Dig-In:

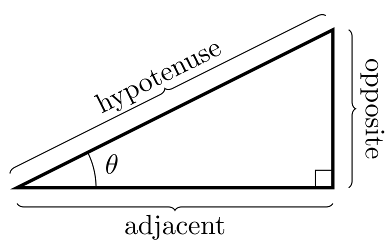
Trigonometric functions

We review trigonometric functions.

What are trigonometric functions?

Definition 1. *A* ***trigonometric function*** *is a function that relates a measure of an angle of a right triangle to a ratio of the triangle's sides.*

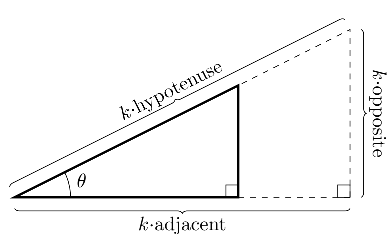
The basic trigonometric functions are cosine and sine. They are called "trigonometric" because they relate measures of angles to measurements of triangles. Given a right triangle



we define

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \text{and} \qquad \sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}.$$

Note, the values of sine and cosine do not depend on the scale of the triangle. Being very explicit, if we scale a triangle by a scale factor k,



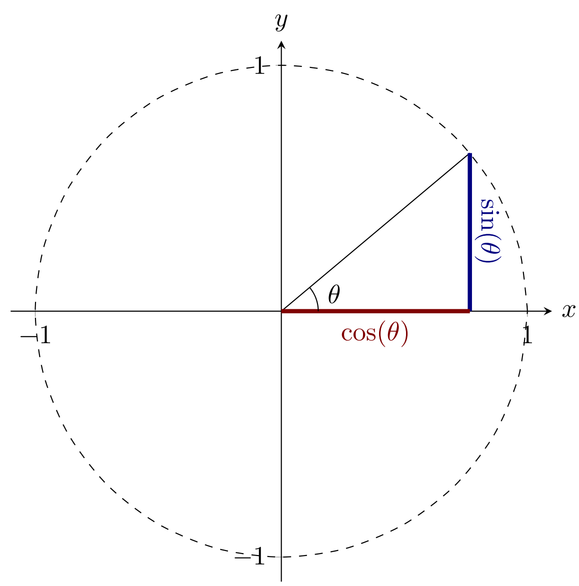
Learning outcomes: Know the graphs and properties of "famous" functions. Understand the properties of trigonometric functions. Evaluate expressions and solve equations involving trigonometric functions and inverse trigonometric functions.

$$\cos(\theta) = \frac{k \cdot \text{adjacent}}{k \cdot \text{hypotenuse}} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

and

$$\sin(\theta) = \frac{k \cdot \text{opposite}}{k \cdot \text{hypotenuse}} = \frac{\text{opposite}}{\text{hypotenuse}}.$$

At this point we could simply assume that whenever we draw a triangle for computing sine and cosine, that the hypotenuse will be 1. We can do this because we are simply scaling the triangle, and as we see above, this makes absolutely no difference when computing sine and cosine. Hence, when the hypotenuse is 1, we find that a convenient way to think about sine and cosine is via the unit circle:



If we consider all possible combinations of ratios of

$$\text{adjacent}, \qquad \text{opposite}, \qquad \text{hypotenuse},$$

(allowing the adjacent and opposite to be negative, as on the unit circle) we obtain all of the trigonometric functions.

Definition 2. *The trigonometric functions are:*

$$\cos(\theta) = \frac{adj}{hyp} \qquad \sin(\theta) = \frac{opp}{hyp}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

where the domain of sine and cosine is all real numbers, and the other trigonometric functions are defined precisely when their denominators are nonzero.

