{23a}$$

$$= (1-z)^2(20(1-z) - 60z) \tag{23b}$$

$$= (1-z)^2(20 - 80z) = 0 \tag{23c}$$

$$z = 1, \frac{1}{4} \tag{23d}$$

Since $\frac{1}{4}$ is the smallest,

$$\frac{20(0.25)(1 - 0.25)^3}{1} = \frac{135}{64} \rightarrow \frac{1}{c} \rightarrow c = \frac{64}{135}$$

3. How many random samples are required?

Let N be the number of iterations required.

$$E(N) = \frac{1000}{c} = \frac{1000 \cdot 135}{64} = \frac{135000}{64}$$

$$N = 1.1 \cdot E(N) = 2321 \text{ random samples}$$

4. How to simulate

- (a) Sim N random samples from $U[0, 1]$ Y
- (b) Sim N random samples from $U[0, 1]$ W
- (c) Make decision. if $W_i < \frac{cf_Z(y)}{f_Y(y)}$ then reject

6.1.3 Normal Distribution

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2), \quad -\infty \leq z \leq \infty$$

Want to simulate a Normal Z but we need to find a candidate distribution with the same support. No other distributions have the same support as the Normal but many have the support $0 \leq y < \infty$.

$$|z| < \infty$$

$$\begin{aligned} P(|z| \leq z) &= P(-z \leq Z \leq z) \\ &= \Phi(z) - \Phi(-z) \\ &= \Phi(z) - (1 - \Phi(z)) \\ &= 2\Phi(z) - 1 \end{aligned} \tag{24}$$

$$f_{|z|}(z) = \frac{2}{\sqrt{2\pi}} \exp(-z^2/2)$$

$$P(z < 0) = P(z \geq 0) = 0.5$$

We will need to simulate random samples from $|z|$.

1. What is the candidate distr. for Y?

$$Y \sim \exp(1), \quad f_Y(z) = \exp(-z), \quad 0 \leq z < \infty$$

2. What is the value of c ?

$$Q = \frac{1}{c} = \frac{f_Z(z)}{f_Y(z)} = \frac{\frac{2}{\sqrt{2\pi}} \exp(-z^2/2)}{\exp(-z)} = \frac{2}{\sqrt{2\pi}} \exp(-z^2/2 + z)$$

Maximize Q which means maximize $-z^2/2 + z$ ($z = 1$)

$$Q = \frac{2}{\sqrt{2\pi}} \exp(0.5) = \sqrt{\frac{2e}{\pi}} = \frac{1}{6}$$

$$c = \sqrt{\frac{\pi}{2e}}$$

$$Q = \frac{1}{c} = \frac{cf_Z(y)}{f_Y(y)} = \sqrt{\frac{2}{\pi}} \exp(-z^2/2 + z) = \exp(-z^2/2 + z - 0.5)$$

$$c \frac{f_Z(y)}{f_Y(y)} = \exp(-\frac{y^2}{2} + y - 0.5) = \exp(-\frac{1}{2}(y - 1)^2)$$

3. How many random samples?

$$E(N) = \frac{1000}{c} \approx 1316$$

$$N = 1.1 \cdot 1316 = 1448 \text{ random samples}$$

4. Simulate

(a) Simulate R.S $U[0, 1]$ W

(b) Simulate S.S from $Y \sim \exp(1)$ Inverse transformation method yields $Y_i = -\log(x_i)$

$$\begin{aligned} F_Y(y) &= 1 - e^{-y} \\ F_Y^{-1}(x) &= -\log(x) \text{ OR } -\log(1 - x_i) \end{aligned} \tag{25}$$

(c) If $W_i \leq \exp(-0.5(y - 1)^2)$, accept.

(d) Simulate R.S $V_i \sim U[0, 1]$

$$Z_k = \text{sign}(V_i - 0.5)Y_i$$

6.2 Brownian Motion Time Differencing

Sometimes instead of a scalar, we want to generate random functions, B .

6.2.1 Properties

- $B(0) = 0$
- $B(\tau)$ and $B(t) - B(\tau)$ are indep for all $0 \leq \tau \leq t$
- $B(t) - B(\tau) \sim N(0, t - \tau)$
- $B(t)$ and $B(\tau)$ are not independent. $cov(t, \tau) = \min(t, \tau) = \tau$
- May be generated at discrete times, $0 = t_0 < t_1 < \dots < t_\alpha = T$
 $B(0) = 0, B(t_k) = B(t_{k-1}) + X_k \sqrt{t_k - t_{k-1}}, k = 1, \dots, d$
 X_1, \dots, X_d are iid.

