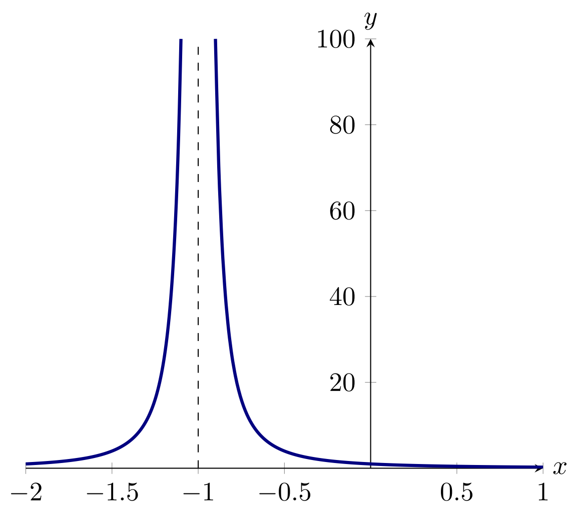
**Dig-In:**

# Vertical asymptotes

*We explore functions that "shoot to infinity" at certain points in their domain.*

Consider the function

$$f(x) = \frac{1}{(x+1)^2}.$$




While the $\lim_{x\to -1} f(x)$ does not exist, something can still be said.

**Definition 1.** *If $f(x)$ grows arbitrarily large as $x$ approaches $a$, we write*

$$\lim_{x\to a} f(x) = \infty$$

*and say that the limit of $f(x)$* ***approaches infinity*** *as $x$ goes to $a$.*

*If $|f(x)|$ grows arbitrarily large as $x$ approaches $a$ and $f(x)$ is negative, we write*

$$\lim_{x\to a} f(x) = -\infty$$

*and say that the limit of $f(x)$* ***approaches negative infinity*** *as $x$ goes to $a$.*

Learning outcomes: Recognize when a limit is indicating there is a vertical asymptote. Evaluate the limit as $x$ approaches a point where there is a vertical asymptote. Match graphs of functions with their equations based on vertical asymptotes. Discuss what it means for a limit to equal $\infty$. Define a vertical asymptote. Understand the relationship between limits and vertical asymptotes. Find vertical asymptotes of famous functions. Find vertical asymptotes by looking at a graph.

**Question 1** *Which of the following are correct?*

***Select All Correct Answers:***

(a) $\lim_{x\to -1} \frac{1}{(x+1)^2} = \infty$ ✓

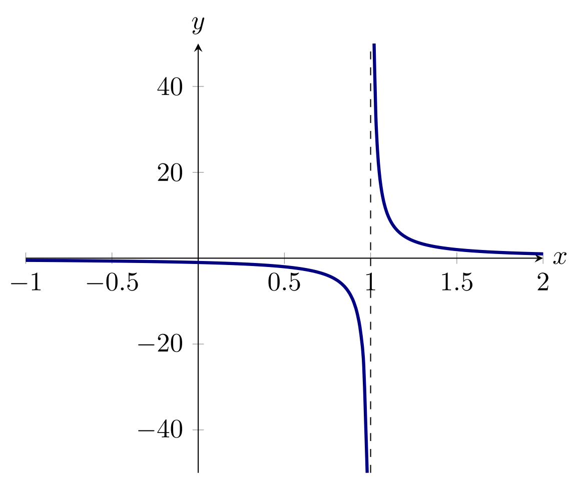
(b) $\lim_{x\to -1} \frac{1}{(x+1)^2} \to \infty$

(c) $f(x) = \frac{1}{(x+1)^2}$, so $f(-1) = \infty$

(d) $f(x) = \frac{1}{(x+1)^2}$, so as $x \to -1$, $f(x) \to \infty$ ✓

On the other hand, consider the function

$$f(x) = \frac{1}{(x-1)}.$$




While the two sides of the limit as $x$ approaches 1 do not agree, we can still consider the one-sided limits. We see $\lim_{x\to 1^+} f(x) = \infty$ and $\lim_{x\to 1^-} f(x) = -\infty$.

**Definition 2.** *If at least one of the following hold:*

- $\lim_{x\to a} f(x) = \pm\infty$,
- $\lim_{x\to a^+} f(x) = \pm\infty$,

- $\lim_{x \to a^-} f(x) = \pm\infty$,

*then the line $x = a$ is a **vertical asymptote** of $f$.*

**Example 1.** *Find the vertical asymptotes of*

$$f(x) = \frac{x^2 - 9x + 14}{x^2 - 5x + 6}.$$

**Explanation.** *Start by factoring both the numerator and the denominator:*

$$\frac{x^2 - 9x + 14}{x^2 - 5x + 6} = \frac{(x-2)(x-7)}{(x-2)(x-3)}$$

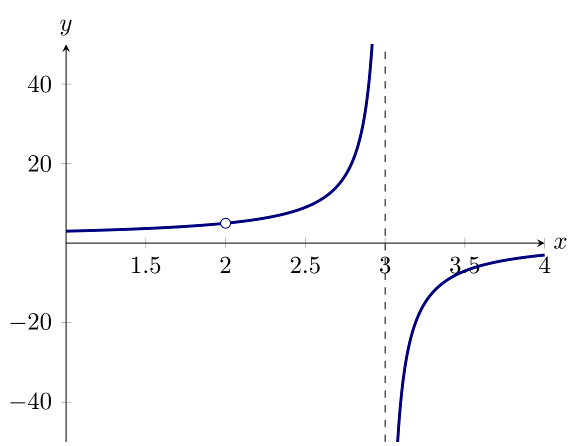
*Using limits, we must investigate when $x \to 2$ and $x \to 3$. Write*

$$\begin{aligned} \lim_{x \to 2} \frac{(x-2)(x-7)}{(x-2)(x-3)} &= \lim_{x \to 2} \frac{(x-7)}{(x-3)} \\ &= \frac{-5}{-1} \\ &= 5. \end{aligned}$$

*Now write*

$$\begin{aligned} \lim_{x \to 3} \frac{(x-2)(x-7)}{(x-2)(x-3)} &= \lim_{x \to 3} \frac{(x-7)}{(x-3)} \\ &= \lim_{x \to 3} \frac{-4}{x-3}. \end{aligned}$$

*Consider the one-sided limits separately. Since $\lim_{x \to 3^+} (x-3)$ approaches $0$ from the right and the numerator is negative, $\lim_{x \to 3^+} f(x) = -\infty$. Since $\lim_{x \to 3^-} (x-3)$ approaches $0$ from the left and the numerator is negative, $\lim_{x \to 3^-} f(x) = \infty$.*




*Hence we have a vertical asymptote at $x = 3$.*