Break-Ground:

Multiplication to addition

Two young mathematicians think about derivatives and logarithms.

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, why is the product rule so much harder than the sum rule?

Riley: Ever since 2nd grade, I've known that multiplication is *harder* than addition.

Devyn: I know! I was reading somewhere that a slide-rule somehow turns "multiplication into addition."

Riley: Wow! I wonder how that works?

Devyn: I *think* it has something to do with logs?

Riley: What? How does this work?

Devyn is right, logarithms are used (and were invented) to convert difficult multiplication problems into simpler addition problems.

Problem 1 *Let* $f(x) = \sin(x) \cdot \cos(x) \cdot e^x$. *Compute*

$$\frac{d}{dx} f(x) = \boxed{\cos(x)\cos(x)e^x - \sin(x)\sin(x)e^x + \sin(x)\cos(x)e^x}$$

Now, let's see what happens if we do the same problem but we take the natural log of both sides first:

$$\begin{aligned} f(x) &= \sin(x) \cdot \cos(x) \cdot e^x \\ \ln(f(x)) &= \ln(\sin(x) \cdot \cos(x) \cdot e^x) \\ \ln(f(x)) &= \ln(\sin(x)) + \ln(\cos(x)) + \ln(e^x) \end{aligned}$$

Now we'll take the derivative of both sides of the equation. By the chain rule

$$\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}$$

Learning outcomes:

Problem 2 *Compute*

$$\frac{d}{dx}\ln(\sin(x)) = \boxed{\frac{\cos(x)}{\sin(x)}}$$

Problem 3 *Compute*

$$\frac{d}{dx}\ln(\cos(x)) = \boxed{\frac{-\sin(x)}{\cos(x)}}$$

Problem 4 *Compute*

$$\frac{d}{dx}\ln(e^x) = \boxed{1}$$

So we have

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)} + 1 \\ f'(x) &= f(x)\left(\frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)} + 1\right) \\ &= \sin(x)\cos(x)e^x\left(\frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)} + 1\right) \end{aligned}$$