Dig-In:

What is a limit?

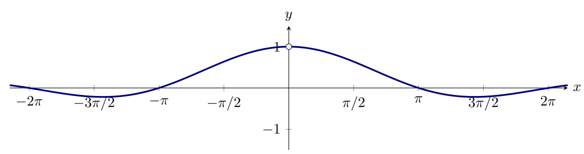
We introduce limits.

The basic idea

Consider the function

$$f(x) = \frac{\sin(x)}{x}.$$

While $f(x)$ is undefined at $x = 0$, we can still plot $f(x)$ at other values near $x = 0$.



Question 1 *Use the graph of* $f(x) = \dfrac{\sin(x)}{x}$ *above to answer the following question: What is* $f(0)$*?*

Multiple Choice:

(a) 0

(b) $f(0)$

(c) 1

(d) $f(0)$ *is undefined* ✓

(e) *it is impossible to say*

Nevertheless, we can see that as x approaches zero, $f(x)$ approaches one. From this setting we come to our definition of a limit.

Learning outcomes: Consider function values nearer and nearer to a given input value. Understand the concept of a limit. Limits as understanding local behavior of functions. Calculate limits from a graph (or state that the limit does not exist). Define a one-sided limit.

Definition 1. *Intuitively,*

*the **limit** of $f(x)$ as x approaches a is L,*

written

$$\lim_{x\to a} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all x sufficiently close, but not equal to, a.

Question 2 *Use the graph of $f(x) = \dfrac{\sin(x)}{x}$ above to finish the following statement: "A good guess is that..."*

Multiple Choice:

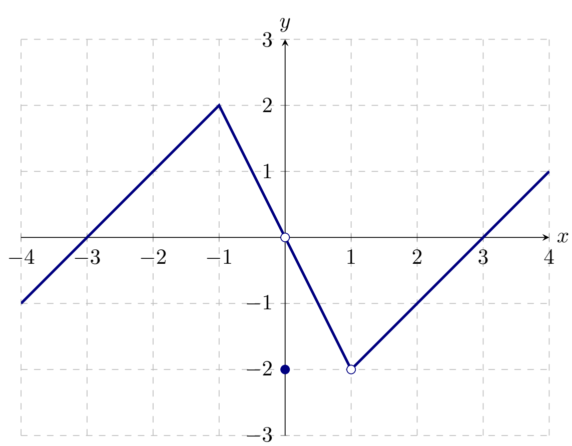
(a) $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$ ✓

(b) $\displaystyle\lim_{x\to 1} \frac{\sin(x)}{x} = 0.$

(c) $\displaystyle\lim_{x\to 1} f(x) = \frac{\sin(1)}{1}.$

(d) $\displaystyle\lim_{x\to 0} f(x) = \frac{\sin(0)}{0} = \infty.$

Question 3 *Consider the following graph of $y = f(x)$*



Use the graph to evaluate the following. Write DNE if the value does not exist.

(a) $f(-2) = \boxed{1}$

(b) $\lim_{x \to -2} f(x) = \boxed{1}$

(c) $f(-1) = \boxed{2}$

(d) $\lim_{x \to -1} f(x) = \boxed{2}$

(e) $f(0) = \boxed{-2}$

(f) $\lim_{x \to 0} f(x) = \boxed{0}$

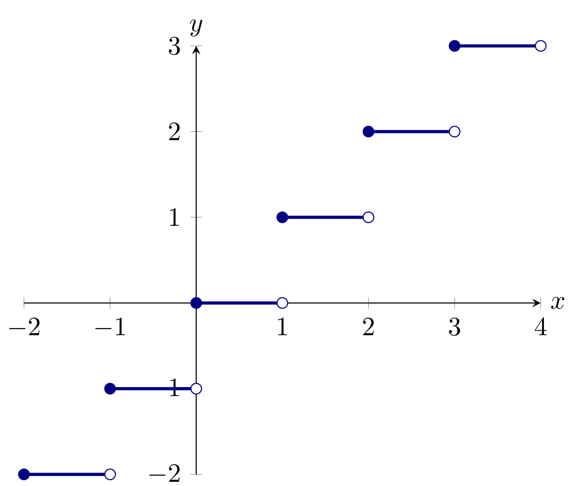
(g) $f(1) = \boxed{DNE}$

(h) $\lim_{x \to 1} f(x) = \boxed{-2}$

Limits might not exist

Limits might not exist. Let's see how this happens.

