One Year Calculus for Colorado State University: Exam 3 Course Content

January 23, 2017

Contents

1 Maximums and minimums

After completing this section, students should be able to do the following.

- Define a critical point.
- Find critical points.
- Define absolute maximum and absolute minimum.
- Find the absolute max or min of a continuous function on a closed interval.
- Define local maximum and local minimum.
- Compare and contrast local and absolute maxima and minima.
- Identify situations in which an absolute maximum or minimum is guaranteed.
- Classify critical points.
- State the First Derivative Test.
- Apply the First Derivative Test.
- State the Second Derivative Test.
- Apply the Second Derivative Test.
- Define inflection points.
- Find inflection points.

Break-Ground:

1.1 More coffee

Two young mathematicians witness the perils of drinking too much coffee.

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley!

Riley: Yes Devyn?

Devyn: Do you like coffee? I like coffee! Sometimes I feel really "bad," sluggish and tired. Then I drink coffee and I feel good! Sometimes I drink a lot of coffee!

Riley: Um?

Devyn: But here's the problem, see: If I drink too much, I become over excited and can't stop talking. I just drink coffee, then talk. Then drink more coffee. Then I start to feel sick. Ugh. I have a love-hate relationship with coffee.

Riley: If only there were a calculus solution to this problem!

Remember, calculus is about studying functions. If we can "see" a function in the work above, maybe we can figure out how to solve it.

Problem 1 *If we were to try to solve Devyn's coffee problem, what would be the best function to know?*

Multiple Choice:

(a) *How many donuts Devyn eats.*

(b) *How "good" Devyn feels after x cups of coffee.*

(c) *How many cups of coffee Devyn drinks when Devyn feels x "good."*

(d) *Impossible to say.*

Problem 2 *If we let $f(x)$ be "How 'good' Devyn feels after x cups of coffee." And we think about what Devyn says above, is there an amount Devyn can drink and feel maximally "good?"*

Multiple Choice:

(a) *yes*

(b) *no*

Dig-In

1.2 Maximums and minimums - Extreme Values and Critical Points

We use derivatives to help locate extrema.

Whether we are interested in a function as a purely mathematical object or in connection with some application to the real world, it is often useful to know what the graph of the function looks like. We can obtain a good picture of the graph using certain crucial information provided by derivatives of the function.

Extrema

Local *extrema* on a function are points on the graph where the y-coordinate is larger (or smaller) than all other y-coordinates on the graph at points "close to" (x, y).

Definition 1.

(a) *A function f has a* ***local maximum*** *at $x = a$, if $f(a) \geq f(x)$ for every x near a.*

(b) *A function f has a* ***local minimum*** *at $x = a$, if $f(a) \leq f(x)$ for every x near a.*

A ***local extremum*** *is either a local maximum or a local minimum.*

Problem 3 *True or false: "All absolute extrema are also local extrema."*

Multiple Choice:

(a) *true*

(b) *false*

Local maximum and minimum points are quite distinctive on the graph of a function, and are therefore useful in understanding the shape of the graph. In many applied problems we want to find the largest or smallest value that a function achieves (for example, we might want to find the minimum cost at which some task can be performed) and so identifying maximum and minimum points will be useful for applied problems as well.

Critical points

If $(x, f(x))$ is a point where f reaches a local maximum or minimum, and if the derivative of f exists at x, then the graph has a tangent line and the tangent line must be horizontal. This is important enough to state as a theorem, though we will not prove it.

Theorem 1 (Fermat's Theorem)**.** *If f has a local extremum at $x = a$ and f is differentiable at a, then $f'(a) = 0$.*

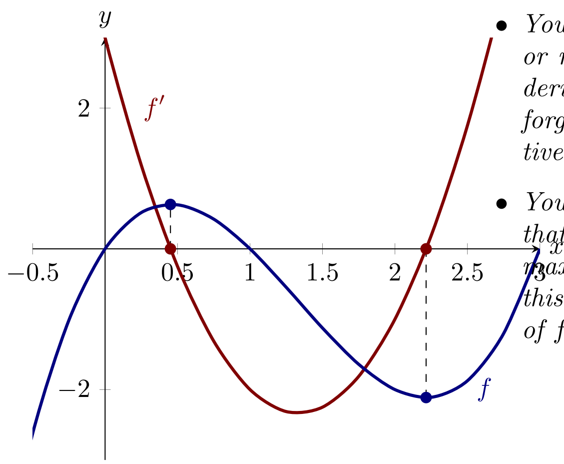
Problem 4 *Does Fermat's Theorem say that if $f'(a) = 0$, then f has a local extrema at $x = a$?*

Multiple Choice:

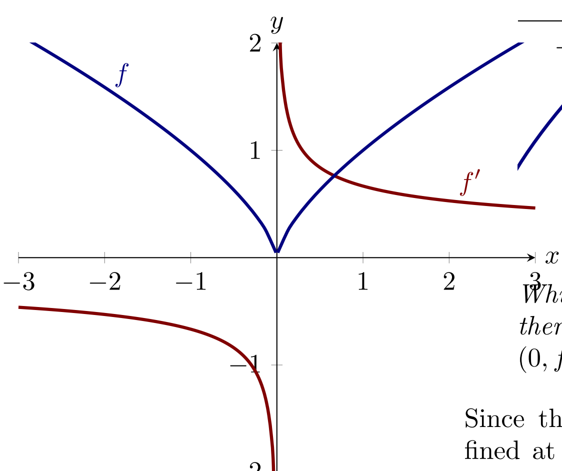
(a) *yes*

(b) *no*

Fermat's Theorem says that the only points at which a function can have a local maximum or minimum are points at which the derivative is zero, consider the plots of $f(x) = x^3 - 4x^2 + 3x$ and $f'(x) = 3x^2 - 8x + 3$,



or the derivative is undefined, as in the plot of $f(x) = x^{2/3}$ and $f'(x) = \dfrac{2}{3x^{1/3}}$:



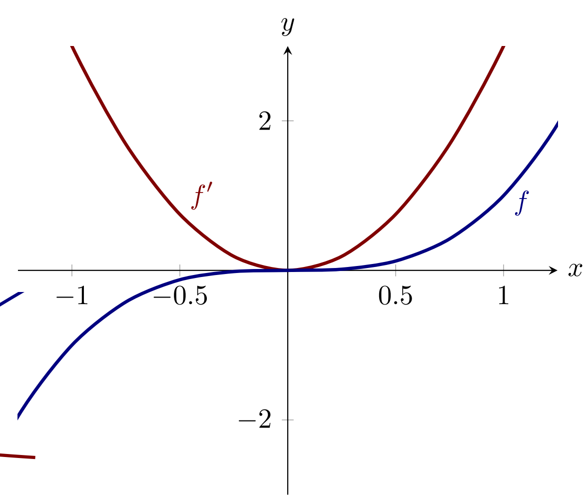
This brings us to our next definition.

