Dig-In:

Basic optimization

Now we put our optimization skills to work.

An **optimization problem** is a problem where you need to maximize or minimize some quantity given some constraints. This can be accomplished using the tools of differential calculus that we have already developed.

Perhaps the most basic optimization problems is generated by the following question:

> Among all rectangles of a fixed perimeter, which has the greatest area?

Let's not do this problem in the abstract, let's do it with numbers.

Example 1. *Of all rectangles of perimeter* 12*, which side lengths give the greatest area?*

Explanation. *If a rectangle has perimeter* 12 *and one side is length* x*, then the length of the other side is* $\underbrace{\boxed{6-x}}_{\text{given}}$*. Hence the area of a rectangle of perimeter* 12 *can be given by*

$$A(x) = \underbrace{\boxed{x(6-x)}}_{\text{given}}.$$

However, for the side lengths to be physically relevant, we must assume that x *is in the interval* $(\underbrace{\boxed{0}}_{\text{given}}, \underbrace{\boxed{6}}_{\text{given}}]$.

So to maximize the area of the rectangle, we need to find the maximum value of $A(x)$ *on the appropriate interval.*

At this point, you should graph the function if you can.

We'll continue on without the aid of a graph, and use the derivative. Write

$$A'(x) = \underbrace{\boxed{6-2x}}_{\text{given}}$$

Learning outcomes: Describe the goals of optimization problems generally. Find all local maximums and minimums using the First and Second Derivative tests. Identify when we can find an absolute maximum or minimum on an open interval. Contrast optimization on open and closed intervals. Describe the objective function and constraints in a given optimization problem. Solve optimization problems by finding the appropriate extreme values.

Now we find the critical points, solving the equation

$$\underset{\text{given}}{\boxed{6-2x}} = 0,$$

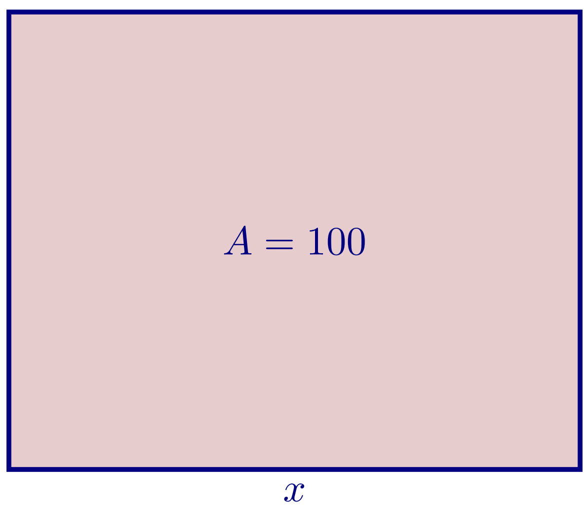
we see that the only critical point of A is at $x = \underset{\text{given}}{\boxed{3}}$

Since $A'(x) = \underset{\text{given}}{\boxed{6-2x}}$ *is* (positive ✓ / negative) *on* $(0,3)$ *and* (positive/ negative ✓) *on* $(3,6]$, $x = 3$ *is where the maximum value of A happens. This is exactly when the rectangle is a square!*

A key step to note, is when we explained why $x = 3$ is actually the maximum. Above we basically used facts about the derivative. Below we use a similar argument.

Example 2. *Of all rectangles of area* 100*, which has the smallest perimeter?*

Explanation. *First we draw a picture, Here is a rectangle with an area of* 100*.*



If x denotes one of the sides of the rectangle, then the adjacent side must be $\underset{\text{given}}{\boxed{100/x}}$.

The perimeter of this rectangle is given by

$$P(x) = \underset{\text{given}}{\boxed{2x + 2 \cdot 100/x}}.$$

We wish to minimize $P(x)$. Note, not all values of x make sense in this problem: lengths of sides of rectangles must be positive, so $x > 0$. If $x > 0$ then so is $100/x$, so we need no second condition on x.

