Dig-In:

The First Fundamental Theorem of Calculus

The rate that accumulated area under a curve grows is described identically by that curve.

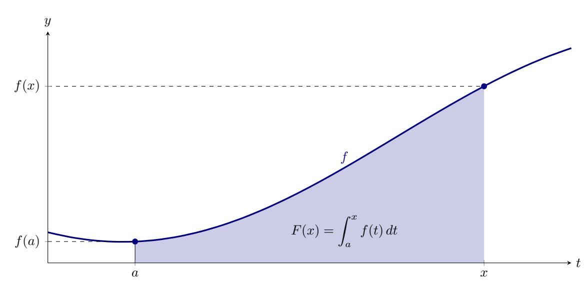
Accumulation functions

While the definite integral computes a signed area, which is a fixed number, there is a way to turn it into a function.

Definition 1. *A given a function f, an* ***accumulation function*** *is*

$$F(x) = \int_a^x f(t)\,dt.$$

One thing that you might notice is that an accumulation function seems to have two variables: x and t. Let's see if we can explain this. Consider the following graph:



An accumulation function F measures the signed area in the region $[a, x]$ between f and the t-axis. Hence t is playing the role of a "place-holder" that allows us to evaluate f. On the other hand, x is the **specific number** that we are using to bound the region that will determine the area between f and the t-axis, and hence the value of F.

Learning outcomes: Define accumulation functions. Calculate and evaluate accumulation functions. State the First Fundamental Theorem of Calculus. Take derivatives of accumulation functions using the First Fundamental Theorem of Calculus. Use accumulation functions to find information about the original function. Understand the relationship between the function and the derivative of its accumulation function.

Question 1 *Given*

$$F(x) = \int_{-3}^{x} 4\, dt,$$

what is $F(5)$?

$$F(5) = \boxed{32}$$

Question 2 *What is $F(-5)$?*

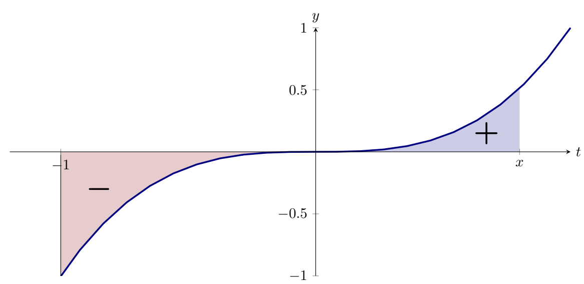
$$F(-5) = \boxed{-8}$$

Example 1. *Consider the following accumulation function for $f(x) = x^3$.*

$$F(x) = \int_{-1}^{x} t^3\, dt.$$

Considering the interval $[-1, 1]$, where is F increasing? Where is F decreasing? When does F have local extrema?

Explanation. *We can see a graph of f along with the signed area measured by the accumulation function below*



The accumulation function starts off at zero, and then as x grows, F is (increasing/decreasing ✓) *as the function accumulates negatively signed area.*

However when $x > 0$, F starts to accumulate positively signed area, and hence is (increasing ✓ /decreasing). *Thus F is increasing on $(0, 1)$, decreasing on $(-1, 0)$ and hence has a local minimum at $(0, 0)$.*

Working with the accumulation function leads us to a question, what is

$$\int_{a}^{x} f(x)\, dx$$

when $x < a$? The general convention is that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

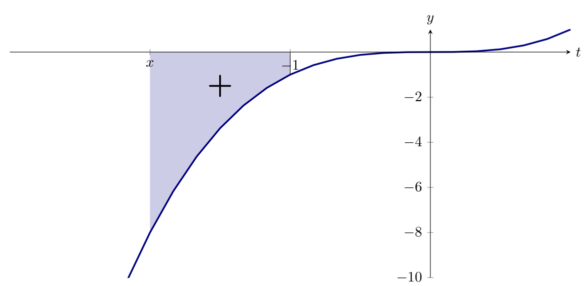
With this in mind, let's consider one more example.

Example 2. *Consider the following accumulation function for $f(x) = x^3$.*

$$F(x) = \int_{-1}^x t^3\,dt.$$

Where is F increasing? Where is F decreasing? When does F have local extrema?

Explanation. *From our previous example, we know that F is increasing on $(0, 1)$. Since f continues to be positive at $t = 1$ and beyond, F is* (increasing ✓ / decreasing) *on $(0, \infty)$. On the other hand, we know from our previous example that F is* (increasing/decreasing ✓) *on $(-1, 0)$.*



For values to the left of $t = -1$, F is still decreasing, as less and less positively signed area is accumulated. Hence F is increasing on $(0, \infty)$, decreasing on $(-\infty, 0)$ and hence has an absolute minimum at $(0, 0)$.

The key point to take from these examples is that an accumulation function

$$F(x) = \int_a^x f(t)\,dt$$

is increasing precisely when f is positive and is decreasing precisely when f is negative. In short, it seems that f is behaving in a similar fashion to F'.

The First Fundamental Theorem of Calculus

Let f be a continuous function on the real numbers and consider

$$F(x) = \int_a^x f(t)\,dt.$$

From our previous work we know that F is increasing when f is positive and F is decreasing when f is negative. Moreover, with careful observation, we can even see that F is concave up when f' is positive and that F is concave down when f' is negative. Thinking about what we have learned about the relationship of a function to its first and second derivatives, it is not too hard to guess that there must be a connection between F' and the function f. This is a good guess, check out our next theorem:

Theorem 1 (First Fundamental Theorem of Calculus)**.** *Suppose that f is continuous on the real numbers and let*

$$F(x) = \int_a^x f(t)\,dt.$$

Then $F'(x) = f(x)$.

The First Fundamental Theorem of Calculus says that an accumulation function of f is an antiderivative of f. Another way of saying this is:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

This could be read as:

> **The rate that accumulated area under a curve grows is described identically by that curve.**

Now that we are working with accumulation functions, let's see what happens when we compose them with other functions.

Example 3. *Find the derivative of*

$$F(x) = \int_2^{x^2} \ln t\,dt.$$

Explanation. *Consider*

$$G(x) = \int_2^x \ln t\,dt$$

and set $h(x) = \underbrace{\boxed{x^2}}_{\text{given}}$. *Now*

$$F(x) = G(h(x)).$$

The First Fundamental Theorem of Calculus states that $G'(x) = \ln x$. *The chain rule gives us*

$$\begin{aligned} F'(x) &= G'(h(x))h'(x) \\ &= \ln(h(x))h'(x) \\ &= \underbrace{\boxed{\ln(x^2)2x}}_{\text{given}}. \end{aligned}$$

Let's practice this once more.

Example 4. *Find the derivative of*

$$F(x) = \int_{\cos x}^{5} t^3\,dt.$$

Explanation. *Consider*

$$G(x) = -\int_{5}^{x} t^3\,dt$$

and set $h(x) = \underbrace{\boxed{\cos(x)}}_{\text{given}}$. *Now*

$$F(x) = G(h(x)).$$

The First Fundamental Theorem of Calculus states that $G'(x) = -x^3$. *The chain rule gives us*

$$\begin{aligned} F'(x) &= G'(h(x))h'(x) \\ &= -h(x)^3 h'(x) \\ &= \underbrace{\boxed{-\cos^3(x)(-\sin(x))}}_{\text{given}}. \end{aligned}$$