Optimization - Homework 1

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1 Problem Statement

Let

$$A = \begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 & 2 & 6 \\ 2 & 7 & 1 & 8 & 2 & 8 & 1 & 8 \\ 1 & 4 & 1 & 4 & 2 & 1 & 3 & 5 \end{bmatrix}, \vec{y} = \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix}$$

Find a 2-Space Vector $\vec{x} \in^8$ so that $\vec{y} = A\vec{x}$
At least one of the following vectors are a solution:

$$\vec{x_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a \\ b \\ 0 \\ 0 \end{bmatrix}, \; \vec{x_2} = \begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; \vec{x_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ a \\ b \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; \vec{x_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a \\ b \end{bmatrix}$$

There are $\binom{8}{2} = 28$ 2-sparse vectors but only the four need to be considered.

2 Solution

2.1 Guiding Principles

$\vec{y} = A\vec{x}$ where $\vec{x}$ contains free variables produces a system of equations which can be organized into a matrix R. Row reduction is applied to R to obtain discrete values a', b' for the free variables. a', b' are then tested using the remaining equation. If the equation is valid, then x_i is a valid solution. Otherwise, it is not.

2.2 $\mathbf{x}_1$

2.2.1 Organize $\vec{y} = Ax_1$ into a System of equations.

$$\begin{aligned}
&\vec{y} = Ax_1 \\
&\rightarrow \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 & 2 & 6 \\ 2 & 7 & 1 & 8 & 2 & 8 & 1 & 8 \\ 1 & 4 & 1 & 4 & 2 & 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a \\ b \\ 0 \\ 0 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 5a + 9b \\ 2a + 8b \\ 2a + b \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 5 & 9 & -7 \\ 2 & 8 & 20 \\ 2 & 1 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}
\end{aligned} \tag{1}$$

2.2.2 Apply Row Reduction to solve free variables.

$$\begin{bmatrix} 5 & 9 & -7 \\ 2 & 8 & 20 \\ 2 & 1 & 8 \end{bmatrix} \xrightarrow[-r_3+r_2]{} \begin{bmatrix} 5 & 9 & -7 \\ 0 & 7 & 12 \\ 2 & 1 & 8 \end{bmatrix} \xrightarrow[-9r_3+r_1]{} \begin{bmatrix} -13 & 0 & -79 \\ 0 & 7 & 12 \\ 2 & 1 & 8 \end{bmatrix} \tag{2}$$

$$-13a + 0 = -79 \tag{3a}$$

$$7b + 0 = 12 \tag{3b}$$

$$2a + b = 8 \tag{3c}$$

Solving for a', b' using (3a) and (3b) yields $a' = \frac{79}{13},\ b' = \frac{12}{7}$

2.2.3 Plug values into the remaining equation

Replacing $a',\ b'$ in (3c) yields $2(\frac{79}{13} + \frac{12}{7} \neq 8)$. Thus $\vec{x}_1$ is **not** a valid solution

2.3 $\mathbf{x}_2$

2.3.1 Organize $\vec{y} = Ax_2$ into a System of equations.

$$\begin{aligned}
&\vec{y} = Ax_2 \\
&\rightarrow \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 & 2 & 6 \\ 2 & 7 & 1 & 8 & 2 & 8 & 1 & 8 \\ 1 & 4 & 1 & 4 & 2 & 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 3a + b \\ 2a + 7b \\ a + 4b \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 3 & 1 & -7 \\ 2 & 7 & 20 \\ 1 & 4 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}
\end{aligned} \tag{4}$$

2.3.2 Apply Row Reduction to solve free variables.

$$\begin{bmatrix} 3 & 1 & -7 \\ 2 & 7 & 20 \\ 1 & 4 & 8 \end{bmatrix} \underset{-2r_3+r_2}{\rightarrow} \begin{bmatrix} 3 & 1 & -7 \\ 0 & -1 & 4 \\ 1 & 4 & 8 \end{bmatrix} \underset{-4r_2+r_3}{\rightarrow} \begin{bmatrix} 3 & 1 & -7 \\ 0 & -1 & 4 \\ 1 & 0 & 24 \end{bmatrix} \tag{5}$$

$$3a + b = -7 \tag{6a}$$

$$0 + -b = 4 \tag{6b}$$

$$a + 0 = 24 \tag{6c}$$

Solving for $a',\ b'$ using (6b) and (6c) yields $a' = 24,\ b' = -4$

2.3.3 Plug values into the remaining equation

Replacing $a',\ b'$ in (6a) yields $3(24) - 4 \neq 7)$. Thus $\vec{x}_2$ is **not** a valid solution