6.2.2 Linear Interpolation

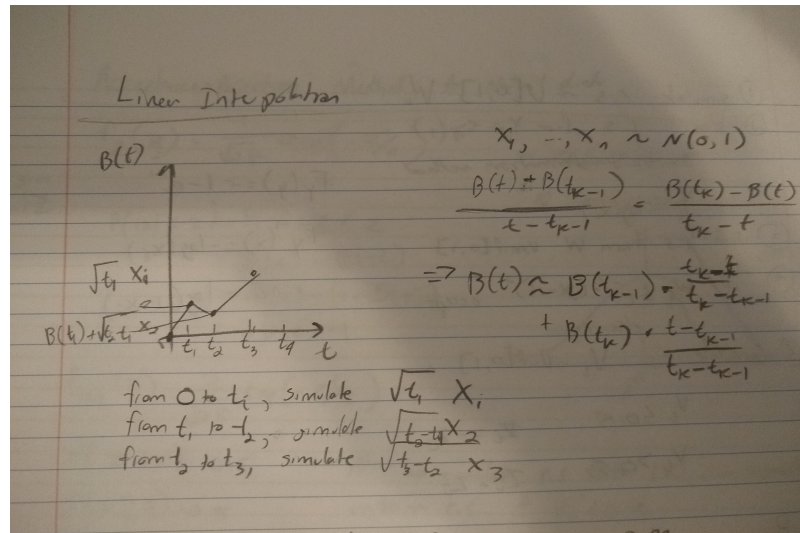


Figure 2: Linear Interpolation

6.2.3 Generating Brownian Sample Paths

d : number of time nodes

$$0, \frac{T}{d}, \frac{2T}{d}, \dots, \frac{(d-1)T}{d}, \frac{dT}{d} = T$$

to generate sample paths of brownian motion.

1. Generate d standard normal random numbers X_1, \dots, X_d
2. Brownian motion at time $\frac{kT}{d}, k = 1, 2, \dots, d$

Relationship between Brownian Motion and Geometric Brownian Motion

They are the same thing.

Brownian Motion

$$S(T) = S(0) \exp(rT - T \frac{\sigma^2}{2} + \sigma B(T))$$

Geometric Brownian Motion

$$S(T) = S(0) \exp(T(r - \frac{\sigma^2}{2}) + \sigma B(T))$$

7 Types of Options (2020/05/14)

7.1 Vanilla European Call/Put Option

Asset path not important for European options

Review

$$S(t) = S(0) \exp(r - \sigma^2/2)t + \sigma B(t) = S(0) \exp(r - \sigma^2/2)t + \sigma \sqrt{t} X$$

$$\sqrt{t} X \approx B(t)$$

where $X \sim N(0, 1)$

$$E(S(t)) = S(0) \exp(rt)$$

$$\begin{aligned} S(\frac{t}{d}) &= S(0) \exp(r - \sigma^2/2) \frac{t}{d} + \sigma \sqrt{\frac{t}{d}} X_1 \\ S(\frac{2t}{d}) &= S(0) \exp(r - \sigma^2/2) \frac{2t}{d} + \sigma \sqrt{\frac{t}{d}} (X_1 + X_2) \\ S(\frac{3t}{d}) &= S(0) \exp(r - \sigma^2/2) \frac{3t}{d} + \sigma \sqrt{\frac{t}{d}} (X_1 + X_2 + X_3) \end{aligned} \tag{26}$$

...

$$S(t) = S(0) \exp(r - \sigma^2/2)t + \sigma B(t) = S(0) \exp(r - \sigma^2/2)t + \sigma \sqrt{t} X$$

More generically,

$$S(\frac{kt}{d}) = S(0)exp(r - \sigma^2/2)\frac{kt}{d} + \sigma\sqrt{\frac{t}{d}}\sum_1^k X_i$$

where $\sum_i^k X_i \sim N(0, \frac{kt}{d})$

d : Number of increments. It can be years, days, hours, etc. In previous discussions, d was *years*. In this case, it is *days*.

Simulation Steps

1. Simulate independent Std. Norm R.V.s $n \times d$
2. Generate Brownian Motion Path for each sample $\sqrt{\frac{t}{d}} \cdot cumsum(x)$
3. Generate Geometric Brownian Motion using formula above (asset price path) payoff = $max(K - S(t), 0)$ discounted payoff = $max(K - S(t), 0)exp(-rt)$

7.2 Asian Call/Put Options

Uses the average price of the asset from purchase to maturity instead of the asset price at maturity.

