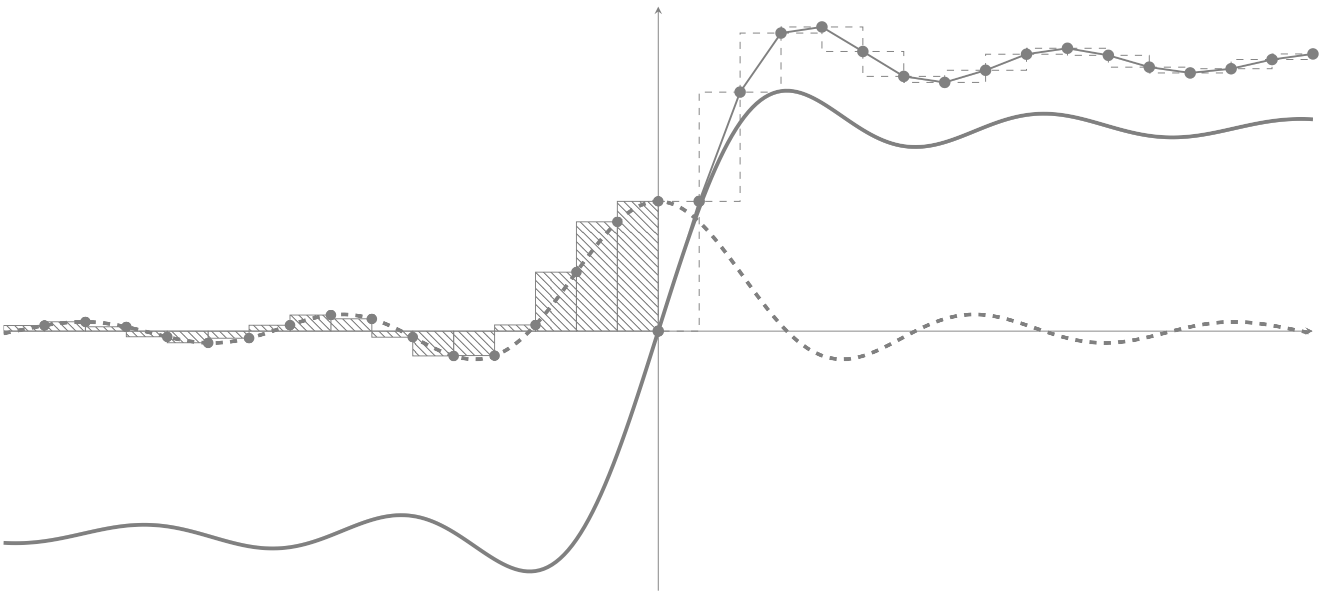
calculus 1

with free online interactive materials



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If you distribute this work or a derivative, include the history of the document. The source code is available at:

http://github.com/mooculus/calculus1

This text contains source material from several other open-source texts:

- *Calculus: Early Transcendentals*. Guichard. Copyright © 2015 Guichard, Creative Commons Attribution-NonCommercial-ShareAlike License 3.0. http://www.whitman.edu/mathematics/calculus/
- *APEX Calculus*. Hartman, Heinold, Siemers, Chalishajar, Bowen (Ed.). Copyright © 2014 Hartman, Creative Commons Attribution-Noncommercial 3.0. http://www.apexcalculus.com/
- *Elementary Calculus: An Infinitesimal Approach*. Keisler. Copyright © 2015 Keisler, Creative Commons Attribution-NonCommercial-ShareAlike License 3.0. http://www.math.wisc.edu/~keisler/calc.html

This book is typeset in the STIX and Gillius fonts.

We will be glad to receive corrections and suggestions for improvement at: ximera@math.osu.edu

Contents

Part I

Functions

1 Understanding functions

After completing this section, students should be able to do the following.

- State the definition of a function.
- Find the domain and range of a function.
- Distinguish between functions by considering their domains.
- Determine where a function is positive or negative.
- Plot basic functions.
- Perform basic operations and compositions on functions.
- Work with piecewise defined functions.
- Determine if a function is one-to-one.
- Recognize different representations of the same function.
- Define and work with inverse functions.
- Plot inverses of basic functions.
- Find inverse functions (algebraically and graphically).
- Find the largest interval containing a given point where the function is invertible.
- Determine the intervals on which a function has an inverse.

Break-Ground:

1.1 Same or different?

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, I have a pressing question.
Riley: Tell me. Tell me everything.
Devyn: Think about the function

$$f(x) = \frac{x^2 - 3x + 2}{x - 2}.$$

Riley: OK.
Devyn: Is this function equal to $g(x) = x - 1$?
Riley: Well if I plot them with my calculator, they look the same.
Devyn: I know!
Riley: And I suppose if I write

$$\begin{aligned} f(x) &= \frac{x^2 - 3x + 2}{x - 2} \\ &= \frac{(x-1)(x-2)}{x-2} \\ &= x - 1 \\ &= g(x). \end{aligned}$$

Devyn: Sure! But what about when $x = 2$? In this case

$$g(2) = 1 \qquad \text{but} \qquad f(2) \text{ is undefined!}$$

Riley: Right, $f(2)$ is undefined because we cannot divide by zero. Hmm. Now I see the problem. Yikes!

Problem 1. *In the context above, are f and g the same function?*

Multiple Choice:

(a) *yes*

(b) *no*

Problem 2. *Suppose f and g are functions but the domain of f is different from the domain of g. Could it be that f and g are actually the same function?*

Multiple Choice:

(a) *yes*

(b) *no*

Problem 3. *Can the same function be represented by different formulas?*

Multiple Choice:

(a) *yes*

(b) *no*

Problem 4. *Are $f(x) = |x|$ and $g(x) = \sqrt{x^2}$ the same function?*

Multiple Choice:

(a) *These are the same function although they are represented by different formulas.*

(b) *These are different functions because they have different formulas.*

Problem 5. *Let $f(x) = \sin^2(x)$ and $g(u) = \sin^2(u)$. The domain of each of these functions is all real numbers. Which of the following statements are true?*

Multiple Choice:

(a) *There is not enough information to determine if $f = g$.*

(b) *The functions are equal.*

(c) *If $x \neq u$, then $f \neq g$.*

(d) *We have $f \neq g$ since f uses the variable x and g uses the variable u.*

Dig-In:

1.2 For each input, exactly one output

Life is complex. Part of this complexity stems from the fact that there are many relationships between seemingly unrelated events. Armed with mathematics, we seek to understand the world. Perhaps the most relevant "real-world" relation is

the position of an object with respect to time.