Question 1 *Which of the following expressions are equal to* $\sec(\theta)$*?*

Select All Correct Answers:

(a) $\dfrac{1}{\cos(\theta)}$ ✓

(b) $\dfrac{1}{\sin(\theta)}$

(c) $\dfrac{adj}{hyp}$

(d) $\dfrac{hyp}{adj}$ ✓

(e) $\dfrac{\tan(\theta)}{\sin(\theta)}$ ✓

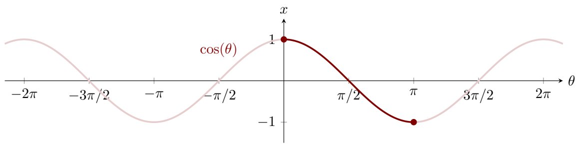
(f) $\dfrac{1}{\sin(\theta)\cdot\cot(\theta)}$ ✓

Connections to inverse functions

Trigonometric functions arise frequently in problems, and often we are interested in finding specific angle measures. For instance, you may want to find some angle θ such that

$$\cos(\theta) = .7$$

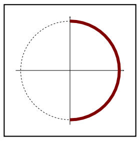
Hence we want to be able to "undo" trigonometric functions. However, since trigonometric functions are not one-to-one, meaning there are are infinitely many angles with $\cos(\theta) = .7$, it is impossible to find a true inverse function for $\cos(\theta)$. Nevertheless, it is useful to have something like an inverse to these functions, however imperfect. The usual approach is to pick out some collection of angles that produce all possible values exactly once. If we "discard" all other angles, the resulting function has a proper inverse. In other words, we are restricting the domain of the trigonometric function in order to find an inverse. The function $\cos(\theta)$ takes on all values between -1 and 1 exactly once on the interval $[0, \pi]$.

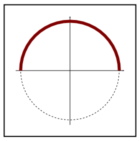


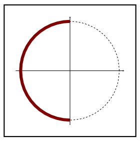
If we restrict the domain of $\cos(\theta)$ to this interval, then this restricted function is one-to-one and hence has an inverse.

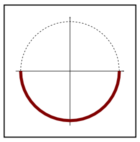
Question 2 *What arc on the unit circle corresponds to the restricted domain described above of* $\cos(\theta)$*?*

Multiple Choice:

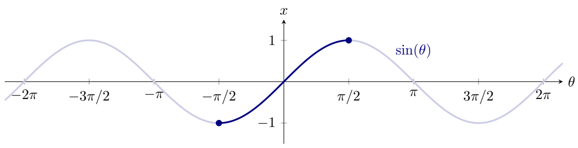
(a) 

(b)  ✓

(c) 

(d) 

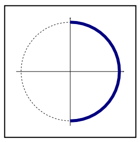
In a similar fashion, we need to restrict the domain of sine to be able to take an inverse. The function $\sin(\theta)$ takes on all values between -1 and 1 exactly once on the interval $[-\pi/2, \pi/2]$.

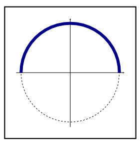


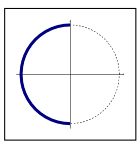
If we restrict the domain of $\sin(\theta)$ to this interval, then this restricted function is one-to-one and thus has an inverse.

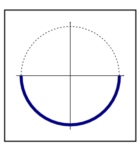
Question 3 *What arc on the unit circle corresponds to the restricted domain described above of* $\sin(\theta)$*?*

Multiple Choice:

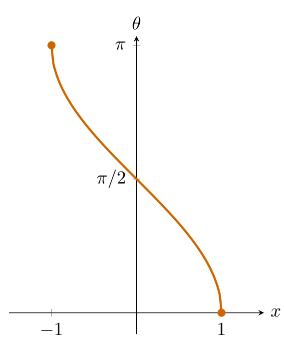
(a)  ✓

(b) 

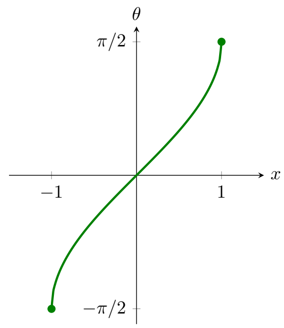
(c) 

(d) 

By examining both sine and cosine on restricted domains, we can now produce functions arcsine and arccosine:



Here we see a plot of $\arccos(x)$, the inverse function of $\cos(\theta)$ when the domain is restricted to the interval $[0, \pi]$.



Here we see a plot of $\arcsin(x)$, the inverse function of $\sin(\theta)$ when the domain is restricted to the interval $[-\pi/2, \pi/2]$.

