Dig-In:

Linear approximation

We use a method called "linear approximation" to estimate the value of a (complicated) function at a given point.

Given a function, a *linear approximation* is a fancy phrase for something you already know:

> **The derivative is the slope of the tangent line.**

Except in this section, the emphasis is on the **line**.

Definition 1. *If f is a differentiable function at $x = a$, then a* ***linear approximation*** *for f at $x = a$ is given by*

$$\ell(x) = f'(a)(x - a) + f(a).$$

Note that $\ell(x)$ is just the tangent line to $f(x)$ at $x = a$.

A linear approximation of f is a "good" approximation as long as x is "not too far" from a. If one "zooms in" on f sufficiently, then f and the linear approximation are nearly indistinguishable. As a first example, we will see how linear approximations allow us to make approximate "difficult" computations.

Example 1. *Use a linear approximation of $f(x) = \sqrt[3]{x}$ at $x = 64$ to approximate $\sqrt[3]{50}$.*

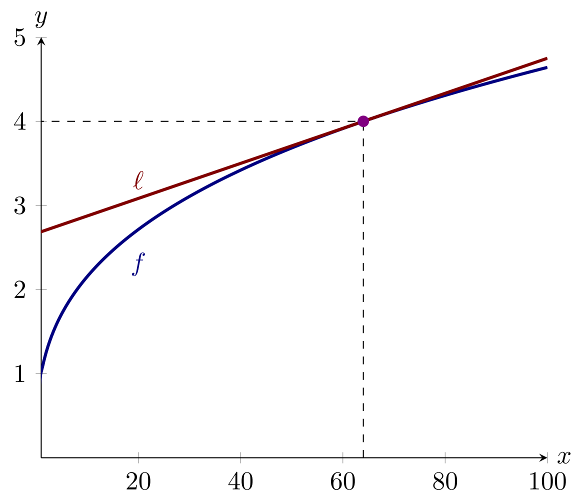
Explanation. *To start, write*

$$\frac{d}{dx}f(x) = \frac{d}{dx}x^{1/3} = \frac{1}{\underset{\text{given}}{\boxed{3x^{2/3}}}}.$$

Learning outcomes: Define linear approximation as an application of the tangent to a curve. Find the linear approximation to a function at a point and use it to approximate the function value. Identify when a linear approximation can be used. Label a graph with the appropriate quantities used in linear approximation. Find the error of a linear approximation. Compute differentials. Contrast the notation and meaning of dy versus Δy. Justify the chain rule via the composition of linear approximations.

So our linear approximation is

$$\begin{aligned}\ell(x) &= \frac{1}{3\cdot 64^{2/3}}(x-64) + \underbrace{\boxed{4}}_{\text{given}} \\ &= \frac{1}{\underbrace{\boxed{48}}_{\text{given}}}(x-64)+4 \\ &= \frac{x}{48}+\frac{8}{3}.\end{aligned}$$



Now we evaluate $\ell(50) \approx 3.71$ *and compare it to* $\sqrt[3]{50} \approx 3.68$*. From this we see that the linear approximation, while perhaps inexact, is computationally* ***easier*** *than computing the cube root.*

With modern calculators and computing software it may not appear necessary to use linear approximations. In fact they are quite useful. In cases requiring an explicit numerical approximation, they allow us to get a quick rough estimate which can be used as a "reality check" on a more complex calculation. In some complex calculations involving functions, the linear approximation makes an otherwise intractable calculation possible, without serious loss of accuracy.

