Dig-In:

The Mean Value Theorem

Here we see a key theorem of calculus.

Here are some interesting questions involving derivatives:

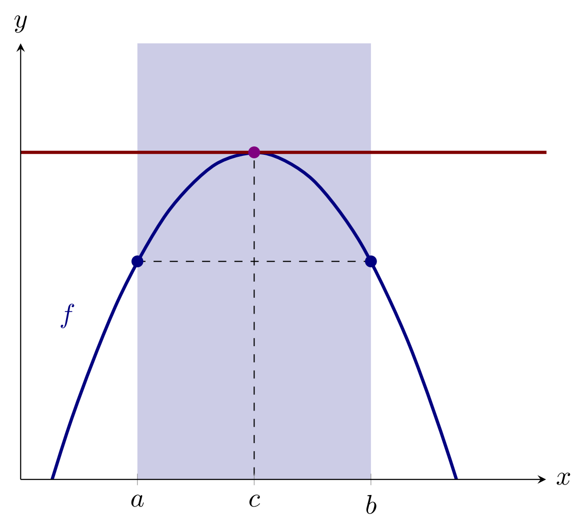
(a) Suppose you toss a ball into the air and then catch it. Must the ball's vertical velocity have been zero at some point?

(b) Suppose you drive a car from toll booth on a toll road to another toll booth 30 miles away in half of an hour. Must you have been driving at 60 miles per hour at some point?

(c) Suppose two different functions have the same derivative. What can you say about the relationship between the two functions?

While these problems sound very different, it turns out that the problems are very closely related. We'll start simply:

Theorem 1 (Rolle's Theorem)**.** *Suppose that f is differentiable on the interval (a, b), is continuous on the interval $[a, b]$, and $f(a) = f(b)$.*



Learning outcomes: Understand the statement of the Mean Value Theorem. Sketch pictures to illustrate why the Mean Value Theorem is true. Determine whether Rolle's Theorem or the Mean Value Theorem can be applied. Find the values guaranteed by Rolle's Theorem or the Mean Value Theorem. Use the Mean Value Theorem to solve word problems. Compare and contrast the Intermediate Value Theorem, Mean Value Theorem, and Rolle's Theorem. Identify calculus ideas which are consequences of the Mean Value Theorem.

Then

$$f'(c) = 0$$

for some $a < c < b$.

We can now answer our first question above.

Example 1. *Suppose you toss a ball into the air and then catch it. Must the ball's vertical velocity have been zero at some point?*

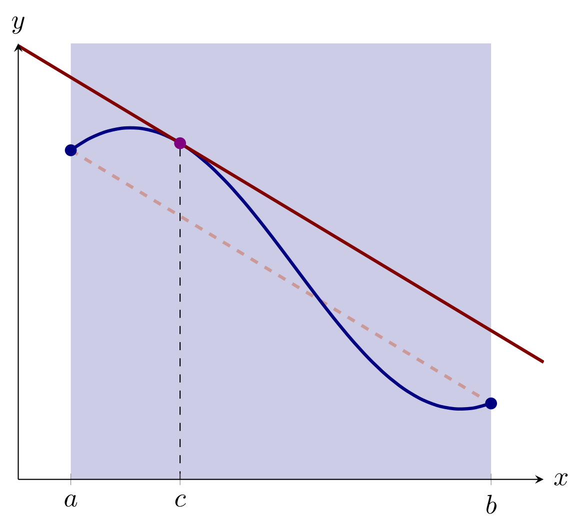
Explanation. *Let $p(t)$ be the position of the ball at time t. Our interval in question will be*

$$[t_{\text{start}}, t_{\text{finish}}]$$

we may assume that p is continuous on $[a, b]$ and differentiable on (a, b). We may now apply Rolle's Theorem to see at some time c, $p'(c) = \underset{\text{given}}{\boxed{0}}$. Hence the velocity must be zero at some point.

Rolle's Theorem is a special case of a more general theorem.

Theorem 2 (Mean Value Theorem)**.** *Suppose that f has a derivative on the interval (a, b) and is continuous on the interval $[a, b]$.*



Then

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for some $a < c < b$.

We can now answer our second question above.