The functions

$$\arccos(x) \qquad \text{and} \qquad \arcsin(x)$$

are called "arc" because they give the angle that cosine or sine used to produce their value. It is quite common to write

$$\arccos(x) = \cos^{-1}(x) \qquad \text{and} \qquad \arcsin(x) = \sin^{-1}(x).$$

However, this notation is misleading as $\cos^{-1}(x)$ and $\sin^{-1}(x)$ are not true inverse functions of cosine and sine. Recall that a function and its inverse undo each other in either order, for example,

$$\sqrt[3]{x^3} = x \qquad \text{and} \qquad \left(\sqrt[3]{x}\right)^3 = x.$$

Since arcsine is the inverse of sine restricted to the interval $[-\pi/2, \pi/2]$, this does not work with sine and arcsine, for example

$$\arcsin(\sin(\pi)) = 0.$$

though it is true that

$$\sin(\arcsin(x)) = x \qquad \text{and} \qquad \cos(\arccos(x)) = x.$$

Question 4 *Which of the following statements is true?*

Multiple Choice:

(a) $\sin^{-1}(x)$ *is the inverse function of* $\sin(x)$

(b) $\sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right) = \frac{1}{2}$ ✓

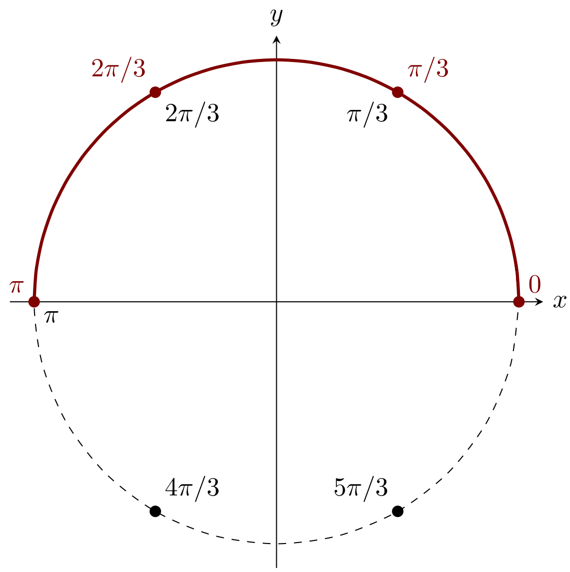
(c) $\sin^{-1}\left(\sin\left(\frac{5\pi}{2}\right)\right) = \frac{5\pi}{2}$

(d) $\sin^{-1}(x) = \frac{1}{\sin(x)}$

Example 1. *Compute:*

$$\arccos(\cos(5\pi/3))$$

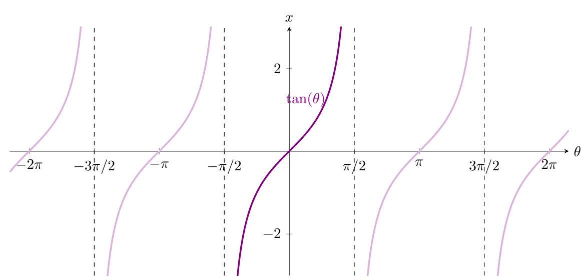
Explanation. *The issue here is that $5\pi/3$ might not be in the range of arccosine. To find our missing number, we'll check with a unit circle that we've decorated with the domain of arccosine:*



Since the points $5\pi/3$ and $\pi/3$ have the same x-coordinate, $\cos(5\pi/3) = \cos(\pi/3)$. Hence

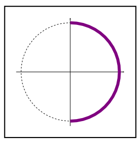
$$\arccos(\cos(5\pi/3)) = \pi/3.$$

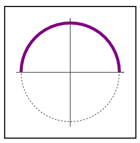
Now that you have a feel for how $\arcsin(x)$ and $\arccos(x)$ behave, let's examine tangent.