Example 1. *Consider the graph of $f(x) = \lfloor x \rfloor$.*



Explain why the limit

$$\lim_{x \to 2} f(x)$$

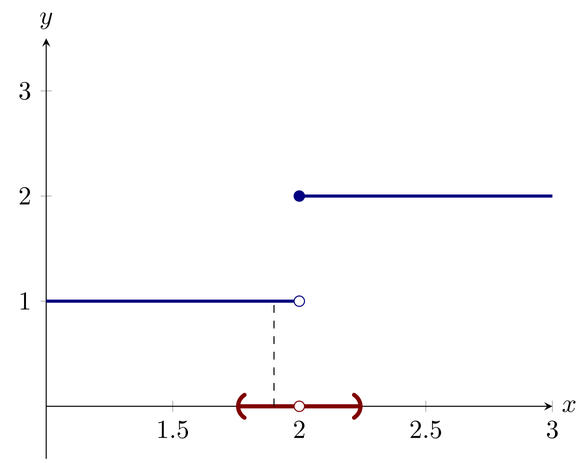
does not exist.

Explanation. *The function $\lfloor x \rfloor$ is the function that returns the greatest integer less than or equal to x. Recall that*

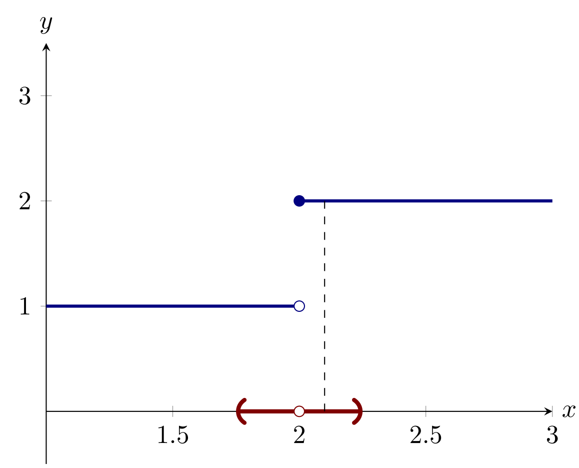
$$\lim_{x \to 2} \lfloor x \rfloor = L$$

if $\lfloor x \rfloor$ can be made arbitrarily close to L by making x sufficiently close, but not equal to, 2. So let's examine x near, but not equal to, 2. Now the question is: What is L?

If this limit exists, then we should be able to look sufficiently close, but not at, $x = 2$, and see that f is approaching some number. Let's look at a graph:



If we look closer and closer to $x = 2$ (on the left of $x = 2$) we see that $f(x) = 1$. However, if we look closer and closer to $x = 2$ (on the right of $x = 2$) we see



So just to the right of $x = 2$, $f(x) = 2$. We cannot find a single number that $f(x)$ approaches as x approaches $x = 2$, and so the limit does not exists.

Tables can be used to help guess limits, but one must be careful.

Question 4 *Consider* $f(x) = \sin\left(\frac{\pi}{x}\right)$. *Fill in the tables below:*

x	$f(x)$
0.1	$\boxed{0}$
0.01	$\boxed{0}$
0.001	$\boxed{0}$
0.0001	$\boxed{0}$

and

x	$f(x)$
0.3	$\boxed{-.866}$
0.03	$\boxed{-.866}$
0.003	$\boxed{-.866}$
0.0003	$\boxed{-.866}$

What do the tables tell us about

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)?$$

Multiple Choice:

(a) $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = 0$

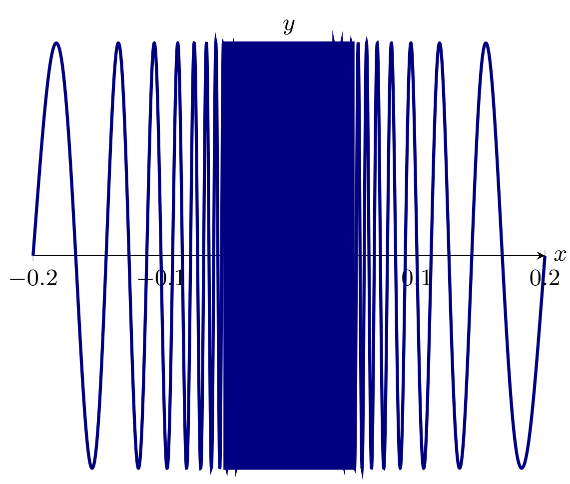
(b) $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = 1$

(c) $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = -.866$

(d) $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = -.433$

(e) *it is unclear what the tables are telling us about the limit* ✓

Feedback (attempt): *Neither tables nor graphs can ever tell us for certain what a limit is. However, sometimes they can help "guess" the limit. In this case the graph of $f(x)$ is somewhat more helpful:*



We see that $f(x)$ oscillates "wildly" as x approaches 0, and hence does not approach any one number.

One-sided limits

While we have seen that $\lim_{x\to 2}\lfloor x\rfloor$ does not exist, more can still be said.

