Dig-In

Maximums and minimums

We use derivatives to help locate extrema.

Whether we are interested in a function as a purely mathematical object or in connection with some application to the real world, it is often useful to know what the graph of the function looks like. We can obtain a good picture of the graph using certain crucial information provided by derivatives of the function.

Extrema

Local *extrema* on a function are points on the graph where the y-coordinate is larger (or smaller) than all other y-coordinates on the graph at points "close to" (x, y).

Definition 1.

(a) *A function f has a **local maximum** at $x = a$, if $f(a) \geq f(x)$ for every x near a.*

(b) *A function f has a **local minimum** at $x = a$, if $f(a) \leq f(x)$ for every x near a.*

*A **local extremum** is either a local maximum or a local minimum.*

Problem 1 *True or false: "All absolute extrema are also local extrema."*

Multiple Choice:

(a) *true* ✓

(b) *false*

Feedback (attempt): *All global extrema are local extrema.*

Learning outcomes: Define a critical point. Find critical points. Define absolute maximum and absolute minimum. Find the absolute max or min of a continuous function on a closed interval. Define local maximum and local minimum. Compare and contrast local and absolute maxima and minima. Identify situations in which an absolute maximum or minimum is guaranteed. Classify critical points. State the First Derivative Test. Apply the First Derivative Test. State the Second Derivative Test. Apply the Second Derivative Test. Define inflection points. Find inflection points.

Local maximum and minimum points are quite distinctive on the graph of a function, and are therefore useful in understanding the shape of the graph. In many applied problems we want to find the largest or smallest value that a function achieves (for example, we might want to find the minimum cost at which some task can be performed) and so identifying maximum and minimum points will be useful for applied problems as well.

Critical points

If $(x, f(x))$ is a point where f reaches a local maximum or minimum, and if the derivative of f exists at x, then the graph has a tangent line and the tangent line must be horizontal. This is important enough to state as a theorem, though we will not prove it.

Theorem 1 (Fermat's Theorem)**.** *If f has a local extremum at $x = a$ and f is differentiable at a, then $f'(a) = 0$.*

Problem 2 *Does Fermat's Theorem say that if $f'(a) = 0$, then f has a local extrema at $x = a$?*

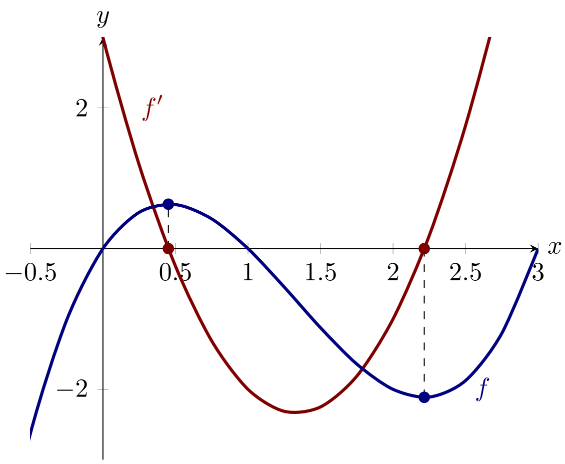
Multiple Choice:

(a) *yes*

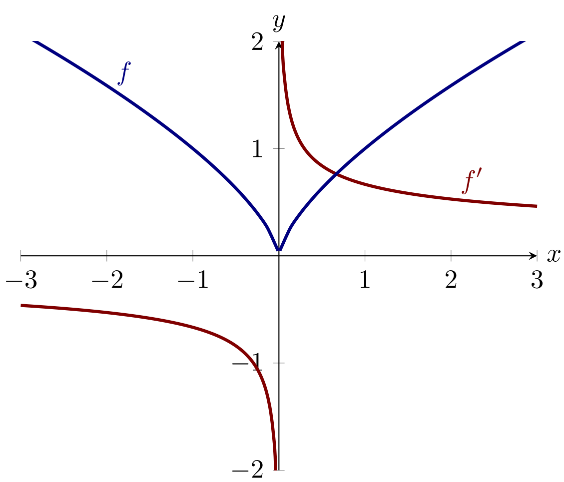
(b) *no* ✓

Feedback (attempt): *Consider $f(x) = x^3$, $f'(0) = 0$, but f does not have a local maximum or minimum at $x = 0$.*

Fermat's Theorem says that the only points at which a function can have a local maximum or minimum are points at which the derivative is zero, consider the plots of $f(x) = x^3 - 4x^2 + 3x$ and $f'(x) = 3x^2 - 8x + 3$,



or the derivative is undefined, as in the plot of $f(x) = x^{2/3}$ and $f'(x) = \dfrac{2}{3x^{1/3}}$:



This brings us to our next definition.

