Homework #5

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Refer to the dataset ConcussionsByTeamAndYear.csv found under the Week Six course content of D2L. Let Y_i be the number of concussions from team i=1,...,32. The model is $Y_i|\lambda_i \sim Poisson(\lambda_i)$ and the prior is $\lambda_i|\theta \sim Gamma(1,\theta)$ where $\theta \sim Gamma(0.1,0.1)$.

1

Derive the full conditional distribution of λ_1 .

$$f(\lambda_{1}|Y,\theta) \propto f(Y,|\lambda_{1},\theta)f(\lambda_{1}|\theta)$$

$$= f(Y|\lambda_{1},\theta)f(\theta)f(\lambda_{1}|\theta)$$

$$= \frac{e^{-\lambda_{1}}\lambda_{1}^{Y}}{Y!} \cdot \theta^{0.1-1}e^{-0.1\theta} \cdot \theta e^{-\theta\lambda_{1}}$$

$$\propto \theta^{0.1}\lambda^{Y}e^{-\lambda_{1}-0.1\theta-\theta\lambda_{1}}$$
(1)

When holding everything but λ_1 , then

$$f(\lambda_1|Y,\theta) \propto \lambda^{Y+1} e^{-\lambda_1(\theta+1)} \sim Gamma(Y+1,\theta+1)$$

 $\mathbf{2}$

Derive the full conditional distribution of θ .

Following the result in (a) and holding θ fixed,

$$P(\theta|Y,\lambda) \propto p(Y|\lambda,\theta)p(\lambda|\theta)p(\theta)$$

$$= \prod_{i=1}^{n} \left[\frac{\lambda_{i}^{y_{i}} e^{-\lambda_{i}}}{y_{i}!} \right] \prod_{i=1}^{n} \left[\theta e^{-\theta \lambda_{i}} \right] \frac{0.1^{0.1}}{\Gamma(0.1)} \theta^{0.1-1} e^{-0.1\theta}$$

$$\propto \theta^{n} e^{-\theta \sum_{i=1}^{n} \lambda_{i}} \theta^{0.1-1} e^{-0.1\theta}$$

$$= \theta^{n+0.1-1} e^{-(0.1+\sum_{i=1}^{n} \lambda_{i})\theta}$$
(2)

$$\theta|Y, \lambda \sim Gamma(n+0.1, 0.1 + \sum_{i=1}^{n} \lambda_i)$$

3

Write Gibbs sampling code to draw samples from the joint distribution of $(\lambda_1, ..., \lambda_{32}, \theta)$.

```
num.teams <- 32
num.params <- num.teams + 1 # for theta</pre>
n.iters <- 30000
res.2012 <- res.2013 <- matrix(0,n.iters, num.params)
lambda.names <- sapply(1:num.teams, function(x) paste0("lambda_", x))</pre>
colnames(res.2012) <- colnames(res.2013) <- c(lambda.names, "theta")</pre>
# initial values for first row
lambdas.init <- concussions %>%
  mutate(mean = (X2012 + X2013) / 2)
theta <- 0.1
lambdas <- lambdas.init</pre>
res.2012[1,] <- res.2013[1,] <- c(lambdas.init$mean, theta)
# The loop to perform Gibbs sampling
for(i in 2:n.iters){
  # Random draw from full conditional distribution of lambdas
  lambdas.2012 <- rgamma(num.teams, lambdas.init$X2012 + 1, theta + 1)
  lambdas.2013 <- rgamma(num.teams, lambdas.init$X2013 + 1, theta + 1)</pre>
  # Random draw from full conditional distribution of theta
  theta.2012 <- rgamma(1, n.iters + 0.1, sum(lambdas.2012) + 0.1)
  theta.2013 <- rgamma(1, n.iters + 0.1, sum(lambdas.2013) + 0.1)
 res.2012[i,] <- c(lambdas.2012, theta.2012)
  res.2013[i,] <- c(lambdas.2013, theta.2013)
res.2012 <- as.data.frame(res.2012)
res.2013 <- as.data.frame(res.2013)
```

4

Show trace plots of the samples for λ_1 and θ . (These plots of the value of the parameter on the y-axis and the iteration number on the x-axis).

