Dig-In:

Compositions of functions

We discuss compositions of functions.

Given two functions, we can compose them. Let's give an example in a "real context."

Example 1. *Let*

$$g(m) = \text{the amount of gas one can buy with } m \text{ dollars,}$$

and let

$$f(g) = \text{how far one can drive with } g \text{ gallons of gas.}$$

What does $f(g(m))$ represent in this setting?

Explanation. *With $f(g(m))$ we first relate how far one can drive with $\underset{\text{given}}{\boxed{g}}$ gallons of gas, and this in turn is determined by how much money $\underset{\text{given}}{\boxed{m}}$ one has. Hence $f(g(m))$ represents how far one can drive with $\underset{\text{given}}{\boxed{m}}$ dollars.*

Composition of functions can be thought of as putting one function inside another. We use the notation

$$(f \circ g)(x) = f(g(x)).$$

Warning 1. *The composition $f \circ g$ only makes sense if*

$$\{\text{the range of } g\} \text{ is contained in or equal to } \{\text{the domain of } f\}$$

Example 2. *Suppose we have*

$$\begin{aligned} f(x) &= x^2 + 5x + 4 && \text{for } -\infty < x < \infty, \\ g(x) &= x + 7 && \text{for } -\infty < x < \infty. \end{aligned}$$

Find $f(g(x))$ and state its domain.

Learning outcomes: Find the domain and range of a function. Distinguish between functions by considering their domains. Perform basic operations and compositions on functions. Work with piecewise defined functions. Recognize different representations of the same function.

Explanation. *The range of g is $-\infty < x < \infty$, which is equal to the domain of f. This means the domain of $f \circ g$ is $-\infty < x < \infty$. Next, we substitute $x + 7$ for each instance of $\underbrace{\boxed{x}}_{\text{given}}$ found in*

$$f(x) = x^2 + 5x + 4$$

and so

$$\begin{aligned} f(g(x)) &= f(x+7) \\ &= \underbrace{\boxed{(x+7)^2 + 5(x+7) + 4}}_{\text{given}}. \end{aligned}$$

Now let's try an example with a more restricted domain.

Example 3. *Suppose we have:*

$$\begin{aligned} f(x) &= x^2 & &\textit{for } -\infty < x < \infty, \\ g(x) &= \sqrt{x} & &\textit{for } 0 \le x < \infty. \end{aligned}$$

Find $f(g(x))$ and state its domain.

Explanation. *The domain of g is $0 \le x < \infty$. From this we can see that the range of g is $\underbrace{\boxed{0}}_{\text{given}} \le x < \infty$. This is contained in the domain of f.*

This means that the domain of $f \circ g$ is $0 \le x < \infty$. Next, we substitute $\underbrace{\boxed{\sqrt{x}}}_{\text{given}}$ for each instance of x found in

$$f(x) = x^2$$

and so

$$\begin{aligned} f(g(x)) &= f(\sqrt{x}) \\ &= \left(\sqrt{x}\right)^2. \end{aligned}$$

We can summarize our results as a piecewise function, which looks somewhat interesting:

$$(f \circ g)(x) = \begin{cases} x & \textit{if } 0 \le x < \infty \\ \textit{undefined} & \textit{otherwise.} \end{cases}$$

Example 4. *Suppose we have:*

$$\begin{aligned} f(x) &= \sqrt{x} & &\textit{for } 0 \le x < \infty, \\ g(x) &= x^2 & &\textit{for } -\infty < x < \infty. \end{aligned}$$

Find $f(g(x))$ and state its domain.

Explanation. *While the domain of g is $-\infty < x < \infty$, its range is only $0 \leq x < \infty$. This is exactly the domain of f. This means that the domain of $f \circ g$ is $-\infty < x < \infty$. Now we may substitute* $\underbrace{\boxed{x^2}}_{\text{given}}$ *for each instance of* $\underbrace{\boxed{x}}_{\text{given}}$ *found in*

$$f(x) = \sqrt{x}$$

and so

$$\begin{aligned} f(g(x)) &= f(x^2) \\ &= \sqrt{x^2}, \\ &= |x|. \end{aligned}$$

Compare and contrast the previous two examples. We used the same functions for each example, but composed them in different ways. The resulting compositions are not only different, they have different domains!