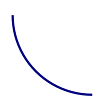
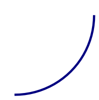
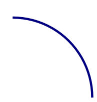
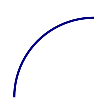
Dig-In

Concavity and the Second Derivative Test

We use second derivatives to help locate extrema.

Concavity

We know that the sign of the derivative tells us whether a function is increasing or decreasing at some point. Likewise, the sign of the second derivative $f''(x)$ tells us whether $f'(x)$ is increasing or decreasing at x. We summarize the consequences of this idea in the table below:

	$f'(x) < 0$	$0 < f'(x)$
$0 < f''(x)$	 Here $y = f(x)$ is decreasing, while the rate itself is increasing. In this case the curve is **concave up**.	 Here $y = f(x)$ is increasing, while the rate itself is increasing. In this case the curve is **concave up**.
$f''(x) < 0$	 Here $y = f(x)$ is decreasing, while the rate itself is decreasing. In this case the curve is **concave down**.	 Here $y = f(x)$ is increasing, while the rate itself is decreasing. In this case the curve is **concave down**.

If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. It is worth summarizing what we have seen already in to a single theorem.

Theorem 1 (Test for Concavity)**.** *Suppose that $f''(x)$ exists on an interval.*

(a) $f''(x) > 0$ *on that interval whenever* $y = f(x)$ *is concave up on that interval.*

Learning outcomes: State the Second Derivative Test. Apply the Second Derivative Test. Define inflection points. Find inflection points.

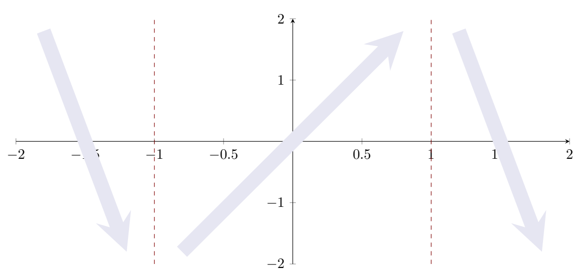
(b) *$f''(x) < 0$ on that interval whenever $y = f(x)$ is concave down on that interval.*

Example 1. *Let f be a continuous function and suppose that:*

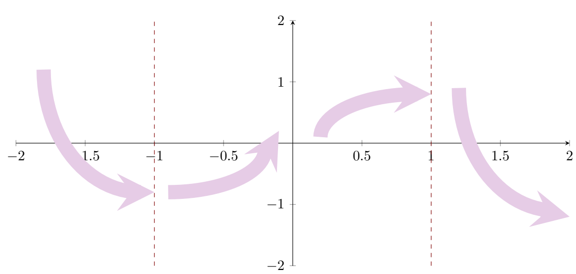
- $f'(x) > 0$ *for* $-1 < x < 1$.
- $f'(x) < 0$ *for* $-2 < x < -1$ *and* $1 < x < 2$.
- $f''(x) > 0$ *for* $-2 < x < 0$ *and* $1 < x < 2$.
- $f''(x) < 0$ *for* $0 < x < 1$.

Sketch a possible graph of f.

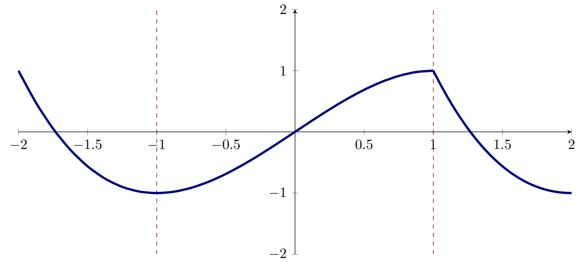
Explanation. *Start by marking where the derivative changes sign and indicate intervals where f is increasing and intervals f is decreasing. The function f has a negative derivative from -2 to $x = \underbrace{\boxed{-1}}_{\text{given}}$. This means that f is* (increasing/ decreasing $\checkmark$) *on this interval. The function f has a positive derivative from $x = \underbrace{\boxed{-1}}_{\text{given}}$ to $x = \underbrace{\boxed{1}}_{\text{given}}$. This means that f is* (increasing $\checkmark$ / decreasing) *on this interval. Finally, The function f has a negative derivative from $x = \underbrace{\boxed{1}}_{\text{given}}$ to 2. This means that f is* (increasing/ decreasing $\checkmark$) *on this interval.*