Our observations seem to indicate that every instant in time is associated to a unique positioning of the objects in the universe. You may have heard the saying,

you cannot be two places at the same time,

and it is this fact that motivates our definition for functions.

Definition. A **function** is a relation between sets where for each input, there is exactly one output.

Question 1. *If our function is the "position with respect to time" of some object, then the input is*

Multiple Choice:

(a) *position*

(b) *time*

(c) *none of the above*

and the output is

Multiple Choice:

(a) *position*

(b) *time*

(c) *none of the above*

Something as simple as a dictionary could be thought of as a relation, as it connects *words* to *definitions*. However, a dictionary is not a function, as there are words with multiple definitions. On the other hand, if each word only had a single definition, then a dictionary would be a function.

Question 2. *Which of the following are functions?*

Select All Correct Answers:

(a) *Mapping words to their definition in a dictionary.*

(b) *Mapping social security numbers of living people to actual living people.*

(c) *Mapping people to their birth date.*

(d) *Mapping mothers to their children.*

What we are hoping to convince you is that the following are true:

(a) The definition of a function is well-grounded in a real context.

(b) The definition of a function is flexible enough that it can be used to model a wide range of phenomena.

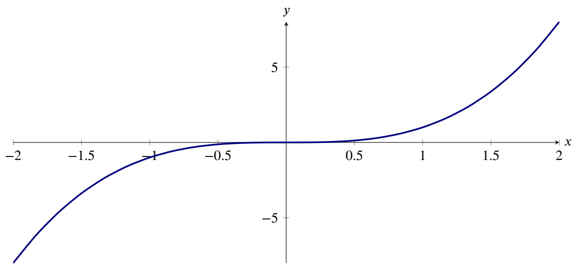
Whenever we talk about functions, we should explicitly state what type of things the inputs are and what type of things the outputs are. In calculus, functions often define a relation from (a subset of) the real numbers (denoted by $\mathbb{R}$) to (a subset of) the real numbers.

Definition. We call the set of the inputs of a function the **domain**, and we call the set of the outputs of a function the **range**.

Example 1. *Consider the function f that maps from the real numbers to the real numbers by taking a number and mapping* *it to its cube:*

$$1 \mapsto 1$$
$$-2 \mapsto -8$$
$$1.5 \mapsto 3.375$$

and so on. This function can be described by the formula $f(x) = x^3$ *or by the graph shown in the plot below:*



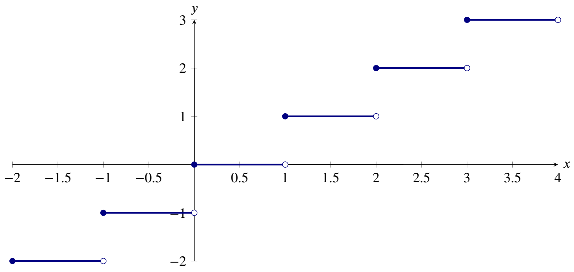
Warning. *A function is a relation (such that for each input, there is exactly one output) between sets. The formula and the graph are merely descriptions of this relation.*

- *A formula* ***describes*** *the relation using symbols.*
- *A graph* ***describes*** *the relation using pictures.*

The ***function is the relation itself****, and is independent of how it is described.*

Our next example may be a function that is new to you. It is the *greatest integer function*.

Example 2. *Consider the* ***greatest integer function****. This function maps any real number x to the greatest integer less than or equal to x. People sometimes write this as* $f(x) = \lfloor x \rfloor$*, where those funny symbols mean exactly the words above describing the function. For your viewing pleasure, here is a graph of the greatest integer function:*



Observe that here we have multiple inputs that give the same output. This is not a problem! To be a function, we merely need to check that for each input, there is exactly one output, and this condition is satisfied.

Question 3. *Compute:*

$$\lfloor 2.4 \rfloor$$

Question 4. *Compute:*

$$\lfloor -2.4 \rfloor$$

Notice that both the functions described above pass the so-called *vertical line test*.

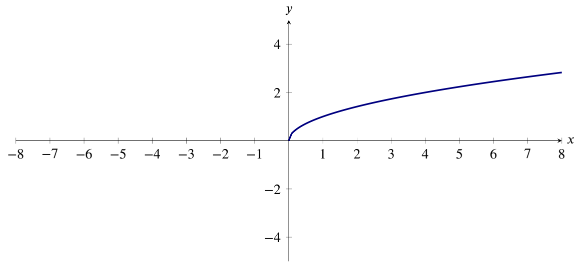
Theorem 1. *The curve* $y = f(x)$ *represents y as a function of x at* $x = a$ *if and only if the vertical line* $x = a$ *intersects the curve* $y = f(x)$ *at exactly one point. This is called the* ***vertical line test****.*

Sometimes the domain and range are the *entire* set of real numbers, denoted by $\mathbb{R}$. In our next examples we show that this is not always the case.

Example 3. *Consider the function that maps non-negative real numbers to their positive square root. This function can be described by the formula*

$$f(x) = \sqrt{x}.$$

The domain is $0 \leq x < \infty$*, which we prefer to write as* $[0, \infty)$ *in interval notation. The range is* $[0, \infty)$*. Here is a graph of* $y = f(x)$*:*



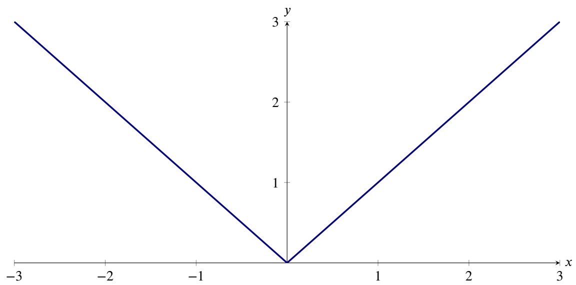
To really tease out the difference between a function and its description, let's consider an example of a function with two different descriptions.