Definition 2. *A function has a **critical point** at $x = a$ if*

$$f'(a) = 0 \qquad \text{or} \qquad f'(a) \text{ does not exist.}$$

Warning 1. *When looking for local maximum and minimum points, you are likely to make two sorts of mistakes:*

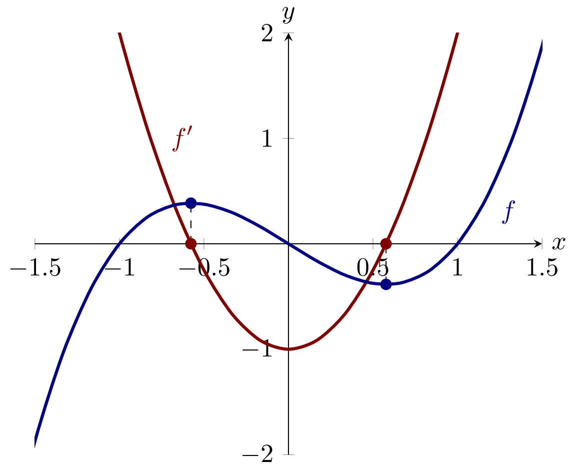
- *You may forget that a maximum or minimum can occur where the derivative does not exist, and so forget to check whether the derivative exists everywhere.*
- *You might assume that any place that the derivative is zero is a local maximum or minimum point, but this is not true, consider the plots of $f(x) = x^3$ and $f'(x) = 3x^2$.*



While $f'(0) = 0$, there is neither a maximum nor minimum at $(0, f(0))$.

Since the derivative is zero or undefined at both local maximum and local minimum points, we need a way to determine which, if either, actually occurs. The most elementary approach is to test directly whether the y coordinates near the potential maximum or minimum are above or below the y coordinate at the point of interest.

It is not always easy to compute the value of a function at a particular point. The task is made easier by the availability of calculators and computers, but they have their own drawbacks: they do not always allow us to distinguish between values that are very close together. Nevertheless, because this method is conceptually simple and sometimes easy to perform, you should always consider it.



Example 1. *Find all local maximum and minimum points for the function $f(x) = x^3 - x$.*

Explanation. *Write*

$$\frac{d}{dx}f(x) = 3x^2 - 1.$$

This is defined everywhere and is zero at $x = \pm\sqrt{3}/3$. Looking first at $x = \sqrt{3}/3$, we see that

$$f(\sqrt{3}/3) = -2\sqrt{3}/9.$$

Now we test two points on either side of $x = \sqrt{3}/3$, making sure that neither is farther away than the nearest critical point; since $\sqrt{3} < 3$, $\sqrt{3}/3 < 1$ and we can use $x = 0$ and $x = 1$. Since

$$f(0) = 0 > -2\sqrt{3}/9 \qquad \text{and} \qquad f(1) = 0 > -2\sqrt{3}/9,$$

there must be a local minimum at $x = \sqrt{3}/3$.

For $x = -\sqrt{3}/3$, we see that $f(-\sqrt{3}/3) = 2\sqrt{3}/9$. This time we can use $x = 0$ and $x = -1$, and we find that $f(-1) = f(0) = 0 < 2\sqrt{3}/9$, so there must be a local maximum at $x = -\sqrt{3}/3$, see the plot below:

Dig-In:

1.3 The Extreme Value Theorem

We examine a fact about continuous functions.

Definition 3.

(a) *A function f has an **global maximum** at $x = a$, if $f(a) \geq f(x)$ for every x in the domain of the function.*

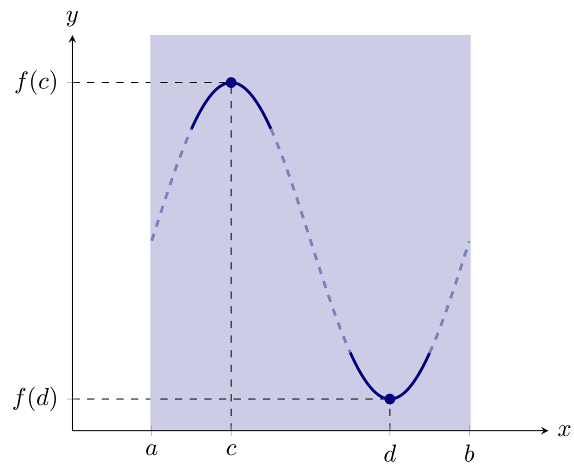
(b) *A function f has an **global minimum** at $x = a$, if $f(a) \leq f(x)$ for every x in the domain of the function.*

*A **global extremum** is either a global maximum or a global minimum.*

If we are working on an finite closed interval, then we have the following theorem.

Theorem 2 (Extreme Value Theorem)**.** *If f is a continuous function for all x in the closed interval $[a, b]$, then there are points c and d in $[a, b]$, such that $(c, f(c))$ is a global maximum and $(d, f(d))$ is a global minimum on $[a, b]$.*

Below, we see a geometric interpretation of this theorem.



Question 5 *Would this theorem hold if we were working on an open interval?*

Multiple Choice:

(a) *yes*

(b) *no*

Dig-In

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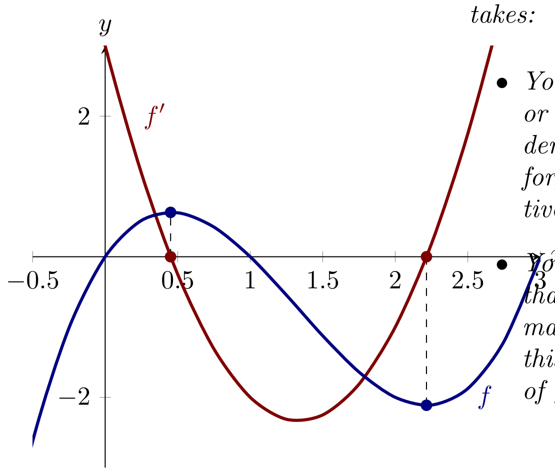

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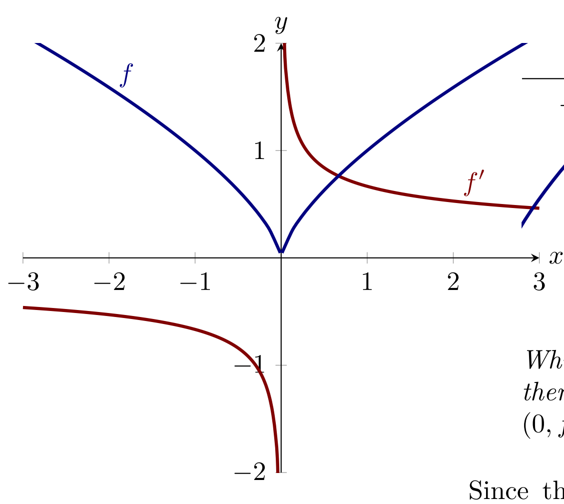

