Break-Ground:

Geometry and substitution

Two students consider substitution geometrically.

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley! We should be able to figure some integrals geometrically using transformations of functions.

Riley: That sounds like a cool idea. Maybe, since we know the graph of $f(x) = \sqrt{1-x^2}$ is a semicircle, we get an ellipse defined on $[-2,2]$ just by stretching the graph of f by a factor of 2 horizontally. The equation of this ellipse would be

$$g(x) = \sqrt{1-\left(\frac{x}{2}\right)^2}$$

Devyn: Exactly! So since we know that

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi}{2}$$

geometrically...

Riley: And we know that the area under g from $[-2,2]$ is twice the under f...

Devyn and Riley: We must have

$$\int_{-2}^{2} \sqrt{1-(x/2)^2}\,dx = \pi!$$

Devyn and Riley: Jinx!

Devyn: It is kind of like we just stretched out our whole coordinate system, and that helped us solve an integral.

Riley: In this case, everything got stretched out by a constant factor of 2 in the horizontal direction. I wonder if we could ever say anything useful about cases where we stretch the x-axis by a different amount at each point?

Devyn: Whao, that is a wild thought. That seems really hard. Since derivatives measure how much a function stretches a little piece of the domain, maybe the derivative will come into play here?

Learning outcomes:

Riley: Hmmmm, but I do not see exactly how. Maybe we should ask our TA about this?

Problem 1 *Say we know that*

$$\int_1^4 f(x)\,dx = 5.$$

Then, using this transformation idea, we can evaluate

$$\int_a^b f(3x+1)\,dx$$

if $a = \boxed{0}$ *and* $b = \boxed{1}$. *The value of the integral on this interval is* $\boxed{5/3}$.