Example 4. *Explain why* $\sqrt{x^2} = |x|$.

Although $\sqrt{x^2}$ may appear to simplify to just x, let's see what happens when we plug in some values.

$$\begin{aligned}\sqrt{3^2} &= \sqrt{9}\\ &= 3,\end{aligned} \qquad \text{and} \qquad \begin{aligned}\sqrt{(-3)^2} &= \sqrt{9}\\ &= 3.\end{aligned}$$

In an entirely similar way, we see that for any positive x, $f(-x) = x$. Hence $\sqrt{x^2} \neq x$. Rather we see that $\sqrt{x^2} = |x|$. The domain of $f(x) = \sqrt{x^2}$ is $(-\infty, \infty)$ and the range is $[0, \infty)$. For your viewing pleasure we've included a graph of $y = f(x)$:



Finally, we will consider a function whose domain is all real numbers except for a single point.

Example 5. *Are*

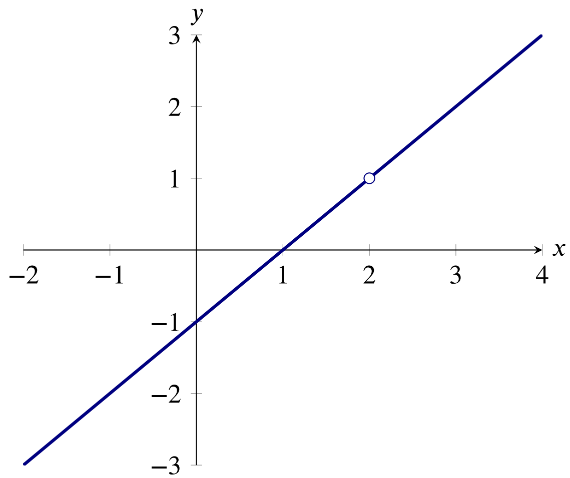
$$f(x) = \frac{x^2 - 3x + 2}{x - 2} = \frac{(x-2)(x-1)}{(x-2)}$$

and

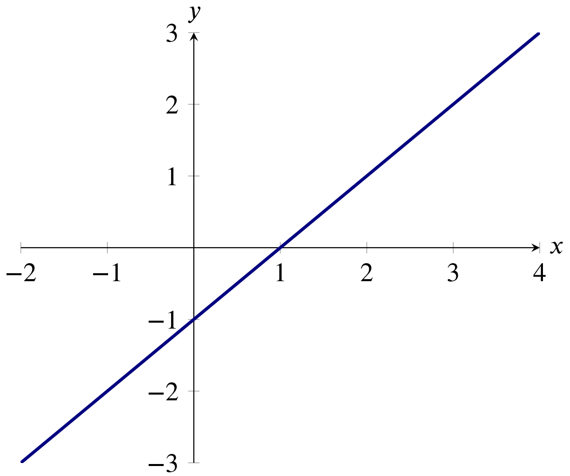
$$g(x) = x - 1$$

the same function?

Let's use a series of steps to think about this question. First, what if we compare graphs? Here we see a graph of f:



On the other hand, here is a graph of g:



Second, what if we compare the domains? We cannot evaluate f at $x = 2$. This is where f is undefined. On the other hand, there is no value of x where we cannot evaluate g. In other words, the domain of g is $(-\infty, \infty)$.

Since these two functions do not have the same graph, and they do not have the same domain, they must not be the same function.

However, if we look at the two functions everywhere except at $x = 2$, we can say that $f(x) = g(x)$. In other words,

$$f(x) = x - 1 \qquad \text{when} \qquad x \neq 2.$$

From this example we see that it is critical to consider the domain and range of a function.

Dig-In:

1.3 Compositions of functions

Given two functions, we can compose them. Let's give an example in a "real context."

Example 6. *Let*

$$g(m) = \textit{the amount of gas one can buy with } m \textit{ dollars,}$$

and let

$$f(g) = \textit{how far one can drive with } g \textit{ gallons of gas.}$$

What does $f(g(m))$ represent in this setting?

With $f(g(m))$ we first relate how far one can drive with g gallons of gas, and this in turn is determined by how much money m one has. Hence $f(g(m))$ represents how far one can drive with m dollars.

Composition of functions can be thought of as putting one function inside another. We use the notation

$$(f \circ g)(x) = f(g(x)).$$

Warning. *The composition $f \circ g$ only makes sense if*

$$\{\textit{the range of } g\} \textit{ is contained in or equal to } \{\textit{the domain of } f\}$$

Example 7. *Suppose we have*

$$\begin{aligned} f(x) &= x^2 + 5x + 4 && \textit{for } -\infty < x < \infty, \\ g(x) &= x + 7 && \textit{for } -\infty < x < \infty. \end{aligned}$$

Find $f(g(x))$ and state its domain.

The range of g is $-\infty < x < \infty$, which is equal to the domain of f. This means the domain of $f \circ g$ is $-\infty < x < \infty$. Next, we substitute $x + 7$ for each instance of x found in

$$f(x) = x^2 + 5x + 4$$

and so

$$\begin{aligned} f(g(x)) &= f(x+7) \\ &= (x+7)^2 + 5(x+7) + 4. \end{aligned}$$

Now let's try an example with a more restricted domain.

Example 8. *Suppose we have:*

$$\begin{aligned} f(x) &= x^2 && \textit{for } -\infty < x < \infty, \\ g(x) &= \sqrt{x} && \textit{for } 0 \le x < \infty. \end{aligned}$$

Find $f(g(x))$ and state its domain.

The domain of g is $0 \le x < \infty$. From this we can see that the range of g is $0 \le x < \infty$. This is contained in the domain of f.

