Dig-In:

Implicit differentiation

In this section we differentiate equations without expressing them in terms of a single variable.

Review of the chain rule

Implicit differentiation is really just an application of the chain rule. So recall:

Theorem 1 (Chain Rule)**.** *If $f(x)$ and $g(x)$ are differentiable, then*

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

Of particular use in this section is the following. If y is a differentiable function of x and if f is a differentiable function, then

$$\frac{d}{dx}(f(y)) = f'(y) \cdot \frac{d}{dx}(y) = f'(y)\frac{dy}{dx}.$$

Implicit differentiation

The functions we've been dealing with so far have been *explicit functions*, meaning that the dependent variable is written in terms of the independent variable. For example:

$$y = 3x^2 - 2x + 1, \qquad y = e^{3x}, \qquad y = \frac{x-2}{x^2-3x+2}.$$

However, there is another type of function, called an *implicit function.* In this case, the dependent variable is not stated explicitly in terms of the independent variable. Some examples are:

$$x^2 + y^2 = 4, \qquad x^3 + y^3 = 9xy, \qquad x^4 + 3x^2 = x^{2/3} + y^{2/3} = 1.$$

Your inclination might be simply to solve each of these for y and go merrily on your way. However this can be difficult and it may require two *branches*,

Learning outcomes: Implicitly differentiate expressions. Find the equation of the tangent line for curves that are not plots of functions. Understand how changing the variable changes how we take the derivative. Understand the derivatives of expressions that are not functions or not "solved for y".

for example to explicitly plot $x^2 + y^2 = 4$, one needs both $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$. Moreover, it may not even be possible to solve for y. To deal with such situations, we use *implicit differentiation*. We'll start with a basic example.

Example 1. *Consider the curve defined by:*

$$x^2 + y^2 = 1$$

(a) *Compute* $\dfrac{dy}{dx}$.

(b) *Find the slope of the tangent line at* $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

Explanation. *Starting with*

$$x^2 + y^2 = 1$$

we apply the differential operator $\dfrac{d}{dx}$ *to both sides of the equation to obtain*

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}1.$$

Applying the sum rule we see

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0.$$

Let's examine each of these terms in turn. To start

$$\frac{d}{dx}x^2 = \underbrace{\boxed{2x}}_{\text{given}}.$$

On the other hand, $\dfrac{d}{dx}y^2$ *is somewhat different. Here you imagine that* $y = f(x)$, *and hence by the chain rule*

$$\begin{aligned}\frac{d}{dx}y^2 &= \frac{d}{dx}(f(x))^2\\ &= 2 \cdot f(x) \cdot f'(x)\\ &= 2y\frac{dy}{dx}.\end{aligned}$$

Putting this together we are left with the equation

$$2x + 2y\frac{dy}{dx} = 0$$

At this point, we solve for $\frac{dy}{dx}$. *Write*

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \underset{\text{given}}{\boxed{\frac{-x}{y}}}. \end{aligned}$$

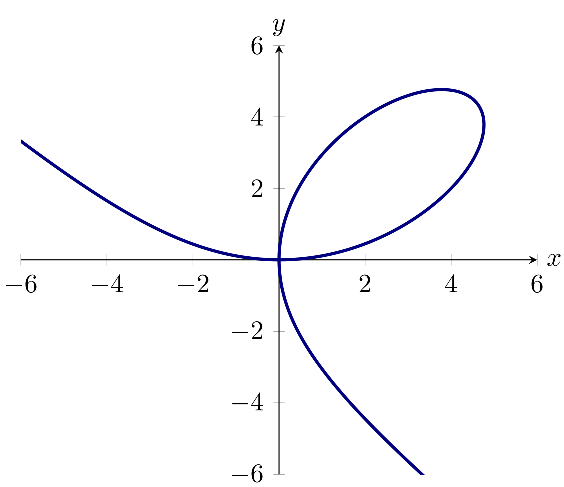
For the second part of the problem, we simply plug $x = \frac{\sqrt{2}}{2}$ *and* $y = \frac{\sqrt{2}}{2}$ *into the formula above, hence the slope of the tangent line at this point is* $\underset{\text{given}}{\boxed{-1}}$. *We can confirm our results by looking at the graph of the curve and our tangent line:*

Graph of $x^2 + y^2 = 1, -x + \sqrt{2}$

Let's see another illustrative example:

Example 2. *Consider the curve defined by:*

$$x^3 + y^3 = 9xy$$



(a) *Compute* $\frac{dy}{dx}$.

(b) *Find the slope of the tangent line at* $(4, 2)$.