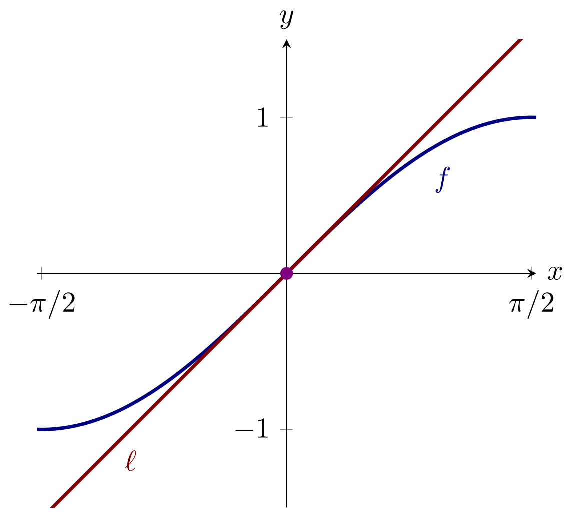
Example 2. *Use a linear approximation of* $f(x) = \sin(x)$ *at* $x = 0$ *to approximate* $\sin(0.3)$.

Explanation. *To start, write*

$$\frac{d}{dx}f(x) = \underbrace{\boxed{\cos(x)}}_{\text{given}},$$

so our linear approximation is

$$\begin{aligned}\ell(x) &= \underbrace{\boxed{\cos(0)}}_{\text{given}} \cdot (x - 0) + 0 \\ &= x.\end{aligned}$$



Hence a linear approximation for $\sin(x)$ *at* $x = 0$ *is* $\ell(x) = x$*, and so* $\ell(0.3) = 0.3$*. Comparing this to* $\sin(.3) \approx 0.295$*, we see that the approximation is quite good. For this reason, it is common to approximate* $\sin(x)$ *with its linear approximation* $\ell(x) = x$ *when* x *is near zero.*

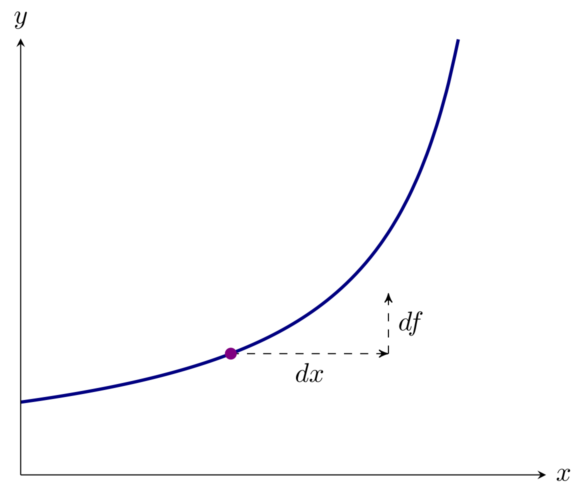
Differentials

The notion of a *differential* goes back to the origins of calculus, though our modern conceptualization of a differential is somewhat different than how they were initially understood.

Definition 2. *Let* f *be a differentiable function. We define a new independent variable* dx*, and a new dependent variable*

$$df = f'(x) \cdot dx.$$

The variables dx *and* df *are called* ***differentials****. Geometrically, differentials can be interpreted via the diagram below*



Note, it is now the case (by definition!) that

$$\frac{df}{dx} = f'(x).$$

Question 1 *The differential dx is:*

Multiple Choice:

(a) *d times x.*

(b) *A single variable.* ✓

Question 2 *The differential df is:*

Multiple Choice:

(a) *d times f.*

(b) *A single variable that is dependent on dx.* ✓

Essentially, differentials allow us to solve the problems presented in the previous examples from a slightly different point of view. Recall, when h is near but not equal zero,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

hence,

$$f'(x)h \approx f(x+h) - f(x).$$

Since h is simply a variable, and dx is simply a variable, we can replace h with dx to write

$$\begin{aligned} f'(x)\cdot dx &\approx f(x+dx) - f(x) \\ df &\approx f(x+dx) - f(x). \end{aligned}$$

Adding $f(x)$ to both sides we see

$$f(x+dx) \approx f(x) + df.$$

While this is something of a "sleight of hand" with variables, there are contexts where the language of differentials is common. Here is the basic strategy:

$$\underbrace{f(x+dx)}_{\text{what you want}} \approx \overbrace{f(x)}^{\text{what you know}} + \underbrace{df}_{\text{what you compute}}$$

