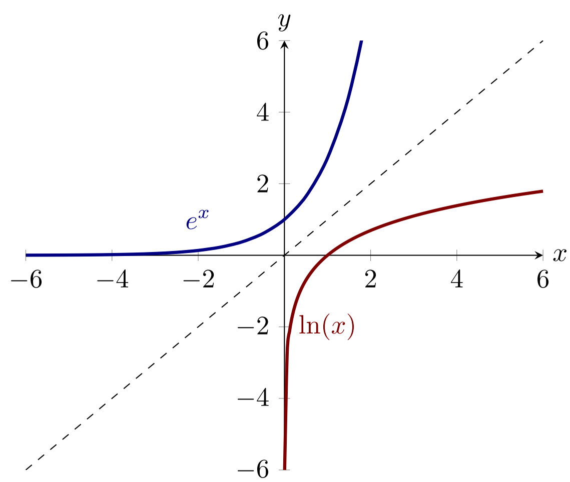
Dig-In:

Derivatives of inverse exponential functions

We derive the derivatives of inverse exponential functions using implicit differentiation.

Geometrically, there is a close relationship between the plots of e^x and $\ln(x)$, they are reflections of each other over the line $y = x$:



One may suspect that we can use the fact that $\frac{d}{dx}e^x = e^x$, to deduce the derivative of $\ln(x)$. We will use implicit differentiation to exploit this relationship computationally.

Theorem 1 (The Derivative of the Natural Logrithm)**.**

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

Explanation. *Recall*

$$\ln(x) = y \qquad \textit{exactly when} \qquad e^y = x \qquad \textit{and} \qquad x > \underset{\text{given}}{\boxed{0}}.$$

Learning outcomes: Find derivatives of inverse functions in general. Understand how the derivative of an inverse function relates to the original derivative. Explain the formula for the derivative of the natural log function. Calculate derivatives of logs in any base.

Hence

$$\begin{aligned} e^y &= x \\ \frac{d}{dx} e^y &= \frac{d}{dx} x && \textit{Differentiate both sides.} \\ e^y \frac{dy}{dx} &= 1 && \textit{Implicit differentiation.} \\ \frac{dy}{dx} &= \frac{1}{e^y} = \underbrace{\boxed{\frac{1}{x}}}_{\text{given}} && \textit{Solve for } \frac{dy}{dx}. \end{aligned}$$

Since $y = \ln(x)$, $\frac{d}{dx} \ln(x) = \underbrace{\boxed{\frac{1}{x}}}_{\text{given}}$.

Question 1 *Compute:*

$$\frac{d}{dx}(-\ln(\cos(x))) = \boxed{\tan(x)}$$

From the derivative of the natural logarithm, we can deduce another fact:

Theorem 2 (The derivative of any logarithm)**.** *Let b be a positive real number. Then*

$$\frac{d}{dx} \log_b(x) = \frac{1}{x \ln(b)}.$$

Explanation. *Here we need to remember that*

$$\log_b(x) = \frac{\ln(x)}{\underbrace{\boxed{\ln(b)}}_{\text{given}}}.$$

So we may write

$$\begin{aligned} \frac{d}{dx} \log_b(x) &= \frac{d}{dx} \frac{\ln(x)}{\underbrace{\boxed{\ln(b)}}_{\text{given}}} \\ &= \underbrace{\boxed{\frac{1}{x \ln(b)}}}_{\text{given}}. \end{aligned}$$

Question 2 *Compute:*

$$\frac{d}{dx} \log_7(x) = \boxed{1/(x \ln(7))}$$

We can also compute the derivative of an arbitrary exponential function.

Theorem 3 (The derivative of an exponential function)**.**

$$\frac{d}{dx}a^x = a^x \cdot \ln(a).$$

Explanation. *Here we need to be slightly sneaky. Note*

$$a^x = e^{\ln(a^x)} = e^{x\ln(a)}.$$

So we may write

$$\begin{aligned} \frac{d}{dx}a^x &= \frac{d}{dx}e^{x\ln(a)} \\ &= e^{x\ln(a)} \cdot \underset{\text{given}}{\boxed{\ln(a)}} \\ &= \underset{\text{given}}{\boxed{a^x \cdot \ln(a)}}. \end{aligned}$$

Question 3 *Compute:*

$$\frac{d}{dx}7^x = \boxed{7^x \ln(7)}$$