Dig-In:

More than one rate

Here we work abstract related rates problems.

Suppose we have two variables x and y which are both changing with respect to time. A *related rates* problem is a problem where we know one rate at a given instant, and wish to find the other.

Here the chain rule is key: If y is written in terms of x, and we are given $\frac{dx}{dt}$, then it is easy to find $\frac{dy}{dt}$ using the chain rule:

$$\frac{dy}{dt} = y'(x(t)) \cdot x'(t).$$

In many cases, particularly the interesting ones, our functions will be related in some other way. Nevertheless, in each case we'll use the power of the chain rule to help us find the desired rate. In this section, we will work several abstract examples, so we can emphasize the mathematical concepts involved. In each of the examples below, we will follow essentially the same plan of attack:

Draw a picture. If possible, draw a schematic picture with all the relevant information.

Find equations. We want equations that relate all relevant functions.

Differentiate the equations. Here we will often use implicit differentiation.

Evaluate and solve. Evaluate each equation at all known desired values and solve for the relevant rate.

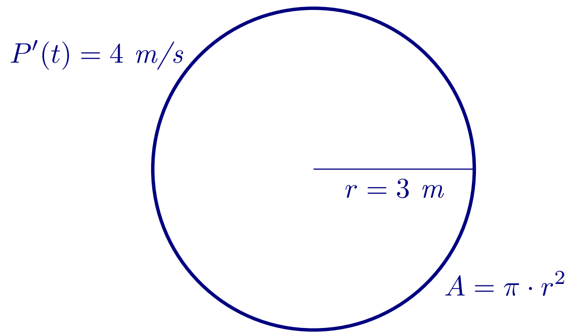
Formulas

One way to combine several functions is with a known formula.

Example 1. *Imagine an expanding circle. If we know that the perimeteter is expanding at a rate of* 4 *m/s, what rate is the area changing when the radius is* 3 *meters?*

Learning outcomes: Solve basic related rates word problems. Understand the process of solving related rates problems. Calculate derivatives of expressions with multiple variables implicitly.

Explanation. *To start, we* ***draw a picture****.*



We must ***find equations*** *that combine relevant functions. Here we use the common formulas for perimeter and area*

$$P = \underbrace{\boxed{2 \cdot \pi \cdot r}}_{\text{given}} \qquad \textit{and} \qquad A = \underbrace{\boxed{\pi \cdot r^2}}_{\text{given}}.$$

Next we imagine that A, r, and P are functions of time

$$P(t) = 2 \cdot \pi \cdot r(t) \qquad \textit{and} \qquad A(t) = \pi \cdot r(t)^2.$$

and we ***differentiate the equations*** *using implicit differentiation, treating all functions as functions of t*

$$P'(t) = 2 \cdot \pi \cdot r'(t) \qquad \textit{and} \qquad A'(t) = 2 \cdot \pi \cdot r(t) \cdot r'(t).$$

Now we ***evaluate and solve****. We know $P'(t) = \underbrace{\boxed{4}}_{\text{given}}$ and that $r(t) = \underbrace{\boxed{3}}_{\text{given}}$. Hence our equations become*

$$4 = 2 \cdot \pi \cdot r'(t) \qquad \textit{and} \qquad A'(t) = 2 \cdot \pi \cdot 3 \cdot r'(t).$$

We see that

$$\begin{aligned} 4 &= 2 \cdot \pi \cdot r'(t) \\ 2/\pi &= r'(t). \end{aligned}$$

and now that

$$\begin{aligned} A'(t) &= 2 \cdot \pi \cdot 3 \cdot 2/\pi \\ &= \underbrace{\boxed{12}}_{\text{given}}. \end{aligned}$$

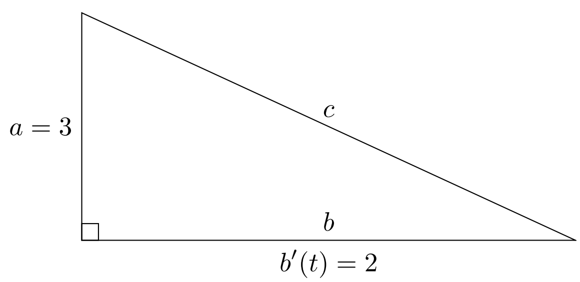
Hence the area is expanding at a rate of $12\ m/s$.