Now we should sketch the concavity: (concave up $\checkmark$ / concave down) *when the second derivative is positive,* (concave up/ concave down $\checkmark$) *when the second derivative is negative.*



Finally, we can sketch our curve:



Inflection points

If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. It is worth summarizing what we have seen already in to a single theorem.

Theorem 2 (Test for Concavity)**.** *Suppose that $f''(x)$ exists on an interval.*

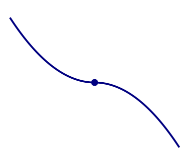
(a) *If $f''(x) > 0$ on an interval, then f is concave up on that interval.*

(b) *If $f''(x) < 0$ on an interval, then f is concave down on that interval.*

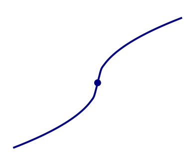
Of particular interest are points at which the concavity changes from up to down or down to up.

Definition 1. *If f is continuous and its concavity changes either from up to down or down to up at $x = a$, then f has an* ***inflection point*** *at $x = a$.*

It is instructive to see some examples of inflection points:

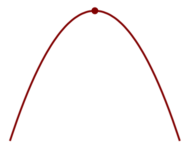


This is an inflection point. The concavity changes from concave up to concave down.

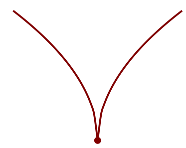


This is an inflection point. The concavity changes from concave up to concave down.

It is also instructive to see some nonexamples of inflection points:



This is **not** an inflection point. The curve is concave down on either side of the point.



This is **not** an inflection point. The curve is concave down on either side of the point.

We identify inflection points by first finding x such that $f''(x)$ is zero or undefined and then checking to see whether $f''(x)$ does in fact go from positive to negative or negative to positive at these points.

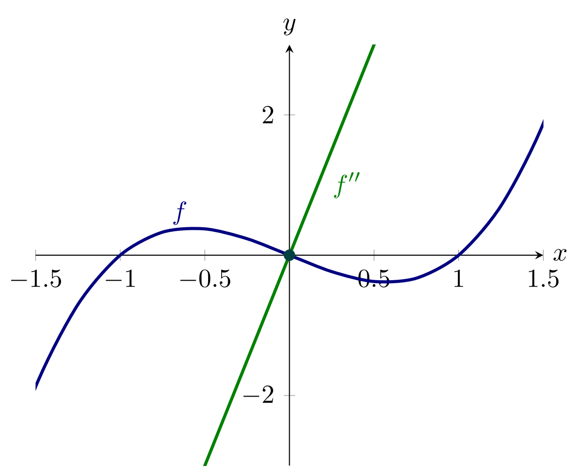
Warning 1. *Even if $f''(a) = 0$, the point determined by $x = a$ might **not** be an inflection point.*

Example 2. *Describe the concavity of $f(x) = x^3 - x$.*

Explanation. *To start, compute the first and second derivative of $f(x)$ with respect to x,*

$$f'(x) = \underbrace{\boxed{3x^2 - 1}}_{\text{given}} \qquad \textit{and} \qquad f''(x) = \underbrace{\boxed{6x}}_{\text{given}}.$$

Since $f''(0) = 0$, there is potentially an inflection point at $x = 0$. Using test points, we note the concavity does change from down to up, hence there is an inflection point at $x = 0$. The curve is concave down for all $x < 0$ and concave up for all $x > 0$, see the graphs of $f(x) = x^3 - x$ and $f''(x) = 6x$.