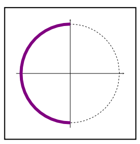


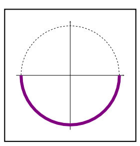
Question 5 *What arc on the unit circle corresponds to the restricted domain described above of* $\tan(\theta)$*?*

Multiple Choice:

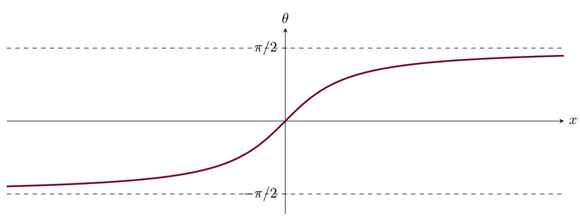
(a)  ✓

(b) 

(c) 

(d) 

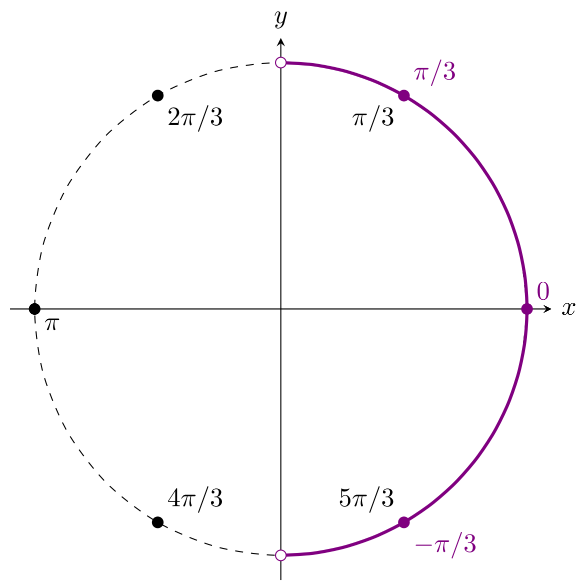
Again, only working on a restricted domain of tangent, we can produce an inverse function, arctangent. Here we see a plot of $\arctan(x)$, the inverse function of $\tan(\theta)$ when its domain is restricted to the interval $(-\pi/2, \pi/2)$.



Example 2. *Compute:*

$$\arctan(\tan(5\pi/3))$$

Explanation. *The issue here is that* $5\pi/3$ *might not be in the range of arctangent. To find our missing number, we'll check with a unit circle that we've decorated with the domain of arctangent:*



Since the points $5\pi/3$ *and* $-\pi/3$ *have the same* x *and* y*-coordinates,*

$$\arctan(\tan(5\pi/3)) = -\pi/3.$$

Now we give some facts of other trigonometric and "inverse" trigonometric functions.

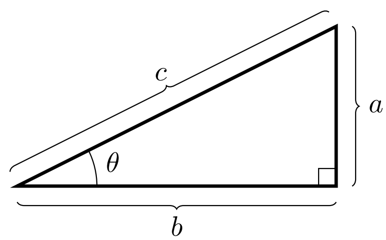
Definition 3.

- $\arccos(x) = \theta$ *means that* $\cos(\theta) = x$ *and* $0 \leq \theta \leq \pi$. *The domain of* $\arccos(x)$ *is* $-1 \leq x \leq 1$.
- $\arcsin(x) = \theta$ *means that* $\sin(\theta) = x$ *and* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. *The domain of* $\arcsin(x)$ *is* $-1 \leq x \leq 1$.
- $\arctan(x) = \theta$ *means that* $\tan(\theta) = x$ *and* $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. *The domain of* $\arctan(x)$ *is* $-\infty < x < \infty$.
- $\operatorname{arccot}(x) = \theta$ *means that* $\cot(\theta) = x$ *and* $0 < \theta < \pi$. *The domain of* $\operatorname{arccot}(x)$ *is* $-\infty < x < \infty$.
- $\operatorname{arcsec}(x) = \theta$ *means that* $\sec(\theta) = x$ *and* $0 \leq \theta \leq \pi$ *with* $\theta \neq \pi/2$. *The domain of* $\operatorname{arcsec}(x)$ *is all* x *with absolute value greater than* 1.
- $\operatorname{arccsc}(x) = \theta$ *means that* $\csc(\theta) = x$ *and* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ *with* $\theta \neq 0$. *The domain of* $\operatorname{arccsc}(x)$ *is all* x *with absolute value greater than* 1.

The power of the Pythagorean Theorem

The Pythagorean Theorem is probably the most famous theorem in all of mathematics.