Example 2. *Suppose you drive a car from toll booth on a toll road to another toll booth* 30 *miles away in half of an hour. Must you have been driving at* 60 *miles per hour at some point?*

Explanation. *If $p(t)$ is the position of the car at time t, and* 0 *hours is the starting time with* 1/2 *hours being the final time, then we may assume that p is continuous on $[0, 1/2]$ and differentiable on $(0, 1/2)$. Now the Mean Value Theorem states there is a time c*

$$p'(c) = \frac{30 - 0}{1/2} = \underbrace{\boxed{60}}_{\text{given}} \qquad \text{where } 0 < c < 1/2.$$

Since the derivative of position is velocity, this says that the car must have been driving at 60 *miles per hour at some point.*

Now we will address the unthinkable, could there be a continuous function f on $[a, b]$ whose derivative is zero on (a, b) that is not constant? As we will see, the answer is "no."

Theorem 3. *If $f'(x) = 0$ for all x in an interval I, then $f(x)$ is constant on I.*

Explanation. *Let $a < b$ be two points in I. Since f is continuous on $[a, b]$ and differentiable on (a, b), by the Mean Value Theorem we know*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some c in the interval (a, b). Since $f'(c) = 0$ we see that $f(b) = f(a)$. Moreover, since a and b were arbitrarily chosen, $f(x)$ must be the constant function.

Now let's answer our third question.

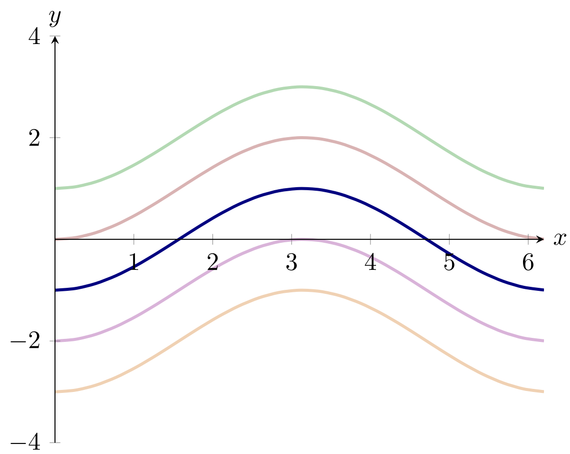
Example 3. *Suppose two different functions have the same derivative. What can you say about the relationship between the two functions?*

Explanation. *Set $h(x) = f(x) - g(x)$, so $h'(x) = f'(x) - g'(x)$. Now $h'(x) = 0$ on the interval (a, b). This means that $h(x) = k$ where k is some constant. Hence*

$$g(x) = f(x) + k.$$

Example 4. *Describe all functions whose derivative is* $\sin(x)$.

Explanation. *One such function is $-\cos(x)$, so all such functions have the form $-\cos(x) + k$,*



Finally, let us investigate two young mathematicians who run to class.

Example 5. *Two students Devyn and Riley raced to class. Was there a point during the race that Devyn and Riley were running at exactly the same velocity?*

Explanation. *Let P_{Devyn} represent Devyn's position with respect to time, and let P_{Riley} represent Riley's position with respect to time. Let t_{start} be the starting time of the race, and t_{finish} be the end of the race. Set*

$$f(t) = P_{\text{Devyn}}(t) - P_{\text{Riley}}(t).$$

Note, we may assume that P_{Devyn} and $P_{\text{Riley}}(t)$ are continuous on $[t_{\text{start}}, t_{\text{finish}}]$ and that they are differentiable on $(t_{\text{start}}, t_{\text{finish}})$. Hence the same is true for f. Since both runners start and finish at the same place,

$$f(t_{\text{start}}) = P_{\text{Devyn}}(t_{\text{start}}) - P_{\text{Riley}}(t_{\text{start}}) = \underset{\text{given}}{\boxed{0}} \qquad \textit{and}$$
$$f(t_{\text{finish}}) = P_{\text{Devyn}}(t_{\text{finish}}) - P_{\text{Riley}}(t_{\text{finish}}) = \underset{\text{given}}{\boxed{0}}.$$

In fact, this shows us that the average rate of change of

$$f(t) = P_{\text{Devyn}}(t) - P_{\text{Riley}}(t) \qquad \textit{on} \qquad [t_{\text{start}}, t_{\text{finish}}]$$

is $\underset{\text{given}}{\boxed{0}}$. *Hence by the mean value theorem, there is a point c*

$$t_{\text{start}} \leq c \leq t_{\text{finish}}$$

with $f'(c) = \underset{\text{given}}{\boxed{0}}$. *However,*

$$\underset{\text{given}}{\boxed{0}} = f'(x) = P'_{\text{Devyn}}(c) - P'_{\text{Riley}}(c).$$

Hence at c,

$$P'_{\text{Devyn}}(c) = P'_{\text{Riley}}(c),$$

this means that there was a time when they were running at exactly the same velocity.