Dig-In:

The derivative of sine

We derive the derivative of sine.

It is now time to visit our two friends who concern themselves periodically with triangles and circles. In particular, we want to show that

$$\frac{d}{d\theta}\sin(\theta) = \cos(\theta).$$

Before we tackle this monster, let's remember a fact, and derive a new fact. You may initially be uncomfortable because you can't quite see why we need these results, but this style of exposition is a fact of technical writing; it is best to get used to it.

First, recall the fact that

$$\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1.$$

Next, we will use this fact to derive our new fact:

Example 1.

$$\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta} = 0.$$

Explanation. *Write with me:*

$$\begin{aligned}
\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta} &= \lim_{\theta\to 0}\left(\frac{\cos(\theta)-1}{\theta}\cdot\frac{\cos(\theta)+1}{\cos(\theta)+1}\right)\\
&= \lim_{\theta\to 0}\frac{\cos^2(\theta)-1}{\theta(\cos(\theta)+1)}\\
&= \lim_{\theta\to 0}\frac{-\sin^2(\theta)}{\theta(\cos(\theta)+1)}\\
&= -\lim_{\theta\to 0}\left(\frac{\sin(\theta)}{\theta}\cdot\frac{\sin(\theta)}{(\cos(\theta)+1)}\right)\\
&= -\lim_{\theta\to 0}\underbrace{\boxed{\frac{\sin(\theta)}{\theta}}}_{\text{given}}\cdot\lim_{\theta\to 0}\frac{\sin(\theta)}{(\cos(\theta)+1)}\\
&= -1\cdot\underbrace{\boxed{\frac{0}{2}}}_{\text{given}} = 0.
\end{aligned}$$

Learning outcomes: Use "shortcut" rules to find and use derivatives. Use the definition of the derivative to develop a shortcut rule to find the derivative of the sine function.

After these delicious appetizers, we are now ready for the main course.

Theorem 1 (The derivative of sine). *For any angle θ measured in radians, the derivative of $\sin(\theta)$ with respect to θ is $\cos(\theta)$. In other words,*

$$\frac{d}{d\theta}\sin(\theta) = \cos(\theta).$$

Explanation. *Using the definition of the derivative, write with me*

$$\frac{d}{d\theta}\sin(\theta) = \lim_{h\to 0}\frac{\sin(\theta+h)-\sin(\theta)}{h}$$

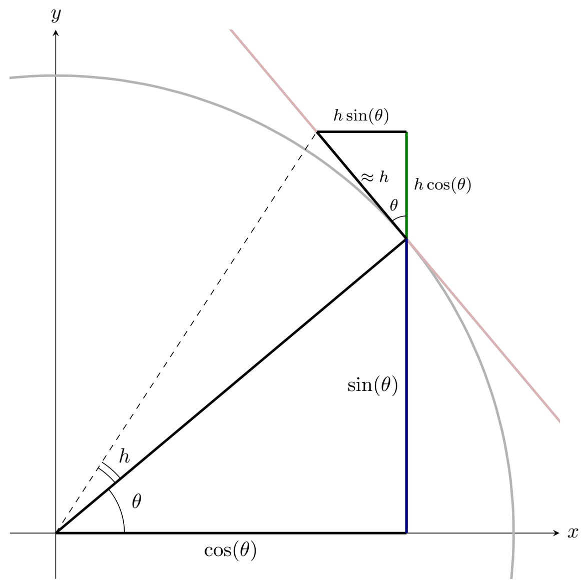
Now we get sneaky and apply the trigonometric addition formula for sine, that says $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)$, to write

$$\begin{aligned}
&= \lim_{h\to 0}\frac{\sin(\theta)\cos(h)+\sin(h)\cos(\theta)-\sin(\theta)}{h}\\
&= \lim_{h\to 0}\left(\frac{\sin(\theta)\cos(h)-\sin(\theta)}{h}+\frac{\sin(h)\cos\theta}{h}\right)\\
&= \lim_{h\to 0}\left(\sin(\theta)\frac{\cos(h)-1}{h}+\cos(\theta)\frac{\sin(h)}{h}\right)\\
&= \sin(\theta)\cdot \underset{\text{given}}{\boxed{0}} + \cos(\theta)\cdot \underset{\text{given}}{\boxed{1}}\\
&= \cos(\theta).
\end{aligned}$$

Question 1 *What is the slope of the line tangent to $\sin(\theta)$ at $\theta = \pi/4$?*

$$\boxed{\sqrt{2}/2}$$

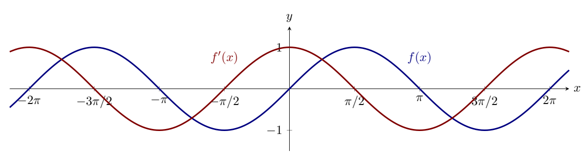
For your intellectual stimulation, consider the following geometric interpretation of the derivative of $\sin(\theta)$.



From this diagram, we see that increasing θ by a small amount h increases $\sin(\theta)$ by approximately $h\cos(\theta)$. Hence,

$$\frac{\Delta y}{\Delta \theta} \approx \frac{h\cos(\theta)}{h} = \cos(\theta).$$

With all of this said, the derivative of a function measures the slope of the plot of a function. If we examine the graphs of the sine and cosine side by side, it should be that the latter appears to accurately describe the slope of the former, and indeed this is true.



Question 2 *Using the graph above, what is the value of x in the interval $[0, 2\pi]$ where the tangent to the graph of $f(x) = \sin(x)$ has slope -1?* *The function $\sin(x)$ has slope -1 at $x = \boxed{\pi}$.*

Pro-tip: When working with trigonometric functions, you should always keep their graphical representations in mind.