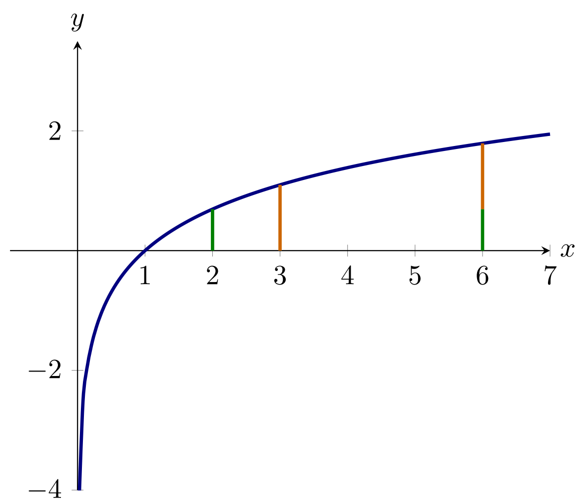
Dig-In:

Logarithmic differentiation

Logarithms were originally developed as a computational tool. The key fact that made this possible is that:

$$\log_b(xy) = \log_b(x) + \log_b(y).$$



Before the days of calculators and computers, this was critical knowledge for anyone in a computational discipline.

Example 1. *Compute* $138 \cdot 23.4$ *using logarithms.*

Explanation. *Start by writing both numbers in scientific notation*

$$\left(1.38 \cdot 10^{\overset{\boxed{2}}{\text{given}}}\right) \cdot \left(2.34 \cdot 10^{\overset{\boxed{1}}{\text{given}}}\right).$$

Next we use a log-table, which gives $\log_{10}(N)$ *for values of* N *ranging between* 0 *and* 9*. We've reproduced part of such a table below.*

N	0	1	2	3	4	5	6	7	8	9
1.3	0.1139	0.1173	0.1206	0.1239	0.1271	0.1303	0.1335	0.1367	0.1399	0.1430
2.3	0.3617	0.3636	0.3655	0.3674	0.3692	0.3711	0.3729	0.3747	0.3766	0.3784
3.2	0.5052	0.5065	0.5079	0.5092	0.5105	0.5119	0.5132	0.5145	0.5159	0.5172

Learning outcomes:

From the table, we see that

$$\log_{10}(1.38) \approx \underbrace{\boxed{0.1399}}_{\text{given}} \qquad \textit{and} \qquad \log_{10}(2.34) \approx \underbrace{\boxed{0.3692}}_{\text{given}}$$

Add these numbers together to get $\underbrace{\boxed{0.5091}}_{\text{given}}$*. Essentially, we know the following at this point:*

$$\begin{array}{ccccc} \log_{10}(?) & = & \log_{10}(1.38) & + & \log_{10}(2.34) \\ \wr\wr & & \wr\wr & & \wr\wr \\ 0.5091 & = & 0.1399 & + & 0.3692 \end{array}$$

Using the table again, we see that $\log_{10}(\underbrace{\boxed{3.23}}_{\text{given}}) \approx 0.5091$*. Since we were working in scientific notation, we need to multiply this by* 10^3*. Our final answer is*

$$3230 \approx 138 \cdot 23.4$$

Since $138 \cdot 23.4 = 3229.2$*, this is a good approximation.*

The moral is:

> **Logarithms allow us to use addition in place of multiplication.**

Logarithmic differentiation

When taking derivatives, both the product rule and the quotient rule can be cumbersome to use. Logarithms will save the day. A key point is the following

$$\frac{d}{dx} \ln(f(x)) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

which follows from the chain rule. Let's look at an illustrative example to see how this is actually used.

Example 2. *Compute:*

$$\frac{d}{dx} \frac{x^9 e^{4x}}{\sqrt{x^2+4}}$$

Recall the properties of logarithms:

- $\log_b(xy) = \log_b(x) + \log_b(y)$

- $\log_b(x/y) = \log_b(x) - \log_b(y)$
- $\log_b(x^y) = y\log_b(x)$

While we could use the product and quotient rule to solve this problem, it would be tedious. Start by taking the logarithm of the function to be differentiated.

$$\begin{aligned}\ln\left(\frac{x^9 e^{4x}}{\sqrt{x^2+4}}\right) &= \ln\Bigg(\underbrace{\boxed{x^9 e^{4x}}}_{\text{given}}\Bigg) - \ln\Bigg(\underbrace{\boxed{\sqrt{x^2+4}}}_{\text{given}}\Bigg)\\ &= \ln\left(x^9\right) + \ln\left(e^{4x}\right) - \ln\left((x^2+4)^{1/2}\right)\\ &= \underbrace{\boxed{9}}_{\text{given}}\ln(x) + 4x - \underbrace{\boxed{\frac{1}{2}}}_{\text{given}}\ln(x^2+4).\end{aligned}$$

Setting $f(x) = \dfrac{x^9 e^{4x}}{\sqrt{x^2+4}}$*, we can write*

$$\ln(f(x)) = 9\ln(x) + 4x - \frac{1}{2}\ln(x^2+4).$$

Differentiating both sides, we find

$$\frac{f'(x)}{f(x)} = \underbrace{\boxed{\frac{9}{x}+4}}_{\text{given}} - \frac{x}{x^2+4}.$$

Finally we solve for $f'(x)$*, write*

$$f'(x) = \left(\frac{9}{x} + 4 - \frac{x}{x^2+4}\right)\Bigg(\underbrace{\boxed{\frac{x^9 e^{4x}}{\sqrt{x^2+4}}}}_{\text{given}}\Bigg).$$

The process above is called *logarithmic differentiation.* Logarithmic differentiation allows us to compute new derivatives too.

Example 3. *Compute:*

$$\frac{d}{dx}x^x$$

Explanation. *The function* x^x *is tricky to differentiate. We cannot use the power rule, as the exponent is not a constant. However, if we set* $f(x) = x^x$ *we can write*

$$\begin{aligned}\ln(f(x)) &= \ln\left(x^x\right)\\ &= x\ln(x).\end{aligned}$$

Differentiating both sides, we find

$$\frac{f'(x)}{f(x)} = \underbrace{\boxed{1 + \ln(x)}}_{\text{given}}.$$

Now we can solve for $f'(x)$,

$$f'(x) = x^x + x^x \ln(x).$$

A general explanation of the power rule

Finally, recall that previously we only explained the power rule for positive exponents. Now we'll use logarithmic differentiation to give a explanation for all real-valued exponents. We restate the power rule for convenience sake:

Theorem 1 (Power Rule)**.** *For any real number n,*

$$\frac{d}{dx} x^n = nx^{n-1}.$$

Explanation. *We will use logarithmic differentiation. Set $f(x) = x^n$. Write*

$$\begin{aligned} \ln(f(x)) &= \ln\left(x^n\right) \\ &= n \ln(x). \end{aligned}$$

Now differentiate both sides, and solve for $f'(x)$

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{n}{x} \\ f'(x) &= \frac{nf(x)}{x} \\ &= \underbrace{\boxed{nx^{n-1}}}_{\text{given}}. \end{aligned}$$

Thus we see that the power rule holds for all real-valued exponents.

While logarithmic differentiation might seem strange and new at first, with a little practice it will seem much more natural to you.