**Dig-In:**

# The definition of the derivative

*We compute the instantaneous growth rate by computing the limit of average growth rates.*

Given a function, it is often useful to know the rate at which the function changes. To give you a feeling why this is true, consider the following:

- If $s(t)$ represents the **displacement** (position relative to an origin) of an object with respect to time, the rate of change gives the **velocity** of the object.
- If $v(t)$ represents the **velocity** of an object with respect to time, the rate of change gives the **acceleration** of the object.
- If $R(x)$ represents the revenue generated by selling $x$ objects, the rate of change gives us the **marginal revenue**, meaning the additional revenue generated by selling one additional unit. Note, there is an implicit assumption that $x$ is quite large compared to 1.
- If $C(x)$ represents the cost to produce $x$ objects, the rate of change gives us the **marginal cost**, meaning the additional cost generated by selling one additional unit. Again, there is an implicit assumption that $x$ is quite large compared to 1.
- If $P(x)$ represents the profit gained by selling $x$ objects, the rate of change gives us the **marginal profit**, meaning the additional cost generated by selling one additional unit. Again, there is an implicit assumption that $x$ is quite large compared to 1.
- The rate of change of a function can help us approximate a complicated function with a simple function.
- The rate of change of a function can be used to help us solve equations that we would not be able to solve via other methods.

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Learning outcomes: Use limits to find the slope of the tangent line at a point. Understand the definition of the derivative at a point. Compute the derivative of a function at a point. Estimate the slope of the tangent line graphically. Write the equation of the tangent line to a graph at a given point. Recognize and distinguish between secant and tangent lines. Recognize different notation for the derivative. Recognize the the tangent line as a local approximation for a differentiable function.

# From slopes of secant lines to slopes of tangent lines

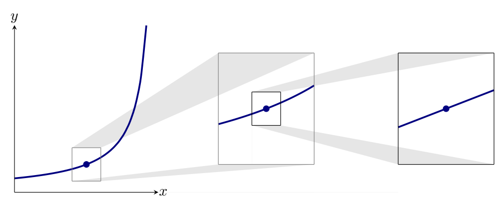
You've been computing average rates of change for a while now, the computation is simply

$$\frac{\text{change in the function}}{\text{change in the input to the function}}.$$

However, the question remains: Given a function that represents an amount, how exactly does one find the function that will give the instantaneous rate of change? Recall that the instantaneous rate of change of a line is the slope of the line. Hence the instantaneous rate of change of a function is the slope of the tangent line. For now, consider the following informal definition of a *tangent line*:

> Given a function $f$ and a number $a$ in the domain of $f$, if one can "zoom in" on the graph at $(a, f(a))$ sufficiently so that it appears to be a straight line, then that line is the **tangent line** to $f(x)$ at the point $(a, f(a))$.

We illustrate this informal definition with the following diagram:




The *derivative* of a function $f$ at $a$, is the instantaneous rate of change, and hence is the slope of the tangent line at $(a, f(a))$.

**Question 1** *What is the instantaneous rate of change of $f(x) = 4x - 3$?*

***Hint:*** *The rate of change is the slope of the tangent line.*

***Hint:*** *The line tangent to $f(x) = 4x - 3$, is simply itself!*

*The derivative is* $\boxed{4}$.

Unfortunately, if $f$ is not a straight line we cannot use the slope formula to calculate this rate of change, since $(a, f(a))$ is the only point on this line that we know. In order to deal with this problem, we consider **secant** lines, lines that locally intersect the curve at two points. One of these points will be $(a, f(a))$, the point at which we are trying to find the rate of change. If we call $h$ the difference between the $x$-coordinates of the two points, then the second point for our secant line is $(a+h, f(a+h))$. The slope of any secant line that passes through the points $(a, f(a))$ and $(a+h, f(a+h))$ is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$

**Example 1.** *If $f(x) = x^2 - 2x$, find the slope of the secant line through $x = 2$ and $x = 2 + h$, in terms of $h$.*

**Explanation.** *Start with the slope formula we just found,*

$$\frac{\Delta y}{\Delta x} = \frac{f(\underbrace{\boxed{2+h}}_{\text{given}}) - f(2)}{\underbrace{\boxed{h}}_{\text{given}}}.$$

*Now substitute in for the function we know,*

$$\frac{\Delta y}{\Delta x} = \frac{\underbrace{\boxed{(2+h)^2 - 2(2+h)}}_{\text{given}} - 0}{h}.$$

