Dig-In:

Continuity of piecewise functions

Here we use limits to ensure piecewise functions are continuous.

In this section we will work a couple of examples involving limits, continuity and piecewise functions.

Example 1. *Consider the following piecewise defined function*

$$f(x) = \begin{cases} \dfrac{x}{x-1} & \text{if } x < 0 \text{ and } x \neq 1, \\ e^{-x} + c & \text{if } x \geq 0. \end{cases}$$

Find c so that f is continuous at $x = 0$.

Explanation. *To find c such that f is continuous at $x = 0$, we need to find c such that*

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(\underbrace{\boxed{0}}_{\text{given}}).$$

In this case

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} \underbrace{\boxed{\frac{x}{x-1}}}_{\text{given}} \\ &= \underbrace{\boxed{\frac{0}{-1}}}_{\text{given}} \\ &= \underbrace{\boxed{0}}_{\text{given}}. \end{aligned}$$

On there other hand

$$\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} \left(\underbrace{\boxed{e^{-x} + c}}_{\text{given}}\right) \\ &= \underbrace{\boxed{e^0 + c}}_{\text{given}} \\ &= \underbrace{\boxed{1 + c}}_{\text{given}} \end{aligned}$$

Learning outcomes: Identify where a function is, and is not, continuous. Understand the connection between continuity of a function and the value of a limit. Make a piece-wise function continuous.

Hence for our function to be continuous, we need

$$1 + c = 0 \qquad \textit{so} \qquad c = \underbrace{\boxed{-1}}_{\text{given}}.$$

Now, $\lim_{x \to 0} f(x) = f(\underbrace{\boxed{0}}_{\text{given}})$, *and so f is continuous.*

Consider the next, more challenging example.

Example 2. *Consider the following piecewise defined function*

$$f(x) = \begin{cases} x + 4 & \textit{if } x < 1, \\ ax^2 + bx + 2 & \textit{if } 1 \leq x < 3, \\ 6x + a - b & \textit{if } x \geq 3. \end{cases}$$

Find a and b so that f is continuous at both $x = 1$ and $x = 3$.

Explanation. *This problem is more challenging because we have more unknowns. However, be brave intrepid mathematician. To find a and b that make f is continuous at $x = 1$, we need to find a and b such that*

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(\underbrace{\boxed{1}}_{\text{given}}).$$

Looking at the limit from the left, we have

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} \left(\underbrace{\boxed{x + 4}}_{\text{given}} \right) \\ &= \underbrace{\boxed{5}}_{\text{given}}. \end{aligned}$$

Looking at the limit from the right, we have

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} \left(\underbrace{\boxed{ax^2 + bx + 2}}_{\text{given}} \right) \\ &= \underbrace{\boxed{a + b + 2}}_{\text{given}}. \end{aligned}$$

Hence for this function to be continuous at $x = 1$, we must have that

$$\begin{aligned} 5 &= a + b + 2 \\ 3 &= \underbrace{\boxed{a + b}}_{\text{given}}. \end{aligned}$$

Hmmmm. More work needs to be done.

To find a and b that make f is continuous at $x = 3$, we need to find a and b such that

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(\underbrace{\boxed{3}}_{\text{given}}).$$

Looking at the limit from the left, we have

$$\begin{aligned}\lim_{x\to 3^-} f(x) &= \lim_{x\to 3^-}\left(\underbrace{\boxed{ax^2+bx+2}}_{\text{given}}\right)\\ &= \underbrace{\boxed{a\cdot 9+b\cdot 3+2}}_{\text{given}}.\end{aligned}$$

Looking at the limit from the right, we have

$$\begin{aligned}\lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+}\left(\underbrace{\boxed{6x+a-b}}_{\text{given}}\right)\\ &= \underbrace{\boxed{18+a-b}}_{\text{given}}.\end{aligned}$$

Hence for this function to be continuous at $x = 3$, we must have that

$$\begin{aligned}a\cdot 9+b\cdot 3+2 &= 18+a-b\\ a\cdot 8+b\cdot 4-16 &= 0\\ a\cdot 2+b-4 &= 0\end{aligned}$$

So now we have two equations and two unknowns:

$$3 = a+b \qquad \textit{and} \qquad a\cdot 2+b-4=0.$$

Set $b = 3-a$ and write

$$\begin{aligned}0 &= a\cdot 2+(3-a)-4\\ &= a-1,\end{aligned}$$

hence

$$a = \underbrace{\boxed{1}}_{\text{given}} \qquad \textit{and so} \qquad b = \underbrace{\boxed{2}}_{\text{given}}.$$

Let's check, so now plugging in values for both a and b we find

$$f(x) = \begin{cases} x+4 & \textit{if } x<1,\\ \underbrace{\boxed{x^2+2x+2}}_{\text{given}} & \textit{if } 1\le x<3,\\ \underbrace{\boxed{6x-1}}_{\text{given}} & \textit{if } x\ge 3.\end{cases}$$

Now

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 5,$$

and

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) = 17.$$

So setting $a = \underbrace{\boxed{1}}_{\text{given}}$ *and* $b = \underbrace{\boxed{2}}_{\text{given}}$ *makes* f *continuous at* $x = 1$ *and* $x = 3$. *We can confirm our results by looking at the graph of* $y = f(x)$*:*

Graph of $y = \{x < 1 : x + 4\}, y = \{1 \leq x < 3 : x^2 + 2x + 2\}, y = \{x \geq 3 : 6x - 1\}$