Dig-In:

Applications of integrals

We give more contexts to understand integrals.

Velocity and displacement, speed and distance

Some values include "direction" that is relative to some fixed point.

Definition 1.

- *$v(t)$ is the **velocity** of an object at time t. This represents the "change in position" at time t.*
- *$s(t)$ is the **position** of an object at time t. This gives location with respect to the origin. If we can assume that $s(a) = 0$, then*

$$s(t) = \int_a^t v(x)\,dx.$$

- *$s(b) - s(a)$ is the **displacement**, the distance between the starting and finishing locations.*

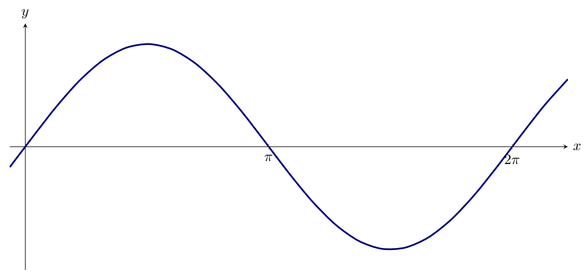
On the other hand *speed* and *distance* are values without "direction."

Definition 2.

- *$|v(t)|$ is the **speed**.*
- *$\int_a^b |v(t)|\,dt$ is the **distance** traveled.*

Question 1 *Consider a particle whose velocity at time t is given by $v(t) = \sin(t)$.*

Learning outcomes: Define the average value of a function. Find the average value of a function. State the Mean Value Theorem for integrals. Use the Mean Value Theorem for integrals.



What is the displacement of the particle from $t = 0$ to $t = \pi$? That is, compute:

$$\int_0^{\pi} \sin(t)\, dt = \underset{\text{given}}{\boxed{2}}$$

What is the displacement of the particle from $t = 0$ to $t = 2\pi$? That is, compute:

$$\int_0^{2\pi} \sin(t)\, dt = \underset{\text{given}}{\boxed{0}}$$

What is the distance traveled by the particle from $t = 0$ to $t = \pi$? That is, compute:

$$\int_0^{\pi} |\sin(t)|\, dt = \underset{\text{given}}{\boxed{2}}$$

What is the distance traveled by the particle from $t = 0$ to $t = 2\pi$? That is, compute:

$$\int_0^{2\pi} |\sin(t)|\, dt = \underset{\text{given}}{\boxed{4}}$$

Average value

Conceptualizing definite integrals as "signed area" works great as long as one can actually visualize the "area." In some cases, a better metaphor for integrals comes from the idea of *average value*. Looking back to your days as an even younger mathematician, you may recall the idea of an *average*:

$$\frac{f_1 + f_2 + \cdots + f_n}{n} = \frac{1}{n}\sum_{k=1}^{n} f_i$$

If we want to know the average value of a function, a naive approach might be to partition the interval $[a, b]$ into n equally spaced subintervals,

$$a = x_0 < x_1 < \cdots < x_n = b$$

and choose any x_k^* in $[x_i, x_{i+1}]$. The average of $f(x_1^*)$, $f(x_2^*)$, ..., $f(x_n^*)$ is:

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n} = \frac{1}{n}\sum_{k=1}^{n} f(x_k^*).$$

Multiply this last expression by $1 = \dfrac{(b-a)}{(b-a)}$:

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} f(x_k^*)\frac{(b-a)}{(b-a)} &= \sum_{k=1}^{n} f(x_k^*)\frac{1}{n}\frac{(b-a)}{(b-a)}\\ &= \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\frac{b-a}{n}\\ &= \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x\end{aligned}$$

where $\Delta x = (b-a)/n$. Ah! On the right we have a Riemann Sum! Now take the limit as $n \to \infty$:

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

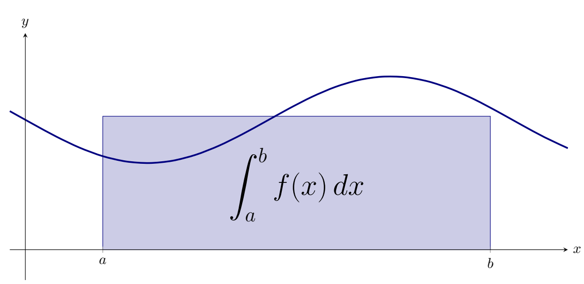
This leads us to our next definition:

Definition 3. *Let f be continuous on $[a,b]$. The **average value** of f on $[a,b]$ is given by*

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

The average value of a function gives the height of a single rectangle whose area is equal to

$$\int_a^b f(x)\,dx$$



An application of this definition is given in the next example.

