Dig-In:

The definite integral

Definite integrals compute signed area.

Definite integrals, often simply called integrals, compute signed area.

Definition 1. *The* ***definite integral***

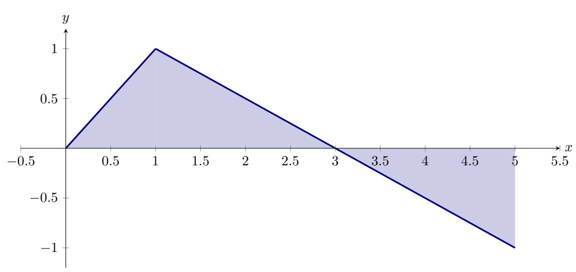
$$\int_a^b f(x)\,dx$$

computes the signed area between $y = f(x)$ and the x-axis on the interval $[a, b]$.

- *If the region is above the x-axis, then the area has positive sign.*
- *If the region is below the x-axis, then the area has negative sign.*

Note, when working with signed area, "positive" and "negative" area cancel each other out.

Question 1 *Consider the following graph of $y = f(x)$:*



Compute:

(a) $\int_0^3 f(x)\,dx = \boxed{1.5}$

Learning outcomes: Use integral notation for both antiderivatives and definite integrals. Compute definite integrals using geometry. Compute definite integrals using the properties of integrals. Justify the properties of definite integrals using algebra or geometry. Understand how Riemann sums are used to find exact area. Define net area. Split the area under a curve into several pieces to aid with calculations. Use symmetry to calculate definite integrals. Explain geometrically why symmetry of a function simplifies calculation of some definite integrals.

(b) $\displaystyle\int_3^5 f(x)\,dx = \boxed{-1}$

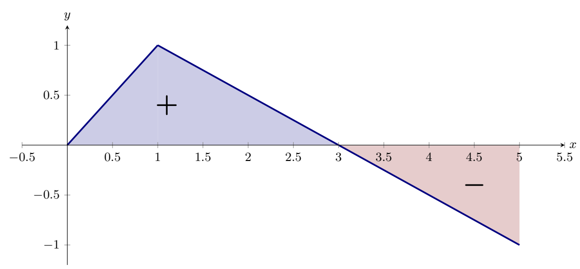
(c) $\displaystyle\int_0^5 f(x)\,dx = \boxed{0.5}$

(d) $\displaystyle\int_0^3 5\cdot f(x)\,dx = \boxed{7.5}$

(e) $\displaystyle\int_1^1 5\cdot f(x)\,dx = \boxed{0}$

Hint: *Use the formula for the area of a triangle.*

Hint: *Remember, we are dealing with "signed" area here:*



Our previous question hopefully gives us enough insight that this next theorem is unsurprising.

Theorem 1 (Properties of the definite integral)**.** *Let f and g be defined on a closed interval $[a, b]$ that contains the value c, and let k be a constant. The following hold:*

(a) $\displaystyle\int_a^a f(x)\,dx = 0$

(b) $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

(c) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

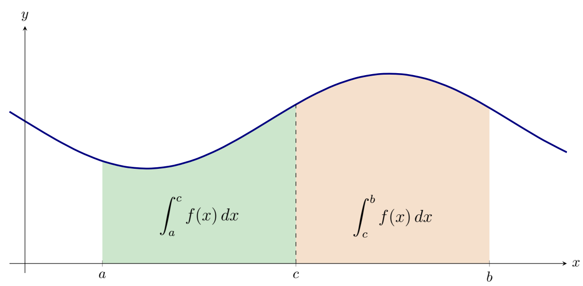
(d) $\displaystyle\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(e) $\displaystyle\int_a^b k\cdot f(x)\,dx = k\cdot\int_a^b f(x)\,dx$

Explanation. *We will address each property in turn:*

(a) *Here, there is no "area under the curve" when the region has no width; hence this definite integral is* 0.

(b) *This states that total area is the sum of the areas of subregions. Here a picture is worth a thousand words:*



It is important to note that this still holds true even if $a < b < c$. *We discuss this in the next point.*

(c) *For now, this property can be viewed a merely a convention to make other properties work well. However, later we will see how this property has a justification all its own.*

(d) *This states that when one scales a function by, for instance,* 7, *the area of the enclosed region also is scaled by a factor of* 7.

