**Dig-In:**

# Falling objects

*We study a special type of differential equation.*

Remember, a **differential equation** is simply an equation with a derivative in it like this:

$$f'(x) = kf(x).$$

When a mathematician solves a differential equation, they are finding a *function* that satisfies the equation.

Recall that the acceleration due to gravity is about $-9.8$ m/s$^2$. Since the first derivative of the function giving the velocity of an object gives the acceleration of the object and the second derivative of a function giving the position of a falling object gives the acceleration, we have the differential equations

$$v'(t) = -9.8,$$
$$p''(t) = -9.8.$$

From these simple equation, we can derive equations for the velocity of the object and for the position using antiderivatives.

**Example 1.** *A ball is tossed into the air with an initial velocity of* 15 *m/s. What is the velocity of the ball after 1 second? How about after 2 seconds?*

**Explanation.** *Knowing that the acceleration due to gravity is* $-9.8$ *m/s*$^2$*, we write*

$$v'(t) = \underbrace{\boxed{-9.8}}_{\text{given}}.$$

*To solve this differential equation, take the antiderivative of both sides*

$$\int v'(t)\,dt = \int \underbrace{\boxed{-9.8}}_{\text{given}}\,dt$$
$$v(t) = \underbrace{\boxed{-9.8t}}_{\text{given}} + C.$$

*Here $C$ represents the initial velocity of the ball. Since it is tossed up with an initial velocity of* 15 *m/s,*

$$\underbrace{\boxed{15}}_{\text{given}} = v(0) = -9.8 \cdot 0 + C,$$

*and we see that* $C = \underbrace{\boxed{15}}_{\text{given}}$*. Hence* $v(t) = -9.8t + 15$*. Now when* $t = 1$*,* $v(1) = 5.2$ *m/s, and the ball is rising, and at* $t = 2$*,* $v(2) = -4.6$ *m/s, and the ball is falling.*

Learning outcomes:

Now let's do a similar problem, but instead of finding the velocity, we will find the position.

**Example 2.** *A ball is tossed into the air with an initial velocity of* 15 *m/s from a height of 2 meters. When does the ball hit the ground?*

**Explanation.** *Knowing that the acceleration due to gravity is* $-9.8$ $m/s^2$*, we write*

$$p''(t) = \underbrace{\boxed{-9.8}}_{\text{given}}.$$

*Start by taking the antiderivative of both sides of the equation*

$$\begin{aligned}\int p''(t)\,dt &= \int \underbrace{\boxed{-9.8}}_{\text{given}}\,dt\\ p'(t) &= \underbrace{\boxed{-9.8t}}_{\text{given}} + C.\end{aligned}$$

*Here* $C$ *represents the initial velocity of the ball. Since it is tossed up with an initial velocity of* 15 $m/s$*,* $C = 15$ *and*

$$p'(t) = -9.8t + 15.$$

*Now let's take the antiderivative again.*

$$\begin{aligned}\int p'(t)\,dt &= \int \underbrace{\boxed{-9.8t + 15}}_{\text{given}}\,dt\\ p(t) &= \underbrace{\boxed{\frac{-9.8t^2}{2} + 15t}}_{\text{given}} + D.\end{aligned}$$

*Since we know the initial height was* 2 *meters, write*

$$2 = p(0) = \frac{-9.8 \cdot 0^2}{2} + 15 \cdot 0 + D.$$

*Hence* $p(t) = \dfrac{-9.8t^2}{2} + 15t + 2$*. We need to know when the ball hits the ground, this is when* $p(t) = 0$*. Solving the equation*

$$\frac{-9.8t^2}{2} + 15t + 2 = 0$$

*we find two solutions* $t \approx -0.1$ *and* $t \approx 3.2$*. Discarding the negative solution, we see the ball will hit the ground after approximately* 3.2 *seconds.*

The power of calculus is that it frees us from rote memorization of formulas and enables us to derive what we need.