Dig-In:

Exponential and logarithmetic functions

Exponential and logarithmic functions illuminated.

Exponential and logarithmic functions may seem somewhat esoteric at first, but they model many phenomena in the real-world.

What are exponential and logarithmic functions?

Definition 1. *An* ***exponential function*** *is a function of the form*

$$f(x) = b^x$$

where $b \neq 1$ is a positive real number. The domain of an exponential function is $(-\infty, \infty)$.

Question 1 *Is b^{-x} an exponential function?*

Multiple Choice:

(a) *yes* ✓

(b) *no*

Feedback (attempt): *Note that*

$$b^{-x} = \left(b^{-1}\right)^x = \left(\frac{1}{b}\right)^x.$$

Definition 2. *A* ***logarithmic function*** *is a function defined as follows*

$$\log_b(x) = y \qquad \text{means that} \qquad b^y = x$$

where $b \neq 1$ is a positive real number. The domain of a logarithmic function is $(0, \infty)$.

In either definition above b is called the **base**.

Learning outcomes:

Connections between exponential functions and logarithms

Let b be a positive real number with $b \neq 1$.

- $b^{\log_b(x)} = x$ for all positive x
- $\log_b(b^x) = x$ for all real x

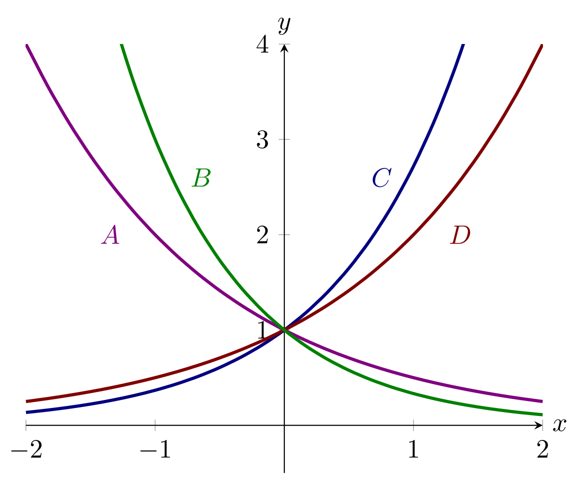
Question 2 *What exponent makes the following expression true?*

$$3^x = e^{\left(x \cdot \boxed{\ln 3}\right)}.$$

What can the graphs look like?

Graphs of exponential functions

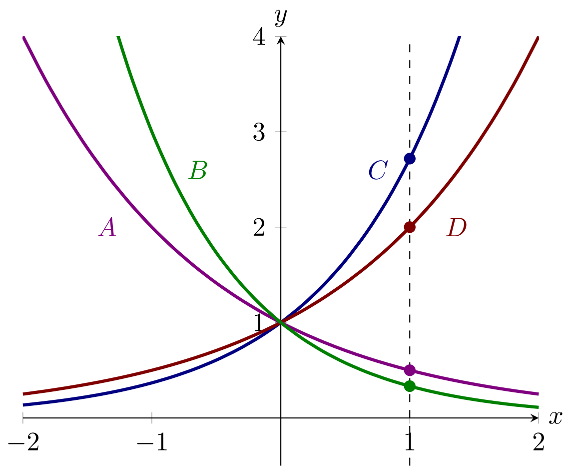
Example 1. *Here we see the the graphs of four exponential functions.*



Match the curves A, B, C, and D with the functions

$$e^x, \qquad \left(\frac{1}{2}\right)^x, \qquad \left(\frac{1}{3}\right)^x, \qquad 2^x.$$

Explanation. *One way to solve these problems is to compare these functions along the vertical line $x = 1$,*



Note

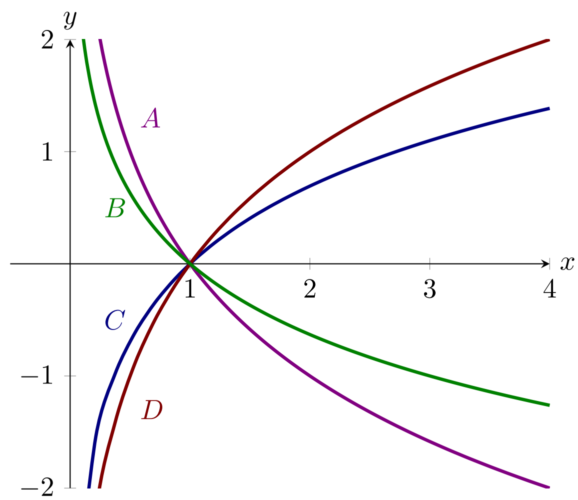
$$\left(\frac{1}{3}\right)^1 < \left(\frac{1}{2}\right)^1 < 2^1 < e^1.$$