This means that the domain of $f \circ g$ is $0 \le x < \infty$. Next, we substitute $\sqrt{x}$ for each instance of x found in

$$f(x) = x^2$$

and so

$$\begin{aligned} f(g(x)) &= f(\sqrt{x}) \\ &= \left(\sqrt{x}\right)^2. \end{aligned}$$

We can summarize our results as a piecewise function, which looks somewhat interesting:

$$(f \circ g)(x) = \begin{cases} x & \text{if } 0 \le x < \infty \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Example 9. *Suppose we have:*

$$\begin{aligned} f(x) &= \sqrt{x} && \textit{for } 0 \le x < \infty, \\ g(x) &= x^2 && \textit{for } -\infty < x < \infty. \end{aligned}$$

Find $f(g(x))$ and state its domain.

> While the domain of g is $-\infty < x < \infty$, its range is only $0 \leq x < \infty$. This is exactly the domain of f. This means that the domain of $f \circ g$ is $-\infty < x < \infty$. Now we may substitute x^2 for each instance of x found in
>
> $$f(x) = \sqrt{x}$$
>
> and so
>
> $$\begin{aligned} f(g(x)) &= f(x^2) \\ &= \sqrt{x^2}, \\ &= |x|. \end{aligned}$$

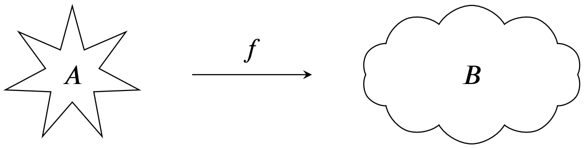
Compare and contrast the previous two examples. We used the same functions for each example, but composed them in different ways. The resulting compositions are not only different, they have different domains!

Dig-In:

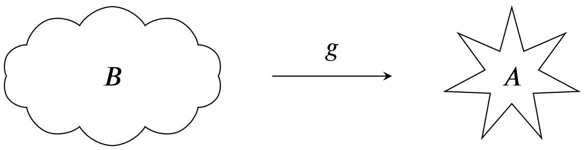
1.4 Inverses of functions

If a function maps every "input" to exactly one "output," an inverse of that function maps every "output" to exactly one "input." We need a more formal definition to actually say anything with rigor.

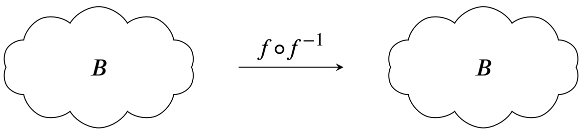
Definition. Let f be a function with domain A and range B:



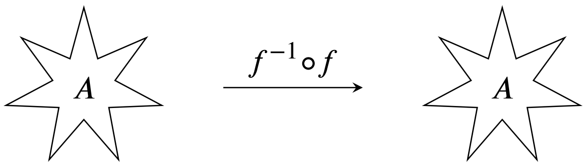
Let g be a function with domain B and range A:



We say that f and g are **inverses** of each other if $f(g(b)) = b$ for all b in B, and also $g(f(a)) = a$ for all a in A. Sometimes we write $g = f^{-1}$ in this case.



and



So, we could rephrase these conditions as

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

These two simple equations are somewhat more subtle than they initially appear.

Question 5. *Let f be a function. If the point $(1,9)$ is on the graph of f, what point must be the the graph of f^{-1}?*

Warning. *This notation can be very confusing. Keep a watchful eye:*

$$f^{-1}(x) = \textit{the inverse function of } f(x).$$
$$f(x)^{-1} = \frac{1}{f(x)}.$$

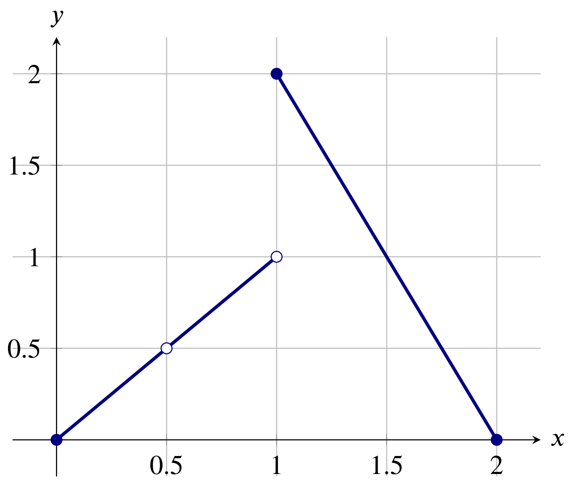
Question 6. *Which of the following is notation for the inverse of the function $\sin(\theta)$ on the interval $[-\pi/2, \pi/2]$?*

Multiple Choice:

(a) $\sin^{-1}(\theta)$

(b) $\sin(\theta)^{-1}$

Question 7. *Consider the graph of $y = f(x)$ below*



Is $f(x)$ invertible at $x = 0.5$?

Multiple Choice:

(a) *yes*

(b) *no*

Question 8.

$$f^{-1}(1) = \boxed{?}$$

So far, we've only dealt with abstract examples. Let's see if we can ground this in a real-life context.

Example 10. *The function*

$$f(t) = \left(\frac{9}{5}\right) t + 32$$

takes a temperature t in degrees Celsius, and converts it into Fahrenheit. The domain of this function is $-\infty < t < \infty$. What does the inverse of this function tell you? What is the inverse of this function?

If f converts Celsius measurements to Fahrenheit measurements of temperature, then f^{-1} converts Fahrenheit measurements to Celsius measurements of temperature.

To find the inverse function, first note that

$$f(f^{-1}(t)) = t \qquad \text{by the definition of inverse functions.}$$

Now write out the left-hand side of the equation

$$f(f^{-1}(t)) = \left(\frac{9}{5}\right) f^{-1}(t) + 32 \qquad \text{by the rule for } f$$

and solve for $f^{-1}(t)$.

$$\begin{aligned} \left(\frac{9}{5}\right) f^{-1}(t) + 32 &= t && \text{by the rule for } f \\ \left(\frac{9}{5}\right) f^{-1}(t) &= t - 32 \\ f^{-1}(t) &= \left(\frac{5}{9}\right)(t - 32). \end{aligned}$$

So $f^{-1}(t) = \left(\frac{5}{9}\right)(t - 32)$ is the inverse function of f, which converts a Fahrenheit measurement back into a Celsius measurement. The domain of this inverse function is $-\infty < t < \infty$.

