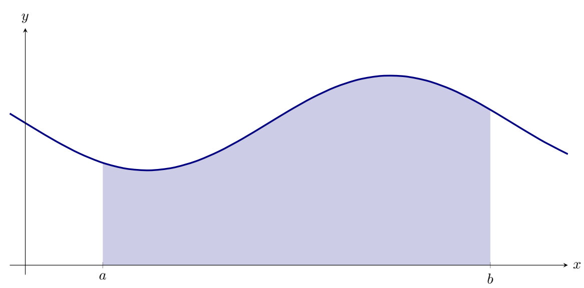
Dig-In:

Approximating area with rectangles

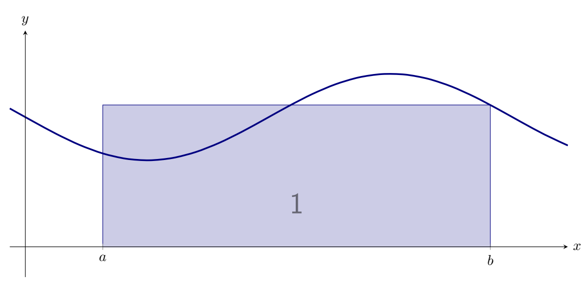
We introduce the basic idea of using rectangles to approximate the area under a curve.

Rectangles and areas

We want to compute the area between the curve $y = f(x)$ and the horizontal axis on the interval $[a, b]$:

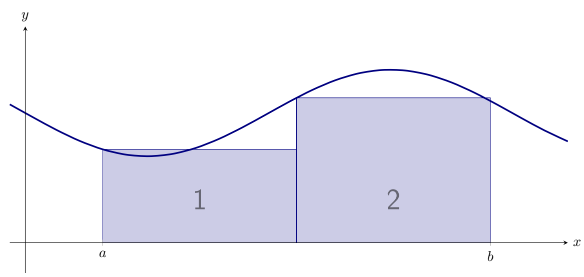


One way to do this would be to approximate the area with rectangles. With one rectangle we get a rough approximation:

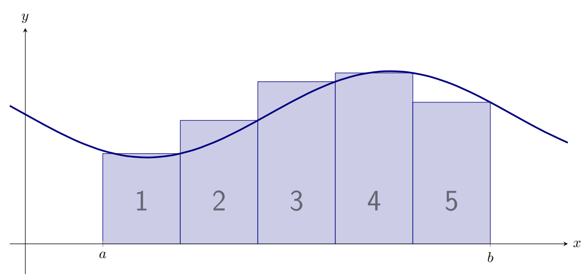


Two rectangles might make a better approximation:

Learning outcomes: Define area. Understand the relationship between area under a curve and sums of rectangles. Approximate area under a curve. Compute left, right, and midpoint Riemann sums with 10 or fewer rectangles.

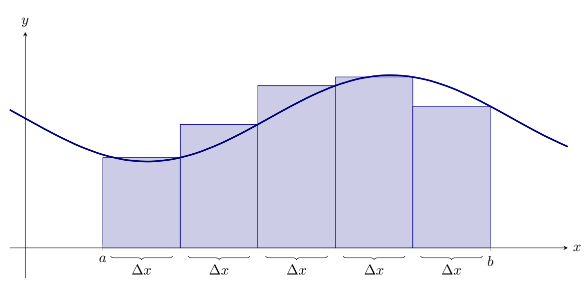


With even more, we get a closer, and closer, approximation:



Definition 1. *If we are approximating the area between a curve and the x-axis on $[a, b]$ with n rectangles of width Δx, then*

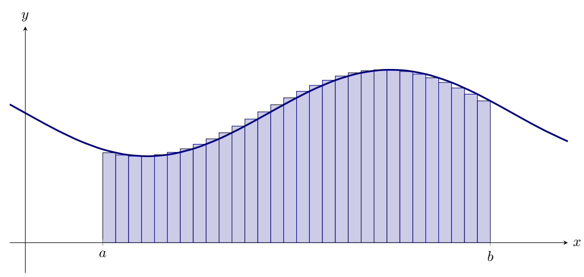
$$\Delta x = \frac{b-a}{n}.$$



Question 1 *Suppose we wanted to approximate area between the curve $y = x^2 + 1$ and the x-axis on the interval $[-1, 1]$, with 8 rectangles. What is Δx?*

$$\Delta x = \boxed{1/4}$$

As we add rectangles, we are more closely approximating the area we are interested in:



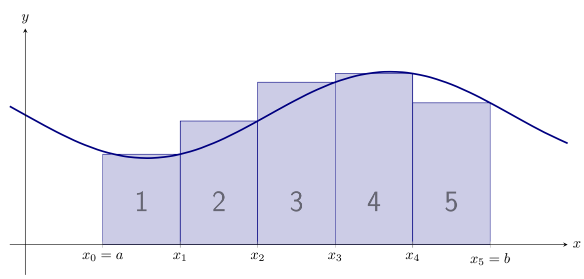
We could find the area exactly if we could compute the limit as the width of the rectangles goes to zero and the number of rectangles goes to infinity.

Let's setup some notation to help with these calculuations:

Definition 2. *When approximating an area with n rectangles, the **grid points***

$$x_0, x_1, x_2, \ldots, x_n$$

are the x-coordinates that determine the edges of the rectangles. In the graph below, we've numbered the rectangles to help you see the relation between the indices of the grid points and the kth rectangle.



Note, if we are approximating the area between a curve and the horizontal axis on $[a, b]$ with n rectangles, then it is always the case that

$$x_0 = a \qquad \text{and} \qquad x_n = b.$$

Question 2 *If we are approximating the area between a curve and the horizontal axis with 11 rectangles, how many grid points will we have?*

Hint: *You can draw it!*

We'll have $\boxed{12}$ *grid points.*

But which set of rectangles?

If we are going to try and actually use many small rectangles to compute the area under a curve, we should decide on exactly *which* rectangles we want to use. We need another definition:

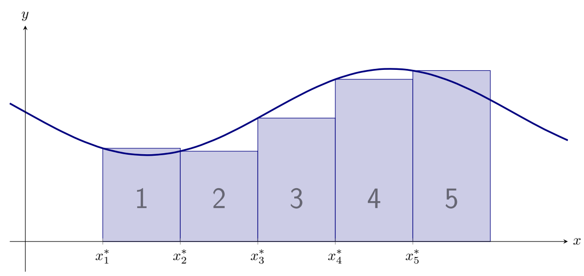
Definition 3. *When approximating an area with rectangles, a **sample point** is the x-coordinate that determines the relevant height of our rectangles. We denote a sample point as:*

$$x_k^*$$

The sample point tells us where the rectangles touch the curve. Here are three options for sample points that we consider:

Rectangles defined by left-endpoints

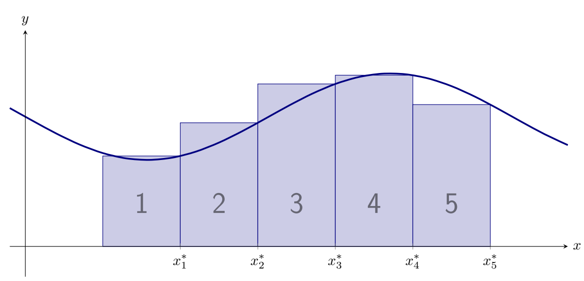
We can set the rectangles up so that the left-endpoint touches the curve.



In the graph above, the kth rectangle's left-endpoint is touching the curve.

Rectangles defined by right-endpoints

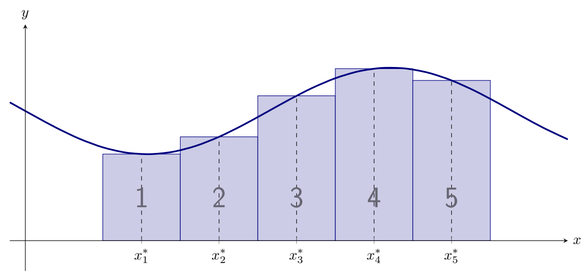
We can set the rectangles up so that the right-endpoint touches the curve.