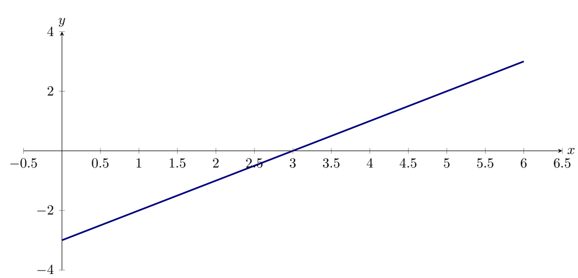
(e) *This states that the integral of the sum is the sum of the integrals.*

Due to the geometric nature of integration, geometric properties of functions can help us compute integrals.

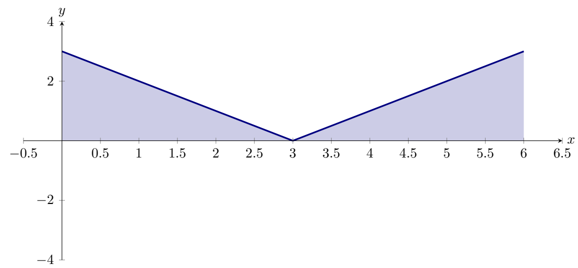
Example 1. *Compute:*

$$\int_0^6 |x-3|\,dx$$

Explanation. *This may seem difficult at first. Perhaps the first thing to do is look at a graph of* $y = x - 3$:



Now we can graph $y = |x - 3|$*:*



Now we see that we really have two triangles, each with base 3 *and height* 3*. Hence*

$$\begin{aligned}\int_0^6 |x-3|\,dx &= \int_0^3 \underbrace{\boxed{3-x}}_{\text{given}}\,dx + \int_3^6 \underbrace{\boxed{x-3}}_{\text{given}}\,dx\\ &= \frac{3\cdot 3}{2} + \frac{3\cdot 3}{2}\\ &= \underbrace{\boxed{9}}_{\text{given}}.\end{aligned}$$

Definition 2. *A function* f *is an* ***odd*** *function if*

$$f(-x) = -f(x),$$

and a function g *is an* ***even*** *function if*

$$g(-x) = g(x).$$

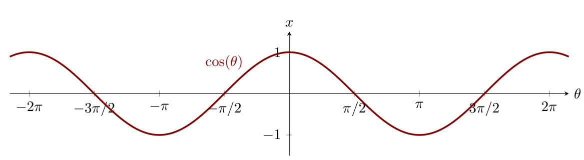
The names *odd* and *even* come from the fact that these properties are shared by functions of the form x^n where n is either odd or even. For example, if $f(x) = x^3$, then

$$f(-7) = -f(7),$$

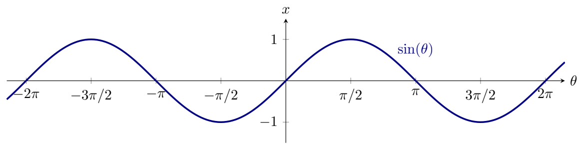
and if $g(x) = x^4$, then

$$g(-7) = g(7).$$

Geometrically, even functions have *horizontal symmetry*. Cosine is an even function:



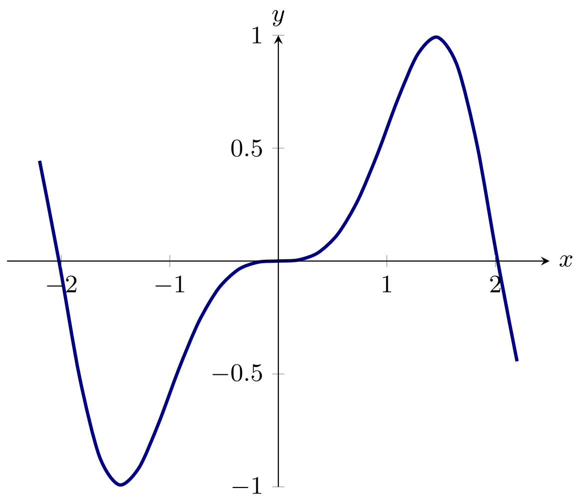
On the other hand, odd functions have 180° *rotational symmetry* around the origin. Sine is an odd function:



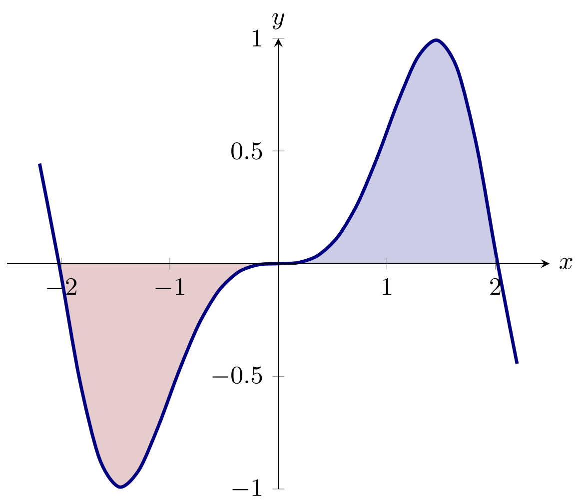
Question 2 *Let f be an odd function defined for all real numbers. Compute:*

$$\int_{-2}^{2} f(x)\,dx = \boxed{0}$$

Hint: *Since our function is odd, it must look something like:*



Hint: *The integral above computes the following (signed) area:*



Question 3 *Let f be an odd function defined for all real numbers. Which of the following are equal to*

$$\int_2^4 f(x)\,dx?$$

Select All Correct Answers:

(a) $\int_4^2 f(x)\,dx$

(b) $\int_{-4}^{-2} f(x)\,dx$

(c) $\int_{-2}^{-4} f(x)\,dx$ ✓

(d) $\int_{-2}^{4} f(x)\,dx$ ✓

(e) $\int_{4}^{-2} f(x)\,dx$

(f) $\int_{2}^{-4} f(x)\,dx$ ✓

(g) $\int_{-4}^{2} f(x)\,dx$

(h) $-\int_{-4}^{2} f(x)\,dx$ ✓

(i) $-\int_{-4}^{-2} f(x)\,dx$ ✓

Signed verses geometric area

We know that the signed area between a curve $y = f(x)$ and the x-axis on $[a, b]$ is given by

$$\int_a^b f(x)\,dx.$$

On the other hand, if we want to know the *geometric area*, meaning the "actual" area, we compute

$$\int_a^b |f(x)|\,dx.$$

Question 4 *True or false:*

$$\int_a^b |f(x)|\,dx = \left|\int_a^b f(x)\,dx\right|$$

Multiple Choice:

(a) *true*

(b) *false* ✓

Feedback (attempt): *Consider* $f(x) = x^3$ *on the interval* $[-1, 1]$. *Here*

$$\int_a^b |f(x)|\,dx = 1/2 \qquad \textit{but} \qquad \left|\int_a^b f(x)\,dx\right| = 0.$$

Integrals and Riemann sums

Exactly how does an integral compute area? It depends on who you ask. If you ask Riemann, then you set

$$\Delta x = \frac{b-a}{n}$$

and look at the following limit of Riemann sums:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_a^b f(x)\,dx.$$

This says, take a curve, slice it up into n pieces on the interval $[a, b]$, add up all the areas of rectangles whose width is determined by the slices and the height is determined by a sample point in one of these pieces.

Example 2. *Compute this limit:*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt{1-\left(-1+\frac{2k}{n}\right)^2}\right)\left(\frac{2}{n}\right)$$

Explanation. *This is a limit of Riemann sums! Specifically, it is a limit of Riemann sums of n rectangles, where*

$$\Delta x = \underbrace{\boxed{\frac{2}{n}}}_{\text{given}}$$

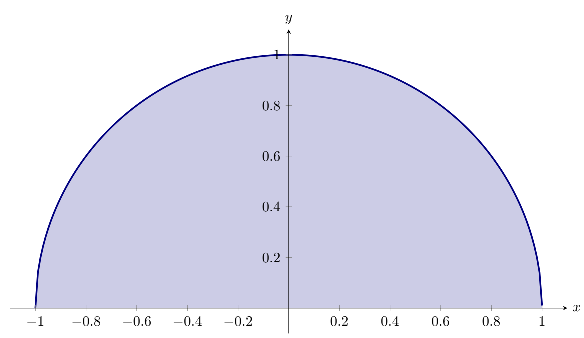
and

$$x_k^* = -1 + \frac{2k}{n}.$$

Hence, we may rewrite this as

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt{1-(x_k^*)^2}\right) \Delta x.$$

Now we see that this computes the area between the x-axis and the curve $y = \sqrt{1-x^2}$. Let's see it:



By geometry, we know that this semicircle has area $\underbrace{\boxed{\pi/2}}_{\text{given}}$. Hence

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt{1-(x_k^*)^2}\right) \Delta x = \underbrace{\boxed{\pi/2}}_{\text{given}}.$$