Dig-In:

Instantaneous velocity

We use limits to compute instantaneous velocity.

When one computes average velocity, we look at

$$\frac{\text{change in displacement}}{\text{change in time}}.$$

To obtain the (instantaneous) velocity, we want the change in time to "go to" zero. By this point we should know that "go to" is a buzz-word for a *limit*. The change in time is often given as an interval whose length goes to zero. However, intervals must always be written

$$[a, b] \qquad \text{where } a < b.$$

Given

$$I = [a, a + h],$$

we see that h cannot be negative, or else it violates the notation for intervals. Hence, if we want smaller, and smaller, intervals around a point a, and we want h to be able to be negative, we write

$$I_h = \begin{cases} [a + h, a] & \text{if } h < 0, \\ [a, a + h] & \text{if } 0 < h. \end{cases}$$

Question 1 *Let* $a = 3$ *and* $h = 0.1$

$$I_h = \left[\boxed{3}, \boxed{3.1}\right]$$

Question 2 *Let* $a = 3$ *and* $h = -0.1$

$$I_h = \left[\boxed{2.9}, \boxed{3}\right]$$

Learning outcomes: Compute limits of families of functions. Compute average velocity. Approximate instantaneous velocity. Compare average and instantaneous velocity. Plot difference quotients for varying approximations of the instantaneous rate of change.

Regardless of the value of h, the average velocity on the interval I_h is computed by

$$\frac{\text{change in displacement}}{\text{change in time}} = \frac{s(t+h)-s(t)}{h}.$$

We will be most interested in this ratio when h approaches zero. Let's put all of this together by working an example.

Example 1. *A group of young mathematicians recently took a road trip from Columbus Ohio to Urbana-Champaign Illinois. The position (west of Columbus, Ohio) of van they drove in is roughly modeled by*

$$s(t) = 36t^2 - 4.8t^3 \qquad \textit{(miles West of Columbus)}$$

on the interval $[0,5]$*, where* t *is measured in hours. What is the average velocity on the interval* $[0,5]$*?*

Additionally, let

$$I_h = \begin{cases} [1+h, 1] & \textit{if } h < 0, \\ [1, 1+h] & \textit{if } 0 < h. \end{cases}$$

What is the average velocity on I_h *when* $h = 0.1$*? What is the average velocity on* I_h *when* $h = -0.1$*?*

Explanation. *The average velocity on the interval* $[0,5]$ *is*

$$\begin{aligned} \frac{s(5)-s(0)}{5-0} &= \frac{36\cdot 5^2 - 4.8\cdot 5^3 - (36\cdot 0^2 - 4.8\cdot 0^3)}{5} \\ &= \frac{\underset{\text{given}}{\boxed{300}}}{5} \\ &= \underset{\text{given}}{\boxed{60}} \qquad \textit{miles per hour.} \end{aligned}$$

On the other hand, consider the interval

$$I_h = \begin{cases} [1+h, 1] & \textit{if } h < 0, \\ [1, 1+h] & \textit{if } 0 < h. \end{cases}$$

When $h = 0.1$*, the average velocity is*

$$\frac{s(1+0.1)-s(1)}{0.1}$$

$$\begin{aligned} &= \frac{36\cdot(1.1)^2 - 4.8\cdot(1.1)^3 - (36\cdot 1^2 - 4.8\cdot 1^3)}{0.1} \\ &= \frac{\underset{\text{given}}{\boxed{5.9712}}}{0.1} \\ &= \underset{\text{given}}{\boxed{59.712}} \qquad \textit{miles per hour.} \end{aligned}$$

On the other hand, when $h = -0.1$, the average velocity is

$$\frac{s(1-0.1)-s(1)}{-0.1}$$

$$= \frac{36\cdot(0.9)^2 - 4.8\cdot(0.9)^3 - (36\cdot 1^2 - 4.8\cdot 1^3)}{-0.1}$$

$$= \frac{\underbrace{\boxed{-5.5392}}_{\text{given}}}{-0.1}$$

$$= \underbrace{\boxed{55.392}}_{\text{given}} \qquad \textit{miles per hour.}$$

In our previous example, we computed *average velocity* on three different intervals. If we let the size of the interval go to zero, we get **instantaneous velocity**. Limits will allow us to compute instantaneous velocity. Let's use the same setting as before.

