Dig-In:

The Squeeze Theorem

The Squeeze theorem allows us to exchange difficult functions for easy functions.

In mathematics, sometimes we can study complex functions by exchanging them for simplier functions. The *Squeeze Theorem* tells us one situation where this is possible.

Theorem 1 (Squeeze Theorem)**.** *Suppose that*

$$g(x) \leq f(x) \leq h(x)$$

for all x close to a but not necessarily equal to a. If

$$\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x),$$

then $\lim_{x\to a} f(x) = L$.

Question 1 *I'm thinking of a function f. I know that for all x*

$$0 \leq f(x) \leq x^2.$$

What is $\lim_{x\to 0} f(x)$?

Multiple Choice:

(a) $f(x)$

(b) $f(0)$

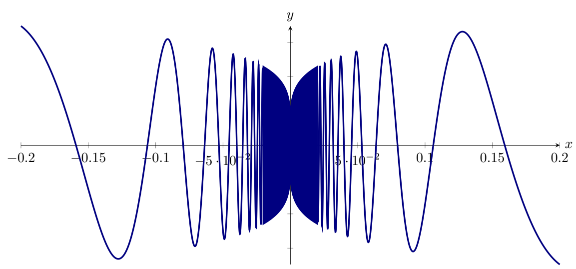
(c) 0 ✓

(d) *impossible to say*

Example 1. *Consider the function*

$$f(x) = \begin{cases} \sqrt[5]{x}\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

Learning outcomes: Understand the Squeeze Theorem and how it can be used to find limit values. Calculate limits using the Squeeze Theorem.



Is this function continuous at $x = 0$?

Explanation. *We must show that* $\lim_{x\to 0} f(x) = \underbrace{\boxed{0}}_{\text{given}}$. *Note*

$$-|\sqrt[5]{x}| \leq f(x) \leq |\sqrt[5]{x}|.$$

Since

$$\lim_{x\to 0} -|\sqrt[5]{x}| = \underbrace{\boxed{0}}_{\text{given}} = \lim_{x\to 0} |\sqrt[5]{x}|,$$

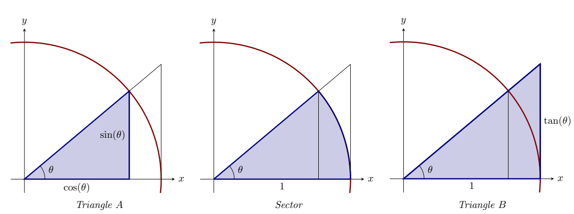
we see by the Squeeze Theorem, Theorem, that $\lim_{x\to 0} f(x) = \underbrace{\boxed{0}}_{\text{given}}$. *Hence $f(x)$ is continuous.*

Here we see how the informal definition of continuity being that you can "draw it" without "lifting your pencil" differs from the formal definition.

Example 2. *Compute:*

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}$$

Explanation. *To compute this limit, use the Squeeze Theorem. First note that we only need to examine* $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ *and for the present time, we'll assume that θ is positive. Consider the diagrams below:*



From our diagrams above we see that

$$\text{Area of Triangle } A \leq \text{Area of Sector} \leq \text{Area of Triangle } B$$

and computing these areas we find

$$\frac{\cos(\theta)\sin(\theta)}{2} \leq \frac{\theta}{2} \leq \frac{\tan(\theta)}{2}.$$

Multiplying through by 2*, and recalling that* $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ *we obtain*

$$\cos(\theta)\sin(\theta) \le \theta \le \frac{\sin(\theta)}{\cos(\theta)}.$$

Dividing through by $\sin(\theta)$ *and taking the reciprocals (reversing the inequalities), we find*

$$\cos(\theta) \le \frac{\sin(\theta)}{\theta} \le \frac{1}{\cos(\theta)}.$$

Note, $\cos(-\theta) = \cos(\theta)$ *and* $\dfrac{\sin(-\theta)}{-\theta} = \dfrac{\sin(\theta)}{\theta}$*, so these inequalities hold for all* $\theta \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$*. Additionally, we know*

$$\lim_{\theta\to 0}\cos(\theta) = \underset{\text{given}}{\boxed{1}} = \lim_{\theta\to 0}\frac{1}{\cos(\theta)},$$

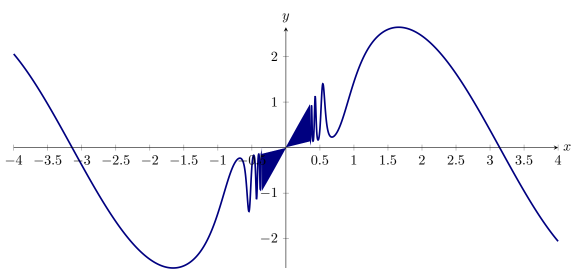
and so we conclude by the Squeeze Theorem, $\displaystyle\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = \underset{\text{given}}{\boxed{1}}$.

When solving a problem with the Squeeze Theorem, one must write a sort of mathematical poem. You have to tell your friendly reader exactly which functions you are using to "squeeze-out" your limit.

Example 3. *Compute:*

$$\lim_{x\to 0}\left(\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}\right)$$

Explanation. *Let's graph this function to see what's going on:*



The function $\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}$ *has two factors:*

$$\overbrace{\sin(x)}^{\textit{goes to zero as } x\to 0}\cdot\underbrace{e^{\cos\left(\frac{1}{x^3}\right)}}_{\textit{bounded between } e^{-1} \textit{ and } e}$$

Hence we have that when $x > 0$

$$\sin(x)\underbrace{\boxed{e^{-1}}}_{\text{given}} \leq \sin(x)e^{\cos\left(\frac{1}{x^3}\right)} \leq \sin(x)\underbrace{\boxed{e}}_{\text{given}}$$

and we see

$$\lim_{x\to 0^+} \sin(x)\underbrace{\boxed{e^{-1}}}_{\text{given}} = \underbrace{\boxed{0}}_{\text{given}} = \lim_{x\to 0^+} \sin(x)\underbrace{\boxed{e}}_{\text{given}}$$

and so by the Squeeze theorem,

$$\lim_{x\to 0^+}\left(\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}\right) = \underbrace{\boxed{0}}_{\text{given}}.$$

In a similar fashion, when $x < 0$,

$$\sin(x)\underbrace{\boxed{e}}_{\text{given}} \leq \sin(x)e^{\cos\left(\frac{1}{x^3}\right)} \leq \sin(x)\underbrace{\boxed{e^{-1}}}_{\text{given}}$$

and so

$$\lim_{x\to 0^-} \sin(x)\underbrace{\boxed{e}}_{\text{given}} = \underbrace{\boxed{0}}_{\text{given}} = \lim_{x\to 0^-} \sin(x)\underbrace{\boxed{e^{-1}}}_{\text{given}},$$

and again by the Squeeze Theorem $\lim_{x\to 0^-}\left(\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}\right) = 0$. *Hence we see that*

$$\lim_{x\to 0}\left(\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}\right) = \underbrace{\boxed{0}}_{\text{given}}.$$