Dig-In:

Basic antiderivatives

We introduce antiderivatives.

Computing derivatives is not too difficult. At this point, you should be able to take the derivative of almost any function you can write down. However, undoing derivatives is much harder. This process of undoing a derivative is called taking an *antiderivative*.

Definition 1. *A function F is called an **antiderivative** of f on an interval if*

$$F'(x) = f(x)$$

for all x in the interval.

Question 1 *How many antiderivatives does $f(x) = 2x$ have?*

Multiple Choice:

(a) *none*

(b) *one*

(c) *infinitely many* ✓

Feedback (attempt): *The functions x^2, x^2+1, x^2-5, and so on, are all antiderivatives of $2x$.*

There are two common ways to notate antiderivatives, either with a capital letter or with a funny symbol:

Definition 2. *The antiderivative is denoted by*

$$\int f(x)\,dx = F(x) + C,$$

*where dx identifies x as the variable and C is a constant indicating that there are many possible antiderivatives, each varying by the addition of a constant. This is often called the **indefinite integral**.*

Learning outcomes:

Fill out these basic antiderivatives. Note each of these examples comes directly from our knowledge of basic derivatives.

Theorem 1 (Basic Antiderivatives)**.**

- $\displaystyle\int k\,dx = kx + C$
- $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$
- $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\int a^x\,dx = \frac{a^x}{\ln(a)} + C$
- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
- $\displaystyle\int \cos(x)\,dx = \sin(x) + C$
- $\displaystyle\int \sin(x)\,dx = -\cos(x) + C$
- $\displaystyle\int \tan(x)\,dx = -\ln|\cos(x)| + C$
- $\displaystyle\int \sec^2(x)\,dx = \tan(x) + C$
- $\displaystyle\int \csc^2(x)\,dx = -\cot(x) + C$
- $\displaystyle\int \sec(x)\tan(x)\,dx = \sec(x) + C$
- $\displaystyle\int \csc(x)\cot(x)\,dx = -\csc(x) + C$
- $\displaystyle\int \frac{1}{x^2+1}\,dx = \arctan(x) + C$
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$

It may seem that one could simply memorize these antiderivatives and antidifferentiating would be as easy as differentiating. This is **not** the case. The issue comes up when trying to combine these functions. When taking derivatives we have the *product rule* and the *chain rule*. The analogues of these two rules are much more difficult to deal with when taking antiderivatives. However, not all is lost. We have the following analogue of the sum rule for derivatives and the constant factor rule.

Theorem 2 (The Sum Rule for Antiderivatives)**.** *If F is an antiderivative of f and G is an antiderivative of g, then $F+G$ is an antiderivative of $f+g$.*

Theorem 3 (The Constant Factor Rule for Antiderivatives)**.** *If F is an antiderivative of f, and k is a constant, then kF is an antiderivative of kf.*

Let's put these rules and our knowledge of basic derivatives to work.

Example 1. *Find the antiderivative of $3x^7$.*

Explanation. *By the theorems above , we see that*

$$\begin{aligned}\int 3x^7\,dx &= 3\int x^7\,dx\\ &= 3\cdot\underbrace{\boxed{\frac{x^8}{8}}}_{\text{given}}+C.\end{aligned}$$

The sum rule for antiderivatives allows us to integrate term-by-term. Let's see an example of this.

Example 2. *Compute:*

$$\int \left(x^4+5x^2-\cos(x)\right)dx$$

Explanation. *Let's start by simplifying the problem using the sum rule for antiderivatives,*

$$\begin{aligned}&\int \left(x^4+5x^2-\cos(x)\right)dx\\ &\quad=\int x^4\,dx+5\int x^2\,dx-\int \cos(x)\,dx.\end{aligned}$$

Now we may integrate term-by-term to find

$$=\underbrace{\boxed{\frac{x^5}{5}+\frac{5x^3}{3}-\sin(x)}}_{\text{given}}+C.$$

Warning 1. *While the sum rule for antiderivatives allows us to integrate term-by-term, we cannot integrate factor-by-factor, meaning that in general*

$$\int f(x)g(x)\,dx \neq \int f(x)\,dx\cdot\int g(x)\,dx.$$

Computing antiderivatives

Unfortunately, we cannot tell you how to compute every antiderivative. We advise that the mathematician view antiderivatives as a sort of *puzzle*. Later we will learn a hand-full of techniques for computing antiderivatives. However, a robust and simple way to compute antiderivatives is guess-and-check.

Tips for guessing antiderivatives

(a) Make a guess for the antiderivative.