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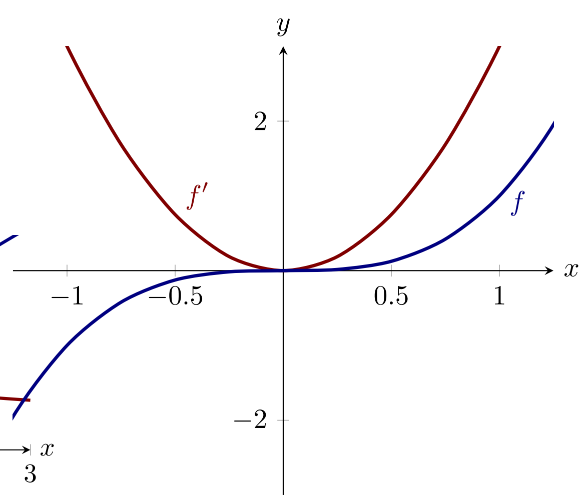

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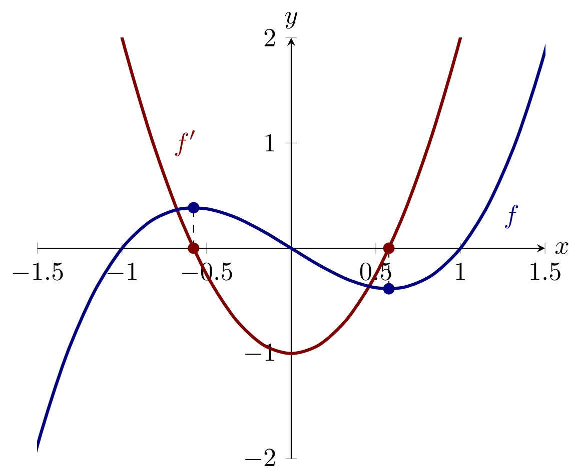


The first derivative test

The method of the previous section for deciding whether there is a local maximum or minimum at a critical point by testing "near-by" points is not always convenient. Instead, since we have already had to compute the derivative to find the critical points, we can use information about the derivative to decide. Recall that

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

So how exactly does the derivative tell us whether there is a maximum, minimum, or neither at a point? Use the *first derivative test*.

Theorem 4 (First Derivative Test)**.** *Suppose that f is continuous on an interval, and that $f'(a) = 0$ for some value of a in that interval.*

- *If $f'(x) > 0$ to the left of a and $f'(x) < 0$ to the right of a, then $f(a)$ is a local maximum.*
- *If $f'(x) < 0$ to the left of a and $f'(x) > 0$ to the right of a, then $f(a)$ is a local minimum.*
- *If $f'(x)$ has the same sign to the left and right of a, then $f(a)$ is not a local extremum.*

Example 3. *Consider the function*

$$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$$

Find the intervals on which f is increasing and decreasing and identify the local extrema of f.

Explanation. *Start by computing*

$$\frac{d}{dx} f(x) = x^3 + x^2 - 2x.$$

Now we need to find when this function is positive and when it is negative. To do this, solve

$$f'(x) = x^3 + x^2 - 2x = 0.$$

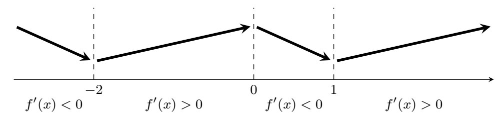
Factor $f'(x)$

$$\begin{aligned} f'(x) &= x^3 + x^2 - 2x \\ &= x(x^2 + x - 2) \\ &= x(x+2)(x-1). \end{aligned}$$

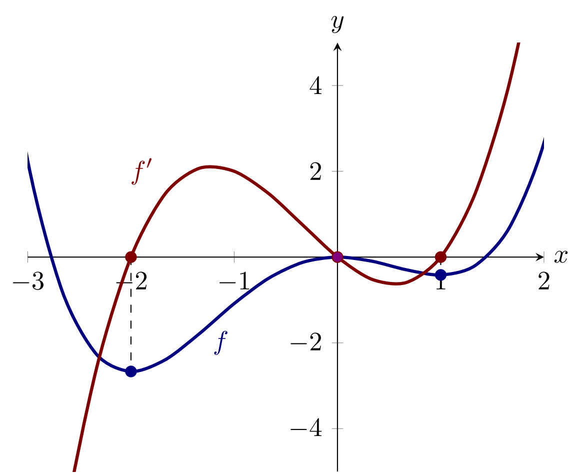
So the critical points (when $f'(x) = 0$) are when $x = -2$, $x = 0$, and $x = 1$. Now we can check points ***between*** *the critical points to find when $f'(x)$ is increasing and decreasing:*

$$\begin{aligned} f'(-3) &= -12, \\ f'(.5) &= -0.625, \\ f'(-1) &= 2, \\ f'(2) &= 8. \end{aligned}$$

From this we can make a sign table:



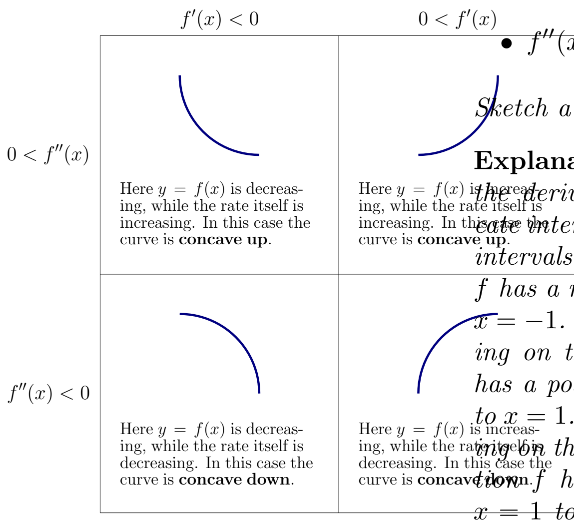
Hence f is increasing on $(-2, 0)$ and $(1, \infty)$ and f is decreasing on $(-\infty, -2)$ and $(0, 1)$. Moreover, from the first derivative test, the local maximum is at $x = 0$ while the local minima are at $x = -2$ and $x = 1$, see the graphs of of $f(x) = x^4/4 + x^3/3 - x^2$ and $f'(x) = x^3 + x^2 - 2x$.



Hence we have seen that if f' is zero and increasing at a point, then f has a local minimum at the point. If f' is zero and decreasing at a point then f has a local maximum at the point. Thus, we see that we can gain information about f by studying how f' changes. This leads us to our next section.

Concavity

We know that the sign of the derivative tells us whether a function is increasing or decreasing at some point. Likewise, the sign of the second derivative $f''(x)$ tells us whether $f'(x)$ is increasing or decreasing at x. We summarize the consequences of this seemingly simple idea in the table below:



If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. It is worth summarizing what we have seen already in to a single theorem.

