Dig-In:

Limits of the form nonzero over zero

We want to solve limits that have the form nonzero over zero.

Let's cut to the chase:

Definition 1. *A limit*

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

is said to be of the form $\frac{\#}{\mathbf{0}}$ *if*

$$\lim_{x\to a} f(x) = k \qquad \textit{and} \qquad \lim_{x\to a} g(x) = 0.$$

where k is some nonzero constant.

Question 1 *Which of the following limits are of the form* $\frac{\#}{\mathbf{0}}$*?*

Select All Correct Answers:

(a) $\displaystyle\lim_{x\to -1} \frac{1}{(x+1)^2}$ ✓

(b) $\displaystyle\lim_{x\to 2} \frac{x^2-3x+2}{x-2}$

(c) $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x}$

(d) $\displaystyle\lim_{x\to 2} \frac{x^2-3x-2}{x-2}$ ✓

(e) $\displaystyle\lim_{x\to 1} \frac{e^x}{\ln(x)}$ ✓

Let's see what is going on with limits of the form $\frac{\#}{\mathbf{0}}$. Consider the function

$$f(x) = \frac{1}{(x+1)^2}.$$

Learning outcomes: Calculate limits of the form number over zero. Identify determinate and indeterminate forms. Distinguish between determinate and indeterminate forms.

While the $\lim_{x \to -1} f(x)$ does not exist, something can still be said. First note that

$$\lim_{x \to -1} \frac{1}{(x+1)^2} \qquad \text{is of the form } \frac{\#}{\mathbf{0}}$$

as

$$\lim_{x \to -1} 1 = 1 \qquad \text{and} \qquad \lim_{x \to -1} (x+1)^2 = 0.$$

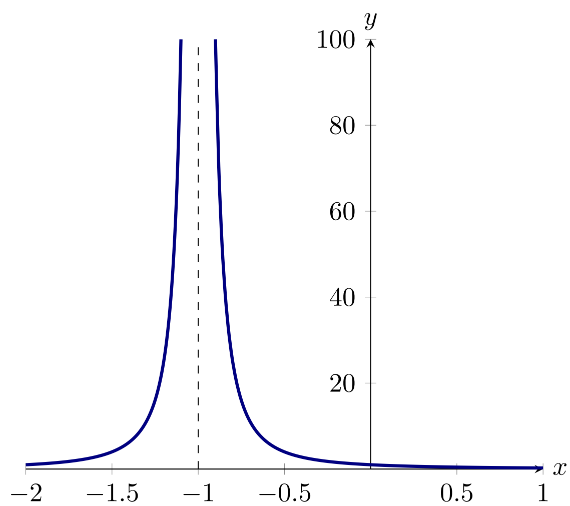
Moreover, as x approaches -1:

- The numerator is positive.
- The denominator approaches zero and is positive.

Hence

$$\lim_{x \to -1} \frac{1}{(x+1)^2}$$

will become arbitrary large, as we can see in the next graph.



We are now ready for our next definition.

Definition 2. *If $f(x)$ grows arbitrarily large as x approaches a, we write*

$$\lim_{x \to a} f(x) = \infty$$

and say that the limit of $f(x)$ ***approaches infinity*** *as x goes to a.*

If $|f(x)|$ grows arbitrarily large as x approaches a and $f(x)$ is negative, we write

$$\lim_{x \to a} f(x) = -\infty$$

and say that the limit of $f(x)$ ***approaches negative infinity*** *as x goes to a.*

Let's consider a few more examples.

Example 1. *Compute:*

$$\lim_{x \to -2} \frac{e^x}{(x+2)^4}$$

Explanation. *First let's look at the form of this limit, we do this by taking the limits of both the numerator and denominator:*

$$\lim_{x \to -2} e^x = \underbrace{\boxed{\frac{1}{e^2}}}_{\text{given}} \qquad \textit{and} \lim_{x \to -2} \left((x+2)^4\right) = 0$$

so this limit is of the form $\frac{\#}{\mathbf{0}}$*. As* x *approaches* -2*:*

- *The numerator is a* (positive ✓ /negative) *number.*
- *The denominator is* (positive ✓ /negative) *and is approaching zero.*

This means that

$$\lim_{x \to -2} \frac{e^x}{(x+2)^4} = \infty.$$

Example 2. *Compute:*

$$\lim_{x \to 3^+} \frac{x^2 - 9x + 14}{x^2 - 5x + 6}$$

Explanation. *First let's look at the form of this limit, which we do by taking the limits of both the numerator and denominator.*

$$\lim_{x \to 3^+} \left(x^2 - 9x + 14\right) = \underbrace{\boxed{-4}}_{\text{given}} \qquad \textit{and} \lim_{x \to 3^+} \left(x^2 - 5x + 6\right) = 0$$

This limit is of the form $\frac{\#}{\mathbf{0}}$*. Next, we should factor the numerator and denominator to see if we can simplify the problem at all.*

$$\begin{aligned} \lim_{x \to 3^+} \frac{x^2 - 9x + 14}{x^2 - 5x + 6} &= \lim_{x \to 3^+} \frac{(x-2)(x-7)}{(x-2)(x-3)} \\ &= \lim_{x \to 3^+} \frac{x-7}{x-3} \end{aligned}$$

Canceling a factor of $x - 2$ *from the numerator and denominator means we can more easily check the behavior of this limit. As* x *approaches* 3 *from the right:*

- *The numerator is a* (positive/negative ✓) *number.*
- *The denominator is* (positive ✓ /negative) *and approaching zero.*

This means that

$$\lim_{x \to 3^+} \frac{x^2 - 9x + 14}{x^2 - 5x + 6} = -\infty.$$

Here is our final example.

Example 3. *Compute:*

$$\lim_{x \to 3} \frac{x^2 - 9x + 14}{x^2 - 5x + 6}$$

Explanation. *We've already considered part of this example, but now we consider the two-sided limit. We already know that*

$$\lim_{x \to 3} \frac{x^2 - 9x + 14}{x^2 - 5x + 6} = \lim_{x \to 3} \frac{x - 7}{x - 3},$$

and that this limit is of the form $\frac{\#}{\mathbf{0}}$. *We also know that as* x *approaches* 3 *from the right,*

- *The numerator is a negative number.*
- *The denominator is positive and approaching zero.*

Hence our function is approaching $-\infty$ *from the right.*

As x *approaches* 3 *from the left,*

- *The numerator is negative.*
- *The denominator is negative and approaching zero.*

Hence our function is approaching ∞ *from the left. This means*

$$\lim_{x \to 3} \frac{x^2 - 9x + 14}{x^2 - 5x + 6} = \underbrace{\boxed{DNE}}_{\text{given}}.$$

Some people worry that the mathematicians are passing into mysticism when we talk about infinity and negative infinity. However, when we write

$$\lim_{x \to a} f(x) = \infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = -\infty$$

all we mean is that as x approaches a, $f(x)$ becomes arbitrarily large and $|g(x)|$ becomes arbitrarily large, with $g(x)$ taking negative values.