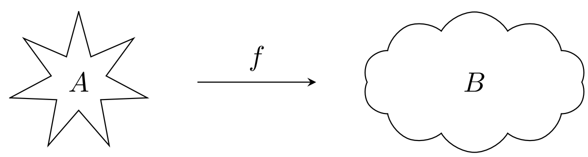
Dig-In:

Inverses of functions

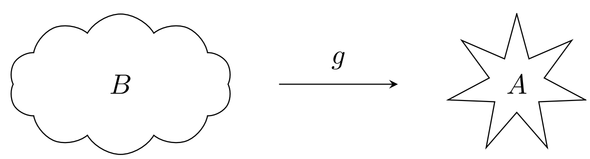
Here we "undo" functions.

If a function maps every "input" to exactly one "output," an inverse of that function maps every "output" to exactly one "input." We need a more formal definition to actually say anything with rigor.

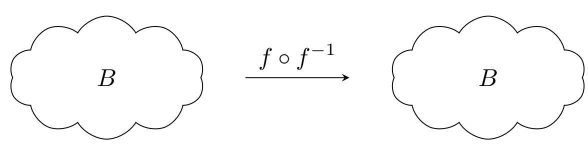
Definition 1. *Let f be a function with domain A and range B:*



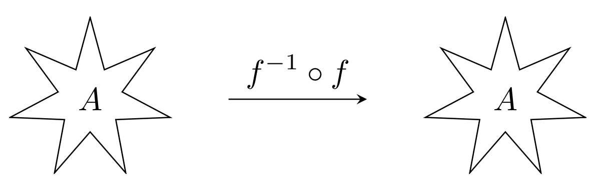
Let g be a function with domain B and range A:



We say that f and g are ***inverses*** *of each other if $f(g(b)) = b$ for all b in B, and also $g(f(a)) = a$ for all a in A. Sometimes we write $g = f^{-1}$ in this case.*



and



Learning outcomes: Find the domain and range of a function. Determine if a function is one-to-one. Perform basic operations and compositions on functions. Define and work with inverse functions. Plot inverses of basic functions. Find inverse functions (algebraically and graphically). Find the largest interval containing a given point where the function is invertible. Determine the intervals on which a function has an inverse.

So, we could rephrase these conditions as

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

These two simple equations are somewhat more subtle than they initially appear.

Question 1 *Let f be a function. If the point $(1, 9)$ is on the graph of f, what point must be the the graph of f^{-1}?*

$$\left(\boxed{9}, \boxed{1}\right)$$

Feedback (attempt): *Since $f(1) = 9$, we must have $f^{-1}(f(1)) = 1$, so $f^{-1}(9) = 1$. Thus $(9, 1)$ is on the graph of f^{-1}. This is a general rule. If (a, b) is on the graph of f, then (b, a) will be on the graph of f^{-1}.*

Warning 1. *This notation can be very confusing. Keep a watchful eye:*

$$f^{-1}(x) = \text{the inverse function of } f(x).$$
$$f(x)^{-1} = \frac{1}{f(x)}.$$

Question 2 *Which of the following is notation for the inverse of the function* $\sin(\theta)$ *on the interval* $[-\pi/2, \pi/2]$*?*

Multiple Choice:

(a) $\sin^{-1}(\theta)$ ✓

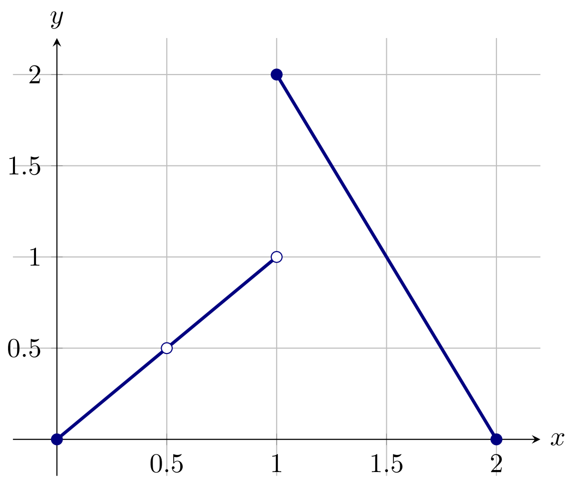
(b) $\sin(\theta)^{-1}$

Feedback (attempt): $\sin^{-1}(\theta)$ *is the inverse function for* $\sin(\theta)$ *on the interval* $[-\pi/2, \pi/2]$.