Theorem 5 (Test for Concavity)**.** *Suppose that $f''(x)$ exists on an interval.*

(a) *$f''(x) > 0$ on that interval whenever $y = f(x)$ is concave up on that interval.*

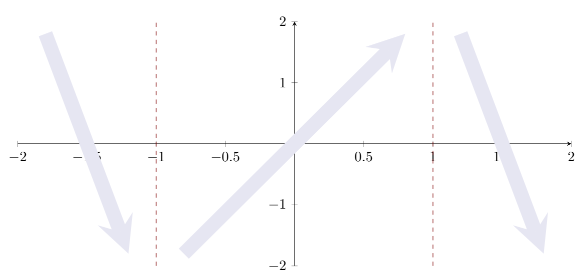
(b) *$f''(x) < 0$ on that interval whenever $y = f(x)$ is concave down on that interval.*

Example 4. *Let f be a continuous function and suppose that:*

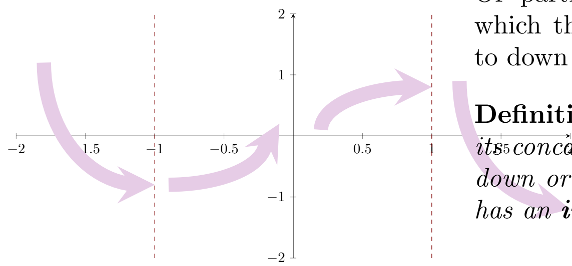
- *$f'(x) > 0$ for $-1 < x < 1$.*
- *$f'(x) < 0$ for $-2 < x < -1$ and $1 < x < 2$.*
- *$f''(x) > 0$ for $-2 < x < 0$ and $1 < x < 2$.*
- *$f''(x) < 0$ for $0 < x < 1$.*

Sketch a possible graph of f.

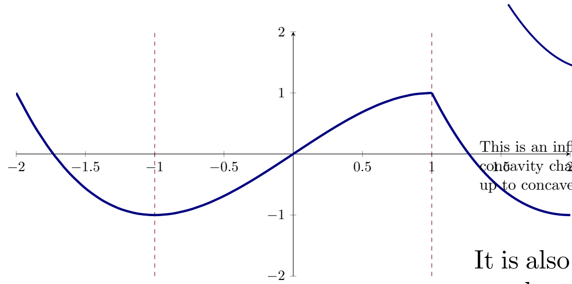
Explanation. *Start by marking where the derivative changes sign and indicate intervals where f is increasing and intervals f is decreasing. The function f has a negative derivative from -2 to $x = -1$. This means that f is decreasing on this interval. The function f has a positive derivative from $x = -1$ to $x = 1$. This means that f is increasing on this interval. Finally, The function f has a negative derivative from $x = 1$ to 2. This means that f is decreasing on this interval.*



Now we should sketch the concavity: concave up when the second derivative is positive, concave down when the second derivative is negative.



Finally, we can sketch our curve:



Inflection points

If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. It is worth summarizing what we have seen already in to a single theorem.

Theorem 6 (Test for Concavity)**.** *Suppose that $f''(x)$ exists on an interval.*

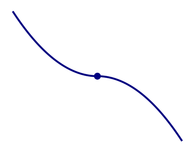
(a) *If $f''(x) > 0$ on an interval, then f is concave up on that interval.*

(b) *If $f''(x) < 0$ on an interval, then f is concave down on that interval.*

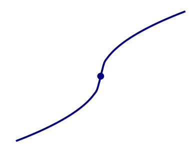
Of particular interest are points at which the concavity changes from up to down or down to up.

Definition 6. *If f is continuous and its concavity changes either from up to down or down to up at $x = a$, then f has an* ***inflection point*** *at $x = a$.*

It is instructive to see some examples of inflection points:

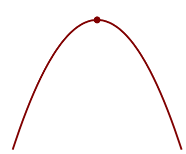


This is an inflection point. The concavity changes from concave up to concave down.

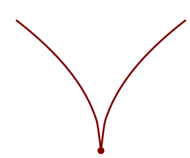


This is an inflection point. The concavity changes from concave up to concave down.

It is also instructive to see some nonexamples of inflection points:



This is **not** an inflection point. The curve is concave down on either side of the point.



This is **not** an inflection point. The curve is concave down on either side of the point.

We identify inflection points by first finding x such that $f''(x)$ is zero or undefined and then checking to see whether $f''(x)$ does in fact go from positive to negative or negative to positive at these points.

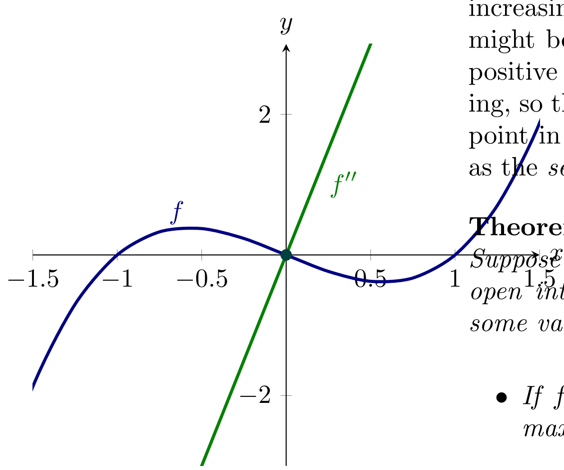
Warning 3. *Even if $f''(a) = 0$, the point determined by $x = a$ might* ***not*** *be an inflection point.*

Example 5. *Describe the concavity of $f(x) = x^3 - x$.*

Explanation. *To start, compute the first and second derivative of $f(x)$ with respect to x,*

$$f'(x) = 3x^2 - 1 \qquad \textit{and} \qquad f''(x) = 6x.$$

Since $f''(0) = 0$, there is potentially an inflection point at $x = 0$. Using test points, we note the concavity does change from down to up, hence there is an inflection point at $x = 0$. The curve is concave down for all $x < 0$ and concave up for all $x > 0$, see the graphs of $f(x) = x^3 - x$ and $f''(x) = 6x$.



Note that we need to compute and analyze the second derivative to understand concavity, so we may as well try to use the second derivative test for maxima and minima. If for some reason this fails we can then try one of the other tests.

The second derivative test

Recall the first derivative test:

- If $f'(x) > 0$ to the left of a and $f'(x) < 0$ to the right of a, then $f(a)$ is a local maximum.
- If $f'(x) < 0$ to the left of a and $f'(x) > 0$ to the right of a, then $f(a)$ is a local minimum.

If f' changes from positive to negative it is decreasing. In this case, f'' might be negative, and if in fact f'' is negative then f' is definitely decreasing, so there is a local maximum at the point in question. On the other hand, if f' changes from negative to positive it is increasing. Again, this means that f'' might be positive, and if in fact f'' is positive then f' is definitely increasing, so there is a local minimum at the point in question. We summarize this as the *second derivative test*.

