Dig-In:

L'Hôpital's rule

We use derivatives to give us a "short-cut" for computing limits.

Derivatives allow us to take problems that were once difficult to solve and convert them to problems that are easier to solve. Let us consider L'Hôpital's rule:

Theorem 1 (L'Hôpital's Rule). *Let $f(x)$ and $g(x)$ be functions that are differentiable near a. If*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \qquad \text{or} \pm\infty,$$

and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, and $g'(x) \neq 0$ for all x near a, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

This theorem is somewhat difficult to prove, in part because it incorporates so many different possibilities, so we will not prove it here.

Remark 1. *L'Hôpital's rule applies even when $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \mp\infty$.*

L'Hôpital's rule allows us to investigate limits of *indeterminate form*.

Definition 1 (List of Indeterminate Forms).

$\mathbf{\frac{0}{0}}$ *This refers to a limit of the form $\lim_{x\to a} \frac{f(x)}{g(x)}$ where $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$.*

$\mathbf{\frac{\infty}{\infty}}$ *This refers to a limit of the form $\lim_{x\to a} \frac{f(x)}{g(x)}$ where $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$.*

$\mathbf{0 \cdot \infty}$ *This refers to a limit of the form $\lim_{x\to a} (f(x) \cdot g(x))$ where $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$.*

Learning outcomes: Recall how to find limits for forms that are not indeterminate. Define an indeterminate form. Convert indeterminate forms to the form zero over zero or infinity over infinity. Define L'Hopital's Rule and identify when it can be used. Use L'Hopital's Rule to find limits.

$\infty - \infty$ *This refers to a limit of the form* $\lim_{x\to a}(f(x) - g(x))$ *where* $f(x) \to \infty$ *and* $g(x) \to \infty$ *as* $x \to a$.

1^∞ *This refers to a limit of the form* $\lim_{x\to a} f(x)^{g(x)}$ *where* $f(x) \to 1$ *and* $g(x) \to \infty$ *as* $x \to a$.

0^0 *This refers to a limit of the form* $\lim_{x\to a} f(x)^{g(x)}$ *where* $f(x) \to 0$ *and* $g(x) \to 0$ *as* $x \to a$.

∞^0 *This refers to a limit of the form* $\lim_{x\to a} f(x)^{g(x)}$ *where* $f(x) \to \infty$ *and* $g(x) \to 0$ *as* $x \to a$.

*In each of these cases, the value of the limit is **not** immediately obvious. Hence, a careful analysis is required!*

Basic indeterminant forms

Our first example is the computation of a limit that was somewhat difficult before.

Example 1. *Compute*

$$\lim_{x\to 0} \frac{\sin(x)}{x}.$$

Explanation. *Set* $f(x) = \sin(x)$ *and* $g(x) = x$. *Since both* $f(x)$ *and* $g(x)$ *are differentiable functions at* 0, *and*

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0,$$

this situation is ripe for L'Hôpital's Rule. Now

$$f'(x) = \underbrace{\boxed{\cos(x)}}_{\text{given}}$$

and

$$g'(x) = \underbrace{\boxed{1}}_{\text{given}}.$$

L'Hôpital's rule tells us that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1.$$

Remark 2. *Note, the astute mathematician will notice that in our example above, we are somewhat cheating. To apply L'Hôpital's rule, we need to know the derivative of sine; however, to know the derivative of sine we must be able to compute the limit:*

$$\lim_{x\to 0} \frac{\sin(x)}{x}$$

Hence using L'Hôpital's rule to compute this limit is a circular argument! We encourage the gentle reader to view L'Hôpital's rule a "reminder" as to what is true, not as the formal derivation of the result.

Our next set of examples will run through the remaining indeterminate forms one is likely to encounter.

Example 2. *Compute*

$$\lim_{x\to\pi/2^+} \frac{\sec(x)}{\tan(x)}.$$

Explanation. *Set $f(x) = \sec(x)$ and $g(x) = \tan(x)$. Both $f(x)$ and $g(x)$ are differentiable near $\pi/2$. Additionally,*

$$\lim_{x\to\pi/2^+} f(x) = \lim_{x\to\pi/2^+} g(x) = -\infty.$$

This situation is ripe for L'Hôpital's Rule. Now

$$f'(x) = \underbrace{\boxed{\sec(x)\tan(x)}}_{\text{given}}$$

and

$$g'(x) = \underbrace{\boxed{\sec^2(x)}}_{\text{given}}.$$

L'Hôpital's rule tells us that

$$\begin{aligned}\lim_{x\to\pi/2^+} \frac{\sec(x)}{\tan(x)} &= \lim_{x\to\pi/2^+} \frac{\sec(x)\tan(x)}{\sec^2(x)}\\ &= \lim_{x\to\pi/2^+} \sin(x)\\ &= \underbrace{\boxed{1}}_{\text{given}}.\end{aligned}$$

Example 3. *Compute*

$$\lim_{x\to 0^+} x\ln x.$$

Explanation. *This doesn't appear to be suitable for L'Hôpital's Rule. As x approaches zero, $\ln x$ goes to $-\infty$, so the product looks like*

$$(\textit{something very small})\cdot(\textit{something very large and negative}).$$

