Dig-In

First Derivative Test

We use derivatives to help locate extrema.

The first derivative test

The method of the previous section for deciding whether there is a local maximum or minimum at a critical point by testing "near-by" points is not always convenient. Instead, since we have already had to compute the derivative to find the critical points, we can use information about the derivative to determine if we have a local maximum or minimum. Recall that

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

So how exactly does the derivative tell us whether there is a maximum, minimum, or neither at a point? Use the *first derivative test*.

Theorem 1 (First Derivative Test)**.** *Suppose that f is continuous on an interval, and that $f'(a) = 0$ for some value of a in that interval.*

- *If $f'(x) > 0$ to the left of a and $f'(x) < 0$ to the right of a, then $f(a)$ is a local maximum.*
- *If $f'(x) < 0$ to the left of a and $f'(x) > 0$ to the right of a, then $f(a)$ is a local minimum.*
- *If $f'(x)$ has the same sign to the left and right of a, then $f(a)$ is not a local extremum.*

Example 1. *Consider the function*

$$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$$

Find the intervals on which f is increasing and decreasing and identify the local extrema of f.

Learning outcomes: Define a critical point. Find critical points. Define absolute maximum and absolute minimum. Find the absolute max or min of a continuous function on a closed interval. Define local maximum and local minimum. Compare and contrast local and absolute maxima and minima. Identify situations in which an absolute maximum or minimum is guaranteed. Classify critical points. State the First Derivative Test. Apply the First Derivative Test. State the Second Derivative Test. Apply the Second Derivative Test. Define inflection points. Find inflection points.

Explanation. *Start by computing*

$$\frac{d}{dx}f(x) = \underset{\text{given}}{\boxed{x^3 + x^2 - 2x}}.$$

Now we need to find when THIS function is positive and when it is negative. To do this, start by asking where the derivative is neither positive nor negative, in other words, when is the derivative zero or undefined. Note that this function is a polynomial, and will always be defined, so we don't have any places to look for where the derivative is undefined. To find when the derivative is zero, solve

$$f'(x) = \underset{\text{given}}{\boxed{x^3 + x^2 - 2x}} = 0.$$

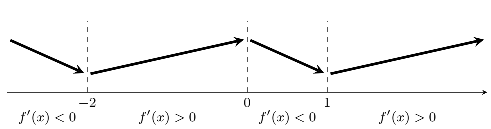
Factor $f'(x)$

$$\begin{aligned} \underset{\text{given}}{\boxed{x^3 + x^2 - 2x}} &= 0 \\ x(\underset{\text{given}}{\boxed{x^2 + x - 2}}) &= 0 \\ x(x+2)(\underset{\text{given}}{\boxed{x - 1}}) &= 0. \end{aligned}$$

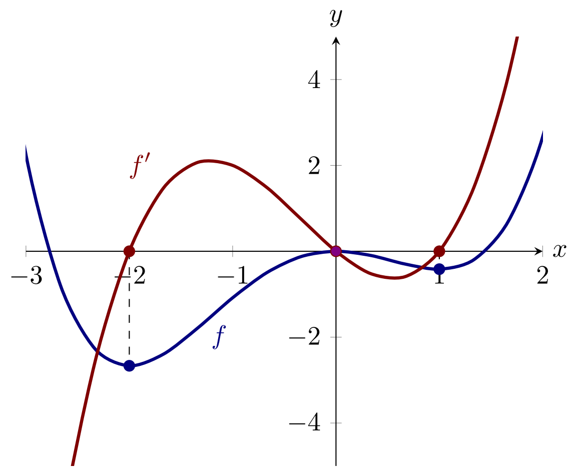
So the critical points (when $f'(x) = 0$) are when $x = -2$, $x = 0$, and $x = 1$. These are the ONLY x-values where our derivative is neither positive nor negative, so the function is neither increasing nor decreasing. The function isn't changing behavior BETWEEN critical points, so we can check a random point ***between*** *the critical points to find the behavior on that interval. In other words, check when $f'(x)$ is increasing and decreasing:*

$$\begin{aligned} f'(-3) &= \underset{\text{given}}{\boxed{-12}}, \\ f'(.5) &= \underset{\text{given}}{\boxed{-0.625}}, \\ f'(-1) &= \underset{\text{given}}{\boxed{2}}, \\ f'(2) &= \underset{\text{given}}{\boxed{8}}. \end{aligned}$$

From this we can make a sign table:



Hence f is increasing on $(-2, 0)$ and $(1, \infty)$ and f is decreasing on $(-\infty, -2)$ and $(0, 1)$. Moreover, from the first derivative test, the local maximum is at $x = 0$ while the local minima are at $x = -2$ and $x = 1$, see the graphs of of $f(x) = x^4/4 + x^3/3 - x^2$ and $f'(x) = x^3 + x^2 - 2x$.



In the graph above we see that if f' is zero and increasing at a point, then f has a local minimum at the point. If f' is zero and decreasing at a point then f has a local maximum at the point. Thus, we see that we can gain information about f by studying how f' changes. This leads us to our next section.