Theorem 7 (Second Derivative Test)**.** *Suppose that $f''(x)$ is continuous on an open interval and that $f'(a) = 0$ for some value of a in that interval.*

- *If $f''(a) < 0$, then f has a local maximum at a.*
- *If $f''(a) > 0$, then f has a local minimum at a.*
- *If $f''(a) = 0$, then the test is inconclusive. In this case, f may or may not have a local extremum at $x = a$.*

The second derivative test is often the easiest way to identify local maximum and minimum points. Sometimes the test fails and sometimes the second derivative is quite difficult to evaluate. In such cases we must fall back on one of the previous tests.

Example 6. *Once again, consider the function*

$$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$$

Use the second derivative test, to locate the local extrema of f.

Multiple Choice:

(a) *The function has a local extrema at $x = a$.*

(b) *The function does not have a local extrema at $x = a$.*

(c) *It gives no information on whether $x = a$ is a local extremum.*

Explanation. *Start by computing*

$$f'(x) = x^3+x^2-2x \qquad \textit{and} \qquad f''(x) = 3x^2+2x-2.$$

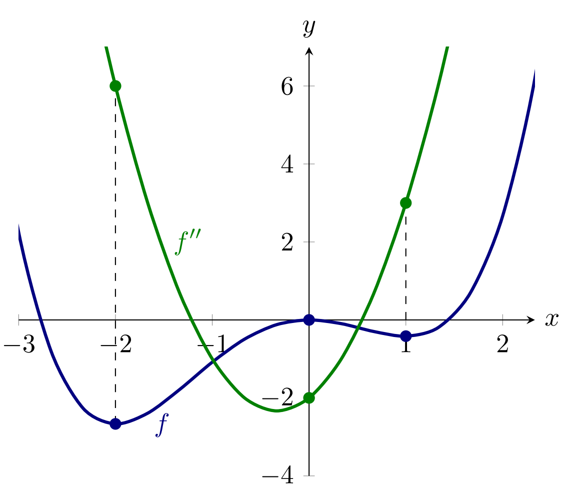
Using the same technique as we used before, we find that

$$f'(-2) = 0, \qquad f'(0) = 0, \qquad f'(1) = 0.$$

Now we'll attempt to use the second derivative test,

$$f''(-2) = 6, \qquad f''(0) = -2, \qquad f''(1) = 3.$$

Hence we see that f has a local minimum at $x = -2$, a local maximum at $x = 0$, and a local minimum at $x = 1$, see below for a plot of $f(x) = x^4/4 + x^3/3 - x^2$ and $f''(x) = 3x^2 + 2x - 2$:



Problem 8 *If $f''(a) = 0$, what does the second derivative test tell us?*

2 Optimization

After completing this section, students should be able to do the following.

- Describe the goals of optimization problems generally.
- Find all local maximums and minimums using the First and Second Derivative tests.
- Identify when we can find an absolute maximum or minimum on an open interval.
- Contrast optimization on open and closed intervals.
- Describe the objective function and constraints in a given optimization problem.
- Solve optimization problems by finding the appropriate extreme values.

Break-Ground:

2.1 A mysterious formula

Two young mathematicians discuss optimization from an abstract point of view.

Check out this dialogue between two calculus students:

Devyn: Riley, what do you think is the maximum value of

$$f(x) = \frac{10}{x^2 - 2.8x + 3}?$$

Riley: Where did that function come from?

Devyn: It's just some, um, random function.

Riley: Wait, does this have to do with coffee?

Devyn: Um, uh, no?

Riley: Well what interval are we on?

Devyn: Let's say $[0, 10]$, I mean there's no way I could possibly drink ten cups of coff...

Riley: I knew this was about coffee.

Here Devyn has made a function, that is supposed to record Devyn's "well-being" with respect to the number of cups of coffee consumed in one day.

Problem 1 *Graph Devyn's function. Where do you estimate the maximum on the interval* $[0, 10]$ *to be?*

Problem 2 *If you wanted to argue that this is the (global) maximum value on* $[0, 10]$ *without plotting, what arguments could you use?*

Dig-In:

2.2 Basic optimization

Now we put our optimization skills to work.

An **optimization problem** is a problem where you need to maximize or minimize some quantity given some constraints. This can be accomplished using the tools of differential calculus that we have already developed.

Perhaps the most basic optimization problems is generated by the following question:

> Among all rectangles of a fixed perimeter, which has the greatest area?

Let's not do this problem in the abstract, let's do it with numbers.

Example 7. *Of all rectangles of perimeter* 12*, which side lengths give the greatest area?*

Explanation. *If a rectangle has perimeter* 12 *and one side is length* x*, then the length of the other side is* $6-x$*. Hence the area of a rectangle of perimeter* 12 *can be given by*

$$A(x) = x(6-x).$$

However, for the side lengths to be physically relevant, we must assume that x *is in the interval* $(0,6]$*.*

So to maximize the area of the rectangle, we need to find the maximum value of $A(x)$ *on the appropriate interval.*

> ***At this point, you should graph the function if you can.***

We'll continue on without the aid of a graph, and use the derivative. Write

$$A'(x) = 6-2x$$

Now we find the critical points, solving the equation

$$6-2x=0,$$

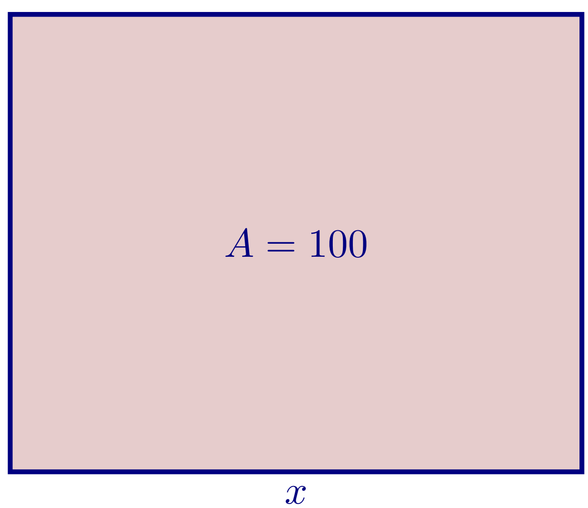
we see that the only critical point of A *is at* $x=3$

Since $A'(x) = 6-2x$ *is positive on* $(0,3)$ *and negative on* $(3,6]$*,* $x=3$ *is where the maximum value of* A *happens. This is exactly when the rectangle is a square!*

A key step to note, is when we explained why $x=3$ is actually the maximum. Above we basically used facts about the derivative. Below we use a similar argument.

Example 8. *Of all rectangles of area* 100*, which has the smallest perimeter?*

Explanation. *First we draw a picture, Here is a rectangle with an area of* 100*.*



If x denotes one of the sides of the rectangle, then the adjacent side must be $100/x$.

The perimeter of this rectangle is given by

$$P(x) = 2x + 2 \cdot 100/x.$$

We wish to minimize $P(x)$. Note, not all values of x make sense in this problem: lengths of sides of rectangles must be positive, so $x > 0$. If $x > 0$ then so is $100/x$, so we need no second condition on x.