2.4 $\mathbf{x}_3$

2.4.1 Organize $\vec{y} = Ax_3$ into a System of equations.

$$\vec{y} = Ax_3$$

$$\rightarrow \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 & 2 & 6 \\ 2 & 7 & 1 & 8 & 2 & 8 & 1 & 8 \\ 1 & 4 & 1 & 4 & 2 & 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ a \\ b \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

$$\rightarrow \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} a + 5b \\ 8a + 2b \\ 4a + 2b \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 5 & -7 \\ 8 & 2 & 20 \\ 4 & 2 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$$

2.4.2 Apply Row Reduction to solve free variables.

$$\begin{bmatrix} 1 & 5 & -7 \\ 8 & 2 & 20 \\ 4 & 2 & 8 \end{bmatrix} \underset{-2r_3+r_2}{\longrightarrow} \begin{bmatrix} 1 & 5 & -7 \\ 0 & -2 & 4 \\ 4 & 2 & 8 \end{bmatrix} \underset{r_2+r_3}{\longrightarrow} \begin{bmatrix} 1 & 5 & -7 \\ 0 & -2 & 4 \\ 4 & 0 & 12 \end{bmatrix} \underset{\frac{1}{4}r_3, -\frac{1}{2}r_2}{\longrightarrow} \begin{bmatrix} 1 & 5 & -7 \\ 0 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix} \tag{8}$$

$$a + 5b = -7 \tag{9a}$$

$$0 + b = -2 \tag{9b}$$

$$a + 0 = 3 \tag{9c}$$

Solving for a', b' using (9b) and (9c) yields $a' = 3, \ b' = -2$

2.4.3 Plug values into the remaining equation

Replacing a', b' in (9a) yields $3 + 5(-2) = -7$. Thus $\vec{x}_3$ **is** a valid solution

2.5 $\mathbf{x}_4$

2.5.1 Organize $\vec{y} = Ax_4$ into a System of equations.

$$\vec{y} = Ax_4$$

$$\rightarrow \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 4 & 1 & 5 & 9 & 2 & 6 \\ 2 & 7 & 1 & 8 & 2 & 8 & 1 & 8 \\ 1 & 4 & 1 & 4 & 2 & 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a \\ b \end{bmatrix} \tag{10}$$

$$\rightarrow \begin{bmatrix} -7 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 2a + 6b \\ a + 8b \\ 3a + 5b \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 6 & -7 \\ 1 & 8 & 20 \\ 3 & 5 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$$

2.5.2 Apply Row Reduction to solve free variables.

$$\begin{bmatrix} 2 & 6 & -7 \\ 1 & 8 & 20 \\ 3 & 5 & 8 \end{bmatrix} \xrightarrow[-3r_2+r_3]{} \begin{bmatrix} 2 & 6 & -7 \\ 1 & 8 & 20 \\ 0 & -19 & -52 \end{bmatrix} \xrightarrow[-\frac{8}{19}r_3+r_2]{} \begin{bmatrix} 2 & 6 & -7 \\ 1 & 0 & \frac{160}{19} \\ 0 & -19 & -52 \end{bmatrix} \tag{11}$$

$$2a + 6b = -7 \tag{12a}$$

$$a + 0 = \frac{160}{19} \tag{12b}$$

$$0 - 19b = -52 \tag{12c}$$

Solving for $a',\ b'$ using (12b) and (12c) yields $a' = \frac{160}{19},\ b' = \frac{52}{19}$

2.5.3 Plug values into the remaining equation

Replacing $a',\ b'$ in (12a) yields $2(\frac{160}{19}) + 6(\frac{52}{19}) = -7$. Thus $\vec{x}_4$ is **not** a valid solution