(b) Take the derivative of your guess.

(c) Note how the above derivative is different from the function whose antiderivative you want to find.

(d) Change your original guess by **multiplying** by constants or by **adding** in new functions.

Template 1. *If the indefinite integral looks* something *like*

$$\int \text{stuff}' \cdot (\text{stuff})^n \, dx$$

guess

$$\text{stuff}^{n+1}$$

where $n \neq -1$.

Example 3. *Compute:*

$$\int \frac{x^3}{\sqrt{x^4 - 6}} \, dx$$

Explanation. *Start by rewriting the indefinite integral as*

$$\int x^3 \left(x^4 - 6\right)^{-1/2} dx.$$

Now start with a guess of

$$\int x^3 \left(x^4 - 6\right)^{-1/2} dx \approx \left(x^4 - 6\right)^{1/2}.$$

Take the derivative of your guess to see if it is correct:

$$\frac{d}{dx} \left(x^4 - 6\right)^{1/2} = (4/2)x^3 \left(x^4 - 6\right)^{-1/2}.$$

We're off by a factor of $2/4$, so multiply our guess by this constant to get the solution,

$$\int \frac{x^3}{\sqrt{x^4 - 6}} \, dx = \underbrace{\boxed{(2/4)(x^4 - 6)^{1/2}}}_{\text{given}} + C.$$

Template 2. *If the indefinite integral looks* something *like*

$$\int \text{junk} \cdot e^{\text{stuff}}\, dx$$

guess

$$e^{\text{stuff}} \;\; \textit{or}\; \text{junk} \cdot e^{\text{stuff}}.$$

Example 4. *Compute:*

$$\int xe^x\, dx$$

Explanation. *We try to guess the antiderivative. Start with a guess of*

$$\int xe^x\, dx \approx xe^x.$$

Take the derivative of your guess to see if it is correct:

$$\frac{d}{dx} xe^x = e^x + xe^x.$$

Ah! So we need only subtract e^x *from our original guess. We now find*

$$\int xe^x\, dx = \underbrace{\boxed{xe^x - e^x}}_{\text{given}} + C.$$

Template 3. *If the indefinite integral looks* something *like*

$$\int \frac{\text{stuff}'}{\text{stuff}}\, dx$$

guess

$$\ln(\text{stuff}).$$

Example 5. *Compute:*

$$\int \frac{2x^2}{7x^3+3}\, dx$$

Explanation. *We'll start with a guess of*

$$\int \frac{2x^2}{7x^3+3}\, dx \approx \ln(7x^3+3).$$

Take the derivative of your guess to see if it is correct:

$$\frac{d}{dx} \ln(7x^3+3) = \frac{21x^2}{7x^3+3}.$$

We are only off by a factor of $2/21$*, so we need to multiply our original guess by this constant to get the solution,*

$$\int \frac{2x^2}{7x^3+3}\, dx = \underbrace{\boxed{(2/21)\ln(7x^3+3)}}_{\text{given}} + C.$$

Template 4. *If the indefinite integral looks* something *like*

$$\int \text{junk} \cdot \sin(\text{stuff})\, dx$$

guess

$$\cos(\text{stuff}) \;\; \textit{or} \; \text{junk} \cdot \cos(\text{stuff}),$$

likewise if you have

$$\int \text{junk} \cdot \cos(\text{stuff})\, dx$$

guess

$$\sin(\text{stuff}) \;\; \textit{or} \; \text{junk} \cdot \sin(\text{stuff}).$$

Example 6. *Compute:*

$$\int x^4 \sin(3x^5 + 7)\, dx$$

Explanation. *Here we simply try to guess the antiderivative. Start with a guess of*

$$\int x^4 \sin(3x^5 + 7)\, dx \approx \cos(3x^5 + 7).$$

To see if your guess is correct, take the derivative of $\cos(3x^5 + 7)$*,*

$$\frac{d}{dx} \cos(3x^5 + 7) = -15x^4 \sin(3x^5 + 7).$$

We are off by a factor of $-1/15$*. Hence we should multiply our original guess by this constant to find*

$$\int x^4 \sin(3x^5 + 7)\, dx = \underbrace{\boxed{\frac{-\cos(3x^5 + 7)}{15}}}_{\text{given}} + C.$$

Final thoughts

Computing antiderivatives is a place where insight and rote computation meet. We cannot teach you a method that will always work. Moreover, merely *understanding* the examples above will probably not be enough for you to become proficient in computing antiderivatives. You must practice, practice, practice!