> ***At this point, you should graph the function if you can.***

We next find $P'(x)$ and set it equal to zero. Write

$$P'(x) = 2 - 200/x^2 = 0.$$

Solving for x gives us $x = \pm 10$. We are interested only in $x > 0$, so only the value $x = 10$ is of interest. Since $P'(x)$ is defined everywhere on the interval $(0, \infty)$, there are no more critical values, and there are no endpoints. Is there a local maximum, minimum, or neither at $x = 10$? The second derivative is

$$P''(x) = 400/x^3,$$

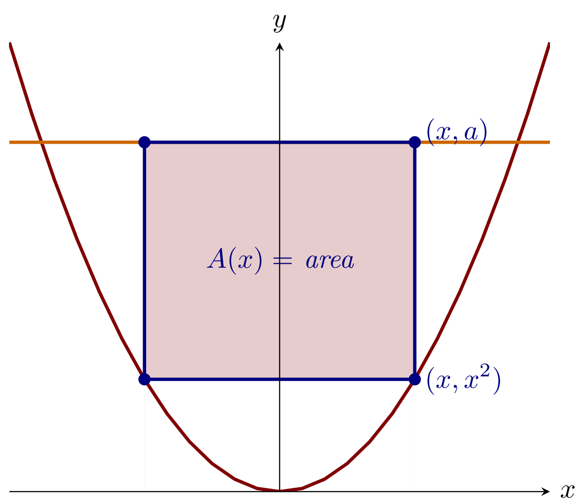
and $P''(10) > 0$, so there is a local minimum. Since there is only one critical point, this is also the global minimum, so the rectangle with smallest perimeter is the 10×10 square.

Hence, calculus gives a **reason** for **why** a square is the rectangle with both

- the largest area for a given perimeter.
- the smallest perimeter for a given area.

We may be done with rectangles, but they aren't done with us. Here is a problem where there are more constraints on the possible side lengths of the rectangle.

Example 9. *Find the rectangle with largest area that fits inside the graph of the parabola $y = x^2$ below the line $y = a$, where a is an unspecified constant value, with the top side of the rectangle on the horizontal line $y = a$. See the figure below:*



Explanation. *We want to maximize value of $A(x)$. The lower right corner of the rectangle is at (x, x^2), and once this is chosen the rectangle is completely determined. Then the area is*

$$A(x) = (2x)(a - x^2) = -2x^3 + 2ax.$$

We want the maximum value of $A(x)$ when x is in $[0, \sqrt{a}]$. You might object to allowing $x = 0$ or $x = \sqrt{a}$, since then the "rectangle" has either *no width or no height, so is not "really" a rectangle. But the problem is somewhat easier if we simply allow such rectangles, which have zero area as we may then apply the Extreme Value Theorem and see that we indeed have a maximum and minimum value.*

> ***At this point, you should graph the function if you can.***

Setting $0 = A'(x) = -6x^2 + 2a$ *we find* $x = \sqrt{a/3}$ *as the only critical point. Testing this and the two endpoints (as the maximum could also be there), we have* $A(0) = A(\sqrt{a}) = 0$ *and* $A(\sqrt{a/3}) = (4/9)\sqrt{3}a^{3/2}$*. Hence, the maximum area occurs when the rectangle has dimensions* $2\sqrt{a/3} \times (2/3)a$*.*

Again, note that above we used the Extreme Value Theorem to guarantee that we found the maximum.

3 Applied optimization

After completing this section, students should be able to do the following.

- Recognize optimization problems.
- Translate a word problem into the problem of finding the extreme values of a function.
- Solve basic word problems involving maxima or minima.
- Interpret an optimization problem as the procedure used to make a system or design as effective or functional as possible.
- Set up an optimization problem by identifying the objective function and appropriate constraints.
- Solve optimization problems by finding the appropriate absolute extremum.
- Identify the appropriate domain for functions which are models of real-world phenomena.

Break-Ground:

3.1 Volumes of aluminum cans

Two young mathematicians discuss optimizing aluminum cans.

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, have you ever noticed aluminum cans?

Riley: So very recyclable!

Devyn: I know! But I've also noticed that there are some that are short and fat, and others that are tall and skinny, and yet they can still have the same volume!

Riley: So very observant!

Devyn: This got me wondering, if we want to make a can with volume V, what shape of can uses the least aluminum?

Riley: Ah! This sounds like a job for calculus! The volume of a cylindrical can is given by

$$V = \pi \cdot r^2 \cdot h$$

where r is the radius of the cylinder and h is the height of the cylinder. Also the surface area is given by

$$\begin{aligned} A &= \underbrace{\pi \cdot r^2}_{\text{bottom}} + \underbrace{2 \cdot \pi \cdot r \cdot h}_{\text{sides}} + \underbrace{\pi \cdot r^2}_{\text{top}} \\ &= 2 \cdot \pi \cdot r^2 + 2 \cdot \pi \cdot r \cdot h. \end{aligned}$$

Somehow we want to minimize the surface area, because that's the amount of aluminum used, but we also want to keep the volume constant.

Devyn: Whoa, we have way too many letters here.

Riley: Yeah, somehow we need only one variable. Yikes. Too many letters.

Problem 1 *Suppose we wish to construct an aluminum can with volume V that uses the least amount of aluminum. In the context above, what do we want to minimize?*

Multiple Choice:

(a) A

(b) V

(c) h

(d) r

Problem 2 *In the context above, what should be considered a constant?*

Select All Correct Answers:

(a) A

(b) V

(c) h

(d) r

As Devyn and Riley noticed, when we work out this type of problem, we need to reduce the problem to a single variable.

Problem 3 *Consider r to be the variable, and express A as a function of r.*

Problem 4 *Now consider h to be the variable, and express A as a function of h.*

Notice that we've reduced (one way or another) this function of two variables to a function of one variable. This process will be a key step in nearly every problem in this next section.

Dig-In:

3.2 Applied optimization

Now we put our optimization skills to work.

In this section, we will present several worked examples of optimization problems. Our method for solving these problems is essentially the following:

Draw a picture. If possible, draw a schematic picture with all the relevant information.

Determine your goal. We need identify what needs to be optimized.

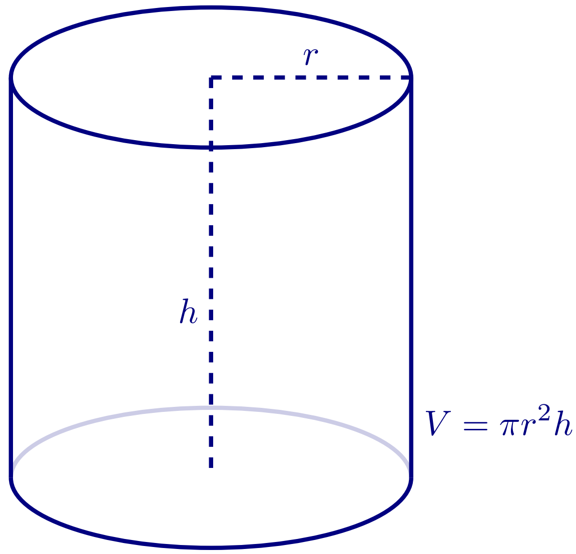
Find constraints. What limitations are set on our optimization?

