Dig-In:

Derivatives of inverse trigonometric functions

We derive the derivatives of inverse trigonometric functions using implicit differentiation.

Now we will derive the derivative of arcsine, arctangent, and arcsecant.

Theorem 1 (The derivative of arcsine)**.**

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}.$$

Explanation. *Recall*

$$\arcsin(x) = \theta$$

means that $\sin(\theta) = x$ *and* $\underbrace{\boxed{\frac{-\pi}{2}}}_{\text{given}} \leq \theta \leq \underbrace{\boxed{\frac{\pi}{2}}}_{\text{given}}$. *Implicitly differentiating with respect* x *we see*

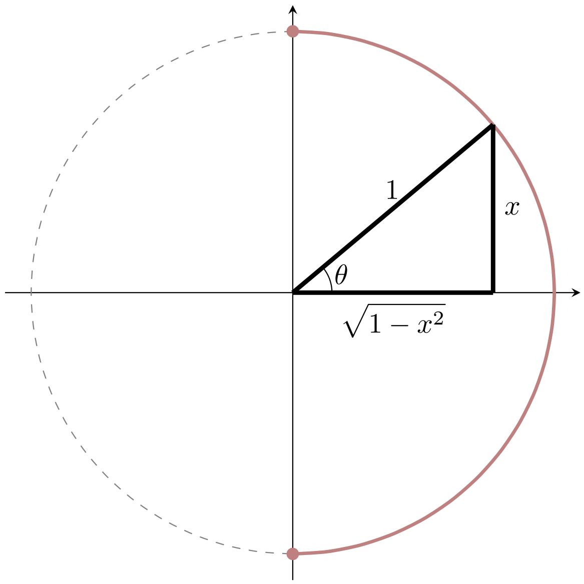
$$\begin{aligned}
\sin(\theta) &= x \\
\frac{d}{dx}\sin(\theta) &= \frac{d}{dx}x && \textit{Differentiate both sides.} \\
\cos(\theta)\cdot\theta' &= 1 && \textit{Implicit differentiation.} \\
\theta' &= \frac{1}{\cos(\theta)} && \textit{Solve for } \theta'.
\end{aligned}$$

While $\theta' = \dfrac{1}{\cos(\theta)}$, *we need our answer written in terms of* x. *Since we are assuming that*

$$\sin(\theta) = x,$$

consider the following triangle with the unit circle:

Learning outcomes: Find derivatives of inverse functions in general. Recall the meaning and properties of inverse trigonometric functions. Derive the derivatives of inverse trigonometric functions. Understand how the derivative of an inverse function relates to the original derivative. Take derivatives which involve inverse trigonometric functions.



From the unit circle above, we see that

$$\begin{aligned}\theta' &= \frac{1}{\cos(\theta)} \\ &= \frac{\text{hyp}}{\text{adj}} \\ &= \underbrace{\boxed{\frac{1}{\sqrt{1-x^2}}}}_{\text{given}}.\end{aligned}$$

To be completely explicit,

$$\frac{d}{dx}\theta = \frac{d}{dx}\arcsin(x) = \underbrace{\boxed{\frac{1}{\sqrt{1-x^2}}}}_{\text{given}}.$$

Question 1 *Compute:*

$$\frac{d}{dx}\sin^{-1}(x) = \boxed{1/\sqrt{1-x^2}}$$

We can do something similar with arctangent.

Theorem 2 (The derivative of arctangent)**.**

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}.$$

Explanation. *Recall*

$$\arctan(x) = \theta$$

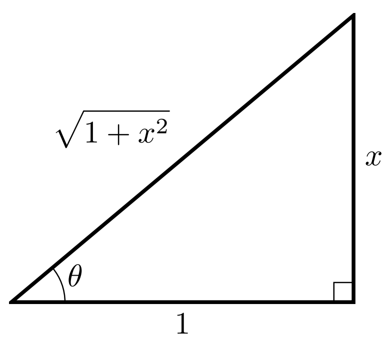
means that $\tan(\theta) = x$ *and* $\underbrace{\boxed{\frac{-\pi}{2}}}_{\text{given}} < \theta < \underbrace{\boxed{\frac{\pi}{2}}}_{\text{given}}$. *Implicitly differentiating with respect* x *we see*

$$\begin{aligned} \tan(\theta) &= x \\ \frac{d}{dx}\tan(\theta) &= \frac{d}{dx}x && \textit{Differentiate both sides.} \\ \sec^2(\theta)\cdot\theta' &= 1 && \textit{Implicit differentiation.} \\ \theta' &= \frac{1}{\sec^2(\theta)} && \textit{Solve for } \theta'. \\ \theta' &= \cos^2(\theta). \end{aligned}$$

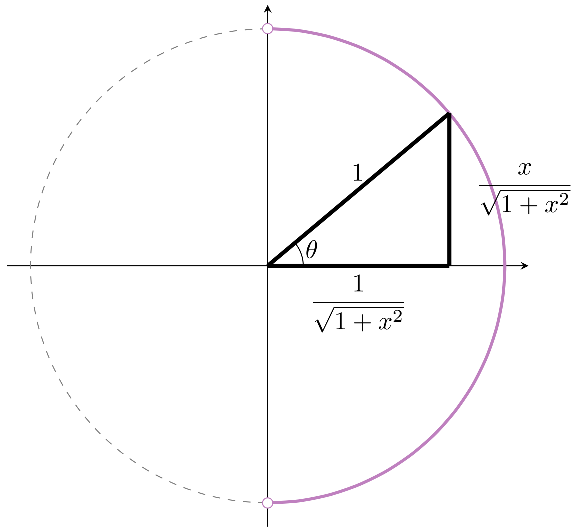
While $\theta' = \cos^2(\theta)$, *we need our answer written in terms of* x. *Since we are assuming that*