Hence we see:

- $\left(\frac{1}{3}\right)^x$ *corresponds to* $\underbrace{\boxed{B}}_{\text{given}}$.
- $\left(\frac{1}{2}\right)^x$ *corresponds to* $\underbrace{\boxed{A}}_{\text{given}}$.
- 2^x *corresponds to* $\underbrace{\boxed{D}}_{\text{given}}$.
- e^x *corresponds to* $\underbrace{\boxed{C}}_{\text{given}}$.

Graphs of logarithmic functions

Example 2. *Here we see the the graphs of four logarithmic functions.*



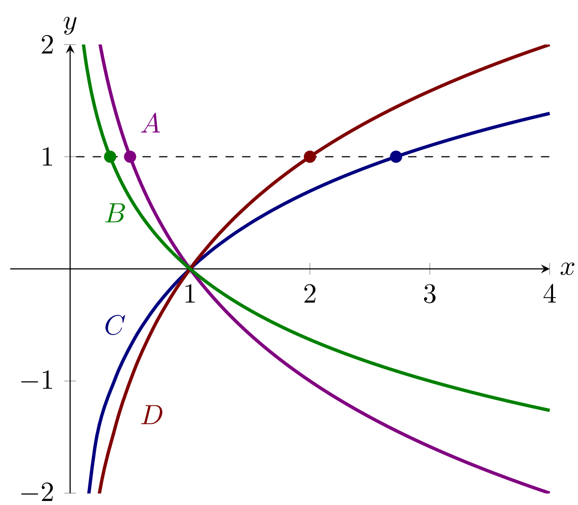
Match the curves A, B, C, *and* D *with the functions*

$$\ln(x), \qquad \log_{1/2}(x), \qquad \log_{1/3}(x), \qquad \log_2(x).$$

Explanation. *First remember what* $\ln(x) = y$ *means:*

$$\ln(x) = y \qquad \textit{means that} \qquad b^y = x.$$

So now examine each of these functions along the horizontal line $y = 1$



Note again (this is from the definition of a logarithm)

$$\left(\frac{1}{3}\right)^1 < \left(\frac{1}{2}\right)^1 < 2^1 < e^1.$$

Hence we see:

- $\log_{1/3}(x)$ *corresponds to* $\underbrace{\boxed{B}}_{\text{given}}$.
- $\log_{1/2}(x)$ *corresponds to* $\underbrace{\boxed{A}}_{\text{given}}$.
- $\log_{2}(x)$ *corresponds to* $\underbrace{\boxed{D}}_{\text{given}}$.
- $\ln(x)$ *corresponds to* $\underbrace{\boxed{C}}_{\text{given}}$.

Properties of exponential functions and logarithms

Working with exponential and logarithmic functions is often simplified by applying properties of these functions. These properties will make appearances throughout our work.

Properties of exponents

Let b be a positive real number with $b \neq 1$.

- $b^m \cdot b^n = b^{m+n}$
- $b^{-1} = \dfrac{1}{b}$
- $(b^m)^n = b^{mn}$

Question 3 *What exponent makes the following true?*

$$2^4 \cdot 2^3 = 2^{\boxed{7}}$$

Hint:

$$(2^4) \cdot (2^3) = (2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2)$$

Properties of logarithms

Let b be a positive real number with $b \neq 1$.

- $\log_b(m \cdot n) = \log_b(m) + \log_b(n)$
- $\log_b(m^n) = n \cdot \log_b(m)$
- $\log_b\left(\frac{1}{m}\right) = \log_b(m^{-1}) = -\log_b(m)$
- $\log_a(m) = \frac{\log_b(m)}{\log_b(a)}$

Question 4 *What value makes the following expression true?*

$$\log_2\left(\frac{8}{16}\right) = 3 - \boxed{4}$$

Question 5 *What makes the following expression true?*

$$\log_3(x) = \frac{\ln(x)}{\boxed{\ln(3)}}$$