Solve for a single variable. Now you should have a function to optimize.

Use calculus to find the extreme values. Be sure to check your answer!

Example 10. *You are making cylindrical containers to contain a given volume. Suppose that the top and bottom are made of a material that is N times as expensive (cost per unit area) as the material used for the lateral side of the cylinder. Find (in terms of N) the ratio of height to base radius of the cylinder that minimizes the cost of making the containers.*

Explanation. *First we draw a picture:*



Letting c represent the cost of the lateral side, we can write an expression for the cost of materials:

$$C = 2\pi crh + 2\pi r^2 Nc.$$

Since we know that $V = \pi r^2 h$, we can use this relationship to eliminate h (we could eliminate r, but it's a little easier if we eliminate h, which appears in only one place in the above formula for cost). We find

$$\begin{aligned} C(r) &= 2c\pi r \frac{V}{\pi r^2} + 2Nc\pi r^2 \\ &= \frac{2cV}{r} + 2Nc\pi r^2. \end{aligned}$$

We want to know the minimum value of this function when r is in $(0, \infty)$. Setting

$$C'(r) = -2cV/r^2 + 4Nc\pi r = 0$$

we find $r = \sqrt[3]{V/(2N\pi)}$. Since $C''(r) = 4cV/r^3 + 4Nc\pi$ is positive when r is positive, there is a local minimum at the critical value, and hence a *global minimum since there is only one critical value.*

Finally, since $h = V/(\pi r^2)$,

$$\begin{aligned}\frac{h}{r} &= \frac{V}{\pi r^3}\\ &= \frac{V}{\pi(V/(2N\pi))}\\ &= 2N,\end{aligned}$$

so the minimum cost occurs when the height h *is* $2N$ *times the radius. If, for example, there is no difference in the cost of materials, the height is twice the radius.*

Example 11. *You want to sell a certain number* n *of items in order to maximize your profit. Market research tells you that if you set the price at* \$1.50*, you will be able to sell* 5000 *items, and for every* 10 *cents you lower the price below* \$1.50 *you will be able to sell another* 1000 *items. Suppose that your fixed costs ("start-up costs") total* \$2000*, and the per item cost of production ("marginal cost") is* \$0.50*. Find the price to set per item and the number of items sold in order to maximize profit, and also determine the maximum profit you can get.*

Explanation. *The first step is to convert the problem into a function maximization problem. The revenue for selling* n *items at* x *dollars is given by*

$$r(x) = nx$$

and the cost of producing n *items is given by*

$$c(x) = 2000 + 0.5n.$$

However, from the problem we see that the number of items sold is itself a function of x,

$$n(x) = 5000 + \frac{1000(1.5 - x)}{0.10}$$

So profit is give by:

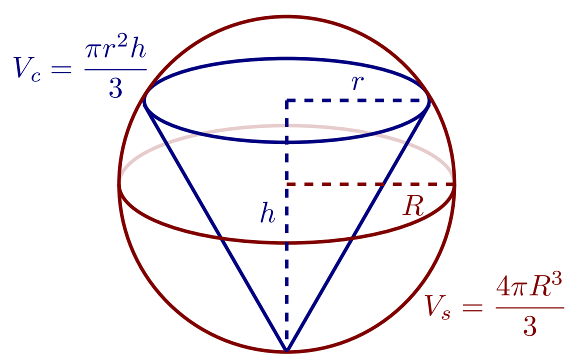
$$\begin{aligned}P(x) &= r(x) - c(x)\\ &= nx - (2000 + 0.5n)\\ &= -10000x^2 + 25000x - 12000.\end{aligned}$$

We want to know the maximum value of this function when x *is between 0 and* 1.5*. The derivative is*

$$P'(x) = -20000x + 25000,$$

which is zero when $x = 1.25$*. Since* $P''(x) = -20000 < 0$*, there must be a local maximum at* $x = 1.25$*, and since this is the only critical value it must be a global maximum as well. Alternately, we could compute* $P(0) = -12000$, $P(1.25) = 3625$*, and* $P(1.5) = 3000$ *and note that* $P(1.25)$ *is the maximum of these. Thus the maximum profit is* \$3625*, attained when we set the price at* \$1.25 *and sell* 7500 *items.*

Example 12. *If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by "cone" we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.)*



Explanation. *Let R be the radius of the sphere, and let r and h be the base radius and height of the cone inside the sphere. Our goal is to maximize the volume of the cone: $V_c = \pi r^2 h/3$. The largest r could be is R and the largest h could be is $2R$.*

Notice that the function we want to maximize, $\pi r^2 h/3$, depends on two variables. Our next step is to find the relationship and use it to solve for one of the variables in terms of the other, so as to have a function of only one variable to maximize. In this problem, the condition is apparent in the figure, as the upper corner of the triangle, whose coordinates are $(r, h-R)$, must be on the circle of radius R. Write

$$r^2 + (h-R)^2 = R^2.$$

Solving for r^2, since r^2 is found in the formula for the volume of the cone, we find

$$r^2 = R^2 - (h-R)^2.$$

Substitute this into the formula for the volume of the cone to find

$$\begin{aligned} V_c(h) &= \pi(R^2 - (h-R)^2)h/3 \\ &= -\frac{\pi}{3}h^3 + \frac{2}{3}\pi h^2 R \end{aligned}$$

We want to maximize $V_c(h)$ when h is between 0 and $2R$. We solve

$$V_c'(h) = -\pi h^2 + (4/3)\pi h R = 0,$$

finding $h = 0$ or $h = 4R/3$. We compute

$$V_c(0) = V_c(2R) = 0$$

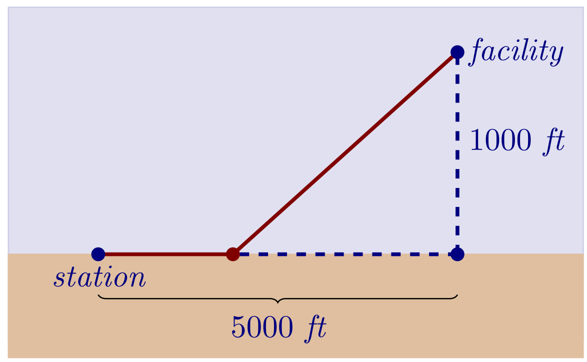
and

$$V_c(4R/3) = (32/81)\pi R^3.$$

The maximum is the latter. Since the volume of the sphere is $(4/3)\pi R^3$, the fraction of the sphere occupied by the cone is

$$\frac{(32/81)\pi R^3}{(4/3)\pi R^3} = \frac{8}{27} \approx 30\%.$$

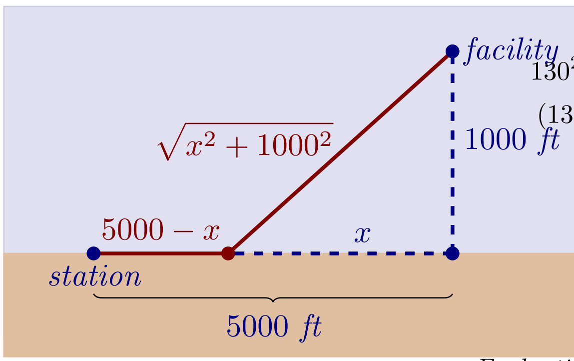
Example 13. *A power line needs to be run from an power station located on the beach to an offshore facility.*



It costs \$50/ft. to run a power line along the land, and \$130/ft. to run a power line under water. How much of the power line should be run along the land to minimize the overall cost? What is the minimal cost?