European Option = Arithmetic or Geometric Asian Option where $d =$

1.

$$\bar{S}_{geo} \leq \bar{S}_{ari}$$

7.2.1 Arithmetic Mean

call payoff = $max(\frac{1}{d} \sum_{j=1}^d S(\frac{jT}{d}) - K, 0)$

put payoff = $max(K - \frac{1}{d} \sum_{j=1}^d S(\frac{jT}{d}), 0)$

European options will be a higher payout because of higher volatility.

7.2.2 Geometric Mean

$$\sqrt{ab}$$

$$\bar{S}_{geo} = [\Pi_1^d S(\frac{jT}{d})]^{1/d}$$

$$\begin{aligned}
S\left(\frac{T}{d}\right) &= S(0) \exp\left((r - \sigma^2/2) \frac{T}{d} + \sigma \sqrt{\frac{T}{d}} X_1\right) \\
S\left(\frac{2T}{d}\right) &= S(0) \exp\left((r - \sigma^2/2) \frac{2T}{d} + \sigma \sqrt{\frac{T}{d}} (X_1 + X_2)\right) \\
&\dots \\
S(T) &= S(0) \exp\left((r - \sigma^2/2) T + \sigma \sqrt{\frac{T}{d}} \sum_{i=1}^d X_i\right)
\end{aligned} \tag{27}$$

$$\begin{aligned}
[\Pi_1^d S(\frac{jT}{d})]^{1/d} &= [S(0) \exp(d(r - \sigma^2/2) \frac{(1+2+\dots+d)T}{d} + \sigma \sqrt{\frac{T}{d}} [X_1 + (d-1)X_2 + (d-2)X_3 + \dots + X_d]) \\
&= S(0) [\exp((r - \sigma^2/2) \frac{d(d+1)}{d} \cdot \frac{T}{d} + \sigma \sqrt{\frac{T}{d}} W)]
\end{aligned} \tag{28}$$

$$\begin{aligned}
W &= dX_1 + (d-1)X_2 + \dots + x_0 d \sim N(0, \frac{d(d+1)(d+2)}{6}) \\
V(W) &= d^2 + (d-1)^2 + \dots + 1^2 = \frac{d(d+1)(d+2)}{6} \\
\text{We replace } W &\text{ with } \sqrt{\frac{d(d+1)(d+2)}{6}} X \\
T &= (\frac{d+1}{2d})T = (\frac{1}{2} + \frac{1}{d})T \leq T \\
\bar{\sigma}^2 &= \frac{\sigma^2(2+\frac{1}{d})}{3} = (\frac{2}{3} + \frac{1}{3d})\sigma^2 \leq \sigma^2 \\
r &= r + (\bar{\sigma}^2 - \sigma^2)/2 \leq r
\end{aligned}$$

7.3 Barrier Options

There may be a barrier price, b , which one must cross to trigger the option (one way or another).

7.3.1 Knock-in Barrier Option

Knock-out is the opposite of Knock-in. I.e. option no longer valid if the barrier is crossed.