Example 2. *The position of van (west of Columbus, Ohio) our young mathematicians drove to Urbana-Champaign, Illinois is roughly modeled by*

$$s(t) = 36t^2 - 4.8t^3 \qquad \textit{for } 0 \le t \le 5,$$

Find a formula for the (instantaneous) velocity of this van.

Explanation. *Again, we are working with the interval*

$$I_h = \begin{cases} [t+h, t] & \textit{if } h < 0, \\ [t, t+h] & \textit{if } 0 < h. \end{cases}$$

To compute the average velocity, we write

$$\frac{s(t+h)-s(t)}{h}$$

but this time, we will let h go to zero. Write with me

$$\lim_{h\to 0} \frac{s(t+h)-s(t)}{h}$$

$$= \lim_{h\to 0} \frac{\underbrace{\boxed{36(t+h)^2 - 4.8(t+h)^3}}_{\text{given}} - \left(36t^2 - 4.8t^3\right)}{h}$$

Now expand the numerator of the fraction and combine like-terms:

$$= \lim_{h\to 0} \frac{72th + 36h^2 - 14.4t^2h - 14.4th^2 - \underbrace{\boxed{4.8h^3}}_{\text{given}}}{h}$$

Factor an h from every term in the numerator:

$$= \lim_{h\to 0} \frac{h\left(\underbrace{\boxed{72t + 36h - 14.4t^2 - 14.4th - 4.8h^2}}_{\text{given}}\right)}{h}$$

Cancel h from the numerator and denominator:

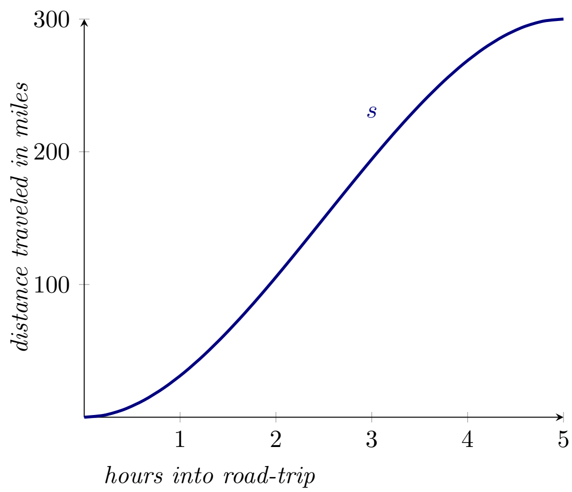
$$= \lim_{h\to 0} \left(\underbrace{\boxed{72t + 36h - 14.4t^2 - 14.4th - 4.8h^2}}_{\text{given}}\right)$$

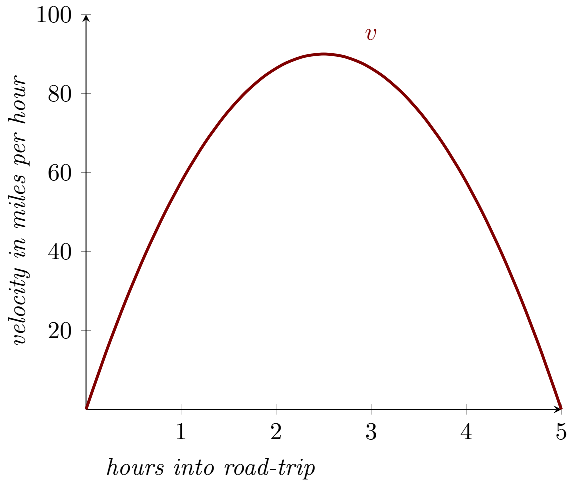
Plug in $h = 0$:

$$= \underbrace{\boxed{72t - 14.4t^2}}_{\text{given}}$$

This gives us a formula for our instantaneous velocity, $v(t) = \underbrace{\boxed{72t - 14.4t^2}}_{\text{given}}$.

For your viewing enjoyment, check out graphs of both $y = s(t)$ and $y = v(t)$:




100
80
60
40
20
velocity in miles per hour
v
1
2
3
4
5
hours into road-trip