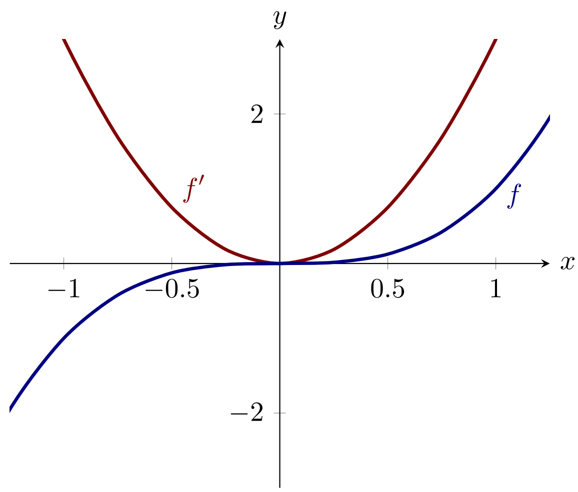
Definition 2. *A function has a* ***critical point*** *at $x = a$ if*

$$f'(a) = 0 \qquad \text{or} \qquad f'(a) \text{ does not exist.}$$

Warning 1. *When looking for local maximum and minimum points, you are likely to make two sorts of mistakes:*

- *You may forget that a maximum or minimum can occur where the derivative does not exist, and so forget to check whether the derivative exists everywhere.*

- *You might assume that any place that the derivative is zero is a local maximum or minimum point, but this is not true, consider the plots of* $f(x) = x^3$ *and* $f'(x) = 3x^2$.



While $f'(0) = 0$*, there is neither a maximum nor minimum at* $(0, f(0))$.

Since the derivative is zero or undefined at both local maximum and local minimum points, we need a way to determine which, if either, actually occurs. The most elementary approach is to test directly whether the y coordinates near the potential maximum or minimum are above or below the y coordinate at the point of interest.

It is not always easy to compute the value of a function at a particular point. The task is made easier by the availability of calculators and computers, but they have their own drawbacks: they do not always allow us to distinguish between values that are very close together. Nevertheless, because this method is conceptually simple and sometimes easy to perform, you should always consider it.

Example 1. *Find all local maximum and minimum points for the function* $f(x) = x^3 - x$.

Explanation. *Write*

$$\frac{d}{dx} f(x) = \underbrace{\boxed{3x^2 - 1}}_{\text{given}}.$$

This is defined everywhere and is zero at $x = \pm\sqrt{3}/3$*. Looking first at* $x = \sqrt{3}/3$*, we see that*

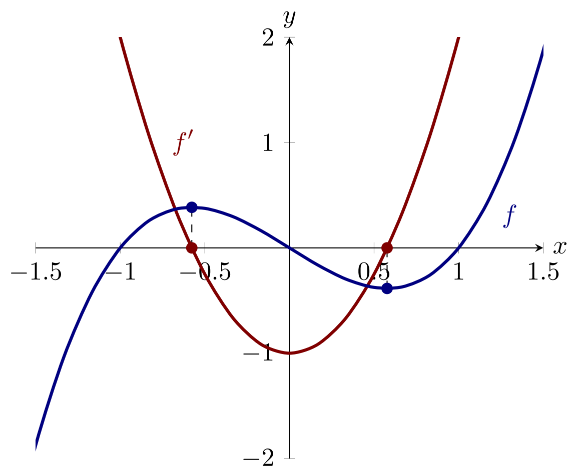
$$f(\sqrt{3}/3) = \underbrace{\boxed{-2\sqrt{3}/9}}_{\text{given}}.$$

Now we test two points on either side of $x = \sqrt{3}/3$*, making sure that neither is farther away than the nearest critical point; since* $\sqrt{3} < 3$*,* $\sqrt{3}/3 < 1$ *and we can use* $x = 0$ *and* $x = 1$*. Since*

$$f(0) = 0 > -2\sqrt{3}/9 \qquad \textit{and} \qquad f(1) = 0 > -2\sqrt{3}/9,$$

there must be a local minimum at $x = \underset{\text{given}}{\boxed{\sqrt{3}/3}}$.