Finally, we could check our work again using the definition of inverse functions. We have already guaranteed that

$$f(f^{-1}(t)) = t,$$

since we solved for f^{-1} in our calculation. On the other hand,

$$\begin{aligned} f^{-1}(f(t)) &= \left(\frac{5}{9}\right)(f(t) - 32) \\ &= \left(\frac{5}{9}\right)(f(t) - 32) \end{aligned}$$

which you should simplify to check that $f^{-1}(f(t)) = t$.

We have examined several functions in order to determine their inverse functions, but there is still more to this story. Not every function has an inverse function, so we must learn how to check for this situation.

Question 9. *Let f be a function, and imagine that the points $(2, 3)$ and $(7, 3)$ are both on its graph. Could f have an inverse function?*

Multiple Choice:

(a) *yes*

(b) *no*

Look again at the last question. If two different inputs for a function have the same output, there is no hope of that function having an inverse function. Why? This is because the inverse function must also be a function, and a function can only have one output for each input. More specifically, we have the next definition.

Definition. A function is called **one-to-one** if each output value corresponds to exactly one input value.

Question 10. *Which of the following are functions that are also one-to-one?*

Select All Correct Answers:

(a) *Mapping words to their meaning in a dictionary.*

(b) *Mapping social security numbers of living people to actual living people.*

(c) *Mapping people to their birthday.*

(d) *Mapping mothers to their children.*

Question 11. *Which of the following functions are one to one? Select all that apply.*

Select All Correct Answers:

(a) $f(x) = x$

(b) $f(x) = x^2$

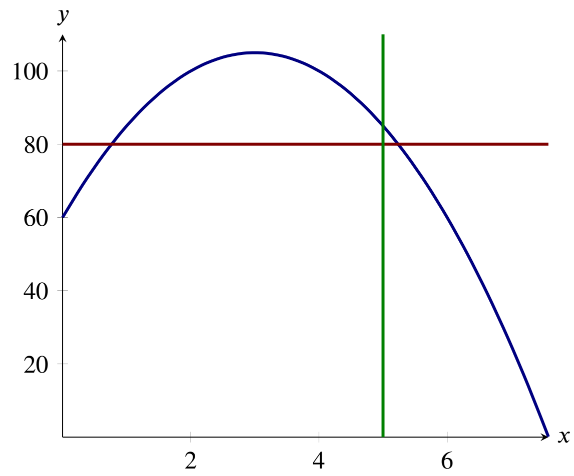
(c) $f(x) = x^3 - 4x$

(d) $f(x) = x^3 + 4$

You may recall that a plot gives y as a function of x if every vertical line crosses the plot at most once, and we called this the **vertical line test**. Similarly, a function is one-to-one if every horizontal line crosses the plot at most once, and we call this the **horizontal line test**.

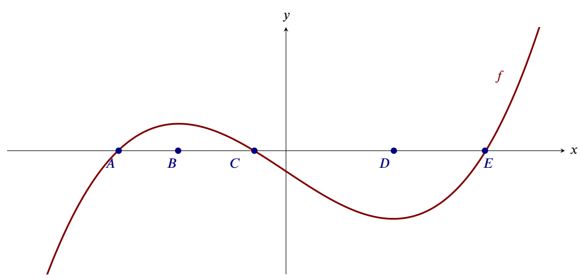
Theorem 2. *A function is one-to-one at $x = a$ if the horizontal line $y = f(a)$ intersects the curve $y = f(x)$ in exactly one point. This is called the* ***horizontal line test****.*

Below, we give a graph of $f(x) = -5x^2 + 30x + 60$. While this graph passes the vertical line test, and hence represents y as a function of x, it does not pass the horizontal line test, so the function is not one-to-one.



As we have discussed, we can only find an inverse of a function when it is one-to-one. If a function is not one-to-one, but we still want an inverse, we must restrict the domain. Let's see what this means in our next examples.

Question 12. *Consider the graph of the function f below:*



On which of the following intervals is f one-to-one?

Select All Correct Answers:

(a) $[A, B]$

(b) $[A, C]$

(c) $[B, D]$

(d) $[C, E]$

(e) $[C, D]$

This idea of restricting the domain is critical for understanding functions like $f(x) = \sqrt{x}$.

Warning. *We define $f(x) = \sqrt{x}$ to be the positive square-root, so that we can be sure that f is a function. Thinking of the square-root as the inverse of the squaring function, we can see the issue a little more clearly. There are two x-values that square to 9.*

$$x^2 = 9 \qquad \textit{means } x = \pm 3$$

*Since we require that **square-root is a function**, we must have only one output value when we plug in 9. We choose the positive square-root, meaning that*

$$\sqrt{9} = 3.$$

Example 11. *Consider the function*

$$f(x) = x^2.$$

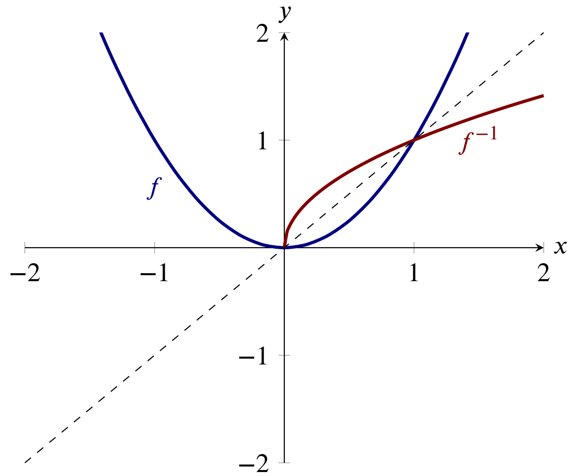
Does f have an inverse? If so, what is it? If not, attempt to restrict the domain of f and find an inverse on the restricted domain.

