Dig-In:

Polynomial functions

Polynomials are some of our favorite functions.

The functions you are most familiar with are probably polynomial functions.

What are polynomial functions?

Definition 1. A polynomial function in the variable x is a function which can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where the a_i 's are all constants (called the **coefficients**) and n is a whole number (called the **degree** when $n \neq 0$). The domain of a polynomial function is $(-\infty, \infty)$.

Question 1 Which of the following are polynomial functions?

Select All Correct Answers:

- (a) f(x) = 0
- (b) $f(x) = -9 \checkmark$
- (c) f(x) = 3x + 1
- (d) $f(x) = x^{1/2} x + 8$
- (e) $f(x) = -4x^{-3} + 5x^{-1} + 7 18x^2$
- (f) $f(x) = (x+1)(x-1) + e^x e^x \checkmark$
- (g) $f(x) = \frac{x^2 3x + 2}{x 2}$
- (h) $f(x) = x^7 32x^6 \pi x^3 + 45/84 \checkmark$

Learning outcomes: Know the graphs and properties of "famous" functions.

The phrase above "in the variable x" can actually change.

$$y^2 - 4y + 1$$

is a polynomial in y, and

$$\sin^2(x) + \sin(x) - 3$$

is a polynomial in sin(x).

What can the graphs look like?

Fun fact:

Theorem 1 (The Fundamental Theorem of Algebra). Every polynomial of the form

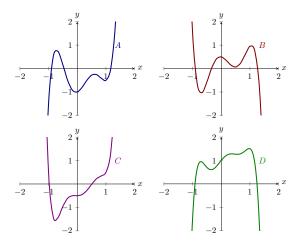
$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where the a_i 's are real (or even complex!) numbers and $a_n \neq 0$ has exactly n (possibly repeated) complex roots.

Remember, a **root** is where a polynomial is zero. The theorem above is a deep fact of mathematics. The great mathematician Gauss proved the theorem in 1799 for his doctoral thesis.

The upshot as far as we are concerned is that when we plot a polynomial of degree n, its graph will cross the x-axis at most n times.

Example 1. Here we see the graphs of four polynomial functions.



For each of the curves, determine if the polynomial has **even** or **odd** degree, and if the leading coefficient (the one next to the highest power of x) of the polynomial is **positive** or **negative**.

Explanation.

- Curve A is defined by an (even/odd ✓) degree polynomial with a (positive ✓ / negative) leading term.
- Curve B is defined by an (even/odd ✓) degree polynomial with a (positive/negative ✓) leading term.
- Curve C is defined by an (even \checkmark /odd) degree polynomial with a (positive \checkmark /negative) leading term.
- Curve D is defined by an (even ✓ /odd) degree polynomial with a (positive/negative ✓) leading term.