# Homework 2

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## Contents



## 1 Problem Statement

Find the point on the sphere $x^2+y^2+z^2=9$ that are closest to and farthest away from the point (2,3,4).

Hint: Construct and use a Jacobian Matrix. i.e gradient vector(s)

## 2 Solution

Let the function to be minimized be represented as

$$f(x,y,z) = (x-2)^2 + (y-3)^2 + (z-4)^2 = d^2$$

where $d^2$ is distance squared. $f(x,y,z)$ represents the distance squared between (x,y,z) and (2,3,4).

Let the constraint function be represented as

$$g(x,y,z) = x^2 + y^2 + z^2 = 9$$

The Jacobian Matrix is a matrix of partial derivatives with $N = 3$ columns and $m = 1$ rows. Let the Jacobian matrix for $f(x,y,z)$ be described as

$$\nabla f(x,y,z) = \begin{bmatrix} \frac{\partial f(x,y,z)}{\partial x} & \frac{\partial f(x,y,z)}{\partial y} & \frac{\partial f(x,y,z)}{\partial z} \end{bmatrix} = \begin{bmatrix} 2(x-2) & 2(y-3) & 2(z-4) \end{bmatrix}$$

and the Jacobian Matrix for $g(x, y, z)$ be

$$\nabla g(x, y, z) = \begin{bmatrix} \frac{\partial g(x,y,z)}{\partial x} & \frac{\partial g(x,y,z)}{\partial y} & \frac{\partial g(x,y,z)}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}$$

at the closest value for $g(x, y, z) = 9$.

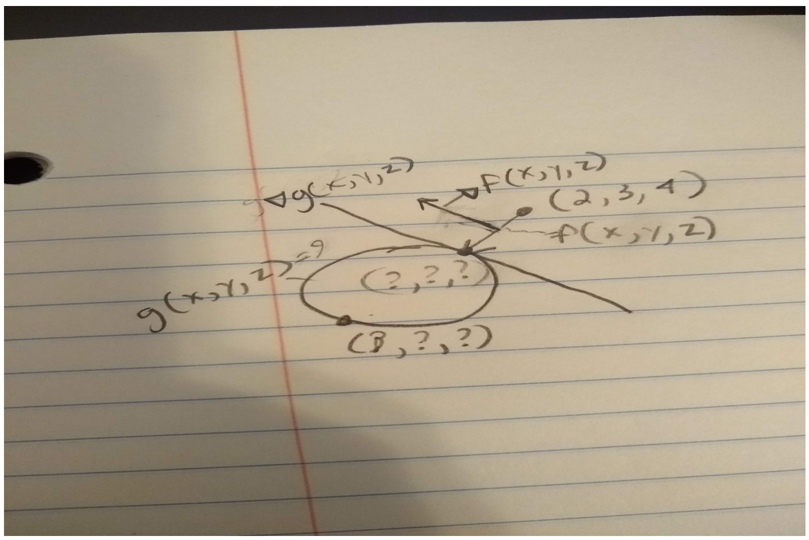


Figure 1: 2-D sketch of the problem

Since a function and its gradient vector are perpendicular, it can be shown that $\nabla f$ and $\nabla g$ are parallel. Thus it can be said, $\nabla f = \lambda \, \nabla g$ where $\lambda$ is some constant.

This can be represented by the system of equations

$$2(x - 2) = 2x\lambda \tag{1a}$$
$$2(y - 3) = 2y\lambda \tag{1b}$$
$$2(z - 4) = 2z\lambda \tag{1c}$$
$$x^2 + y^2 + z^2 = 9 \tag{1d}$$

Solving (1a) – (1c) in terms of $\lambda$ will then be plugged into (1d) to determine the points on $g(x, y, z) = 9$ closest and furthest from (2,3,4).

**1a**

$$
\begin{aligned}
2(x-2) =& 2x\lambda \\
x-2 =& x\lambda \\
\frac{2}{1-\lambda} =& x
\end{aligned} \tag{2}
$$

**1b**

$$
\begin{aligned}
2(y-3) =& 2y\lambda \\
y-3 =& y\lambda \\
\frac{3}{1-\lambda} =& y
\end{aligned} \tag{3}
$$

**1c**

$$
\begin{aligned}
2(z-4) =& 2z\lambda \\
z-4 =& z\lambda \\
\frac{4}{1-\lambda} =& z
\end{aligned} \tag{4}
$$

**1d**

$$
\begin{aligned}
(\frac{2}{1-\lambda})^2 + (\frac{3}{1-\lambda})^2 + (\frac{4}{1-\lambda})^2 =& 9 \\
\frac{4}{(1-\lambda)^2} + \frac{9}{(1-\lambda)^2} + \frac{16}{(1-\lambda)^2} =& 9 \\
\frac{29}{(1-\lambda)^2} =& 9 \\
(1-\lambda)^2 =& \frac{29}{9} \\
1-\lambda = \frac{\sqrt{29}}{3} \text{ OR } 1-\lambda = -\frac{\sqrt{29}}{3} & \\
1 \pm \frac{\sqrt{29}}{3} =& \lambda
\end{aligned} \tag{5}
$$

For (2,3,4) and $\lambda = 1 - \frac{\sqrt{29}}{3}$,

$$\begin{aligned}\frac{2}{1-\lambda} =& x \\ \frac{2}{1-(1+\frac{\sqrt{29}}{3})} =& x \\ \frac{-6}{\sqrt{29}} =& x\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{2}{1-\lambda} =& x \\ \frac{2}{1-(1-\frac{\sqrt{29}}{3})} =& x \\ \frac{6}{\sqrt{29}} =& x\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{3}{1-\lambda} =& y \\ \frac{3}{1-(1+\frac{\sqrt{29}}{3})} =& y \\ \frac{-9}{\sqrt{29}} =& y\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{3}{1-\lambda} =& y \\ \frac{3}{1-(1-\frac{\sqrt{29}}{3})} =& y \\ \frac{9}{\sqrt{29}} =& y\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{4}{1-\lambda} =& z \\ \frac{4}{1-(1+\frac{\sqrt{29}}{3})} =& z \\ \frac{-12}{\sqrt{29}} =& z\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{4}{1-\lambda} &= z \\ \frac{4}{1-(1-\frac{\sqrt{29}}{3})} &= z \\ \frac{12}{\sqrt{29}} &= z\end{aligned} \tag{11}$$

Let P be the point when $\lambda = 1 - \frac{\sqrt{29}}{3}$, then $P(\frac{6}{\sqrt{29}}, \frac{9}{\sqrt{29}}, \frac{12}{\sqrt{29}}))$

Let Q be the point when $\lambda = 1 + \frac{\sqrt{29}}{3}$, then $Q(\frac{-6}{\sqrt{29}}, \frac{-9}{\sqrt{29}}, \frac{-12}{\sqrt{29}}))$

Since all values for P are positive and all values of Q are negative, it follows that P is the closest point to (2,3,4) on the sphere $x^2 + y^2 + z^2 = 9$ and Q is the furthest.