For $x = -\sqrt{3}/3$*, we see that* $f(-\sqrt{3}/3) = 2\sqrt{3}/9$*. This time we can use* $x = 0$ *and* $x = -1$*, and we find that* $f(-1) = f(0) = 0 < 2\sqrt{3}/9$*, so there must be a local maximum at* $x = \underset{\text{given}}{\boxed{-\sqrt{3}/3}}$*, see the plot below:*



The first derivative test

The method of the previous section for deciding whether there is a local maximum or minimum at a critical point by testing "near-by" points is not always convenient. Instead, since we have already had to compute the derivative to find the critical points, we can use information about the derivative to decide. Recall that

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

So how exactly does the derivative tell us whether there is a maximum, minimum, or neither at a point? Use the *first derivative test*.

Theorem 2 (First Derivative Test)**.** *Suppose that f is continuous on an interval, and that $f'(a) = 0$ for some value of a in that interval.*

- *If $f'(x) > 0$ to the left of a and $f'(x) < 0$ to the right of a, then $f(a)$ is a local maximum.*
- *If $f'(x) < 0$ to the left of a and $f'(x) > 0$ to the right of a, then $f(a)$ is a local minimum.*
- *If $f'(x)$ has the same sign to the left and right of a, then $f(a)$ is not a local extremum.*

Example 2. *Consider the function*

$$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$$

Find the intervals on which f is increasing and decreasing and identify the local extrema of f.

Explanation. *Start by computing*

$$\frac{d}{dx} f(x) = \underbrace{\boxed{x^3 + x^2 - 2x}}_{\text{given}}.$$

Now we need to find when this function is positive and when it is negative. To do this, solve

$$f'(x) = \underbrace{\boxed{x^3 + x^2 - 2x}}_{\text{given}} = 0.$$

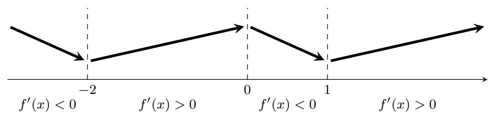
Factor $f'(x)$

$$\begin{aligned} f'(x) &= \underbrace{\boxed{x^3 + x^2 - 2x}}_{\text{given}} \\ &= x(\underbrace{\boxed{x^2 + x - 2}}_{\text{given}}) \\ &= x(x+2)\underbrace{\boxed{(x-1)}}_{\text{given}}. \end{aligned}$$

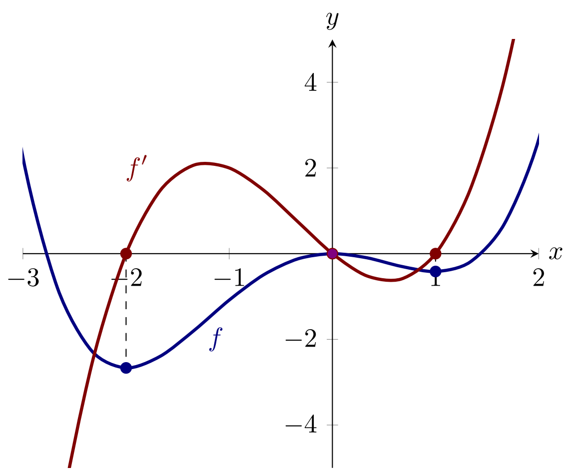
*So the critical points (when $f'(x) = 0$) are when $x = -2$, $x = 0$, and $x = 1$. Now we can check points **between** the critical points to find when $f'(x)$ is increasing and decreasing:*

$$\begin{aligned} f'(-3) &= \underbrace{\boxed{-12}}_{\text{given}}, \\ f'(.5) &= \underbrace{\boxed{-0.625}}_{\text{given}}, \\ f'(-1) &= \underbrace{\boxed{2}}_{\text{given}}, \\ f'(2) &= \underbrace{\boxed{8}}_{\text{given}}. \end{aligned}$$