Right triangles

A common way to combine functions is through facts related to right triangles.

Example 2. *Imagine an expanding right triangle. If one leg has a fixed length of 3 m, one leg is increasing with a rate of 2 m/s, and the hypotenuse is expanding to accomidate the expanding leg, at what rate is the hypotenuse expanding when both legs are 3 m long?*

Explanation. *To start, we **draw a picture**.*



*We must **find equations** that combines relevant functions. Here we use the Pythagorean Theorem.*

$$c^2 = a^2 + b^2$$

Imagining c and b as being functions of time

$$c(t)^2 = a^2 + b(t)^2$$

*we are now able to **differentiate the equation** using implicit differentiation, treating all functions as functions of t, note a is constant,*

$$2 \cdot c(t) \cdot c'(t) = 2 \cdot b(t) \cdot b'(t).$$

*Now we **evaluate and solve**. We know that $b'(t) = 2$ and that $b(t) = 3$*

$$2 \cdot c(t) \cdot c'(t) = \underset{\text{given}}{\boxed{12}}$$

However, we still need to know $c(t)$ when $b(t) = 3$. Here we use the Pythagorean Theorem,

$$\begin{aligned} c(t)^2 &= 3^2 + 3^2 \\ &= \underset{\text{given}}{\boxed{18}}, \end{aligned}$$

and so we see that $c(t) = 3\sqrt{2}$. We may now write

$$\begin{aligned} 6\sqrt{2} \cdot c'(t) &= 12 \\ c'(t) &= \sqrt{2}. \end{aligned}$$

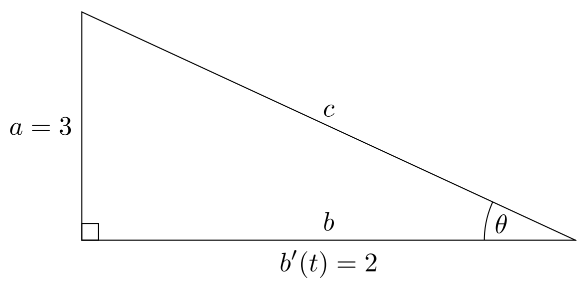
Hence $c(t)$ is growing at a rate of $\underbrace{\boxed{\sqrt{2}}}_{\text{given}}$ *m/s.*

Angular rates

We can also investigate problems involving angular rates.

Example 3. *Imagine an expanding right triangle. If one leg has a fixed length of 3 m, one leg is increasing with a rate of 2 m/s, and the hypotenuse is expanding to accomidate the expanding leg, at what rate is the angle attached to the fixed leg changing when both legs are 3 m long?*

Explanation. *To start, we **draw a picture**.*



*We must **find equations** that combines relevant functions. Here we note that*

$$\tan(\theta) = \underbrace{\boxed{\frac{a}{b}}}_{\text{given}}$$

Imagining θ and b as being functions of time

$$\tan(\theta(t)) = \frac{a}{b(t)}$$

*we are now able to **differentiate the equation** using implicit differentiation, treating all functions as functions of t, note a is constant,*

$$\sec^2(\theta(t))\theta'(t) = \frac{-a \cdot b'(t)}{b(t)^2}.$$

*Now we **evaluate and solve**. We know that $a = 3$, $b'(t) = 2$, and that $b(t) = 3$*

$$\begin{aligned}\sec^2(\theta(t)) \cdot \theta'(t) &= \frac{-3 \cdot 2}{3^2} \\ &= \frac{-6}{9} \\ &= \frac{-2}{3}.\end{aligned}$$

However, we still need to know $\sec^2(\theta)$*. Here we use the Pythagorean Theorem,*

$$\begin{aligned} c^2(t) &= 3^2 + 3^2 \\ &= \underset{\text{given}}{\boxed{18}}, \end{aligned}$$

and so we see that $c(t) = 3\sqrt{2}$*. Now*

$$\begin{aligned} \sec^2(\theta) &= \frac{\text{hypotenuse}^2}{\text{adjacent}^2} \\ &= \frac{\left(3\sqrt{2}\right)^2}{3^2} \\ &= \underset{\text{given}}{\boxed{2}}. \end{aligned}$$