At this point, you should graph the function if you can.

We next find $P'(x)$ and set it equal to zero. Write

$$P'(x) = \underbrace{\boxed{2 - 200/x^2}}_{\text{given}} = 0.$$

Solving for x gives us $x = \pm\underbrace{\boxed{10}}_{\text{given}}$. We are interested only in $x > 0$, so only the value $x = \underbrace{\boxed{10}}_{\text{given}}$ is of interest. Since $P'(x)$ is defined everywhere on the interval $(0, \infty)$, there are no more critical values, and there are no endpoints. Is there a local maximum, minimum, or neither at $x = 10$? The second derivative is

$$P''(x) = \underbrace{\boxed{400/x^3}}_{\text{given}},$$

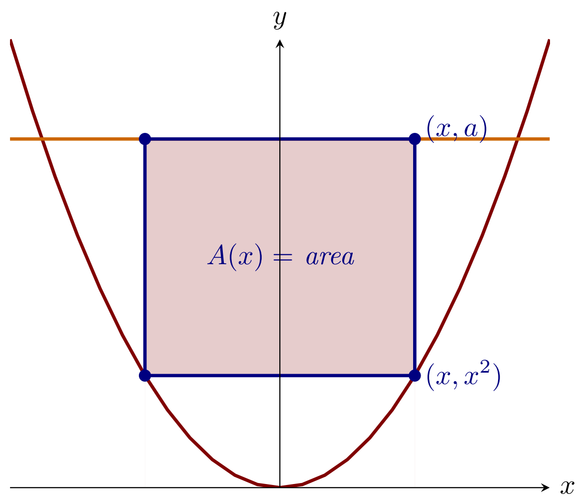
and $P''(10) > 0$, so there is a local minimum. Since there is only one critical point, this is also the global minimum, so the rectangle with smallest perimeter is the 10×10 square.

Hence, calculus gives a **reason** for **why** a square is the rectangle with both

- the largest area for a given perimeter.
- the smallest perimeter for a given area.

We may be done with rectangles, but they aren't done with us. Here is a problem where there are more constraints on the possible side lengths of the rectangle.

Example 3. *Find the rectangle with largest area that fits inside the graph of the parabola $y = x^2$ below the line $y = a$, where a is an unspecified constant value, with the top side of the rectangle on the horizontal line $y = a$. See the figure below:*



Explanation. *We want to maximize value of* $A(x)$*. The lower right corner of the rectangle is at* (x, x^2)*, and once this is chosen the rectangle is completely determined. Then the area is*

$$A(x) = \underbrace{\boxed{(2x)(a - x^2) = -2x^3 + 2ax}}_{\text{given}}.$$

We want the maximum value of $A(x)$ *when* x *is in* $[0, \sqrt{a}]$*. You might object to allowing* $x = 0$ *or* $x = \sqrt{a}$*, since then the "rectangle" has either no width or no height, so is not "really" a rectangle. But the problem is somewhat easier if we simply allow such rectangles, which have zero area as we may then apply the Extreme Value Theorem and see that we indeed have a maximum and minimum value.*

At this point, you should graph the function if you can.

Setting $0 = A'(x) = \underbrace{\boxed{-6x^2 + 2a}}_{\text{given}}$ *we find* $x = \underbrace{\boxed{\sqrt{a/3}}}_{\text{given}}$ *as the only critical point. Testing this and the two endpoints (as the maximum could also be there), we have* $A(0) = A(\sqrt{a}) = \underbrace{\boxed{0}}_{\text{given}}$ *and* $A(\sqrt{a/3}) = \underbrace{\boxed{(4/9)\sqrt{3}a^{3/2}}}_{\text{given}}$*. Hence, the maximum area occurs when the rectangle has dimensions* $2\sqrt{a/3} \times (2/3)a$*.*

Again, note that above we used the Extreme Value Theorem to guarantee that we found the maximum.