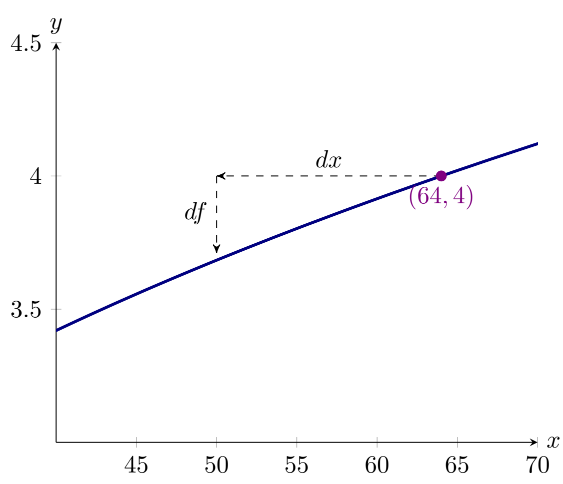
We will repeat our previous examples using differentials.

Example 3. *Use differentials to approximate* $\sqrt[3]{50}$.

Explanation. *Set* $f(x) = \sqrt[3]{x}$. *We want to know* $\sqrt[3]{50}$. *Since* $4^3 = 64$, *we set* $x = 64$. *Setting* $dx = -14$, *we have*

$$\begin{aligned} \sqrt[3]{50} = f(x+dx) &\approx f(x) + df \\ &\approx \sqrt[3]{64} + df. \end{aligned}$$

Here we see a plot of $y = \sqrt[3]{x}$ *with the differentials above marked:*



Now we must compute df:

$$\begin{aligned} df &= f'(x)\cdot df \\ &= \underbrace{\boxed{\frac{1}{3x^{2/3}}}}_{\text{given}}\cdot df \\ &= \frac{1}{3\cdot 64^{2/3}}\cdot(\underbrace{\boxed{-14}}_{\text{given}}) \\ &= \frac{1}{3\cdot 64^{2/3}}\cdot(-14) \\ &= \frac{-7}{24} \end{aligned}$$

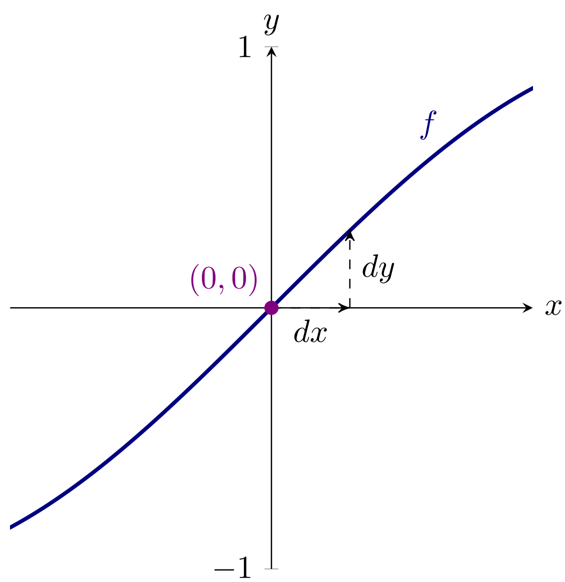
Hence $f(50)\approx f(64)+\dfrac{-7}{24}\approx 3.71$.

Example 4. *Use differentials to approximate* $\sin(0.3)$.

