Dig-In:

Derivatives of trigonometric functions

We use the chain rule to unleash the derivatives of the trigonometric functions.

Up until this point of the course we have been ignoring a large class of functions: Trigonometric functions other than $\sin(x)$. We know that

$$\frac{d}{dx}\sin(x) = \cos(x).$$

Armed with this fact we will discover the derivatives of all of the standard trigonometric functions.

Theorem 1 (The derivative of cosine)**.**

$$\frac{d}{dx}\cos(x) = -\sin(x).$$

Explanation. *Recall that*

- $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$, *and*
- $\sin(x) = \cos\left(\frac{\pi}{2} - x\right)$.

Now

$$\begin{aligned}\frac{d}{dx}\cos(x) &= \frac{d}{dx}\sin\left(\frac{\pi}{2} - x\right)\\ &= -\cos\left(\frac{\pi}{2} - x\right)\\ &= -\sin(x).\end{aligned}$$

Example 1. *Compute:*

$$\left[\frac{d}{dx}\cos\left(\frac{x^3}{2}\right)\right]_{x=\sqrt[3]{\pi}}$$

Explanation. *Now that we know the derivative of cosine, we may combine this with the chain rule, so we have that*

$$\frac{d}{dx}\cos\left(\frac{x^3}{2}\right) = \underset{\text{given}}{\boxed{\frac{3x^2}{2}}}\left(-\sin\left(\frac{x^3}{2}\right)\right)$$

Learning outcomes: Apply chain rule to relate quantities expressed with different units. Compute derivatives of trigonometric functions.

and so

$$\left[\frac{d}{dx}\cos\left(\frac{x^3}{2}\right)\right]_{x=\sqrt[3]{\pi}}$$

$$\begin{aligned}
&= \left[\left(\frac{3}{2}x^2\left(-\sin\left(\frac{x^3}{2}\right)\right)\right)\right]_{x=\sqrt[3]{\pi}}\\
&= -\frac{3}{2}(\sqrt[3]{\pi})^2\sin\left(\frac{\pi}{2}\right)\\
&= -\frac{3}{2}\pi^{\frac{2}{3}}\cdot\underbrace{\boxed{1}}_{\text{given}}\\
&= \underbrace{\boxed{\frac{-3\pi^{\frac{2}{3}}}{2}}}_{\text{given}}.
\end{aligned}$$

Next we have:

Theorem 2 (The derivative of tangent)**.**

$$\frac{d}{dx}\tan(x) = \sec^2(x).$$

Explanation. *We'll rewrite* $\tan(x)$ *as* $\dfrac{\sin(x)}{\cos(x)}$ *and use the quotient rule. Write with me:*

$$\begin{aligned}
\frac{d}{dx}\tan(x) &= \frac{d}{dx}\frac{\sin(x)}{\cos(x)}\\
&= \frac{\cos^2(x) + \underbrace{\boxed{\sin^2(x)}}_{\text{given}}}{\cos^2(x)}\\
&= \frac{\underbrace{\boxed{1}}_{\text{given}}}{\cos^2(x)}\\
&= \sec^2(x).
\end{aligned}$$

Example 2. *Compute:*

$$\frac{d}{dx}\left(\frac{5x\tan(x)}{x^2-3}\right)$$

Explanation. *Applying the quotient rule, and the product rule, and the deriva-*

tive of cosine:

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{5x\tan(x)}{x^2-3}\right)\\
&\quad= \frac{(x^2-3)\cdot\frac{d}{dx}(\underbrace{\boxed{5x\tan(x)}}_{\text{given}}) - 5x\tan(x)\cdot\frac{d}{dx}(\underbrace{\boxed{x^2-3}}_{\text{given}})}{(x^2-3)^2}\\
&\quad= \frac{(x^2-3)(5\tan(x)+5x\underbrace{\boxed{\sec^2(x)}}_{\text{given}}) - 5x\tan(x)\cdot 2x}{(x^2-3)^2}\\
&\quad= \frac{5(x^2-3)(\tan(x)+x\sec^2(x)) - 10x^2\tan(x)}{(x^2-3)^2}
\end{aligned}$$

Finally, we have:

Theorem 3 (The derivative of secant)**.**

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x).$$

Explanation. *We'll rewrite* $\sec(x)$ *as* $(\cos(x))^{-1}$ *and use the power rule and the chain rule. Write*

$$\begin{aligned}
\frac{d}{dx}\sec(x) &= \frac{d}{dx}(\cos(x))^{-1}\\
&= -1(\cos(x))^{-2}(\underbrace{\boxed{-\sin(x)}}_{\text{given}})\\
&= \frac{\sin(x)}{\cos^2(x)}\\
&= \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)}\\
&= \sec(x)\tan(x).
\end{aligned}$$

The derivatives of the cotangent and cosecant are similar and left as exercises. Putting this all together, we have:

Theorem 4 (The Derivatives of Trigonometric Functions)**.**

- $\dfrac{d}{dx}\sin(x) = \cos(x)$.
- $\dfrac{d}{dx}\cos(x) = -\sin(x)$.
- $\dfrac{d}{dx}\tan(x) = \sec^2(x)$.

- $\dfrac{d}{dx}\sec(x) = \sec(x)\tan(x)$.
- $\dfrac{d}{dx}\csc(x) = -\csc(x)\cot(x)$.
- $\dfrac{d}{dx}\cot(x) = -\csc^2(x)$.

Example 3. *Compute:*

$$\left[\frac{d}{dx}(\csc(x)\cot(x))\right]_{x=\frac{\pi}{3}}$$

Explanation. *Applying the product rule the facts above, we know that*

$$\frac{d}{dx}(\csc(x)\cot(x)) = -\csc^3(x) - \cot^2(x)\underbrace{\boxed{\csc(x)}}_{\text{given}}$$

and so

$$\left[\frac{d}{dx}(\csc(x)\cot(x))\right]_{x=\frac{\pi}{3}}$$

$$\begin{aligned} &= \left[-\csc^3(x) - \cot^2(x)\underbrace{\boxed{\csc(x)}}_{\text{given}}\right]_{x=\frac{\pi}{3}} \\ &= -\frac{8}{3\sqrt{3}} - \frac{1}{3}\cdot\underbrace{\boxed{2/\sqrt{3}}}_{\text{given}} \end{aligned}$$

Warning 1. *When working with derivatives of trigonometric functions, we suggest you use **radians** for angle measure. For example, while*

$$\sin\left((90^\circ)^2\right) = \sin\left(\left(\frac{\pi}{2}\right)^2\right),$$

one must be careful with derivatives as

$$\left[\frac{d}{dx}\sin\left(x^2\right)\right]_{x=90^\circ} \neq \underbrace{2\cdot 90\cdot\cos(90^2)}_{\textit{incorrect}}$$

Alternatively, one could think of x° as meaning $\dfrac{x\cdot\pi}{180}$, as then $90^\circ = \dfrac{90\cdot\pi}{180} = \dfrac{\pi}{2}$. In this case

$$2\cdot 90^\circ\cdot\cos((90^\circ)^2) = 2\cdot\frac{\pi}{2}\cdot\cos\left(\left(\frac{\pi}{2}\right)^2\right).$$