1. Up-and-in If barrier is met during the time to maturity, the option is activated.
2. Down-and-out

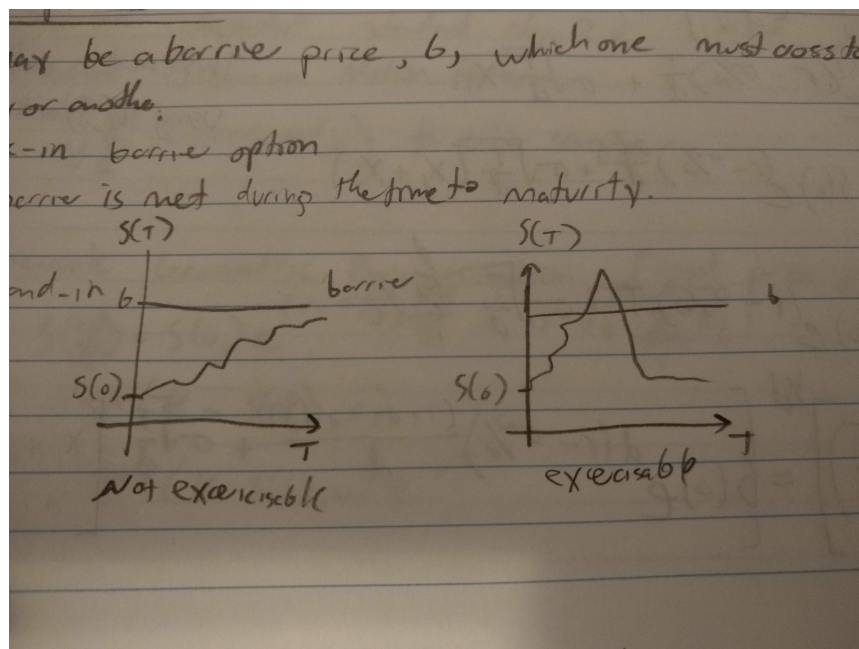


Figure 3: Barrier Options: Up and In

7.4 Lookback Options

Minimum or maximum value of the asset price before expiry acts as a strike price.

Call Payoff

$$S(T) - \min_{j=1, \dots, d} S(jT/d) \text{ or } S(T) - \min_{0 \leq t \leq T} S(t)$$

Put Payoff

$$\max_{j=1, \dots, d} S(jT/d) - S(T) \text{ or } \max_{0 \leq t \leq T} S(t) - S(T)$$

8 Control-Variate Method for Efficiency (2020/05/21)

Suppose we want to estimate estimate $\mu = E(Y)$.

Possible estimators

- \bar{Y}

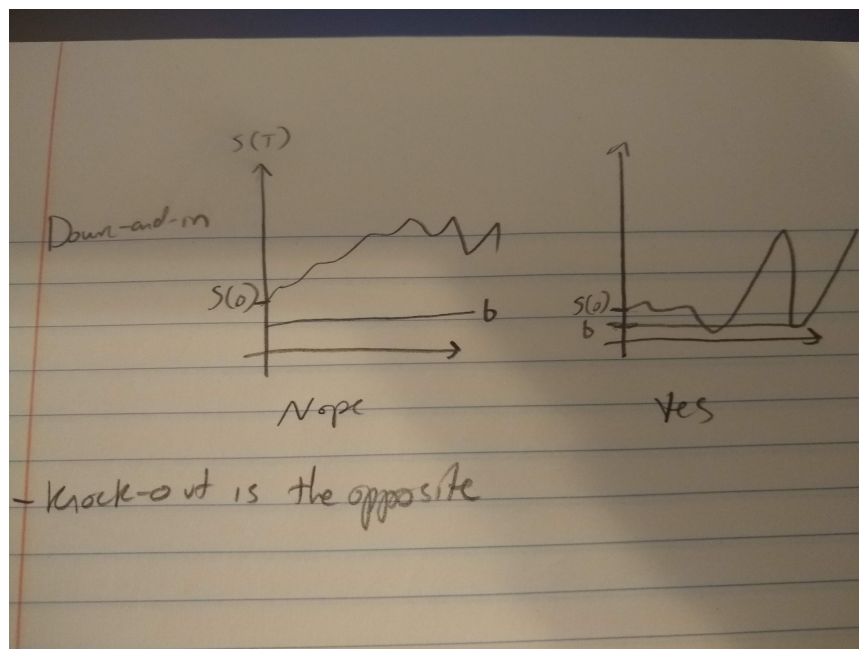


Figure 4: Barrier Options: Down and Out

- Y_1
- $Y_1 + \frac{Y_2 + Y_3}{2}$

$\hat{\mu}$ is a point estimator of μ .

How do we evaluate a point estimator? MSE

$$\begin{aligned} MSE &= E((\hat{\mu} - \mu)^2) \\ &= Var(\hat{\mu}) + Bias^2(\hat{\mu}) \end{aligned} \quad (29)$$

Let Y be a R.V. whose μ you want to estimate. Let X be a R.V. whose μ_x you already know.

Example

A student takes two tests. We would like to estimate the score of the second test. We know the distribution and score of the first test. The second test is *correlated* with the score of the first. For example, if they do well on the first exam, it is likely that they will do well on the second exam.

$$X \sim N(85, 5^2), W \sim N(0, 3^2)$$

$$Y = X + W$$

X in this case is the **variate**. The **Control Variate** means Control X .

