Dig-In:

The derivative of the natural exponential function

We derive the derivative of the natural exponential function.

We don't know anything about derivatives that allows us to compute the derivatives of exponential functions without getting our hands dirty. Let's do a little work with the definition of the derivative:

Explanation.

$$\begin{aligned}
\frac{d}{dx}a^x &= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h}\\
&= \lim_{h\to 0}\frac{a^x a^h - a^x}{h}\\
&= \lim_{h\to 0} a^x \frac{\underset{\text{given}}{\boxed{a^h-1}}}{h}\\
&= a^x \lim_{h\to 0}\frac{a^h-1}{h}\\
&= a^x \cdot \underbrace{\textit{(constant)}}_{\lim_{h\to 0}\frac{a^h-1}{h}}
\end{aligned}$$

There are two interesting things to note here: We are left with a limit that involves h but not x, which means that whatever $\lim_{h\to 0}(a^h-1)/h$ is, we know that it is a number, or in other words, a constant. This means that a^x has a remarkable property:

> **The derivative of an exponential function is a constant times itself.**

Unfortunately it is beyond the scope of this text to compute the limit

$$\lim_{h\to 0}\frac{a^h-1}{h}.$$

However, we can look at some examples. Consider $(2^h-1)/h$ and $(3^h-1)/h$:

Learning outcomes: Use "shortcut" rules to find and use derivatives. Use the definition of the derivative to develop a shortcut rule to find the derivative of the natural exponential function.

h	$(2^h-1)/h$
-1	$.5$
-0.1	≈ 0.6700
-0.01	≈ 0.6910
-0.001	≈ 0.6929
-0.0001	≈ 0.6931
-0.00001	≈ 0.6932

h	$(2^h-1)/h$
1	1
0.1	≈ 0.7177
0.01	≈ 0.6956
0.001	≈ 0.6934
0.0001	≈ 0.6932
0.00001	≈ 0.6932

h	$(3^h-1)/h$
-1	≈ 0.6667
-0.1	≈ 1.0404
-0.01	≈ 1.0926
-0.001	≈ 1.0980
-0.0001	≈ 1.0986
-0.00001	≈ 1.0986

h	$(3^h-1)/h$
1	2
0.1	≈ 1.1612
0.01	≈ 1.1047
0.001	≈ 1.0992
0.0001	≈ 1.0987
0.00001	≈ 1.0986

While these tables don't prove that we have a pattern, it turns out that

$$\lim_{h\to 0}\frac{2^h-1}{h}\approx .7 \qquad \text{and} \qquad \lim_{h\to 0}\frac{3^h-1}{h}\approx 1.1.$$

Moreover, if you do more examples, choosing other values for the base a, you will find that the limit varies directly with the value of a: bigger a, bigger limit; smaller a, smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between $a=2$ and $a=3$ the limit will be exactly 1. This happens when

$$a = e = 2.718281828459045\ldots.$$

We will define the number e by this property, see the next definition:

Definition 1. *The number denoted by e, called **Euler's number**, is defined to be the number satisfying the following relation*

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

Using this definition, we see that the function e^x has the following truly remarkable property.

Theorem 1 (The derivative of the natural exponential function)**.** *The derivative of the natural exponential function is the natural exponential function itself. In other words,*

$$\frac{d}{dx}e^x = e^x.$$

Explanation. *From the limit definition of the derivative, write*

$$\begin{aligned}
\frac{d}{dx}e^x &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} \\
&= \lim_{h\to 0}\frac{e^x e^h - e^x}{h} \\
&= \lim_{h\to 0}\underbrace{\boxed{e^x}}_{\text{given}}\frac{e^h-1}{h} \\
&= e^x\lim_{h\to 0}\frac{e^h-1}{h} \\
&= e^x.
\end{aligned}$$

Hence e^x is its own derivative. In other words, the slope of the plot of e^x is the same as its height, or the same as its second coordinate. Said another way, the function $f(x) = e^x$ goes through the point (a, e^a) and has slope e^a at that point, no matter what a is.

Question 1 *What is the slope of the tangent line to the function $f(x) = e^x$ at $x = 5$? The slope is* $\underset{\text{given}}{\boxed{e^5}}$.

Example 1. *Compute:*

$$\frac{d}{dx}\left(8\sqrt{x} + 7e^x\right)$$

Explanation. *Write with me:*

$$\begin{aligned}\frac{d}{dx}\left(8\sqrt{x} + 7e^x\right) &= 8\frac{d}{dx}x^{1/2} + 7\frac{d}{dx}e^x\\ &= 4x^{-1/2} + 7\underset{\text{given}}{\boxed{e^x}}.\end{aligned}$$