In this case f is not one-to-one. However, it is one-to-one on the interval $[0, \infty)$. Hence we can find an inverse of $f(x) = x^2$ on this interval. We plug $f^{-1}(x)$ into f and write

$$\begin{aligned} f(f^{-1}(x)) &= \left(f^{-1}(x)\right)^2 \\ x &= \left(f^{-1}(x)\right)^2. \end{aligned}$$

Since the domain of f is $[0, \infty)$, we know that x is positive. This means we can take the square-root of each side of the equation to find that

$$\sqrt{x} = f^{-1}(x).$$



Example 12. *Consider the function*

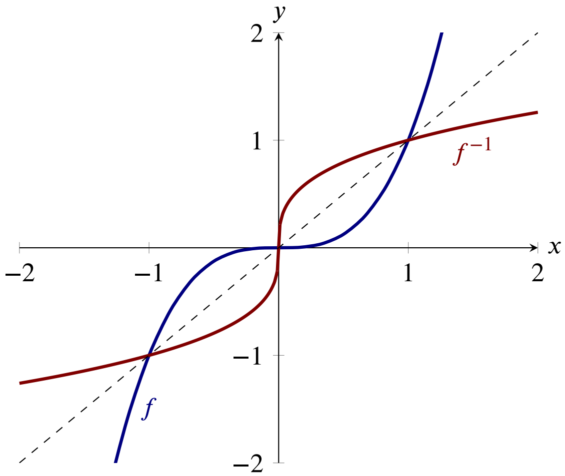
$$f(x) = x^3.$$

Does $f(x)$ have an inverse? If so, what is it? If not, attempt to restrict the domain of $f(x)$ and find an inverse on the restricted domain.

In this case $f(x)$ is one-to-one. We may write

$$\begin{aligned} f(f^{-1}(x) &= \left(f^{-1}(x)\right)^3 \\ x &= \left(f^{-1}(x)\right)^3 \\ \sqrt[3]{x} &= f^{-1}(x). \end{aligned}$$

For your viewing pleasure we give a graph of $y = f(x) = x^3$ and $y = f^{-1}(x) = \sqrt[3]{x}$. Note, the graph of f^{-1} is the image of f after being flipped over the line $y = x$.


y
2
1
−2
−1
1
2
x
−1
−2
f^{-1}
f

2 Review of famous functions

After completing this section, students should be able to do the following.

- Know the graphs and properties of “famous” functions.
- Know and use the properties of exponential and logarithmic functions.
- Understand the relationship between exponential and logarithmic functions.
- Understand the definition of a rational function.
- Understand the properties of trigonometric functions.
- Evaluate expressions and solve equations involving trigonometric functions and inverse trigonometric functions.

Break-Ground:

2.1 How crazy could it be?

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, do you remember when we first starting graphing functions? Like with a "T-chart?"
Riley: I remember everything.
Devyn: I used to get so excited to plot stuff! I would wonder: "What crazy curve would be drawn this time? What crazy picture will I see?"
Riley: Then we learned about the slope-intercept form of a line. Good-old

$$y = mx + b.$$

Devyn: Yeah, but lines are really boring. What about polynomials? What could you tell me about

$$y = 5x^6 - 5x^5 - 5x^4 + 5x^3 + x^2 - 1$$

just by looking at the equation?
Riley: Hmmmm. I'm not sure…

Problem 1. *When x is a large number (furthest from zero), which term of* $5x^6 - 5x^5 - 5x^4 + 5x^3 + x^2 - 1$ *is largest (furthest from zero)?*

Multiple Choice:

(a) -1

(b) x^2

(c) $5x^3$

(d) $-5x^4$

(e) $-5x^5$

(f) $5x^6$

Problem 2. *When x is a small number (near zero), which term of* $5x^6 - 5x^5 - 5x^4 + 5x^3 + x^2 - 1$ *is largest (furthest from zero)?*

Multiple Choice:

(a) -1

(b) x^2

(c) $5x^3$

(d) $-5x^4$

(e) $-5x^5$

(f) $5x^6$

Problem 3. *Very roughly speaking, what does the graph of* $y = 5x^6 - 5x^5 - 5x^4 + 5x^3 + x^2 - 1$ *look like?*

Multiple Choice:

(a) *The graph starts in the lower left and ends in the upper right of the plane.*

(b) *The graph starts in the lower right and ends in the upper left of the plane.*

(c) *The graph looks something like the letter "U."*

(d) *The graph looks something like an upside down letter "U."*

Dig-In:

2.2 Polynomial functions

The functions you are most familiar with are probably polynomial functions.

What are polynomial functions?

Definition. A **polynomial function** in the variable x is a function which can be written in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i's are all constants (called the **coefficients**) and n is a whole number (called the **degree** when $n \neq 0$). The domain of a polynomial function is $(-\infty, \infty)$.

Question 13. *Which of the following are polynomial functions?*

Select All Correct Answers:

(a) $f(x) = 0$

(b) $f(x) = -9$

(c) $f(x) = 3x + 1$

(d) $f(x) = x^{1/2} - x + 8$

(e) $f(x) = -4x^{-3} + 5x^{-1} + 7 - 18x^2$

(f) $f(x) = (x+1)(x-1) + e^x - e^x$

(g) $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$

(h) $f(x) = x^7 - 32x^6 - \pi x^3 + 45/84$

The phrase above "in the variable x" can actually change.

$$y^2 - 4y + 1$$

is a polynomial in y, and

$$\sin^2(x) + \sin(x) - 3$$

is a polynomial in $\sin(x)$.

What can the graphs look like?

Fun fact:

Theorem 3 (The Fundamental Theorem of Algebra)**.** *Every polynomial of the form*

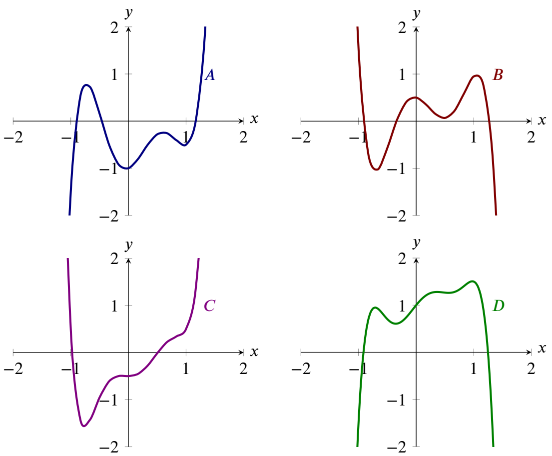
$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i's are real (or even complex!) numbers and $a_n \neq 0$ has exactly n (possibly repeated) complex roots.