Let (X_i, Y_i) , $i = 1, \dots, n$ be iid draws of (X, Y) which are not independent of each other.

$$\hat{Y} = \bar{Y} + \beta(\mu_X - \bar{X})E(\hat{Y}) = \mu$$

Without X , \bar{Y} is an unbiased estimator for μ .

$$V(\bar{Y}) = \frac{\sigma^2}{n}$$

$$V(\hat{Y}) = V(\bar{Y}) + \beta(\mu_X - \bar{X})$$

We would like to choose β such that $Var(\hat{Y})$ is minimized.

$$\begin{aligned}\hat{Y} &= \frac{Y_1 + \dots + Y_n}{n} + \frac{\beta(\mu_X - (x_1 + \dots + x_n))}{n} \\ &= \frac{1}{n} \sum_1^n Y_i + \beta(\mu_X - X_i)\end{aligned}\tag{30}$$

Adjust $\mu_X - X_i$ to be closer to the mean. This reduces variance of the point estimator.

$$\begin{aligned}V(\hat{Y}) &= V(\bar{Y} + \beta(\mu_X - \bar{X})) \\ &= V(\bar{Y}) + V(\beta(\mu_X - \bar{X})) + 2cov(\hat{Y}, \beta(\mu_X - \bar{X})) \\ &= \frac{\sigma^2}{n} + \beta^2 \frac{\sigma_X^2}{n} - 2\beta cov(\bar{Y}, \bar{X} - \mu_X)\end{aligned}\tag{31}$$

$$\begin{aligned}cov(\bar{Y}, \bar{X}) &= cov\left(\frac{Y_1 + \dots + Y_n}{n}, \frac{X_1 + \dots + X_n}{n}\right) \\ &= \frac{n}{n^2} cov(X, Y) \\ &= \frac{cov(X, Y)}{n} = \frac{corr(X, Y)\sigma_x\sigma_y}{n}\end{aligned}\tag{32}$$

$$\frac{\partial V(\hat{Y})}{\partial \beta} = \frac{2\beta\sigma_X^2}{n} - \frac{2corr(X, Y)\sigma_x\sigma_Y}{n} = 0$$

$$\beta = \frac{corr(X, Y)\sigma_X\sigma_Y}{\sigma_X^2} = \frac{cov(X, Y)}{\sigma_X^2} = \frac{\sigma_{xy}^2}{\sigma_x^2}$$

$$\begin{aligned}
V(\hat{Y}) &= \frac{\sigma_Y^2}{n} + \left(\frac{\sigma_{xy}^2}{\sigma_X^2}\right)^2 \frac{\sigma_X^2}{n} - \frac{2\sigma_{xy}^2}{\sigma_X^2} - \frac{\sigma_{xy}^2}{n} \\
&= \frac{\sigma_Y^2}{n} + \frac{\sigma_{xy}^4}{\sigma_X^4} \cdot \frac{\sigma_X^2}{n} - \frac{2\sigma_{xy}^4}{n\sigma_X^2} \\
&= \frac{\sigma_Y^2}{n} + \frac{\sigma_{xy}^4}{n\sigma_X^2} \\
&= \frac{\sigma_Y^2}{n} \left[1 - \frac{\sigma_{xy}^4}{\sigma_X^2 \sigma_Y^2}\right] = \frac{\sigma_Y^2}{n} [1 - \text{corr}^2(X, Y)]
\end{aligned} \tag{33}$$

The larger the correlation, the **better**.

$$V(\bar{Y}) = \frac{\sigma_Y^2}{n}$$

$$RMSE(\hat{Y}) = \sqrt{Var(\hat{Y})} = \frac{\sigma_Y}{\sqrt{n}} [1 - \text{corr}^2(X, Y)]^{1/2}$$

Choosing the Control Variate, X, to make $\text{Corr}(X, Y)$ as close to ± 1 as possible.

$$\begin{aligned}
\hat{Y} &= \bar{Y} + \hat{\beta}(\mu_X - \bar{X}) = \bar{Y} + \frac{(\mu_X - \bar{X}) \sum_1^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_1^n (X_i - \bar{X})^2} \\
&= \sum_1^n w_i Y_i \text{ where } w_i = \frac{1}{n} + \frac{(\mu_X - \bar{X})(X_i - \bar{X})}{\sum_1^n (X_i - \bar{X})^2}
\end{aligned} \tag{34}$$

If $X_i < \bar{X}$ and $\mu_X > \bar{X}$, then opposite side of \bar{X} has a negative weight. Otherwise, its positive.

Remarks

- X, Y must be highly correlated
- All pairs of X, Y must be independent
- Reduces MSE but doesn't necessarily lead to a more accurate estimator.