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = x,$$

we may consider the following triangle:



We may now scale this triangle by a factor of $\dfrac{1}{\sqrt{1+x^2}}$ *to place it on the unit circle:*



From the unit circle above, we see that

$$\begin{aligned}\theta' &= \cos^2(\theta)\\ &= \left(\frac{\text{adj}}{\text{hyp}}\right)^2\\ &= \underbrace{\boxed{\frac{1}{1+x^2}}}_{\text{given}}.\end{aligned}$$

To be completely explicit,

$$\frac{d}{dx}\theta = \frac{d}{dx}\arctan(x) = \underbrace{\boxed{\frac{1}{1+x^2}}}_{\text{given}}.$$

Question 2 *Compute:*

$$\frac{d}{dx}\tan^{-1}(\sqrt{x}) = \boxed{1/(2\sqrt{x}(1+x))}$$

Finally, we investigate the derivative of arcsecant.

Theorem 3 (The derivative of arcsecant)**.**

$$\frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \textit{for } |x| > 1.$$

Explanation. *Recall*

$$\operatorname{arcsec}(x) = \theta$$

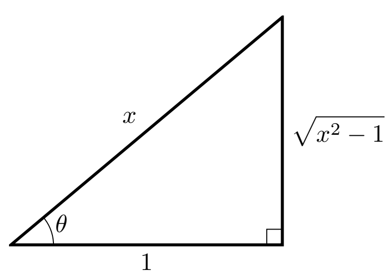
means that $\sec(\theta) = x$ *and* $\underbrace{\boxed{0}}_{\text{given}} \leq \theta \leq \underbrace{\boxed{\pi}}_{\text{given}}$ *with* $x \neq \underbrace{\boxed{\pi/2}}_{\text{given}}$. *Implicitly differentiating with respect* x *we see*

$$\begin{aligned}
\sec(\theta) &= x \\
\frac{d}{dx}\sec(\theta) &= \frac{d}{dx}x && \textit{Differentiate both sides.} \\
\sec(\theta)\tan(\theta)\cdot\theta' &= 1 && \textit{Implicit differentiation.} \\
\theta' &= \frac{1}{\sec(\theta)\tan(\theta)} && \textit{Solve for } \theta'. \\
\theta' &= \frac{\cos^2(\theta)}{\sin(\theta)}.
\end{aligned}$$

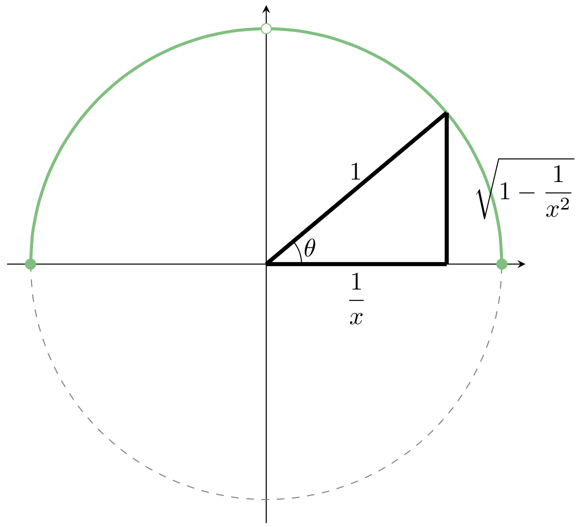
While $\theta' = \dfrac{\cos^2(\theta)}{\sin(\theta)}$, *we need our answer written in terms of* x. *Since we are assuming that*

$$\sec(\theta) = \frac{1}{\cos(\theta)} = x,$$

we may consider the following triangle:



We may now scale this triangle by a factor of $\dfrac{1}{x}$ *to place it on the unit circle:*



From the unit circle above, we see that

$$\begin{aligned}
\theta' &= \frac{\cos^2(\theta)}{\sin(\theta)} \\
&= \frac{\left(\frac{\text{adj}}{\text{hyp}}\right)^2}{\frac{\text{opp}}{\text{hyp}}} \\
&= \frac{(\text{adj})^2}{\text{opp}} && \textit{Note, } \text{hyp} = 1. \\
&= \frac{\underset{\text{given}}{\boxed{1/x^2}}}{\sqrt{1-1/x^2}} \\
&= \frac{1}{|x|\sqrt{x^2-1}},
\end{aligned}$$

To be completely explicit,

$$\frac{d}{dx}\theta = \frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \textit{for } |x| > 1.$$

Question 3 *Compute:*

$$\frac{d}{dx}\sec^{-1}(3x) = \boxed{\frac{1}{|x|\sqrt{(3x)^2-1}}}$$

We leave it to you, the reader, to investigate the derivatives of cosine, arccosecant, and arccotangent. However, as a gesture of friendship, we now present you with a list of derivative formulas for inverse trigonometric functions.

Theorem 4 (The Derivatives of Inverse Trigonometric Functions)**.**

- $\dfrac{d}{dx} \arcsin(x) = \dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx} \arccos(x) = \dfrac{-1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx} \arctan(x) = \dfrac{1}{1+x^2}$
- $\dfrac{d}{dx} \operatorname{arcsec}(x) = \dfrac{1}{|x|\sqrt{x^2-1}}$ *for* $|x| > 1$
- $\dfrac{d}{dx} \operatorname{arccsc}(x) = \dfrac{-1}{|x|\sqrt{x^2-1}}$ *for* $|x| > 1$
- $\dfrac{d}{dx} \operatorname{arccot}(x) = \dfrac{-1}{1+x^2}$