On the other hand,

$$\sin(\theta)^{-1} = \frac{1}{\sin(\theta)} \neq \sin^{-1}(\theta).$$

Question 3 *Consider the graph of $y = f(x)$ below*



Is $f(x)$ invertible at $x = 0.5$?

Multiple Choice:

(a) *yes* ✓

(b) *no*

Question 4

$$f^{-1}(1) = \boxed{1.5}$$

So far, we've only dealt with abstract examples. Let's see if we can ground this in a real-life context.

Example 1. *The function*

$$f(t) = \left(\frac{9}{5}\right) t + 32$$

takes a temperature t in degrees Celsius, and converts it into Fahrenheit. The domain of this function is $-\infty < t < \infty$. What does the inverse of this function tell you? What is the inverse of this function?

Explanation. *If f converts Celsius measurements to Fahrenheit measurements of temperature, then f^{-1} converts Fahrenheit measurements to Celsius measurements of temperature.*

To find the inverse function, first note that

$$f(f^{-1}(t)) = t \qquad \textit{by the definition of inverse functions.}$$

Now write out the left-hand side of the equation

$$f(f^{-1}(t)) = \left(\frac{9}{5}\right) f^{-1}(t) + 32 \qquad \textit{by the rule for } f$$

and solve for $f^{-1}(t)$.

$$\begin{aligned} \left(\frac{9}{5}\right) f^{-1}(t) + 32 &= t && \textit{by the rule for } f \\ \left(\frac{9}{5}\right) f^{-1}(t) &= t - 32 \\ f^{-1}(t) &= \underbrace{\boxed{\left(\frac{5}{9}\right)(t-32)}}_{\text{given}}. \end{aligned}$$

So $f^{-1}(t) = \left(\frac{5}{9}\right)(t-32)$ *is the inverse function of* f*, which converts a Fahrenheit measurement back into a Celsius measurement. The domain of this inverse function is* $-\infty < t < \infty$.

Finally, we could check our work again using the definition of inverse functions. We have already guaranteed that

$$f(f^{-1}(t)) = t,$$

since we solved for f^{-1} *in our calculation. On the other hand,*

$$\begin{aligned} f^{-1}(f(t)) &= \left(\frac{5}{9}\right)(f(t) - 32) \\ &= \left(\frac{5}{9}\right)(f(t) - 32) \end{aligned}$$

which you should simplify to check that $f^{-1}(f(t)) = t$.

We have examined several functions in order to determine their inverse functions, but there is still more to this story. Not every function has an inverse function, so we must learn how to check for this situation.

Question 5 *Let* f *be a function, and imagine that the points* $(2, 3)$ *and* $(7, 3)$ *are both on its graph. Could* f *have an inverse function?*

Multiple Choice:

(a) *yes*

(b) *no* ✓

Feedback (attempt): *The function f could* ***not*** *have an inverse function. Imagine that it did. Then $f^{-1}(f(2)) = 2$ and $f^{-1}(f(7)) = 7$. Then we have both $f^{-1}(3) = 2$ and $f^{-1}(3) = 7$. Since a* ***function*** *cannot send the same input to two different outputs, f must not have an inverse function.*

Look again at the last question. If two different inputs for a function have the same output, there is no hope of that function having an inverse function. Why? This is because the inverse function must also be a function, and a function can only have one output for each input. More specifically, we have the next definition.

