Dig-In:

The Product rule and quotient rule

Here we compute derivatives of products and quotients of functions

The product rule

Consider the product of two simple functions, say

$$f(x) \cdot g(x)$$

where $f(x) = x^2 + 1$ and $g(x) = x^3 - 3x$. An obvious guess for the derivative of $f(x)g(x)$ is the product of the derivatives:

$$\begin{aligned} f'(x)g'(x) &= (2x)(3x^2 - 3) \\ &= 6x^3 - 6x. \end{aligned}$$

Is this guess correct? We can check by rewriting f and g and doing the calculation in a way that is known to work. Write with me

$$\begin{aligned} f(x)g(x) &= (x^2 + 1)(x^3 - 3x) \\ &= x^5 - 3x^3 + x^3 - 3x \\ &= x^5 - 2x^3 - 3x. \end{aligned}$$

Hence

$$\frac{d}{dx} f(x)g(x) = \frac{d}{dx}(x^5 - 2x^3 - 3x) = 5x^4 - 6x^2 - 3,$$

so we see that

$$\frac{d}{dx} f(x)g(x) \neq f'(x)g'(x).$$

So the derivative of $f(x)g(x)$ is **not** as simple as $f'(x)g'(x)$. Never fear, we have a rule for exactly this situation.

Theorem 1 (The product rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + f'(x)g(x).$$

Learning outcomes: Identify products of functions. Use the product rule to calculate derivatives. Identify quotients of functions. Use the quotient rule to calculate derivatives. Combine derivative rules to take derivatives of more complicated functions. Explain the signs of the terms in the numerator of the quotient rule. Multiply tangent lines to justify the product rule. Use the product and quotient rule to calculate derivatives from a table of values.

Let's return to the example with which we started.

Example 1. *Let* $f(x) = (x^2 + 1)$ *and* $g(x) = (x^3 - 3x)$*. Compute:*

$$\frac{d}{dx} f(x)g(x)$$

Explanation. *Write with me*

$$\begin{aligned}\frac{d}{dx} f(x)g(x) &= f(x)g'(x) + f'(x)g(x)\\ &= (x^2+1)(\underset{\text{given}}{\boxed{3x^2-3}}) + (\underset{\text{given}}{\boxed{2x}})(x^3-3x).\end{aligned}$$

We could stop here, but we should show that expanding this out recovers our previous result. Write with me

$$\begin{aligned}&(x^2+1)(3x^2-3) + 2x(x^3-3x)\\ &\qquad = 3x^4 - 3x^2 + 3x^2 - 3 + 2x^4 - 6x^2\\ &\qquad = \underset{\text{given}}{\boxed{5x^4 - 6x^2 - 3}},\end{aligned}$$

which is precisely what we obtained before.

Now that we are pros, let's try one more example.

Example 2. *Compute:*

$$\frac{d}{dx}(xe^x - e^x)$$

Explanation. *Using the product rule and the sum rule, write with me*

$$\frac{d}{dx}\left(xe^x - e^x\right) = \underset{\text{given}}{\boxed{xe^x}}.$$

The quotient rule

We'd like to have a formula to compute

$$\frac{d}{dx}\frac{f(x)}{g(x)}$$

if we already know $f'(x)$ and $g'(x)$. Instead of attacking this problem head-on, let's notice that we've already done part of the problem: $f(x)/g(x) = f(x)\cdot(1/g(x))$, that is, this is really a product, and we can compute the derivative if we know $f'(x)$ and $(1/g(x))'$. This brings us to our next derivative rule.

Theorem 2 (The quotient rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Example 3. *Compute:*

$$\frac{d}{dx}\frac{x^2+1}{x^3-3x}$$

Explanation. *Write with me*

$$\frac{d}{dx}\frac{x^2+1}{x^3-3x} = \frac{2x(\underbrace{\boxed{x^3-3x}}_{\text{given}}) - (\underbrace{\boxed{x^2+1}}_{\text{given}})(3x^2-3)}{(\underbrace{\boxed{x^3-3x}}_{\text{given}})^2}$$

$$= \frac{-x^4-6x^2+3}{(\underbrace{\boxed{x^3-3x}}_{\text{given}})^2}.$$

It is often possible to calculate derivatives in more than one way, as we have already seen. Since every quotient can be written as a product, it is always possible to use the product rule to compute the derivative, though it is not always simpler.

Example 4. *Compute:*

$$\frac{d}{dx}\frac{625-x^2}{\sqrt{x}}$$

in two ways. First using the quotient rule and then using the product rule.

Explanation. *First, we'll compute the derivative using the quotient rule. Write with me*

$$\frac{d}{dx}\frac{625-x^2}{\sqrt{x}} = \frac{(-2x)\left(\underbrace{\boxed{\sqrt{x}}}_{\text{given}}\right) - (\underbrace{\boxed{625-x^2}}_{\text{given}})\left(\frac{1}{2}x^{-1/2}\right)}{\underbrace{\boxed{x}}_{\text{given}}}.$$

Second, we'll compute the derivative using the product rule:

$$\frac{d}{dx}\frac{625-x^2}{\sqrt{x}} = \frac{d}{dx}\left(625-x^2\right)x^{-1/2}$$

$$= \left(625-x^2\right)\left(\underbrace{\boxed{\frac{-x^{-3/2}}{2}}}_{\text{given}}\right) + (\underbrace{\boxed{-2x}}_{\text{given}})\left(x^{-1/2}\right).$$

With a bit of algebra, both of these simplify to

$$-\frac{3x^2+625}{2x^{3/2}}.$$