From this we can make a sign table:



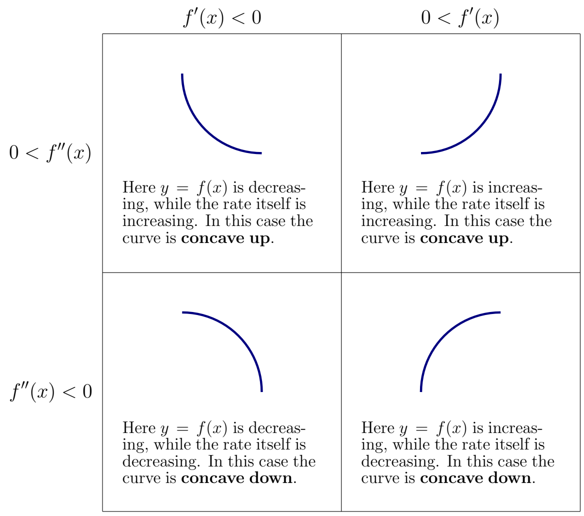
Hence f is increasing on $(-2, 0)$ and $(1, \infty)$ and f is decreasing on $(-\infty, -2)$ and $(0, 1)$. Moreover, from the first derivative test, the local maximum is at $x = 0$ while the local minima are at $x = -2$ and $x = 1$, see the graphs of of $f(x) = x^4/4 + x^3/3 - x^2$ and $f'(x) = x^3 + x^2 - 2x$.



Hence we have seen that if f' is zero and increasing at a point, then f has a local minimum at the point. If f' is zero and decreasing at a point then f has a local maximum at the point. Thus, we see that we can gain information about f by studying how f' changes. This leads us to our next section.

Concavity

We know that the sign of the derivative tells us whether a function is increasing or decreasing at some point. Likewise, the sign of the second derivative $f''(x)$ tells us whether $f'(x)$ is increasing or decreasing at x. We summarize the consequences of this seemingly simple idea in the table below:



If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. It is worth summarizing what we have seen already in to a single theorem.

Theorem 3 (Test for Concavity)**.** *Suppose that $f''(x)$ exists on an interval.*

(a) $f''(x) > 0$ *on that interval whenever* $y = f(x)$ *is concave up on that interval.*

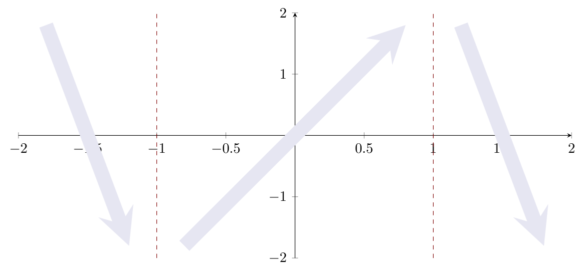
(b) $f''(x) < 0$ *on that interval whenever* $y = f(x)$ *is concave down on that interval.*

Example 3. *Let f be a continuous function and suppose that:*

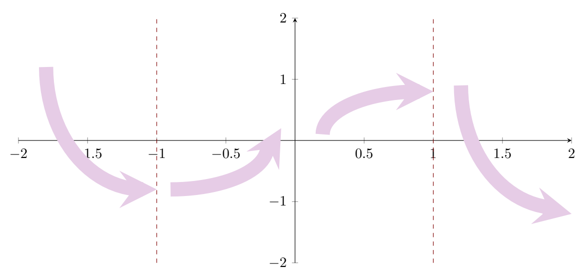
- $f'(x) > 0$ *for* $-1 < x < 1$.
- $f'(x) < 0$ *for* $-2 < x < -1$ *and* $1 < x < 2$.
- $f''(x) > 0$ *for* $-2 < x < 0$ *and* $1 < x < 2$.
- $f''(x) < 0$ *for* $0 < x < 1$.

Sketch a possible graph of f.