Definition 2. *A function is called* ***one-to-one*** *if each output value corresponds to exactly one input value.*

Question 6 *Which of the following are functions that are also one-to-one?*

Select All Correct Answers:

(a) *Mapping words to their meaning in a dictionary.*

(b) *Mapping social security numbers of living people to actual living people.* ✓

(c) *Mapping people to their birthday.*

(d) *Mapping mothers to their children.*

Feedback (attempt):

- *Since words may have more than one definition, "relating words to their definition in a dictionary" is not a function.*
- *Since every social security number corresponds exactly to one person, "relating social security numbers of living people to actual living people" is a function. Also, since each person has exactly one social security number, it is one-to-one.*
- *Since every person only has one birth date, "relating people to their birth date" is a function. However, many people have the same birth date, hence this function is not one-to-one.*
- *Since mothers can have more (or less) than one child, "relating mothers to their children" is not a function.*

Question 7 *Which of the following functions are one to one? Select all that apply.*

Select All Correct Answers:

(a) $f(x) = x$ ✓

(b) $f(x) = x^2$

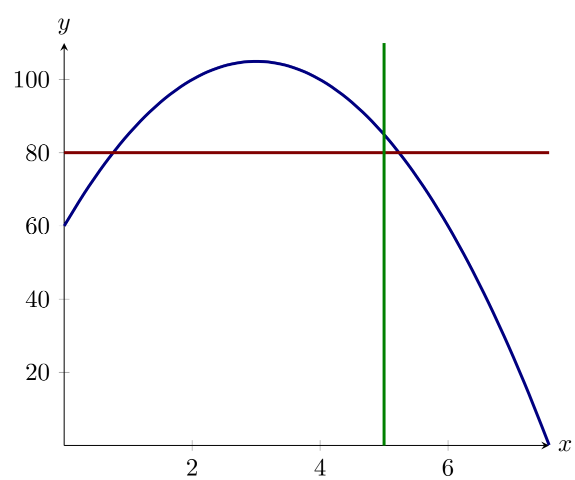
(c) $f(x) = x^3 - 4x$

(d) $f(x) = x^3 + 4$ ✓

You may recall that a plot gives y as a function of x if every vertical line crosses the plot at most once, and we called this the **vertical line test**. Similarly, a function is one-to-one if every horizontal line crosses the plot at most once, and we call this the **horizontal line test**.

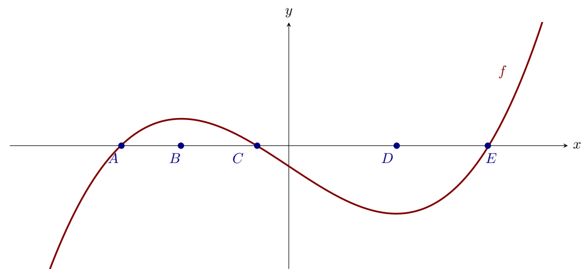
Theorem 1. *A function is one-to-one at $x = a$ if the horizontal line $y = f(a)$ intersects the curve $y = f(x)$ in exactly one point. This is called the* ***horizontal line test***.

Below, we give a graph of $f(x) = -5x^2 + 30x + 60$. While this graph passes the vertical line test, and hence represents y as a function of x, it does not pass the horizontal line test, so the function is not one-to-one.



As we have discussed, we can only find an inverse of a function when it is one-to-one. If a function is not one-to-one, but we still want an inverse, we must restrict the domain. Let's see what this means in our next examples.

Question 8 *Consider the graph of the function f below:*



On which of the following intervals is f one-to-one?

Select All Correct Answers:

(a) $[A, B]$ ✓

(b) $[A, C]$

(c) $[B, D]$ ✓

(d) $[C, E]$

(e) $[C, D]$ ✓

This idea of restricting the domain is critical for understanding functions like $f(x) = \sqrt{x}$.