Remember, a **root** is where a polynomial is zero. The theorem above is a deep fact of mathematics. The great mathematician Gauss proved the theorem in 1799 for his doctoral thesis.

The upshot as far as we are concerned is that when we plot a polynomial of degree n, its graph will cross the x-axis at most n times.

Example 13. *Here we see the the graphs of four polynomial functions.*



*For each of the curves, determine if the polynomial has **even** or **odd** degree, and if the leading coefficient (the one next to the highest power of x) of the polynomial is **positive** or **negative**.*

- Curve A is defined by an odd degree polynomial with a positive leading term.
- Curve B is defined by an odd degree polynomial with a negative leading term.
- Curve C is defined by an even degree polynomial with a positive leading term.
- Curve D is defined by an even degree polynomial with a negative leading term.

Dig-In:

2.3 Rational functions

What are rational functions?

Definition. A **rational function** in the variable x is a function the form

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions. The domain of a rational function is all real numbers except for where the denominator is equal to zero.

Question 14. *Which of the following are rational functions?*

Select All Correct Answers:

(a) $f(x) = 0$

(b) $f(x) = \dfrac{3x+1}{x^2-4x+5}$

(c) $f(x) = e^x$

(d) $f(x) = \dfrac{\sin(x)}{\cos(x)}$

(e) $f(x) = -4x^{-3} + 5x^{-1} + 7 - 18x^2$

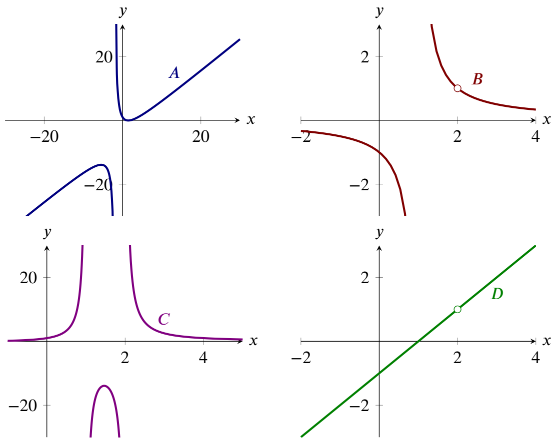
(f) $f(x) = x^{1/2} - x + 8$

(g) $f(x) = \dfrac{\sqrt{x}}{x^3 - x}$

What can the graphs look like?

There is a somewhat wide variation in the graphs of rational functions.

Example 14. *Here we see the the graphs of four rational functions.*



Match the curves A, B, C, and D with the functions

$$\frac{x^2-3x+2}{x-2}, \qquad \frac{x^2-3x+2}{x+2},$$
$$\frac{x-2}{x^2-3x+2}, \qquad \frac{x+2}{x^2-3x+2}.$$

Consider $\dfrac{x^2-3x+2}{x-2}$. This function is undefined only at $x = 2$. Of the curves that we see above, D is undefined exactly at $x = 2$.

Now consider $\dfrac{x^2-3x+2}{x+2}$. This function is undefined only at $x = -2$. The only function above that undefined exactly at $x = -2$ is curve A.

Now consider $\dfrac{x-2}{x^2-3x+2}$. This function is undefined at the roots of

$$x^2 - 3x + 2 = (x-2)(x-1).$$

Hence it is undefined at $x = 2$ and $x = 1$. It looks like both curves B and C would work. Distinguishing between these two curves is easy enough if we evaluate at $x = -2$. Check it out.

$$\begin{aligned}\left[\frac{x-2}{x^2-3x+2}\right]_{x=-2} &= \frac{-2-2}{(-2)^2-3(-2)+2}\\ &= \frac{-4}{4+6+2}\\ &= \frac{-4}{12}.\end{aligned}$$

Since this is negative, we see that $\dfrac{x-2}{x^2-3x+2}$ corresponds to curve $\boldsymbol{B}$.

Finally, it must be the case that curve $\boldsymbol{C}$ corresponds to $\dfrac{x+2}{x^2-3x+2}$. We should note that if this function is evaluated at $x = -2$, the output is zero, and this corroborates our work above.

Dig-In:

2.4 Exponential and logarithmetic functions

Exponential and logarithmic functions may seem somewhat esoteric at first, but they model many phenomena in the real-world.

What are exponential and logarithmic functions?

Definition. An **exponential function** is a function of the form

$$f(x) = b^x$$

where $b \neq 1$ is a positive real number. The domain of an exponential function is $(-\infty, \infty)$.

Question 15. *Is b^{-x} an exponential function?*

Multiple Choice:

(a) *yes*

(b) *no*

Definition. A **logarithmic function** is a function defined as follows

$$\log_b(x) = y \qquad \text{means that} \qquad b^y = x$$

where $b \neq 1$ is a positive real number. The domain of a logarithmic function is $(0, \infty)$.

In either definition above b is called the **base**.

Connections between exponential functions and logarithms Let b be a positive real number with $b \neq 1$.

- $b^{\log_b(x)} = x$ for all positive x
- $\log_b(b^x) = x$ for all real x

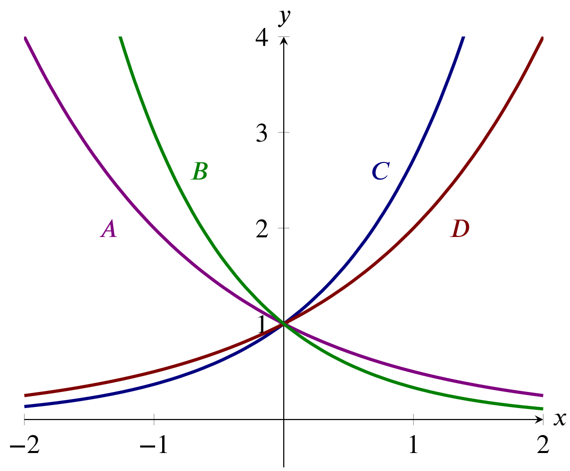
Question 16. *What exponent makes the following expression true?*

$$3^x = e^{\left(x \cdot \boxed{?}\right)}.$$

What can the graphs look like?

