Dig-In:

The Second Fundamental Theorem of Calculus

The accumulation of a rate is given by the change in the amount.

There is a another common form of the Fundamental Theorem of Calculus:

Theorem 1 (Second Fundamental Theorem of Calculus)**.** *Let f be continuous on $[a, b]$. If F is* ***any*** *antiderivative of f, then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Explanation. *Let $a \leq c \leq b$ and write*

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\\ &= \int_c^b f(x)\,dx - \int_c^a f(x)\,dx.\end{aligned}$$

By the First Fundamental Theorem of Calculus, we have

$$F(b) = \int_c^b f(x)\,dx \qquad \text{and} \qquad F(a) = \int_c^a f(x)\,dx$$

for some antiderivative F of f. So

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

for this antiderivative. However, ***any*** *antiderivative could have be chosen, as antiderivatives of a given function differ only by a constant, and this constant always cancels out of the expression when evaluating $F(b) - F(a)$.*

From this you should see that the two versions of the Fundamental Theorem are very closely related. In reality, the two forms are **equivalent**, just differently stated. Hence people often simply call them both "The Fundamental Theorem of Calculus." One way of thinking about the Second Fundamental Theorem of Calculus is:

Learning outcomes: State the Second Fundamental Theorem of Calculus. Evaluate definite integrals using the Second Fundamental Theorem of Calculus. Understand how the area under a curve is related to the antiderivative. Understand the relationship between indefinite and definite integrals.

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

This could be read as:

> **The accumulation of a rate is given by the change in the amount.**

When we compute a definite integral, we first find an antiderivative and then evaluate at the limits of integration. It is convenient to first display the antiderivative and then evaluate. A special notation is often used in the process of evaluating definite integrals using the Fundamental Theorem of Calculus. Instead of explicitly writing $F(b) - F(a)$, we often write

$$\Big[F(x)\Big]_a^b$$

meaning that one should evaluate $F(x)$ at b and then subtract $F(x)$ evaluated at a

$$\Big[F(x)\Big]_a^b = F(b) - F(a).$$

Let's see some examples of the fundamental theorem in action.

Example 1. *Compute:*

$$\int_{-2}^{2} x^3\,dx$$

Explanation. *We start by finding an antiderivative of x^3. A correct choice is $\frac{x^4}{4}$, one could verify this by taking the derivative. Hence*

$$\begin{aligned}\int_{-2}^{2} x^3\,dx &= \left[\frac{x^4}{4}\right]_{-2}^{2}\\ &= \frac{2^4}{4} - \frac{(-2)^4}{4}\\ &= 0.\end{aligned}$$

Example 2. *Compute:*

$$\int_0^\pi \sin\theta\,d\theta$$

Explanation. *We start by finding an antiderivative of* $\sin\theta$. *A correct choice is* $-\cos\theta$, *one could verify this by taking the derivative. Hence*

$$\begin{aligned}\int_0^\pi \sin\theta\, d\theta &= \Big[-\cos\theta\Big]_0^\pi \\ &= \underbrace{\boxed{-\cos(\pi)}}_{\text{given}} - (-\cos(0)) \\ &= \underbrace{\boxed{2}}_{\text{given}}.\end{aligned}$$

This is interesting: It says that the area under one "hump" of a sine curve is 2.

Example 3. *Compute:*

$$\int_0^5 e^t\, dt$$

Explanation. *We start by finding an antiderivative of* e^t. *A correct choice is* $\underbrace{\boxed{e^t}}_{\text{given}}$, *one could verify this by taking the derivative. Hence*

$$\begin{aligned}\int_0^5 e^t\, dt &= \Big[e^t\Big]_0^5 \\ &= e^5 - \underbrace{\boxed{e^0}}_{\text{given}} \\ &= e^5 - \underbrace{\boxed{1}}_{\text{given}}.\end{aligned}$$

Example 4. *Compute:*

$$\int_1^2 \left(x^9 + \frac{1}{x}\right) dx$$

Explanation. *We start by finding an antiderivative of* $x^9 + \dfrac{1}{x}$. *A correct choice is* $\dfrac{x^{10}}{10} + \ln(x)$, *one could verify this by taking the derivative. Hence*

$$\begin{aligned}\int_1^2 \left(x^9 + \frac{1}{x}\right) dx &= \left[\frac{x^{10}}{10} + \ln(x)\right]_1^2 \\ &= \frac{2^{10}}{10} + \ln(2) - \underbrace{\boxed{\frac{1}{10}}}_{\text{given}}.\end{aligned}$$

Understanding motion with the Fundamental Theorem of Calculus

We know that

- The derivative of a position function is a velocity function.
- The derivative of a velocity function is an acceleration function.

Now consider definite integrals of velocity and acceleration functions. Specifically, if $v(t)$ is a velocity function, what does $\int_a^b v(t)\,dt$ mean?

The Second Fundamental Theorem of Calculus states that

$$\int_a^b v(t)\,dt = V(b) - V(a),$$

where $V(t)$ is any antiderivative of $v(t)$. Since $v(t)$ is a velocity function, $V(t)$ must be a position function, and $V(b) - V(a)$ measures a **change in position**, or **displacement**.

Example 5. *A ball is thrown straight up with velocity given by $v(t) = -32t + 20ft/s$, where t is measured in seconds. Find, and interpret, $\int_0^1 v(t)\,dt$.*

Explanation. *Using the Second Fundamental Theorem of Calculus, we have*

$$\begin{aligned}\int_0^1 v(t)\,dt &= \int_0^1 (-32t + 20)\,dt \\ &= \left[\underbrace{\boxed{-16t^2 + 20t}}_{\text{given}}\right]_0^1 \\ &= 4.\end{aligned}$$

Thus if a ball is thrown straight up into the air with velocity

$$v(t) = \underbrace{\boxed{-32t + 20}}_{\text{given}},$$

the height of the ball, 1 second later, will be 4 feet above the initial height. Note that the ball has traveled much farther. It has gone up to its peak and is falling down, but the difference between its height at $t = 0$ and $t = 1$ is 4ft.

Now we know that to solve certain kinds of problems, those that involve accumulation of some form, we "merely" find an antiderivative and substitute two values and subtract. Unfortunately, finding antiderivatives can be quite difficult. While there are a small number of rules that allow us to compute the derivative of any common function, there are no such rules for antiderivatives. There are some techniques that frequently prove useful, but we will never be able to reduce the problem to a completely mechanical process.