Break-Ground:

Patterns in derivatives

Two young mathematicians think about "short cuts" for differentiation.

Check out this dialogue between two calculus students (based on a true story):

Devyn: I hate the limit definition of derivative. I wish there were a shorter way.

Riley: I think I might have found a pattern for taking derivatives.

Devyn: Really? I love patterns!

Riley: I know! Check this out, I've made a chart

$f(x)$	$f'(x)$
x^2	$2 \cdot x^1$
x^3	$3 \cdot x^2$
x^4	$4 \cdot x^3$

So maybe if we have a function

$$f(x) = x^n \quad \text{then} \qquad f'(x) = n \cdot x^{n-1}.$$

Devyn: Hmmm does it work with square roots?

Riley: Oh that's right, a square root is a power, just write

$$f(x) = \sqrt{x} = x^{1/2}.$$

So a square root is of the form x^n.

Learning outcomes: Use "shortcut" rules to find derivatives

Devyn: Let's check it. If $f(x) = \sqrt{x}$,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0} \left(\frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \right) \\ &= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{1}{\sqrt{x}+\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \\ &= \frac{1}{2} \cdot x^{-1/2}. \end{aligned}$$

Riley: Holy Cat Fur! It works! In this case $f'(x) = n \cdot x^{n-1}$.

Devyn: I wonder if it *always* works? If so I want to know *why* it works! I wonder what other patterns we can find?

The pattern

$$\text{if} \qquad f(x) = x^n \quad \text{then} \qquad f'(x) = n \cdot x^{n-1}$$

holds whenever n is a constant. Explaining why it works in generality will take some time. For now, let's see if we can use the problem to squash some derivatives with ease.

Problem 1 *Using the pattern found above, compute:*

$$\frac{d}{dx} x^{101} = \boxed{101x^{100}}$$

Problem 2 *Using the pattern found above, compute:*

$$\frac{d}{dx} \frac{1}{x^{77}} = \boxed{-77x^{-78}}$$

Problem 3 *Using the pattern found above, compute:*

$$\frac{d}{dx}\sqrt[5]{x} = \boxed{x^{-4/5}/5}$$

Problem 4 *Using the pattern found above, compute:*

$$\frac{d}{dx}x^{e} = \boxed{ex^{e-1}}$$