Graphs of exponential functions

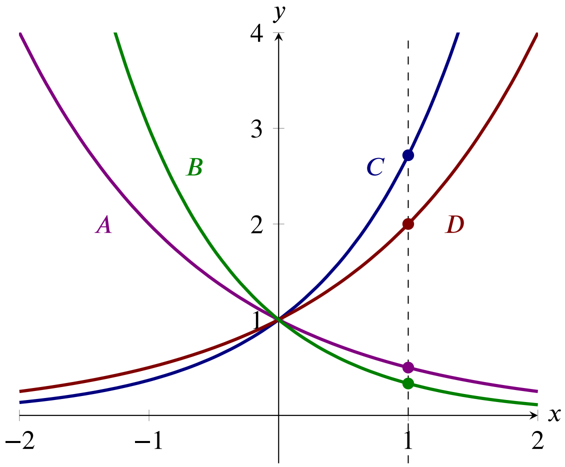
Example 15. *Here we see the the graphs of four exponential functions.*



Match the curves A, B, C, and D with the functions

$$e^x, \qquad \left(\frac{1}{2}\right)^x, \qquad \left(\frac{1}{3}\right)^x, \qquad 2^x.$$

> One way to solve these problems is to compare these functions along the vertical line $x = 1$,



Note

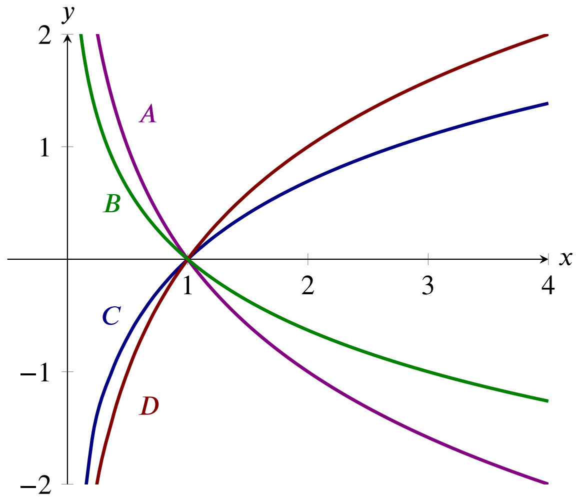
$$\left(\frac{1}{3}\right)^1 < \left(\frac{1}{2}\right)^1 < 2^1 < e^1.$$

Hence we see:

- $\left(\frac{1}{3}\right)^x$ corresponds to B.
- $\left(\frac{1}{2}\right)^x$ corresponds to A.
- 2^x corresponds to D.
- e^x corresponds to C.

Graphs of logarithmic functions

Example 16. *Here we see the the graphs of four logarithmic functions.*



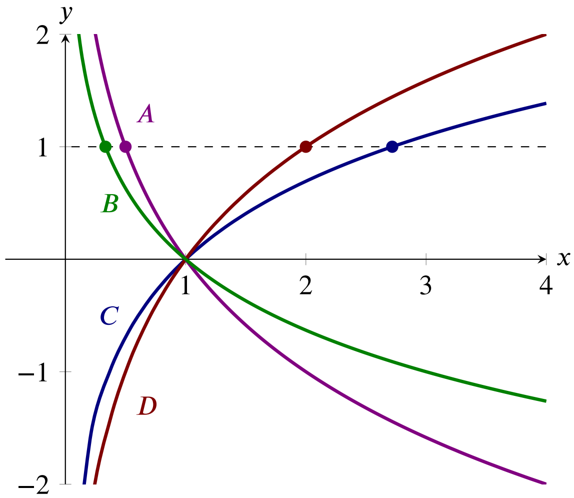
Match the curves A, B, C, and D with the functions

$$\ln(x), \qquad \log_{1/2}(x), \qquad \log_{1/3}(x), \qquad \log_2(x).$$

First remember what $\ln(x) = y$ means:

$$\ln(x) = y \qquad \text{means that} \qquad b^y = x.$$

So now examine each of these functions along the horizontal line $y = 1$



Note again (this is from the definition of a logarithm)

$$\left(\frac{1}{3}\right)^1 < \left(\frac{1}{2}\right)^1 < 2^1 < e^1.$$

Hence we see:

- $\log_{1/3}(x)$ corresponds to B.
- $\log_{1/2}(x)$ corresponds to A.
- $\log_2(x)$ corresponds to D.
- $\ln(x)$ corresponds to C.

Properties of exponential functions and logarithms

Working with exponential and logarithmic functions is often simplified by applying properties of these functions. These properties will make appearances throughout our work.

Properties of exponents Let b be a positive real number with $b \neq 1$.

- $b^m \cdot b^n = b^{m+n}$
- $b^{-1} = \dfrac{1}{b}$
- $(b^m)^n = b^{mn}$

Question 17. *What exponent makes the following true?*

$$2^4 \cdot 2^3 = 2^{\boxed{?}}$$

Properties of logarithms Let b be a positive real number with $b \neq 1$.

- $\log_b(m \cdot n) = \log_b(m) + \log_b(n)$
- $\log_b(m^n) = n \cdot \log_b(m)$
- $\log_b\left(\dfrac{1}{m}\right) = \log_b(m^{-1}) = -\log_b(m)$
- $\log_a(m) = \dfrac{\log_b(m)}{\log_b(a)}$

Question 18. *What value makes the following expression true?*

$$\log_2\left(\frac{8}{16}\right) = 3 - \boxed{?}$$

Question 19. *What makes the following expression true?*

$$\log_3(x) = \frac{\ln(x)}{\boxed{?}}$$

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