Dig-In:

Slant asymptotes

We explore functions that "shoot to infinity" at certain points in their domain.

If we think of an asymptote as a "line that a function resembles when the input or output is large," then there are three types of asymptotes, just as there are three types of lines:

Vertical Asymptotes	$\leftrightarrow$	Vertical Lines
Horizontal Aymptotes	$\leftrightarrow$	Horizontal Lines
Slant Asymptotes	$\leftrightarrow$	Slant Lines

Here we've made up a new term "slant" line, meaning a line whose slope is neither zero, nor is it undefined. Let's do a quick review of the different types of asymptotes:

Vertical asymptotes Recall, a function f has a vertical asymptote at $x = a$ if at least one of the following hold:

- $\displaystyle\lim_{x\to a} f(x) = \pm\infty$,
- $\displaystyle\lim_{x\to a^+} f(x) = \pm\infty$,
- $\displaystyle\lim_{x\to a^-} f(x) = \pm\infty$.

In this case, the asymptote is the vertical line

$$x = a.$$

Horizontal asymptotes We have also seen that a function f has a horizontal asymptote if

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L,$$

and in this case, the asymptote is the horizontal line

$$\ell(x) = L.$$

Learning outcomes: Define a slant asymptote. Approximate a slant asymptote from the graph of a function. Find slant asymptotes algebraically and graphically.

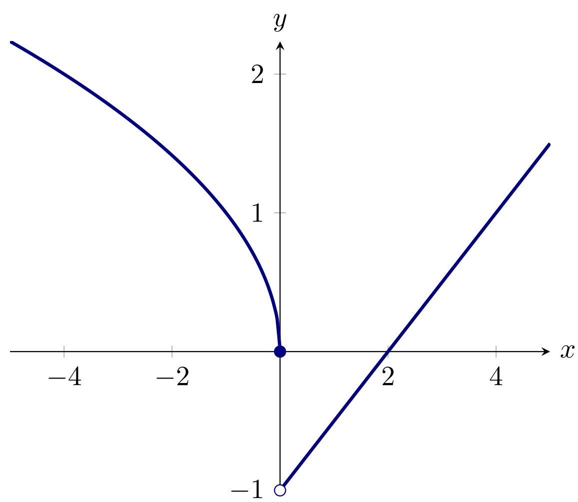
Slant asymptotes On the other hand, a *slant asymptote* is a somewhat different beast.

Definition 1. *If there is a nonhorizontal line $\ell(x) = m \cdot x + b$ such that*

$$\lim_{x\to\infty} (f(x) - \ell(x)) = 0 \qquad \textit{or} \qquad \lim_{x\to-\infty} (f(x) - \ell(x)) = 0,$$

*then ℓ is a **slant asymptote** for f.*

Question 1 *Consider the graph of the following function.*



What is the slant asymptote of this function?

$$\ell(x) = \boxed{x/2 - 1}$$

To analytically find slant asymptotes, one must find the required information to determine a line:

- The slope.
- The y-intercept.

While there are several ways to do this, we will give a method that is fairly general.

Example 1. *Find the slant asymptote of*

$$f(x) = \frac{3x^2 + x + 2}{x + 2}.$$

Explanation. *We are looking to see if there is a line ℓ such that*

$$\lim_{x\to\infty}(f(x)-\ell(x))=0 \qquad \text{or} \qquad \lim_{x\to-\infty}(f(x)-\ell(x))=0.$$

First, let's consider the limit as x approaches positive infinity. We will imagine that we have such a line

$$\ell(x)=m\cdot x+b$$

and attempt to find the correct values for m and b. Let's look again at our limit. We are assuming:

$$\lim_{x\to\infty}\left(\frac{3x^2+x+2}{x+2}-(mx+b)\right)=0.$$

We know that $f(x)$ is continuous everywhere except at $x=-2$ and $\ell(x)$ is continuous everywhere, so we can apply our limit laws away from $x=-2$. We're looking at large values of x, so this is no problem. We use the fact that the sum of the limits is the limit of the sums.

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{3x^2+x+2}{x+2}\right)-\lim_{x\to\infty}(mx+b)&=0\\ \lim_{x\to\infty}\frac{3x^2+x+2}{x+2}&=\lim_{x\to\infty}(mx+b)\end{aligned}$$

We are assuming these two limits are equal. Dividing by x on the right hand side makes the limit equal to m:

$$m=\lim_{x\to\infty}\left(\frac{mx}{x}+\frac{b}{x}\right)=\lim_{x\to\infty}\left(\frac{mx+b}{x}\right).$$

To find the value of m, then, we can divide the left hand side by x and evaluate the limit. We see the following.

$$\begin{aligned}m&=\lim_{x\to\infty}\frac{\frac{3x^2+x+2}{x+2}}{x}\\ &=\lim_{x\to\infty}\frac{\frac{3x^2+x+2}{x+2}}{x}\\ &=\lim_{x\to\infty}\frac{3x^2+x+2}{x^2+2x}\\ &=\lim_{x\to\infty}\left(\frac{3x^2+x+2}{x^2+2x}\cdot\frac{1/x^2}{1/x^2}\right)\\ &=\lim_{x\to\infty}\frac{3+1/x+2/x^2}{1+2/x}\\ &=\underset{\text{given}}{\boxed{3}}.\end{aligned}$$

So $m = 3$. *We now know that*

$$\lim_{x\to\infty} \frac{3x^2+x+2}{x+2} = \lim_{x\to\infty}(3x+b)$$

for some value of b. *To find the* y*-intercept* b*, we use a similar method. Notice that*

$$\lim_{x\to\infty}(3x+b-3x) = \lim_{x\to\infty} b = b,$$

so if we subtract $3x$ *from the right hand side, we are left with just* b. *Since the two sides are equal, subtracting* $3x$ *from the left hand side and evaluating the limit will give us the value for* b. *We write the following.*

$$\begin{aligned}
b &= \lim_{x\to\infty}\left(\frac{3x^2+x+2}{x+2} - 3x\right) \\
&= \lim_{x\to\infty}\left(\frac{3x^2+x+2}{x+2} - \frac{3x^2+6x}{x+2}\right) \\
&= \lim_{x\to\infty}\frac{3x^2+x+2-3x^2-6x}{x+2} \\
&= \lim_{x\to\infty}\frac{-5x+2}{x+2} \\
&= \lim_{x\to\infty}\left(\frac{-5x+2}{x+2}\cdot\frac{1/x}{1/x}\right) \\
&= \lim_{x\to\infty}\frac{-5+2/x}{1+2/x} \\
&= \underset{\text{given}}{\boxed{-5}}.
\end{aligned}$$

By this method, we have determined that

$$\lim_{x\to\infty}\left(\frac{3x^2+x+2}{x+2} - (3x-5)\right) = 0.$$

In other words, $\ell(x) = \underset{\text{given}}{\boxed{3x-5}}$ *is a slant asymptote for our function* f. *You should check that we get the same slant asymptote* $\ell(x) = \underset{\text{given}}{\boxed{3x-5}}$ *when we take the limit to negative infinity as well.* *We can confirm our results by looking at the graph of* $y = f(x)$ *and* $y = \ell(x)$*:*

Graph of $\dfrac{3x^2+x+2}{x+2}, 3x-5$