Dig-In:

The precise definition of a limit

We give a mathematically precise definition of a limit.

Recall that intuitively, the *limit* of $f(x)$ as x approaches a is L, written

$$\lim_{x\to a} f(x) = L,$$

if the value $f(x)$ can be made as close as one wishes to L for all x sufficiently close, but not equal to, a. This leads us to a precise definition of a *limit*.

The definition of a limit

Definition 1. *The **limit** of $f(x)$ as x goes to a is L,*

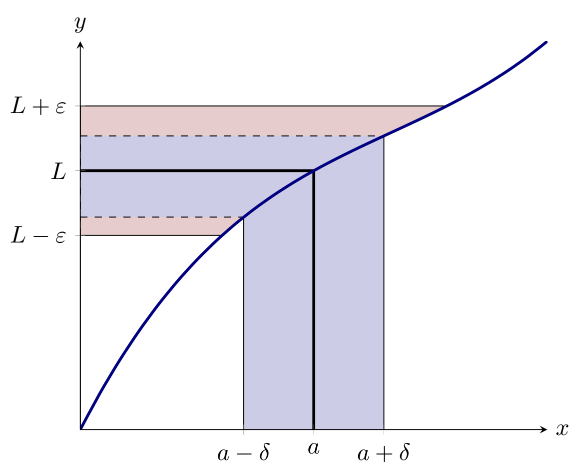
$$\lim_{x\to a} f(x) = L,$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever

$$0 < |x - a| < \delta, \qquad \textit{we have} \qquad |f(x) - L| < \varepsilon.$$

*If no such value of L can be found, then we say that the **limit does not exist**.*

In the figure below, we see a geometric interpretation of this definition.



Learning outcomes:

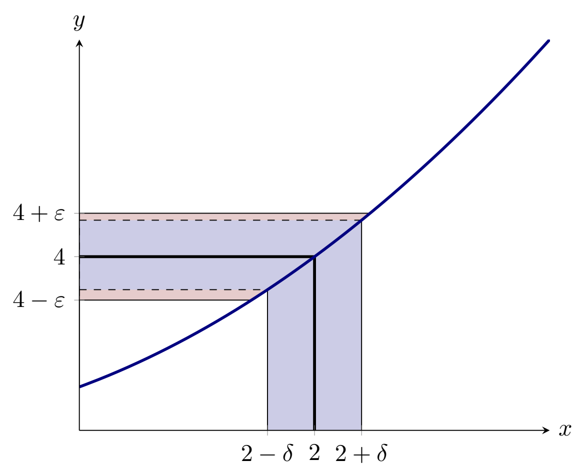
Now we are going to get our hands dirty, and really use the definition of a limit.

Example 1. *Show that* $\lim_{x\to 2} x^2 = 4$.

Explanation. *We want to show that for any given* $\varepsilon > 0$*, we can find a* $\delta > 0$ *such that*

$$|x^2 - 4| < \varepsilon$$

whenever $0 < |x - 2| < \delta$.



Start by factoring the left-hand side of the inequality above

$$|x + 2||x - 2| < \varepsilon.$$

Since we are going to assume that $0 < |x - 2| < \delta$*, we will focus on the factor* $|x + 2|$*. Since* x *is assumed to be close to* 2*, suppose that* $x \in [1, 3]$*. In this case*

$$|x + 2| \leq 3 + 2 = 5,$$

and so we want

$$\begin{aligned} 5 \cdot |x - 2| &< \varepsilon \\ |x - 2| &< \frac{\varepsilon}{5} \end{aligned}$$

Recall, we assumed that $x \in [1, 3]$*, which is equivalent to* $|x - 2| \leq 1$*. Hence we must set* $\delta = \min\left(\frac{\varepsilon}{5}, 1\right)$.

When dealing with limits of polynomials, the general strategy is always the same. Let $p(x)$ be a polynomial. If showing

$$\lim_{x\to a} p(x) = L,$$

one must first factor out $|x-a|$ from $|p(x)-L|$. Next bound $x \in [a-1, a+1]$ and estimate the largest possible value of

$$\left|\frac{p(x)-L}{x-a}\right|$$

for $x \in [a-1, a+1]$, call this estimation M. Finally, one must set $\delta = \min\left(\frac{\varepsilon}{M}, 1\right)$.

Tolerance problems