Explanation. *Starting with*

$$x^3 + y^3 = 9xy,$$

we apply the differential operator $\frac{d}{dx}$ *to both sides of the equation to obtain*

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}9xy.$$

Applying the sum rule we see

$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx}9xy.$$

Let's examine each of these terms in turn. To start

$$\frac{d}{dx}x^3 = \underbrace{\boxed{3x^2}}_{\text{given}}.$$

On the other hand $\frac{d}{dx}y^3$ *is somewhat different. Here you imagine that* $y = f(x)$*, and hence by the chain rule*

$$\begin{aligned}\frac{d}{dx}y^3 &= \frac{d}{dx}(y(x))^3\\ &= 3(f(x))^2 \cdot f'(x)\\ &= 3y^2\frac{dy}{dx}.\end{aligned}$$

Considering the final term $\frac{d}{dx}9xy$*, we again imagine that* $y = f(x)$*. Hence*

$$\begin{aligned}\frac{d}{dx}9xy &= 9\frac{d}{dx}x \cdot f(x)\\ &= 9\left(x \cdot y'(x) + f(x)\right)\\ &= 9x\frac{dy}{dx} + 9y.\end{aligned}$$

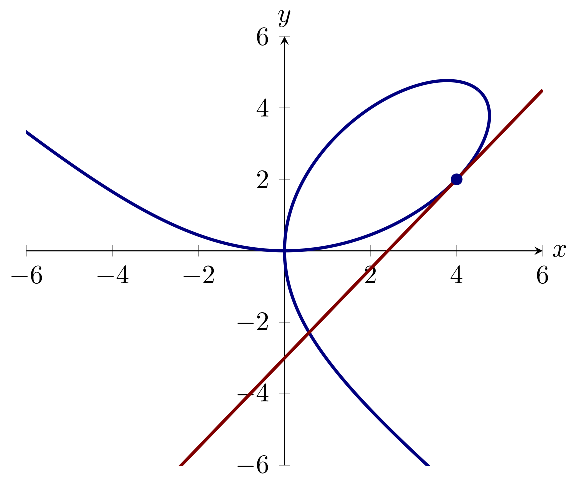
Putting this all together we are left with the equation

$$3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y.$$

At this point, we solve for $\frac{dy}{dx}$*. Write*

$$\begin{aligned}3x^2 + 3y^2\frac{dy}{dx} &= 9x\frac{dy}{dx} + 9y\\ 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} &= 9y - 3x^2\\ \frac{dy}{dx}\left(3y^2 - 9x\right) &= 9y - 3x^2\\ \frac{dy}{dx} &= \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}.\end{aligned}$$

For the second part of the problem, we simply plug $x = 4$ and $y = 2$ into the formula above, hence the slope of the tangent line at $(4, 2)$ is $\frac{5}{4}$. We've included a plot for your viewing pleasure:



You might think that the step in which we solve for $\frac{dy}{dx}$ could sometimes be difficult. In fact, *this never happens.* All occurrences $\frac{dy}{dx}$ arise from applying the chain rule, and whenever the chain rule is used it deposits a single $\frac{dy}{dx}$ multiplied by some other expression. Hence our expression is linear in $\frac{dy}{dx}$, it will always be possible to group the terms containing $\frac{dy}{dx}$ together and factor out the $\frac{dy}{dx}$, just as in the previous examples.

One more last example:

Example 3. *Consider the curve defined by*

$$\cos(xy) - \frac{y}{x} = 4x^2y^3.$$

Compute $\frac{dx}{dy}$.

Explanation. *First, notice that the problem asks for* $\frac{dx}{dy}$, ***not*** $\frac{dy}{dx}$. *So we are considering x as a function of y. This means the variables have changed places! Not to worry, everything is exactly the same. We apply $\frac{d}{dy}$ to both sides of the equation to get*

$$\frac{d}{dy}\left(\cos(xy) - \frac{y}{x}\right) = \frac{d}{dy}(4x^2y^3)$$

which gives us

$$-\underbrace{\boxed{\sin(xy)}}_{\text{given}}\left(y\frac{dx}{dy}+x\right)-\frac{x-y\frac{dx}{dy}}{x^2}=8xy^3\frac{dx}{dy}+12x^2y^2.$$

Distributing and multiplying by x^2 yields

$$\begin{aligned}-x^2y\sin(xy)\frac{dx}{dy}-x^3\sin(xy)-x+y\frac{dx}{dy}&\\=8x^3y^3\frac{dx}{dy}+12x^4y^2.&\end{aligned}$$

Grouping terms, factoring, and dividing finally gives us

$$\begin{aligned}-x^2y\sin(xy)\frac{dx}{dy}+y\frac{dx}{dy}-8x^3y^3\frac{dx}{dy}&\\=x^3\sin(xy)+x+12x^4y^2&\end{aligned}$$

so,

$$\left(y-x^2y\sin(xy)-8x^3y^3\right)\frac{dx}{dy}=x^3\sin(xy)+x+12x^4y^2$$

and now we see

$$\frac{dx}{dy}=\frac{x^3\sin(xy)+x+12x^4y^2}{y-x^2y\sin(xy)-8x^3y^3}.$$