Theorem 1 (Pythagorean Theorem)**.** *Given a right triangle:*



We have that:

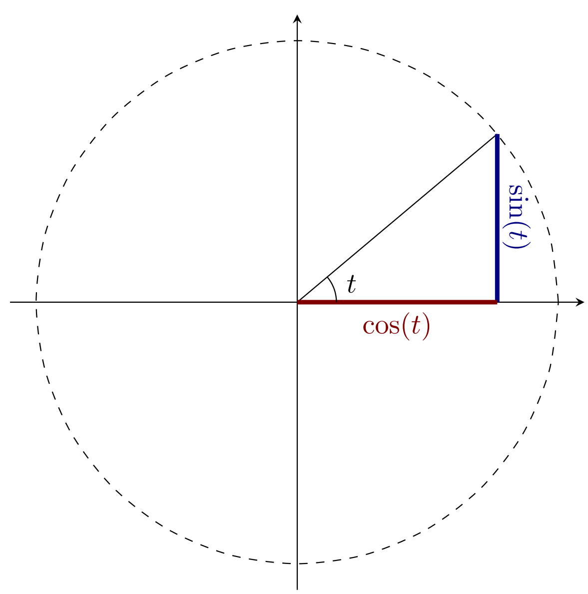
$$a^2 + b^2 = c^2$$

The Pythagorean Theorem gives several key trigonometric identities.

Theorem 2 (Pythagorean Identities)**.** *The following hold:*

$$\cos^2\theta + \sin^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

Explanation. *From the unit circle we can see*



via the Pythagorean Theorem that

$$\cos^2(t) + \sin^2(t) = 1.$$

If we divide this expression by $\underbrace{\boxed{\cos^2(t)}}_{\text{given}}$ *we obtain*

$$1 + \tan^2(t) = \sec^2(t)$$

and if we divide $\cos^2(t) + \sin^2(t) = 1$ *by* $\underbrace{\boxed{\sin^2(t)}}_{\text{given}}$ *we obtain*

$$\cot^2(t) + 1 = \csc^2(t).$$

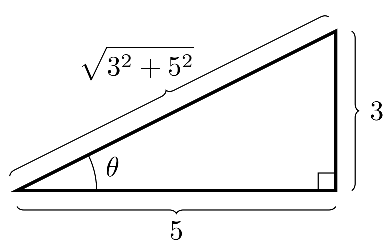
We can simplify expressions using the Pythagorean Theorem

Example 3. *Suppose that* $\arctan(3/5) = \theta$. *Compute* $\sin(\theta)$.

Explanation. *If* $\arctan(3/5) = \theta$, *then*

$$\begin{aligned} \tan(\arctan(3/5)) &= \tan(\theta) \\ \underbrace{\boxed{3/5}}_{\text{given}} &= \tan(\theta). \end{aligned}$$

Now we will use the Pythagorean Theorem to deduce $\sin(\theta)$. *If* $\tan(\theta) = 3/5$, *the triangle in question must be similar to this triangle:*



From this triangle and our work above, we see that

$$\sin(\theta) = \underbrace{\boxed{3/\sqrt{3^2+5^2}}}_{\text{given}}.$$

We'll also use the Pythagorean Theorem to help us simplify abstract expressions into ones we can compute with ease.

Example 4. *Simplify*

$$\tan(\arccos(x))$$

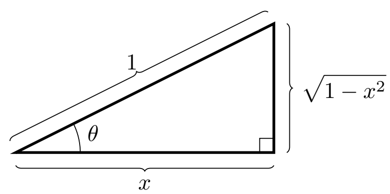
Explanation. *Start by saying*

$$\theta = \arccos(x)$$

This means $\tan(\arccos(x)) = \tan(\theta)$. *Apply cosine to both sides of the equation above,*

$$\begin{aligned}\cos(\theta) &= \cos(\arccos(x))\\ \cos(\theta) &= \underbrace{\boxed{x}}_{\text{given}}.\end{aligned}$$

Now we will use the Pythagorean Theorem to deduce $\tan(\theta)$. *If* $\cos(\theta) = x$, *the triangle in question must be similar to this triangle:*



From this triangle and our work above, we see that

$$\tan(\arccos(x)) = \tan(\theta) = \underbrace{\boxed{\frac{\sqrt{1-x^2}}{x}}}_{\text{given}}.$$