5

Plot the estimated posterior mean of λ_i versus Y_i and comment on whether the code is returning reasonable estimates

```
lambdas.mean.2012 <- colMeans(res.2012) %>% head(-1)
concussions %>%
  bind_cols(Posterior.Mean = lambdas.mean.2012) %>%
  select(X, Actual = X2012, Posterior.Mean) %>%
  pivot_longer(-X, names_to = "Category", values_to = "value") %>%
  mutate(X = str_remove(X, "\xa0")) %>%
  ggplot(aes(x = X, color = Category, y = value)) +
  geom_point() +
```

```
theme(axis.text.x = element_text(angle = 90, vjust = 0.5, hjust=1)) +
     labs(x = "Team", y = "Number of Concussions")
     12.5
Number of Concussions
     10.0
                                                                                                                                                                                                         Category
       7.5
                                                                                                                                                                                                              Actual
                                                                                                                                                                                                               Posterior.Mean
      2.5
                    Atlanta Falcons -
              Arizona Cardinals
                          Baltimore Ravens -
                                          Chicago Bears
                                                                                                         Miami Dolphins
                                                                                                                                                                  San Diego Chargers
                               Buffalo Bills
                                     Carolina Panthers
                                                Cincinnati Bengals
                                                            Dallas Cowboys
                                                                  Denver Broncos
                                                                             Green Bay Packers
                                                                                   Houston Texans
                                                                                        Indianapolis Colts
                                                                                                                    New England Patriots
                                                                                                                                                 Philadelphia Eagles
                                                                                                                                                            Saint Louis Rams
                                                                                                                                                                       San Francisco 49ers
                                                                                                                                                                                              Washington Redskins
                                                      Cleveland Browns
                                                                       Detroit Lions
                                                                                             lacksonville Jaguars
                                                                                                    Kansas City Chiefs
                                                                                                               Minnesota Vikings
                                                                                                                          New Orleans Saints
                                                                                                                                New York Giants
                                                                                                                                     New York Jets
                                                                                                                                           Oakland Raiders
                                                                                                                                                       Pittsburgh Steelers
                                                                                                                                                                                   ampa Bay Buccaneers
lambdas.mean.2013 <- colMeans(res.2013) %>% head(-1)
concussions %>%
     bind_cols(Posterior.Mean = lambdas.mean.2013) %>%
     select(X, Actual = X2013, Posterior.Mean) %>%
     pivot_longer(-X, names_to = "Category", values_to = "value") %>%
     mutate(X = str_remove(X, "\xa0")) %>%
     ggplot(aes(x = X, color = Category, y = value)) +
     geom point() +
     theme(axis.text.x = element_text(angle = 90, vjust = 0.5, hjust=1)) +
     labs(x = "Team", y = "Number of Concussions")
     12.5
     10.0
Number of Concussions
       7.5
                                                                                                                                                                                                         Category
                                                                                                                                                                                                              Actual
      5.0
                                                                                                                                                                                                               Posterior.Mean
      0.0 -
              Arizona Cardinals -
                    Atlanta Falcons -
                          Baltimore Ravens -
                                Buffalo Bills -
                                     Carolina Panthers -
                                                            Dallas Cowboys -
                                                                 Denver Broncos -
                                          Chicago Bears
                                                Cincinnati Bengals
                                                      Cleveland Browns
                                                                       Detroit Lions
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                                                                                   Houston Texans
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                                                                                                         Miami Dolphins
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                                                                                                                                New York Giants
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                                                                                                                                                                             Seattle Seahawks
                                                                                                                                                                                   fampa Bay Buccaneers
                                                                                                                                                                                        Tennessee Titans
                                                                                                                                                                                              Washington Redskins
                                                                                                    Team
```

The posterior mean is pretty close to the actual estimates for each year indicating that this is not a bad model.