## **Break-Ground:**

## Volumes of aluminum cans

Two young mathematicians discuss optimizing aluminum cans.

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, have you ever noticed aluminum cans?

Riley: So very recyclable!

**Devyn:** I know! But I've also noticed that there are some that are short and fat, and others that are tall and skinny, and yet they can still have the same volume!

Riley: So very observant!

**Devyn:** This got me wondering, if we want to make a can with volume V, what shape of can uses the least aluminum?

**Riley:** Ah! This sounds like a job for calculus! The volume of a cylindrical can is given by

$$V = \pi \cdot r^2 \cdot h$$

where r is the radius of the cylinder and h is the height of the cylinder. Also the surface area is given by

$$A = \underbrace{\pi \cdot r^2}_{\text{bottom}} + \underbrace{2 \cdot \pi \cdot r \cdot h}_{\text{sides}} + \underbrace{\pi \cdot r^2}_{\text{top}}$$
$$= 2 \cdot \pi \cdot r^2 + 2 \cdot \pi \cdot r \cdot h.$$

Somehow we want to minimize the surface area, because that's the amount of aluminum used, but we also want to keep the volume constant.

**Devyn:** Whoa, we have way too many letters here.

Riley: Yeah, somehow we need only one variable. Yikes. Too many letters.

**Problem 1** Suppose we wish to construct an aluminum can with volume V that uses the least amount of aluminum. In the context above, what do we want to minimize?

Learning outcomes: Interpret an optimization problem as the procedure used to make a system or design as effective or functional as possible. Set up an optimization problem by identifying the objective function and appropriate constraints.

Multiple Choice:

- (a)  $A \checkmark$
- (b) V
- (c) h
- (d) r

**Problem 2** In the context above, what should be considered a constant?

Select All Correct Answers:

- (a) A
- (b) *V* ✓
- (c) h
- (d) r

As Devyn and Riley noticed, when we work out this type of problem, we need to reduce the problem to a single variable.

**Problem 3** Consider r to be the variable, and express A as a function of r.

**Hint:** First, let's eliminate the variable h. We know that V is a constant and that

$$V = \pi \cdot r^2 \cdot h$$

so  $h = V/(\pi r^2)$ . Substitute this expression for h in the equation

$$A = 2 \cdot \pi \cdot r^2 + 2 \cdot \pi \cdot r \cdot h.$$

$$A = \boxed{2 \cdot \pi \cdot r^2 + 2 \cdot \pi \cdot r \cdot V / (\pi r^2)}$$

**Problem 4** Now consider h to be the variable, and express A as a function of h.

**Hint:** First, let's eliminate the variable r. We know that V is a constant and that

$$V = \pi \cdot r^2 \cdot h$$

so  $r = \sqrt{V/(\pi h)}$ . Substitute this expression for h in the equation

$$A = 2 \cdot \pi \cdot r^2 + 2 \cdot \pi \cdot r \cdot h.$$

$$A = \boxed{2 \cdot \pi \cdot V/(\pi h) + 2 \cdot \pi \cdot \sqrt{V/(\pi h)} \cdot h}$$

Notice that we've reduced (one way or another) this function of two variables to a function of one variable. This process will be a key step in nearly every problem in this next section.