In the graph above, the kth rectangle's right-endpoint is touching the curve.

Rectangles defined by midpoints

We can set the rectangles up so that the midpoint of one of the horizontal sides touches the curve.



In the graph above, the midpoint of the horizontal side of the kth rectangle is touching the curve.

Riemann sums and approximating area

Once we know how to identify our rectangles, we can compute some approximate areas. If we are approximating area with n rectangles, then

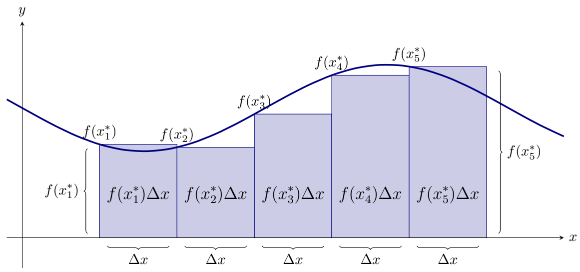
$$\begin{aligned}\text{Area} &\approx \sum_{k=1}^{n} (\text{height of } k\text{th rectangle}) \times (\text{width of } k\text{th rectangle}) \\ &= \sum_{k=1}^{n} f(x_k^*)\Delta x \\ &= f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + \cdots + f(x_n^*)\Delta x.\end{aligned}$$

Definition 4. *A sum of the form:*

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

is called a ***Riemann sum****, pronounced "ree-mahn" sum.*

A Riemann sum computes an approximation of the area between a curve and the x-axis on the interval $[a, b]$. It can be defined several different ways. In our class, it will be defined via left-endpoints, right-endpoints, or midpoints. Here we see the explicit connection between a Riemann sum defined by left-endpoints and the area between a curve and the x-axis on the interval $[a, b]$:



and here is the associated Riemann sum

$$\sum_{k=1}^{5} f(x_k^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x + f(x_5^*)\Delta x.$$

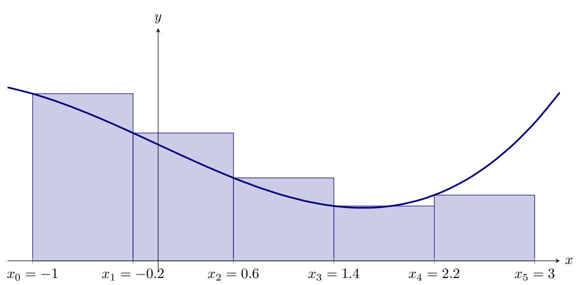
Left Riemann sums

Example 1. *Consider $f(x) = x^3/8 - x + 2$. Approximate the area between f and the x-axis on the interval $[-1, 3]$ using a left-endpoint Riemann sum with $n = 5$ rectangles.*

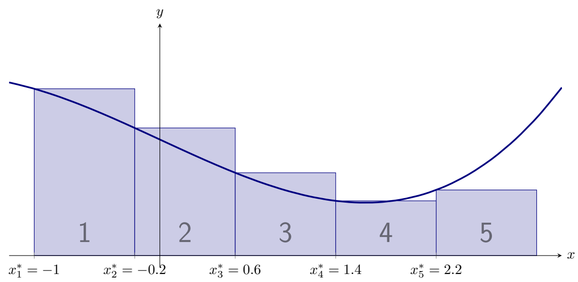
Explanation. *First note that the width of each rectangle is*

$$\Delta x = \frac{3-(-1)}{5} = \underbrace{\boxed{4/5}}_{\text{given}}.$$

The grid points define the edges of the rectangle and are seen below:



On the other hand, the sample points identify which endpoints we use:



It is helpful to collect all of this data into a table:

k	x_k	x_k^*	$f(x_k^*)$
0	$\underset{\text{given}}{\boxed{-1}}$	NA	NA
1	$\underset{\text{given}}{\boxed{-0.2}}$	$\underset{\text{given}}{\boxed{-1}}$	$\underset{\text{given}}{\boxed{2.875}}$
2	$\underset{\text{given}}{\boxed{0.6}}$	$\underset{\text{given}}{\boxed{-0.2}}$	$\underset{\text{given}}{\boxed{2.199}}$
3	$\underset{\text{given}}{\boxed{1.4}}$	$\underset{\text{given}}{\boxed{0.6}}$	$\underset{\text{given}}{\boxed{1.427}}$
4	$\underset{\text{given}}{\boxed{2.2}}$	$\underset{\text{given}}{\boxed{1.4}}$	$\underset{\text{given}}{\boxed{0.943}}$
5	$\underset{\text{given}}{\boxed{3}}$	$\underset{\text{given}}{\boxed{2.2}}$	$\underset{\text{given}}{\boxed{1.131}}$

Now we may write a left Riemann sum and approximate the area

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x + f(x_5^*)\Delta x,$$

which evaluates to

$$\begin{aligned} &= 2.875 \cdot (4/5) + 2.199 \cdot (4/5) + 1.427 \cdot (4/5) \\ &\qquad + 0.943 \cdot (4/5) + 1.131 \cdot (4/5) \end{aligned}$$

and we find

$$= \underset{\text{given}}{\boxed{6.86}}.$$

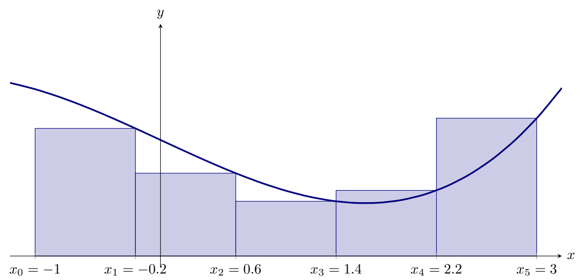
Right Riemann sums

Example 2. *Consider $f(x) = x^3/8 - x + 2$. Approximate the area between f and the x-axis on the interval $[-1, 3]$ using a right-endpoint Riemann sum with $n = 5$ rectangles.*

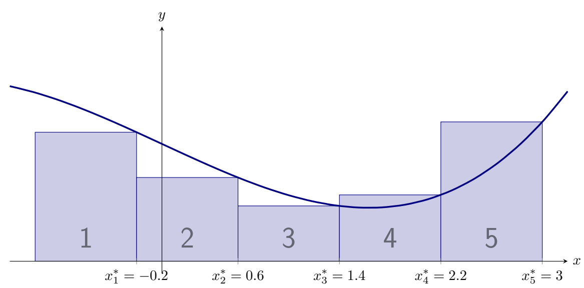
Explanation. *First note that the width of each rectangle is*

$$\Delta x = \frac{3-(-1)}{5} = \underbrace{\boxed{4/5}}_{\text{given}}.$$

The grid points define the edges of the rectangle and are seen below:



On the other hand, the sample points identify which endpoints we use:



It is helpful to collect all of this data into a table:

k	x_k	x_k^*	$f(x_k^*)$
0	$\underbrace{\boxed{-1}}_{\text{given}}$	*NA*	*NA*
1	$\underbrace{\boxed{-0.2}}_{\text{given}}$	$\underbrace{\boxed{-0.2}}_{\text{given}}$	$\underbrace{\boxed{2.199}}_{\text{given}}$
2	$\underbrace{\boxed{0.6}}_{\text{given}}$	$\underbrace{\boxed{0.6}}_{\text{given}}$	$\underbrace{\boxed{1.427}}_{\text{given}}$
3	$\underbrace{\boxed{1.4}}_{\text{given}}$	$\underbrace{\boxed{1.4}}_{\text{given}}$	$\underbrace{\boxed{0.943}}_{\text{given}}$
4	$\underbrace{\boxed{2.2}}_{\text{given}}$	$\underbrace{\boxed{2.2}}_{\text{given}}$	$\underbrace{\boxed{1.131}}_{\text{given}}$
5	$\underbrace{\boxed{3}}_{\text{given}}$	$\underbrace{\boxed{3}}_{\text{given}}$	$\underbrace{\boxed{2.375}}_{\text{given}}$