Note that we need to compute and analyze the second derivative to understand concavity, so we may as well try to use the second derivative test for maxima and minima. If for some reason this fails we can then try one of the other tests.

The second derivative test

Recall the first derivative test:

- If $f'(x) > 0$ to the left of a and $f'(x) < 0$ to the right of a, then $f(a)$ is a local maximum.
- If $f'(x) < 0$ to the left of a and $f'(x) > 0$ to the right of a, then $f(a)$ is a local minimum.

If f' changes from positive to negative it is decreasing. In this case, f'' might be negative, and if in fact f'' is negative then f' is definitely decreasing, so there is a local maximum at the point in question. On the other hand, if f' changes from negative to positive it is increasing. Again, this means that f'' might be positive, and if in fact f'' is positive then f' is definitely increasing, so there is a local minimum at the point in question. We summarize this as the *second derivative test*.

Theorem 3 (Second Derivative Test)**.** *Suppose that $f''(x)$ is continuous on an open interval and that $f'(a) = 0$ for some value of a in that interval.*

- *If $f''(a) < 0$, then f has a local maximum at a.*
- *If $f''(a) > 0$, then f has a local minimum at a.*
- *If $f''(a) = 0$, then the test is inconclusive. In this case, f may or may not have a local extremum at $x = a$.*

The second derivative test is often the easiest way to identify local maximum and minimum points. Sometimes the test fails and sometimes the second derivative is quite difficult to evaluate. In such cases we must fall back on one of the previous tests.

Example 3. *Once again, consider the function*

$$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$$

Use the second derivative test, to locate the local extrema of f.

Explanation. *Start by computing*

$$f'(x) = \underbrace{\boxed{x^3 + x^2 - 2x}}_{\text{given}} \qquad \textit{and} \qquad f''(x) = \underbrace{\boxed{3x^2 + 2x - 2}}_{\text{given}}.$$

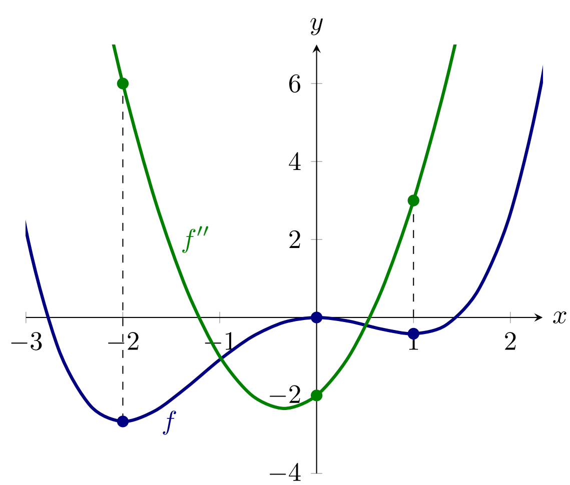
Using the same technique as we used before, we find that

$$f'(-2) = \underbrace{\boxed{0}}_{\text{given}}, \qquad f'(0) = \underbrace{\boxed{0}}_{\text{given}}, \qquad f'(1) = \underbrace{\boxed{0}}_{\text{given}}.$$

Now we'll attempt to use the second derivative test,

$$f''(-2) = \underbrace{\boxed{6}}_{\text{given}}, \qquad f''(0) = \underbrace{\boxed{-2}}_{\text{given}}, \qquad f''(1) = \underbrace{\boxed{3}}_{\text{given}}.$$

Hence we see that f has a local minimum at $x = -2$, a local maximum at $x = 0$, and a local minimum at $x = 1$, see below for a plot of $f(x) = x^4/4 + x^3/3 - x^2$ and $f''(x) = 3x^2 + 2x - 2$:



Problem 1 *If $f''(a) = 0$, what does the second derivative test tell us?*

Multiple Choice:

(a) *The function has a local extrema at $x = a$.*

(b) *The function does not have a local extrema at $x = a$.*

(c) *It gives no information on whether $x = a$ is a local extremum.* ✓