This product could be anything. A careful analysis is required. Write

$$x\ln x = \frac{\ln x}{x^{-1}}.$$

Set $f(x) = \ln(x)$ and $g(x) = x^{-1}$. Since both functions are differentiable near zero and

$$\lim_{x\to 0+} \ln(x) = -\infty \qquad \textit{and} \qquad \lim_{x\to 0+} x^{-1} = \infty,$$

we may apply L'Hôpital's rule. Write with me

$$f'(x) = \underbrace{\boxed{x^{-1}}}_{\text{given}}$$

and

$$g'(x) = \underbrace{\boxed{-x^{-2}}}_{\text{given}},$$

so

$$\begin{aligned} \lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} \\ &= \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} \\ &= \lim_{x\to 0^+} -x \\ &= 0. \end{aligned}$$

One way to interpret this is that since $\lim\limits_{x\to 0^+} x\ln x = 0$*, the function* x *approaches zero much faster than* $\ln x$ *approaches* $-\infty$*.*

Indeterminate forms involving subtraction

There are two basic cases here, we'll do an example of each.

Example 4. *Compute*

$$\lim_{x\to 0} \left(\cot(x) - \csc(x)\right).$$

Explanation. *Here we simply need to write each term as a fraction,*

$$\begin{aligned} \lim_{x\to 0} \left(\cot(x) - \csc(x)\right) &= \lim_{x\to 0} \left(\frac{\cos(x)}{\sin(x)} - \frac{1}{\sin(x)}\right) \\ &= \lim_{x\to 0} \frac{\cos(x) - 1}{\sin(x)} \end{aligned}$$

Setting $f(x) = \cos(x) - 1$ *and* $g(x) = \sin(x)$*, both functions are differentiable near zero and*

$$\lim_{x\to 0}(\cos(x) - 1) = \lim_{x\to 0} \sin(x) = 0.$$

We may now apply L'Hôpital's rule. Write with me

$$f'(x) = \underbrace{\boxed{-\sin(x)}}_{\text{given}}$$

and

$$g'(x) = \underbrace{\boxed{\cos(x)}}_{\text{given}},$$

so

$$\begin{aligned}
\lim_{x\to 0}\left(\cot(x)-\csc(x)\right) &= \lim_{x\to 0}\frac{\cos(x)-1}{\sin(x)}\\
&= \lim_{x\to 0}\frac{-\sin(x)}{\cos(x)}\\
&= 0.
\end{aligned}$$

Sometimes one must be slightly more clever.

Example 5. *Compute*

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right).$$

Explanation. *Again, this doesn't appear to be suitable for L'Hôpital's Rule. A bit of algebraic manipulation will help. Write with me*

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right) &= \lim_{x\to\infty}\left(x\left(\sqrt{1+1/x}-1\right)\right)\\
&= \lim_{x\to\infty}\frac{\sqrt{1+1/x}-1}{x^{-1}}
\end{aligned}$$

Now set $f(x)=\sqrt{1+1/x}-1$, $g(x)=x^{-1}$. *Since both functions are differentiable for large values of* x *and*

$$\lim_{x\to\infty}(\sqrt{1+1/x}-1)=\lim_{x\to\infty}x^{-1}=0,$$

we may apply L'Hôpital's rule. Write with me

$$f'(x)=\underbrace{\boxed{(1/2)(1+1/x)^{-1/2}\cdot(-x^{-2})}}_{\text{given}}$$

and

$$g'(x)=\underbrace{\boxed{-x^{-2}}}_{\text{given}}$$

so

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right) &= \lim_{x\to\infty}\frac{\sqrt{1+1/x}-1}{x^{-1}}\\
&= \lim_{x\to\infty}\frac{(1/2)(1+1/x)^{-1/2}\cdot(-x^{-2})}{-x^{-2}}\\
&= \lim_{x\to\infty}\frac{1}{2\sqrt{1+1/x}}\\
&= \frac{1}{2}.
\end{aligned}$$

Exponential Indeterminate Forms

There is a standard trick for dealing with the indeterminate forms

$$1^\infty, \qquad 0^0, \qquad \infty^0.$$

Given $u(x)$ and $v(x)$ such that

$$\lim_{x\to a} u(x)^{v(x)}$$

falls into one of the categories described above, rewrite as

$$\lim_{x\to a} e^{v(x)\ln(u(x))}$$

and then examine the limit of the exponent

$$\lim_{x\to a} v(x)\ln(u(x)) = \lim_{x\to a} \frac{\ln(u(x))}{v(x)^{-1}}$$

using L'Hôpital's rule. Since these forms are all very similar, we will only give a single example.

Example 6. *Compute*

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x.$$

Explanation. *Write*

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)}.$$

So now look at the limit of the exponent

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}}.$$

Setting $f(x) = \ln\left(1+\frac{1}{x}\right)$ *and* $g(x) = x^{-1}$, *both functions are differentiable for large values of* x *and*

$$\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} x^{-1} = 0.$$

We may now apply L'Hôpital's rule. Write

$$f'(x) = \underbrace{\boxed{\frac{-x^{-2}}{1+\frac{1}{x}}}}_{\text{given}}$$

and

$$g'(x) = \underbrace{\boxed{-x^{-2}}}_{\text{given}},$$

so

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}} &= \lim_{x\to\infty} \frac{\frac{-x^{-2}}{1+\frac{1}{x}}}{-x^{-2}} \\ &= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} \\ &= 1.\end{aligned}$$

Hence,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)} = e^1 = e.$$