Warning 2. *We define $f(x) = \sqrt{x}$ to be the positive square-root, so that we can be sure that f is a function. Thinking of the square-root as the inverse of the squaring function, we can see the issue a little more clearly. There are two x-values that square to 9.*

$$x^2 = 9 \qquad \text{means } x = \pm 3$$

Since we require that ***square-root is a function****, we must have only one output value when we plug in 9. We choose the positive square-root, meaning that*

$$\sqrt{9} = 3.$$

Example 2. *Consider the function*

$$f(x) = x^2.$$

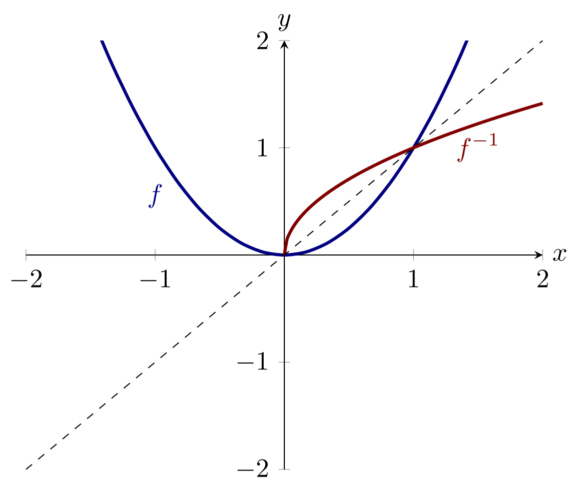
Does f have an inverse? If so, what is it? If not, attempt to restrict the domain of f and find an inverse on the restricted domain.

Explanation. *In this case f is not one-to-one. However, it is one-to-one on the interval $[0, \infty)$. Hence we can find an inverse of $f(x) = x^2$ on this interval. We plug $f^{-1}(x)$ into f and write*

$$\begin{aligned} f(f^{-1}(x)) &= \left(f^{-1}(x)\right)^2 \\ x &= \left(f^{-1}(x)\right)^2. \end{aligned}$$

Since the domain of f is $[0, \infty)$, we know that x is positive. This means we can take the square-root of each side of the equation to find that

$$\sqrt{x} = f^{-1}(x).$$



Example 3. *Consider the function*

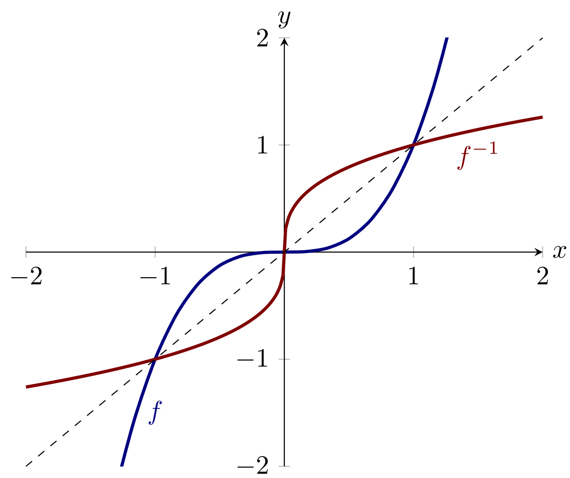
$$f(x) = x^3.$$

Does $f(x)$ have an inverse? If so, what is it? If not, attempt to restrict the domain of $f(x)$ and find an inverse on the restricted domain.

Explanation. *In this case $f(x)$ is one-to-one. We may write*

$$\begin{aligned} f(f^{-1}(x) &= \left(f^{-1}(x)\right)^3 \\ x &= \left(f^{-1}(x)\right)^3 \\ \sqrt[3]{x} &= f^{-1}(x). \end{aligned}$$

For your viewing pleasure we give a graph of $y = f(x) = x^3$ and $y = f^{-1}(x) = \sqrt[3]{x}$. Note, the graph of f^{-1} is the image of f after being flipped over the line $y = x$.


y
2
1
x
−2
−1
1
2
−1
−2
f^{-1}
f