Dig-In:

The sine integral

Here we will use facts about calculus to investigate a "slippary" function, one that is hard to get our hands on.

Definition 1. *The **sine integral** is most commonly defined as*

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\,dt.$$

What does the graph look like?

We'll use all of our curve sketching techniques to try to understand this function on the interval $[-2\pi, 2\pi]$, as gesture of friendship, we'll tell you that $\mathrm{Si}(x)$ is continuous on $[-2\pi, 2\pi]$. The first thing we should do is compute the first derivative of $\mathrm{Si}(x)$.

Example 1. *Compute:*

$$\frac{d}{dx}\,\mathrm{Si}(x)$$

Explanation. *By the First Fundamental Theorem of Calculus,*

$$\begin{aligned}\frac{d}{dx}\,\mathrm{Si}(x) &= \frac{d}{dx}\int_0^x \frac{\sin(t)}{t}\,dt\\ &= \frac{\sin(x)}{x}.\end{aligned}$$

Now we'll compute the second derivative of $\mathrm{Si}(x)$.

Example 2. *Compute:*

$$\frac{d^2}{dx^2}\,\mathrm{Si}(x)$$

Explanation. *By our previous work and the quotient rule we see*

$$\begin{aligned}\frac{d^2}{dx^2}\,\mathrm{Si}(x) &= \frac{d}{dx}\frac{\sin(x)}{x}\\ &= \frac{x\cos(x) - \sin(x)}{x^2}.\end{aligned}$$

Learning outcomes:

Now we should find the y-intercept.

Example 3. *Compute* $\mathrm{Si}(0)$*.*

Explanation. *Here* $\mathrm{Si}(0) = 0$*, as* Si *is an accumulation function, and at* $x = 0$*, no area has been accumulated.*

Now we'll look for critical points, where the derivative is zero or undefined.

Example 4. *Find the critical points of* $\mathrm{Si}(x)$*.*

Explanation. *The critical points are where* $\mathrm{Si}'(x) = 0$ *or it does not exist. Since*

$$\mathrm{Si}'(x) = \frac{\sin(x)}{x}.$$

We see that this derivative does not exist at $x = 0$*, and for* $x \neq 0$*,* $\mathrm{Si}'(x) = 0$ *precisely when* $\sin(x)$ *is zero. Since* $\sin(x)$ *is zero at* $x = -2\pi, -\pi, \pi, 2\pi$*, we see that the critical points are where*

$$x = -2\pi, -\pi, 0, \pi, 2\pi.$$

We'll identify which of these are maximums and minimums.

Example 5. *Find the local extrema of* $\mathrm{Si}(x)$ *on the interval* $[-2\pi, 2\pi]$*.*

Explanation. *The critical points are at*

$$x = -2\pi, -\pi, 0, \pi, 2\pi.$$

We will use the first derivative test to identifiy which of these are local extrema.

Inflection points are harder. Let's try our hand.

Example 6. *Find the inflection points of* $\mathrm{Si}(x)$*.*

Explanation. *We start by looking at the second derivative of* $\mathrm{Si}(x)$*,*

$$\mathrm{Si}''(x) = \frac{x\cos(x) - \sin(x)}{x^2}.$$

The first candidate for an infection point is $x = 0$*, since the second derivative does not exist. To find other inflection points on* $[-2\pi, 2\pi]$*, we need to find when*

$$x\cos(x) - \sin(x) = 0.$$

This is zero when $x = 0$*, anywhere else?*