Dig-In:

The Inverse Function Theorem

We see the theoretical underpinning of finding the derivative of an inverse function at a point.

There is one catch to all the explanations given above where we computed derivatives of inverse functions. To write something like

$$\frac{d}{dx}(e^y) = e^y \cdot y'$$

we need to know that the function y *has* a derivative. All we have shown is that *if* it has a derivative then that derivative must be $1/x$. The *Inverse Function Theorem* guarantees this.

Theorem 1 (Inverse Function Theorem)**.** *If f is a differentiable function that is one-to-one near a and $f'(a) \neq 0$, then*

(a) *$f^{-1}(x)$ is **defined** for x near $b = f(a)$,*

(b) *$f^{-1}(x)$ is **differentiable** near $b = f(a)$,*

(c) *last, but not least:*

$$\left[\frac{d}{dx} f^{-1}(x)\right]_{x=b} = \frac{1}{f'(a)} \qquad \text{where} \qquad b = f(a).$$

Explanation. *We will only explain the last result. We know*

$$f(f^{-1}(x)) = x,$$

and now we use implicit differentiation (and the chain rule) to write

$$\begin{aligned}\frac{d}{dx} f(f^{-1}(x)) &= f'(f^{-1}(x))(f^{-1})'(x) \\ &= 1.\end{aligned}$$

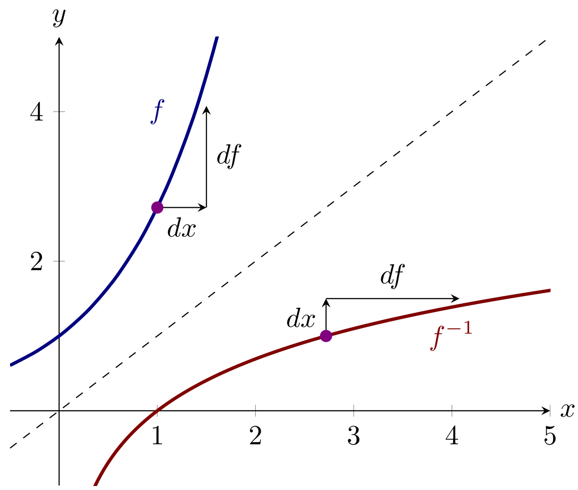
Solving for $(f^{-1})'(x)$ we see

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

This is what we have written above.

Learning outcomes: Understand how the derivative of an inverse function relates to the original derivative.

It is worth giving one more piece of evidence for the formula above, this time based on differentials. Consider this plot of a function f and its inverse:



Since the inverse of a function is the reflection of the function over the line $y = x$, we see that the differentials are "switched" when reflected. Hence we see that

$$\frac{df^{-1}}{dx} = \frac{dx}{df}.$$

The inverse function theorem gives us a recipe for computing the derivatives of inverses of functions at points.

Example 1. *Let f be a differentiable function that has an inverse. In the table below we give several values for both f and f':*

x	f	f'
2	0	2
3	1	-2
4	-3	0

Compute

$$\frac{d}{dx} f^{-1}(x) \text{ at } x = 1.$$

Explanation. *From the table above we see that*

$$1 = f(\underbrace{\boxed{3}}_{\text{given}}).$$

Hence, by the inverse function theorem

$$\left(f^{-1}(1)\right)' = \frac{1}{f'(\underbrace{\boxed{3}}_{\text{given}})} = \underbrace{\boxed{\frac{-1}{2}}}_{\text{given}}.$$

If one example is good, two are better:

Example 2. *Let f be a differentiable function that has an inverse. In the table below we give several values for both f and f':*

x	f	f'
2	0	2
3	1	-2
4	-3	0

Compute

$$\left(f^{-1}(0)\right)'$$

Explanation. *Note,*

$$\left(f^{-1}(0)\right)' = \frac{d}{dx} f^{-1}(x) \text{ at } x = 0.$$

From the table above we see that

$$0 = f(\underbrace{\boxed{2}}_{\text{given}}).$$

Hence, by the inverse function theorem

$$\left(f^{-1}(0)\right)' = \frac{1}{f'(\underbrace{\boxed{2}}_{\text{given}})} = \underbrace{\boxed{\frac{1}{2}}}_{\text{given}}.$$

Finally, let's see an example where the theorem does not apply.

Example 3. *Let f be a differentiable function that has an inverse. In the table below we give several values for both f and f':*

x	f	f'
2	0	2
3	1	-2
4	-3	0

Compute

$$\left[\frac{d}{dx} f^{-1}(x)\right]_{x=-3}$$

Explanation. *From the table above we see that*

$$-3 = f(\underbrace{\boxed{4}}_{\text{given}}).$$

Ah! But here, $f'(\underbrace{\boxed{4}}_{\text{given}}) = \underbrace{\boxed{0}}_{\text{given}}$, so we have no guarantee that the inverse exists near the point $x = -3$, but even if it did the inverse would not be differentiable there.