Example 1. *An object moves back and forth along a straight line with a velocity given by $v(t) = (t-1)^2$ on $[0,3]$, where t is measured in seconds and $v(t)$ is measured in ft/s.*

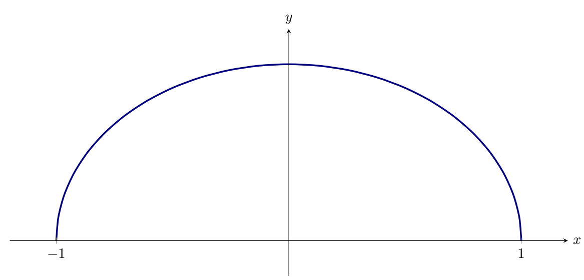
What is the average velocity of the object?

Explanation. *By our definition, the average velocity is:*

$$\begin{aligned}\frac{1}{3-0}\int_0^3 (t-1)^2\,dt &= \frac{1}{3}\int_0^3 t^2 - 2t + 1\,dt \\ &= \frac{1}{3}\left[\frac{t^3}{3} - t^2 + t\right]_0^3 \\ &= 1 \textit{ ft/s}.\end{aligned}$$

When we take the average of a finite set of values, it does not matter how we order those values. When we are taking the average value of a function, however, we need to be more careful.

For instance, there are at least two different ways to make sense of a vague phrase like "The average height of a point on the unit semi circle"



One way we can make sense of "The average height of a point on the unit semi circle" is to compute the average value of the function

$$f(x) = \sqrt{1-x^2}$$

on the interval $[-1, 1]$.

Example 2. *Compute the average value of the function*

$$f(x) = \sqrt{1-x^2}$$

on the interval $[-1, 1]$.

Explanation. *By definition, we wish to compute*

$$\frac{1}{2}\int_{-1}^{1} \sqrt{1-x^2}\,dx.$$

Computing this integral geometrically, we find the average value to be $\underset{\text{given}}{\boxed{\frac{\pi}{4}}}$.

Another way we can make sense of "The average height of a point on the unit semi circle" is the average value of the function

$$g(\theta) = \sin(\theta)$$

on $[0, \pi]$, since $\sin(\theta)$ is the height of the point on the unit circle at the angle θ.

Example 3. *Compute the average value of the function*

$$g(\theta) = \sin(\theta)$$

on the interval $[0, \pi]$.

Explanation. *By definition, we wish to compute*

$$\frac{1}{\pi}\int_0^\pi \sin(\theta)\, d\theta.$$

Computing this integral geometrically, we find the average value to be $\underbrace{\boxed{\frac{2}{\pi}}}_{\text{given}}$.

See if you can understand intuitively why the average using f should be larger than the average using g.

Mean value theorem for integrals

Just as we have a Mean Value Theorem for Derivatives, we also have a Mean Value Theorem for Integrals.

Theorem 1 (The Mean Value Theorem for integrals)**.** *Let f be continuous on $[a, b]$. There exists a value c in $[a, b]$ such that*

$$\int_a^b f(x)\, dx = f(c)(b - a).$$

This is an *existential* statement. The Mean Value Theorem for Integrals tells us:

> **The average value of a continuous function is in the range of the function.**

We demonstrate the principles involved in this version of the Mean Value Theorem in the following example.

Example 4. *Consider* $\displaystyle\int_0^\pi \sin x\, dx$. *Find a value c guaranteed by the Mean Value Theorem.*

Explanation. *We first need to evaluate* $\int_0^\pi \sin x \, dx$.

$$\int_0^\pi \sin x \, dx = \Big[-\cos x \Big]_0^\pi = 2.$$

Thus we seek a value c *in* $[0, \pi]$ *such that* $\pi \sin c = 2$.

$$\pi \sin c = 2 \quad \Rightarrow \quad \sin c = 2/\pi \quad \Rightarrow \quad c = \arcsin(2/\pi) \approx 0.69.$$

A graph of $\sin x$ *is sketched along with a rectangle with height* $\sin(0.69)$ *are pictured below. The area of the rectangle is the same as the area under* $\sin x$ *on* $[0, \pi]$.