Now we may write a right Riemann sum and approximate the area

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x + f(x_5^*)\Delta x,$$

which evaluates to

$$\begin{aligned} &= 2.199 \cdot (4/5) + 1.427 \cdot (4/5) + 0.943 \cdot (4/5) \\ &\quad + 1.131 \cdot (4/5) + 2.375 \cdot (4/5) \end{aligned}$$

and we find

$$= \underbrace{\boxed{6.46}}_{\text{given}}.$$

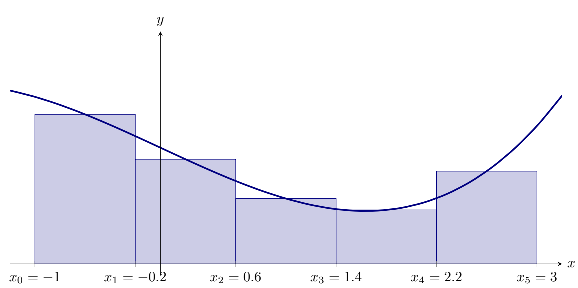
Midpoint Riemann sums

Example 3. *Consider $f(x) = x^3/8 - x + 2$. Approximate the area between f and the x-axis on the interval $[-1, 3]$ using a midpoint Riemann sum with $n = 5$ rectangles.*

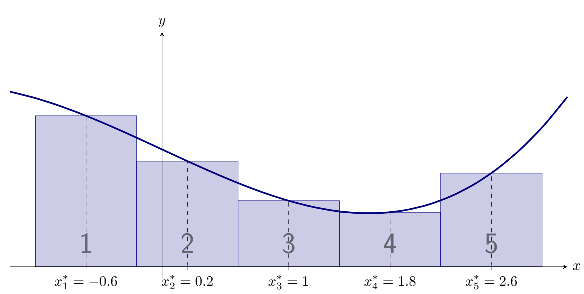
Explanation. *First note that the width of each rectangle is*

$$\Delta x = \frac{3-(-1)}{5} = \underbrace{\boxed{4/5}}_{\text{given}}.$$

The grid points define the edges of the rectangle and are seen below:



On the other hand, the sample points identify which endpoints we use:



It is helpful to collect all of this data into a table:

k	x_k	x_k^*	$f(x_k^*)$
0	$\underset{\text{given}}{\boxed{-1}}$	NA	NA
1	$\underset{\text{given}}{\boxed{-0.2}}$	$\underset{\text{given}}{\boxed{-0.6}}$	$\underset{\text{given}}{\boxed{2.573}}$
2	$\underset{\text{given}}{\boxed{0.6}}$	$\underset{\text{given}}{\boxed{0.2}}$	$\underset{\text{given}}{\boxed{1.801}}$
3	$\underset{\text{given}}{\boxed{1.4}}$	$\underset{\text{given}}{\boxed{1}}$	$\underset{\text{given}}{\boxed{1.125}}$
4	$\underset{\text{given}}{\boxed{2.2}}$	$\underset{\text{given}}{\boxed{1.8}}$	$\underset{\text{given}}{\boxed{0.929}}$
5	$\underset{\text{given}}{\boxed{3}}$	$\underset{\text{given}}{\boxed{2.6}}$	$\underset{\text{given}}{\boxed{1.597}}$

Now we may write a midpoint Riemann sum and approximate the area

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x + f(x_5^*)\Delta x,$$

which evaluates to

$$\begin{aligned} &= 2.573 \cdot (4/5) + 1.801 \cdot (4/5) + 1.125 \cdot (4/5) \\ &\qquad + 0.929 \cdot (4/5) + 1.597 \cdot (4/5) \end{aligned}$$

and we find

$$= \underset{\text{given}}{\boxed{6.42}}.$$

Summary

Riemann sums approximate the area between curves and the x-axis via rectangles. When computing this area via rectangles, there are several things to know:

- What interval are we on? In our discussion above we call this $[a, b]$.
- How many rectangles will be used? In our discussion above we called this n.
- What is the width of each individual rectangle? In our discussion above we called this Δx.
- What points will determine the height of the rectangle? In our discussion above we called these sample points, x_k^*, and they can be left-endpoints, right-endpoints, or midpoints.
- What is the actual height of the rectangle? This will always be $f(x_k^*)$.