*Now expand the numerator of the fraction,*

$$\frac{\Delta y}{\Delta x} = \frac{4 + 4h + h^2 - 4 - 2h}{h}.$$

*Now combine like-terms,*

$$\frac{\Delta y}{\Delta x} = \frac{2h + h^2}{h}.$$

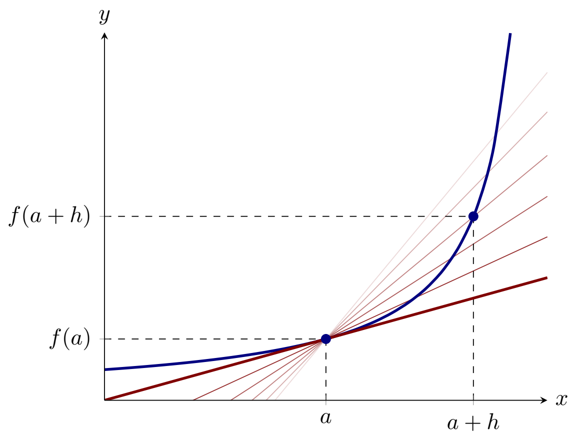
*Factor an $h$ from every term in the numerator,*

$$\frac{\Delta y}{\Delta x} = \frac{h(\underbrace{\boxed{2+h}}_{\text{given}})}{h}.$$

*Cancel $h$ from the numerator and denominator,*

$$\frac{\Delta y}{\Delta x} = \underbrace{\boxed{2+h}}_{\text{given}}.$$

The following diagram shows the secant lines for several values of $h$, as well as the tangent line at $(a, f(a))$.




Notice that as $a+h$ approaches $a$, the slopes of the secant lines are approaching the slope of the tangent line. This leads to the *definition of the derivative*:

**Definition 1.** *The **derivative** of $f$ at $a$ is*

$$\left[\frac{d}{dx}f(x)\right]_{x=a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

*If this limit exists, then we say that $f$ is **differentiable** at $a$. If this limit does not exist for a given value of $a$, then $f$ is **non-differentiable** at $a$.*

**Question 2** *Which of the following computes the derivative, $\left[\frac{d}{dx}f(x)\right]_{x=a}$?*

***Select All Correct Answers:***

(a) $\lim_{h\to 0}\frac{(f(a)+h)-f(a)}{(a+h)-a}$

(b) $\lim_{h\to 0}\frac{f(a+h)-f(a)}{(a+h)-a}$ ✓

(c) $\lim_{h\to 0}\frac{(f(a)-h)-f(a)}{(a-h)-a}$

(d) $\lim_{h\to 0}\frac{f(a-h)-f(a)}{(a-h)-a}$ ✓

(e) $\lim_{h\to 0}\frac{f(a)-(f(a)+h)}{a-(a+h)}$

(f) $\lim_{h\to 0}\frac{f(a)-f(a+h)}{a-(a+h)}$ ✓

(g) $\displaystyle\lim_{h\to 0} \frac{f(a)-(f(a)-h)}{a-(a-h)}$

(h) $\displaystyle\lim_{h\to 0} \frac{f(a)-f(a-h)}{a-(a-h)}$ ✓

**Definition 2.** *There are several different notations for the derivative. The two we'll mainly be using are*

$$\left[\frac{d}{dx} f(x)\right]_{x=a} = f'(a).$$

Now we will give a number of examples.

**Example 2.** *If $f(x) = x^2 - 2x$, find the derivative of $f$ at 2.*

**Explanation.** *Start with the definition of the derivative,*

$$\left[\frac{d}{dx} f(x)\right]_{x=2} = \lim_{h\to 0} \frac{f(\underbrace{\boxed{2+h}}_{\text{given}}) - f(2)}{h}.$$

*Now substitute in for the function we know,*

$$\left[\frac{d}{dx} f(x)\right]_{x=2} = \lim_{h\to 0} \frac{\underbrace{\boxed{(2+h)^2 - 2(2+h)}}_{\text{given}} - 0}{h}.$$

*Now expand the numerator of the fraction,*

$$\left[\frac{d}{dx} f(x)\right]_{x=2} = \lim_{h\to 0} = \frac{4+4h+h^2-4-2h}{h}.$$