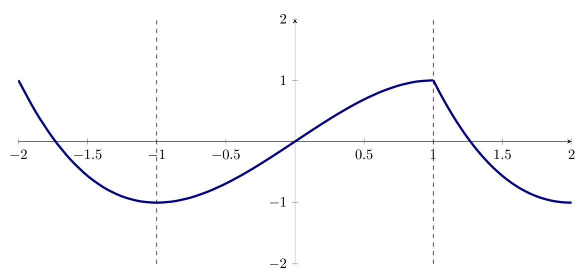
Explanation. *Start by marking where the derivative changes sign and indicate intervals where f is increasing and intervals f is decreasing. The function f has a negative derivative from -2 to $x = \underset{\text{given}}{\boxed{-1}}$. This means that f is* (increasing/ decreasing ✓) *on this interval. The function f has a positive derivative from* $x = \underset{\text{given}}{\boxed{-1}}$ *to* $x = \underset{\text{given}}{\boxed{1}}$. *This means that f is* (increasing ✓ /decreasing) *on this* *interval. Finally, The function f has a negative derivative from* $x = \underset{\text{given}}{\boxed{1}}$ *to* 2. *This means that f is* (increasing/ decreasing ✓) *on this interval.*



Now we should sketch the concavity: (concave up ✓ / concave down) *when the second derivative is positive,* (concave up/ concave down ✓) *when the second derivative is negative.*



Finally, we can sketch our curve:



Inflection points

If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. It is worth summarizing what we have seen already in to a single theorem.

Theorem 4 (Test for Concavity)**.** *Suppose that $f''(x)$ exists on an interval.*

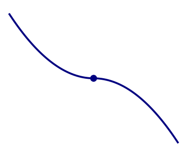
(a) *If $f''(x) > 0$ on an interval, then f is concave up on that interval.*

(b) *If $f''(x) < 0$ on an interval, then f is concave down on that interval.*

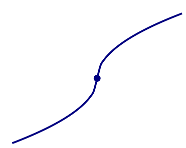
Of particular interest are points at which the concavity changes from up to down or down to up.

Definition 3. *If f is continuous and its concavity changes either from up to down or down to up at $x = a$, then f has an* ***inflection point*** *at $x = a$.*

It is instructive to see some examples of inflection points:

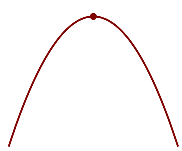


This is an inflection point. The concavity changes from concave up to concave down.

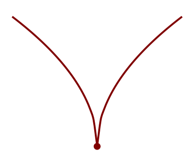


This is an inflection point. The concavity changes from concave up to concave down.

It is also instructive to see some nonexamples of inflection points:



This is **not** an inflection point. The curve is concave down on either side of the point.



This is **not** an inflection point. The curve is concave down on either side of the point.

We identify inflection points by first finding x such that $f''(x)$ is zero or undefined and then checking to see whether $f''(x)$ does in fact go from positive to negative or negative to positive at these points.

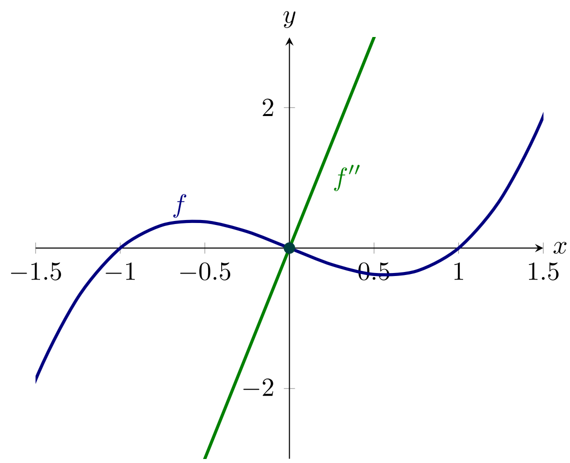
Warning 2. *Even if $f''(a) = 0$, the point determined by $x = a$ might* ***not*** *be an inflection point.*

Example 4. *Describe the concavity of $f(x) = x^3 - x$.*

Explanation. *To start, compute the first and second derivative of $f(x)$ with respect to x,*

$$f'(x) = \underset{\text{given}}{\boxed{3x^2 - 1}} \qquad \textit{and} \qquad f''(x) = \underset{\text{given}}{\boxed{6x}}.$$

Since $f''(0) = 0$, there is potentially an inflection point at $x = 0$. Using test points, we note the concavity does change from down to up, hence there is an inflection point at $x = 0$. The curve is concave down for all $x < 0$ and concave up for all $x > 0$, see the graphs of $f(x) = x^3 - x$ and $f''(x) = 6x$.



