Dig-In:

Differential equations

We study equations with that relate functions with their rates.

A *differential equation* is simply an equation with a derivative in it. Here is an example:

$$a \cdot f''(x) + b \cdot f'(x) + c \cdot f(x) = g(x).$$

Question 1 *What is a differential equation?*

Multiple Choice:

(a) *An equation that you take the derivative of.*

(b) *An equation that relates the rate of a function to other values.* ✓

(c) *It is a formula for the slope of a tangent line at a given point.*

When a mathematician solves a differential equation, they are finding *functions* satisfying the equation.

Question 2 *Which of the following functions solve the differential equation*

$$f''(x) = -f(x)?$$

Select All Correct Answers:

(a) e^x

(b) $\sin(x)$ ✓

(c) $\cos(x)$ ✓

Learning outcomes: Define a differential equation. Verify solutions to differential equations.

Exponential growth and decay

A function f exhibits **exponential growth** if its growth rate is proportional to its value. As a differential equation, this means

$$f'(x) = kf(x) \qquad \text{for some constant of proportionality } k.$$

We claim that this differential equation is solved by

$$f(x) = Ae^{kx},$$

where A and k are constants. Check it out, if $f(x) = Ae^{kx}$, then

$$\begin{aligned} f'(x) &= Ake^{kx} \\ &= k\left(Ae^{kx}\right) \\ &= kf(x). \end{aligned}$$

Example 1. *A culture of yeast starts with* 100 *cells. After* 160 *minutes, there are* 350 *cells. Assuming that the growth rate of the yeast is proportional to the number of yeast cells present, estimate when the culture will have* 1000 *cells.*

Explanation. *Since the growth rate of the yeast is proportional to the number of yeast cells present, we have the following differential equation*

$$p'(t) = kp(t)$$

where $p(t)$ is the population of the yeast culture at time t with t measured in minutes. We know that this differential equation is solved by the function

$$p(t) = Ae^{kt}$$

where A and k are yet to be determined constants. Since

$$100 = p(0) = Ae^{k\cdot 0}$$

we see that $A = 100$. So

$$p(t) = 100e^{kt}.$$

Now we must find k. Since we know that

$$\underbrace{\boxed{350}}_{\text{given}} = p(160) = 100e^{k\cdot 160}$$

we need to solve for k. Write

$$\begin{aligned} 350 &= 100e^{k\cdot 160} \\ 3.5 &= e^{k\cdot 160} \\ \ln(3.5) &= k \cdot 160 \\ \ln(3.5)/160 &= k. \end{aligned}$$

Hence

$$p(t) = 100e^{t\ln(3.5)/160} = 100 \cdot 3.5^{t/160}.$$

To find out when the culture has 1000 cells, write

$$\begin{aligned} 1000 &= 100 \cdot 3.5^{t/160} \\ 10 &= 3.5^{t/160} \\ \ln(10) &= \frac{t\ln(3.5)}{160} \\ \frac{160\ln(10)}{\ln(3.5)} &= t. \end{aligned}$$

From this we find that after approximately 294 *minutes, there are around* 1000 *yeast cells present.*

It is worth seeing an example of exponential decay as well. Consider this: Living tissue contains two types of carbon, a stable isotope carbon-12 and a radioactive (unstable) isotope carbon-14. While an organism is alive, the ratio of one isotope of carbon to the other is always constant. When the organism dies, the ratio changes as the radioactive isotope decays. This is the basis of radiocarbon dating.

Example 2. *The half-life of carbon-14 (the time it takes for half of an amount of carbon-14 to decay) is about 5730 years. Moreover, the rate of decay of carbon-14 is proportional to the amount of carbon-14.*

If we find a bone with $1/70$*th of the amount of carbon-14 we would expect to find in a living organism, approximately how old is the bone?*

Explanation. *Since the rate of decay of carbon-14 is proportional to the amount of carbon-14 present, we can model this situation with the differential equation*

$$f'(t) = kf(t).$$

We know that this differential equation is solved by the function defined by

$$f(t) = Ae^{kt}$$

where A and k are yet to be determined constants. Since the half-life of carbon-14 is about 5730 *years we write*

$$\frac{1}{2} = e^{k5730}.$$

Solving this equation for k, gives

$$k = \frac{-\ln(2)}{5730}.$$

Since we currently have $1/70th$ *of the original amount of carbon-14 we write*

$$\underbrace{\boxed{\frac{1}{70}}}_{\text{given}} = 1 \cdot e^{\frac{-\ln(2)t}{5730}}.$$

Solving this equation for t*, we find* $t \approx -35121$*. This means that the bone is approximately* 35121 *years old.*

Infectious diseases

There are many models for the spread of infectious diseases. Perhaps the most basic is the following:

$$\text{infect}'(t) = k \cdot \text{infect}(t) \cdot (P - \text{infect}(t))$$

where k is a constant, $\text{infect}(t)$ is the number of people infected by the disease on day t, and P is the size of the population vulnerable to the disease.

What this is saying is that the rate that the infectious disease spreads is proportional to the product of the infected by the uninfected:

$$\underbrace{\text{infect}'(t)}_{\text{rate the disease spreads}} = \overbrace{\ k\ }^{\text{is proportional to}} \cdot \underbrace{\text{infect}(t) \cdot (P - \text{infect}(t))}_{\text{this product}}$$

Why might this make a good model? We expect the rate that disease is spreading to be largest when

$$\text{infect}(t) \approx P/2.$$

The product

$$\text{infect}(t) \cdot (P - \text{infect}(t))$$

is largest when $\text{infect}(t) = P/2$. Finally we add the constant of proportionality as a scale factor.

Example 3. *Suppose your calculus class has had a freak outbreak of the math-philia. Some facts: We have around* 200 *students in our class, we are now on the* 23*rd day of the outbreak, and currently* 100 *students are infected. Using the differential equation*

$$\text{infect}'(t) = k \cdot \text{infect}(t) \cdot (P - \text{infect}(t))$$

we can model the spread of math-philia by setting $k = 0.001$*. What is* $\text{infect}'(23)$*?*

Explanation. *Here all we need to do is substitute all of the necessary information into the differential equation. We know*

$$\begin{aligned} t &= \underbrace{\boxed{23}}_{\text{given}}, \\ P &= \underbrace{\boxed{200}}_{\text{given}}, \\ \text{infect}(23) &= \underbrace{\boxed{100}}_{\text{given}}, \\ k &= 0.001. \end{aligned}$$

So

$$\begin{aligned} \text{infect}'(t) &= k \cdot \text{infect}(t) \cdot (P - \text{infect}(t)) \\ &= 0.001 \cdot 100 \cdot (200 - 100) \\ &= \underbrace{\boxed{10}}_{\text{given}}. \end{aligned}$$

Hence on day 23*, we expect the disease to be spreading at a rate of* $\underbrace{\boxed{10}}_{\text{given}}$ *newly infected people per day.*