*Now combine like-terms,*

$$\left[\frac{d}{dx} f(x)\right]_{x=2} = \lim_{h\to 0} \frac{2h+h^2}{h}.$$

*Factor an $h$ from every term in the numerator,*

$$\left[\frac{d}{dx} f(x)\right]_{x=2} = \lim_{h\to 0} \frac{h(\underbrace{\boxed{2+h}}_{\text{given}})}{h}.$$

*Cancel $h$ from the numerator and denominator,*

$$\left[\frac{d}{dx} f(x)\right]_{x=2} = \lim_{h\to 0} \underbrace{\boxed{2+h}}_{\text{given}}.$$

*Take the limit as $h$ goes to 0,*

$$\left[\frac{d}{dx} f(x)\right]_{x=2} = \underbrace{\boxed{2}}_{\text{given}}.$$

**Example 3.** *Find an equation for the line tangent to* $f(x) = \dfrac{1}{3-x}$ *at the point* $(2,1)$*.*

**Explanation.** *To find an equation for a line, we need two pieces of information. We need to know a point on the line, and we need to know the slope. In this question, we are given that* $(2,1)$ *is on the line. That means we need to find the slope of the tangent line. Finding the slope of the tangent line at the point* $(2,1)$ *means finding* $f'(2)$*.*

*Start by writing out the definition of the derivative,*

$$f'(2) = \lim_{h\to 0} \frac{f(\underbrace{\boxed{2+h}}_{\text{given}}) - f(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{1-h} - 1}{h}.$$

*Multiply by* $\dfrac{1-h}{1-h}$ *to clear the fraction in the numerator,*

$$f'(2) = \lim_{h\to 0} \frac{\underbrace{\boxed{1-(1-h)}}_{\text{given}}}{h(1-h)}.$$

*Combine like-terms in the numerator,*

$$f'(2) = \lim_{h\to 0} \frac{h}{h(1-h)},$$

*Cancel* $h$ *from the numerator and denominator,*

$$f'(2) = \lim_{h\to 0} \frac{1}{1-h},$$

*Take the limit as* $h$ *goes to* $0$*,*

$$f'(2) = \underbrace{\boxed{1}}_{\text{given}}.$$

*We are looking for an equation of the line through the point* $(2,1)$ *with slope* $m = f'(2) = 1$*. The point-slope formula tells us that the line has equation given by*

$$y = \underbrace{\boxed{(x-2)+1}}_{\text{given}}.$$

**Example 4.** *An object moving along a straight line has displacement given by* $s(t) = \sqrt{t+3}$*. Find the velocity of the object at time* $t = 6$*.*

**Explanation.** *Velocity is the rate of change of displacement with respect to time. We are being asked to find* $\left[\dfrac{d}{dt}s(t)\right]_{t=6}$*. The definition of the derivative gives*

$$\left[\frac{d}{dt}s(t)\right]_{t=6} = \lim_{h\to 0} \frac{s(\underbrace{\boxed{6+h}}_{\text{given}}) - s(6)}{h} = \lim_{h\to 0} \frac{\sqrt{\underbrace{\boxed{9+h}}_{\text{given}}} - 3}{h}.$$

*Multiply by* $\dfrac{\sqrt{9+h}+3}{\sqrt{9+h}+3}$,

$$\left[\frac{d}{dt}s(t)\right]_{t=6} = \lim_{h\to 0}\left(\frac{\sqrt{9+h}-3}{h}\right)\left(\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}\right).$$

*Now expand the numerator,*

$$\left[\frac{d}{dt}s(t)\right]_{t=6} = \lim_{h\to 0}\frac{\underset{\text{given}}{\boxed{9+h}}-9}{h\left(\sqrt{9+h}+3\right)}.$$

*Combine like-terms,*

$$\left[\frac{d}{dt}s(t)\right]_{t=6} = \lim_{h\to 0}\frac{\underset{\text{given}}{\boxed{h}}}{h\left(\sqrt{9+h}+3\right)}.$$

*Cancel $h$ from the numerator and denominator,*

$$\left[\frac{d}{dt}s(t)\right]_{t=6} = \lim_{h\to 0}\frac{1}{\sqrt{9+h}+3}.$$

*Take the limit as $h$ tends to $0$,*

$$\left[\frac{d}{dt}s(t)\right]_{t=6} = \underset{\text{given}}{\boxed{\frac{1}{6}}}.$$

*The object has velocity* $\underset{\text{given}}{\boxed{\frac{1}{6}}}$ *at time* $t=6$.