Explanation. *Set* $y=\sin(x)$. *We want to know* $\sin(0.3)$. *Since* $\sin(0)=0$, *we will set* $x=0$ *and* $dx=0.3$. *Write with me*

$$\begin{aligned} \sin(0.3)=\sin(x+dx) &\approx \sin(x)+dy \\ &\approx 0+dy. \end{aligned}$$

Here we see a plot of $y=\sin(x)$ *with the differentials above marked:*



Now we must compute dy*:*

$$\begin{aligned} dy &= \left(\frac{d}{dx}\sin(x)\right)\cdot dx \\ &= \underbrace{\boxed{\cos(0)}}_{\text{given}}\cdot dx \\ &= 1\cdot(\underbrace{\boxed{0.3}}_{\text{given}}) \\ &= 0.3 \end{aligned}$$

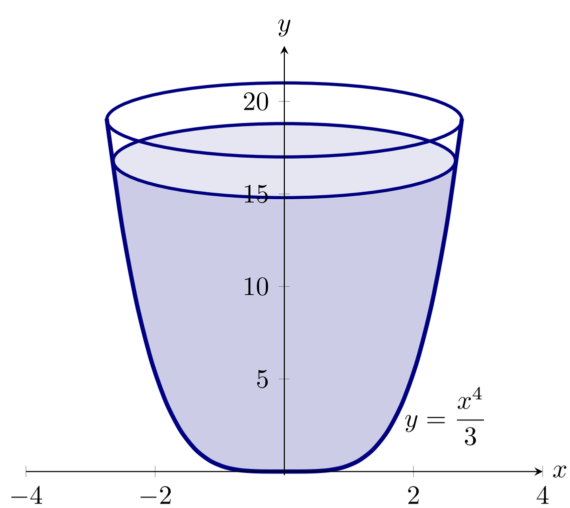
Hence $\sin(0.3) \approx \sin(0) + 0.3 \approx 0.3$.

The upshot is that linear approximations and differentials are simply two slightly different ways of doing the exact same thing.

Error approximation

Differentials also help us estimate error in real life settings.

Example 5. *The cross-section of a* 250 *ml glass can be modeled by the function* $r(x) = \dfrac{x^4}{3}$*:*



At 16.8 *cm from the base of the glass, there is a mark indicating when the glass is filled to* 250 *ml. If the glass is filled within* ± 2 *millimeters of the mark, what are the bounds on the volume? As a gesture of friendship, we will tell you that* *the volume in milliliters, as a function of the height of water in centimeters, y, is given by*

$$V(y) = \frac{2\pi y^{3/2}}{\sqrt{3}}.$$

Note: If you persist in your quest to learn calculus, you will be able to derive the formula above like it's no-big-deal.

Explanation. *We want to know what a small change in the height, y does to the volume V. These small changes can be modeled by the differentials dV and dy. Since*

$$dV = V'(y)\, dy$$

and $V'(x) = \underbrace{\boxed{\pi\sqrt{3y}}}_{\text{given}}$ *we use the fact that* $dy = \pm 0.2$ *with* $y = \underbrace{\boxed{16.8}}_{\text{given}}$ *to see*

$$dV = \underbrace{\boxed{\pi\sqrt{3 \cdot 16.8}}}_{\text{given}} \cdot 0.2.$$

Hence the volume will vary by at most $\pm\underbrace{\boxed{4.46062}}_{\text{given}}$ *milliliters.*

New and old friends

You might be wondering, given a plot $y = f(x)$,

> What's the difference between Δx and dx? What about Δy and dy?

Regardless, it is now a pressing question. Here's the deal:

$$\frac{\Delta y}{\Delta x}$$

is the **average rate of change** of $y = f(x)$ with respect to x. On the other hand:

$$\frac{dy}{dx}$$

is the **instantaneous rate of change** of $y = f(x)$ with respect to x. Essentially, Δx and dx are the same type of thing, they are (usually small) changes in x. However, Δy and dy are very different things.

- Δy is the change of y associated to Δx.
- dy is the change in y needed to make the following relation true:

$$dy = f'(x)\, dx$$

Question 3 *Suppose $f(x) = x^2$. If $\Delta x = dx = 0.1$, what is Δy? What is dy?*

$$\Delta y = \boxed{0.01} \qquad dy = \boxed{0.02}$$

Differentials can be confusing at first. However, when you master them, you will have a powerful tool at your disposal.