Hence

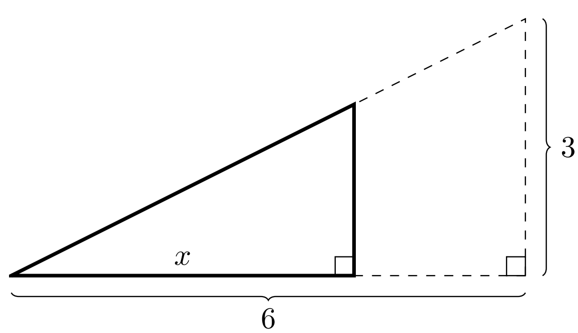
$$\begin{aligned} \sec^2(\theta(t)) \cdot \theta'(t) &= \frac{-2}{3} \\ 2 \cdot \theta'(t) &= \frac{-2}{3} \\ \theta'(t) &= \frac{-1}{3}. \end{aligned}$$

So when $a = b = 3$*, the angle is changing at* $\underset{\text{given}}{\boxed{-1/3}}$ *radians per second.*

Similar triangles

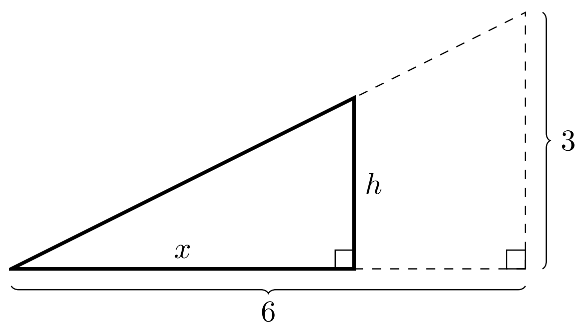
Finally, facts about similar triangles are often useful when solving related rates problems.

Example 4. *Imagine two right triangles that share an angle:*



If x is growing from the vertex with a rate of 3 m/s, what rate is the area of the smaller triangle changing when $x = 5$?

Explanation. *Despite the fact that a nice picture is given, we should start as we always do and* ***draw a picture****. Note, we've added information to the picture:*



We must ***find equations*** *that combines relevant functions. In this case there are two. The first is the formula for the area of a triangle:*

$$A = \underbrace{\boxed{(1/2) \cdot x \cdot h}}_{\text{given}}$$

The second uses the fact that the larger triangle is similar to the smaller triangle, meaning that the proportions of the sides are the same,

$$\frac{x}{h} = \underbrace{\boxed{\frac{6}{3}}}_{\text{given}} \qquad \textit{so} \qquad x = \underbrace{\boxed{2}}_{\text{given}} \cdot h$$

Imagining A, x, and h as functions of time we may write

$$A(t) = (1/2) \cdot x(t) \cdot h(t) \qquad \textit{and} \qquad x(t) = 2 \cdot h(t).$$

We are now able to ***differentiate the equations*** *using implicit differentiation, treating all functions as functions of t,*

$$\begin{aligned} A'(t) &= (1/2) \cdot x'(t) \cdot h(t) + (1/2) \cdot x(t) \cdot h'(t), \\ x'(t) &= 2 \cdot h'(t). \end{aligned}$$

Now we ***evaluate and solve****. We know that $x(t) = 5$ and that $x'(t) = 3$. Since*

$$5 = x(t) = 2 \cdot h(t) \qquad \textit{and} \qquad 3 = x'(t) = 2 \cdot h'(t)$$

we see that $h(t) = 5/2$ and $h'(t) = 3/2$. Hence

$$\begin{aligned} A'(t) &= (1/2) \cdot 3 \cdot (5/2) + (1/2) \cdot 5 \cdot (3/2) \\ &= 15/4 + 15/4 \\ &= \underbrace{\boxed{15/2}}_{\text{given}}. \end{aligned}$$

Hence the area is changing at a rate of $15/2\ m^2/s$.