Note that we need to compute and analyze the second derivative to understand concavity, so we may as well try to use the second derivative test for maxima and minima. If for some reason this fails we can then try one of the other tests.

The second derivative test

Recall the first derivative test:

- If $f'(x) > 0$ to the left of a and $f'(x) < 0$ to the right of a, then $f(a)$ is a local maximum.
- If $f'(x) < 0$ to the left of a and $f'(x) > 0$ to the right of a, then $f(a)$ is a local minimum.

If f' changes from positive to negative it is decreasing. In this case, f'' might be negative, and if in fact f'' is negative then f' is definitely decreasing, so there is a local maximum at the point in question. On the other hand, if f' changes from negative to positive it is increasing. Again, this means that f'' might be positive, and if in fact f'' is positive then f' is definitely increasing, so there is a local minimum at the point in question. We summarize this as the *second derivative test*.

Theorem 5 (Second Derivative Test)**.** *Suppose that $f''(x)$ is continuous on an open interval and that $f'(a) = 0$ for some value of a in that interval.*

- *If $f''(a) < 0$, then f has a local maximum at a.*
- *If $f''(a) > 0$, then f has a local minimum at a.*
- *If $f''(a) = 0$, then the test is inconclusive. In this case, f may or may not have a local extremum at $x = a$.*

The second derivative test is often the easiest way to identify local maximum and minimum points. Sometimes the test fails and sometimes the second derivative is quite difficult to evaluate. In such cases we must fall back on one of the previous tests.

Example 5. *Once again, consider the function*

$$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$$

Use the second derivative test, to locate the local extrema of f.

Explanation. *Start by computing*

$$f'(x) = \underbrace{\boxed{x^3 + x^2 - 2x}}_{\text{given}} \qquad \text{and} \qquad f''(x) = \underbrace{\boxed{3x^2 + 2x - 2}}_{\text{given}}.$$

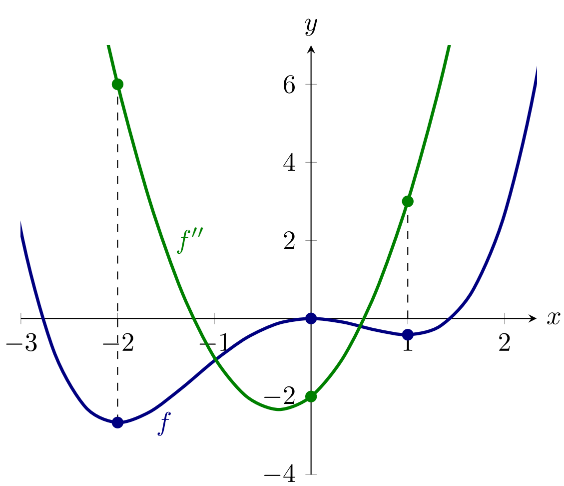
Using the same technique as we used before, we find that

$$f'(-2) = \underbrace{\boxed{0}}_{\text{given}}, \qquad f'(0) = \underbrace{\boxed{0}}_{\text{given}}, \qquad f'(1) = \underbrace{\boxed{0}}_{\text{given}}.$$

Now we'll attempt to use the second derivative test,

$$f''(-2) = \underbrace{\boxed{6}}_{\text{given}}, \qquad f''(0) = \underbrace{\boxed{-2}}_{\text{given}}, \qquad f''(1) = \underbrace{\boxed{3}}_{\text{given}}.$$

Hence we see that f has a local minimum at $x = -2$, a local maximum at $x = 0$, and a local minimum at $x = 1$, see below for a plot of $f(x) = x^4/4 + x^3/3 - x^2$ and $f''(x) = 3x^2 + 2x - 2$:



Problem 3 *If $f''(a) = 0$, what does the second derivative test tell us?*

Multiple Choice:

(a) *The function has a local extrema at $x = a$.*

(b) *The function does not have a local extrema at $x = a$.*

(c) *It gives no information on whether $x = a$ is a local extremum.* ✓