Definition 2. *Intuitively,*

*the **limit from the right** of f as x approaches a is L,*

written

$$\lim_{x\to a^+} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all $x > a$ sufficiently close, but not equal to, a.

Similarly,

*the **limit from the left** of $f(x)$ as x approaches a is L,*

written

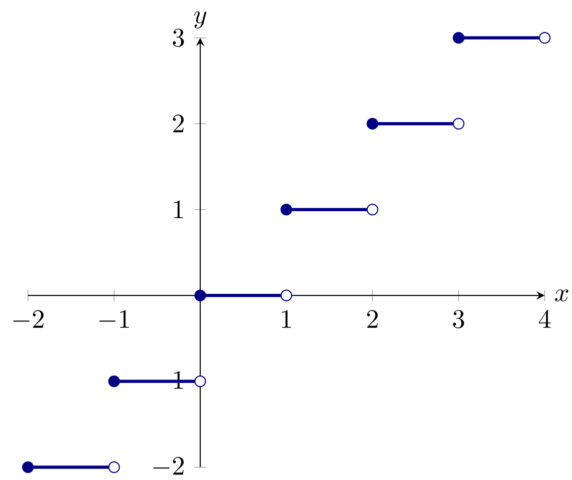
$$\lim_{x\to a^-} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all $x < a$ sufficiently close, but not equal to, a.

Example 2. *Compute:*

$$\lim_{x\to 2^-} f(x) \qquad \text{and} \qquad \lim_{x\to 2^+} f(x)$$

by using the graph below



Explanation. *From the graph we can see that as x approaches 2 from the left, $\lfloor x \rfloor$ remains at $y = 1$ up until the exact point that $x = 2$. Hence*

$$\lim_{x \to 2^-} f(x) = 1.$$

Also from the graph we can see that as x approaches 2 from the right, $\lfloor x \rfloor$ remains at $y = 2$ up to $x = 2$. Hence

$$\lim_{x \to 2^+} f(x) = 2.$$

When you put this all together

One-sided limits help us talk about limits.

Theorem 1. *A limit*

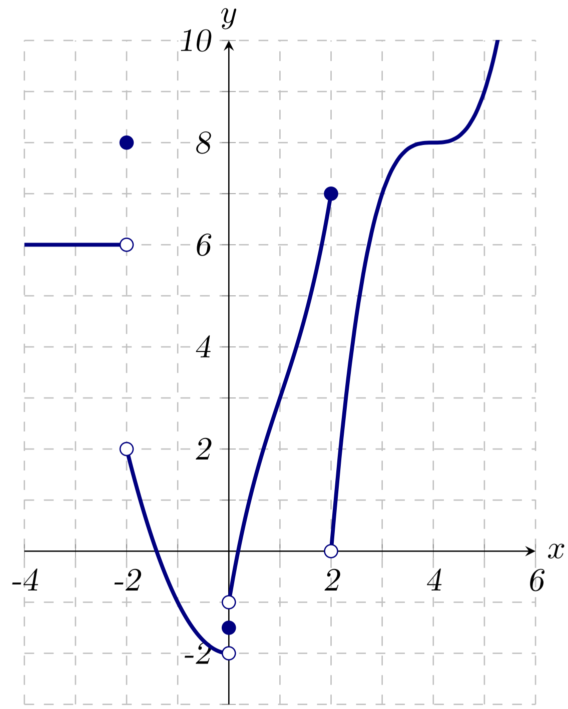
$$\lim_{x \to a} f(x)$$

exists if and only if

- $\lim_{x \to a^-} f(x)$ *exists*
- $\lim_{x \to a^+} f(x)$ *exists*
- $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

In this case, $\lim_{x \to a} f(x)$ is equal to the common value of the two one sided limits.

Question 5 *Evaluate the expressions by referencing the graph below. Write DNE if the limit does not exist.*



(a) $\lim_{x\to 4} f(x) = \boxed{8}$

(b) $\lim_{x\to -3} f(x) = \boxed{6}$

(c) $\lim_{x\to 0} f(x) = \boxed{DNE}$

(d) $\lim_{x\to 0^-} f(x) = \boxed{-2}$

(e) $\lim_{x\to 0^+} f(x) = \boxed{-1}$

(f) $f(-2) = \boxed{8}$

(g) $\lim_{x\to 2^-} f(x) = \boxed{7}$

(h) $\lim_{x\to -2^-} f(x) = \boxed{6}$

(i) $\lim_{x\to 0} f(x+1) = \boxed{3}$

(j) $f(0)=\boxed{-3/2}$

(k) $\lim\limits_{x\to 1^-} f(x-4)=\boxed{6}$

(l) $\lim\limits_{x\to 0^+} f(x-2)=\boxed{2}$