Explanation. *There are two immediate solutions that we could consider, each of which we will reject through "common sense." First, we could minimize the distance by directly connecting the two locations with a straight line. However, this requires that all the wire be laid underwater, the most costly option. Second, we could minimize the underwater length by running a wire all 5000 ft. along the beach, directly across from the offshore facility. This has the undesired effect of having the longest distance of all, probably ensuring a nonminimal cost.*

The optimal solution likely has the line being run along the ground for a *while, then underwater, as the figure implies. We need to label our unknown distances: the distance run along the ground and the distance run underwater. Recognizing that the underwater distance can be measured as the hypotenuse of a right triangle, we can label our figure as follows*



By choosing x as we did, we make the expression under the square root simple. We now create the cost function:

$$\begin{array}{ccc} \textit{Cost} = & & \\ \textit{land cost} & + & \textit{water cost} \\ \textit{\$50} \times \textit{land distance} & + & \textit{\$130} \times \textit{water distance} \\ 50(5000-x) & + & 130\sqrt{x^2+1000^2}. \end{array}$$

So we have

$$c(x) = 50(5000-x) + 130\sqrt{x^2+1000^2}.$$

This function only makes sense on the interval $[0, 5000]$. While we are fairly certain the endpoints will not give a minimal cost, we still evaluate $c(x)$ at each to verify.

$$c(0) = 380000 \qquad c(5000) \approx 662873.$$

We now find the critical points of $c(x)$. We compute $c'(x)$ as

$$c'(x) = -50 + \frac{130x}{\sqrt{x^2+1000^2}}.$$

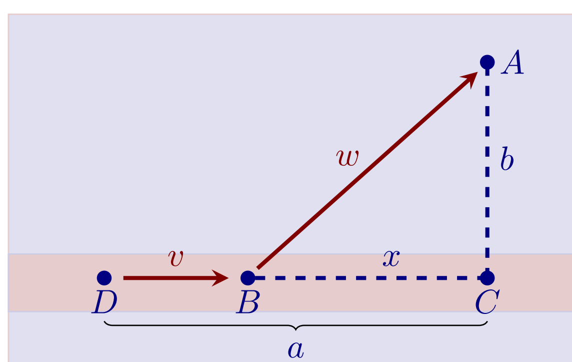
Recognize that this is never undefined. Setting $c'(x) = 0$ and solving for x, we have:

$$\begin{aligned} -50 + \frac{130x}{\sqrt{x^2+1000^2}} &= 0 \\ \frac{130x}{\sqrt{x^2+1000^2}} &= 50 \\ \frac{130^2x^2}{x^2+1000^2} &= 50^2 \\ 130^2x^2 &= 50^2(x^2+1000^2) \\ 130^2x^2 - 50^2x^2 &= 50^2 \cdot 1000^2 \\ (130^2-50^2)x^2 &= 50000^2 \\ x^2 &= \frac{50000^2}{130^2-50^2} \\ x &= \frac{50000}{\sqrt{130^2-50^2}} \\ x &= \frac{50000}{120}, \end{aligned}$$

Evaluating $c(x)$ at $x = 416.67$ gives a cost of about \$370000. *The distance the power line is laid along land is* $5000 - 416.67 = 4583.33$ ft *and the underwater distance is* $\sqrt{416.67^2 + 1000^2} \approx 1083$ ft.

We now work a similar problem without concrete numbers.

Example 14. *Suppose you want to reach a point A that is located across the sand from a nearby road.*



Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?

Explanation. *Let x be the distance short of C where you turn off, the distance from B to C. We want to minimize the total travel time. Recall that when traveling at constant velocity, time is distance divided by velocity.*

You travel the distance from D to B at speed v, and then the distance from B to A at speed w. The distance from D to B is $a - x$. By the Pythagorean theorem, the distance from B to A is

$$\sqrt{x^2 + b^2}.$$

Hence the total time for the trip is

$$T(x) = \frac{a - x}{v} + \frac{\sqrt{x^2 + b^2}}{w}.$$

We want to find the minimum value of T when x is between 0 and a. As usual we set $T'(x) = 0$ and solve for x. Write

$$T'(x) = -1/v + \frac{x}{w\sqrt{x^2 + b^2}} = 0.$$

We find that

$$x = \frac{wb}{\sqrt{v^2 - w^2}}$$

Notice that a does not appear in the last expression, but a is not irrelevant, since we are interested only in critical values that are in $[0, a]$, and $wb/\sqrt{v^2 - w^2}$ is either in this interval or not. If it is, we can use the second derivative to test it:

$$T''(x) = \frac{b^2}{(x^2 + b^2)^{3/2} w}.$$

Since this is always positive there is a local minimum at the critical point, and so it is a global minimum as well.

If the critical value is not in $[0, a]$ it is larger than a. In this case the minimum must occur at one of the endpoints. We can compute

$$T(0) = \frac{a}{v} + \frac{b}{w}$$
$$T(a) = \frac{\sqrt{a^2 + b^2}}{w}$$

but it is difficult to determine which of these is smaller by direct comparison. If, as is likely in practice, we know the values of v, w, a, and b, then it is easy to determine this. With a little cleverness, however, we can determine the minimum in general. We have seen that $T''(x)$ is always positive, so the derivative $T'(x)$ is always increasing. We know that at $wb/\sqrt{v^2 - w^2}$ the derivative is zero, so for values of x less than that critical value, the derivative is negative. This means that $T(0) > T(a)$, so the minimum occurs when $x = a$.

So the upshot is this: If you start farther away from C than $wb/\sqrt{v^2 - w^2}$ then you always want to cut across the sand when you are a distance $wb/\sqrt{v^2 - w^2}$ from point C. If you start closer than this to C, you should cut directly across the sand.

With optimization problems you will see a variety of situations that require you to combine problem solving skills with calculus. Focus on the *process.*

One must learn how to form equations from situations that can be manipulated into what you need. Forget memorizing how to do "this kind of problem" as opposed to "that kind of problem."

> **Learning a process will benefit one far more than memorizing a specific technique.**