Dig-In:

The chain rule

Here we compute derivatives of compositions of functions

So far we have seen how to compute the derivative of a function built up from other functions by addition, subtraction, multiplication and division. There is another very important way that we combine functions: composition. The *chain rule* allows us to deal with this case. Consider

$$h(x) = \sin(1 + 2x).$$

While there are several different ways to differentiate this function, if we let $f(x) = \sin(x)$ and $g(x) = 1 + 2x$, then we can express $h(x) = f(g(x))$. The question is, can we compute the derivative of a composition of functions using the derivatives of the constituents $f(x)$ and $g(x)$? To do so, we need the *chain rule.*

Theorem 1 (Chain Rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

It will take a bit of practice to make the use of the chain rule come naturally, it is more complicated than the earlier differentiation rules we have seen. Let's return to our motivating example.

Example 1. *Compute:*

$$\frac{d}{dx} \sin(1 + 2x)$$

Explanation. *Set $f(x) = \sin(x)$ and $g(x) = 1 + 2x$, now*

$$f'(x) = \underbrace{\boxed{\cos(x)}}_{\text{given}} \qquad \textit{and} \qquad g'(x) = \underbrace{\boxed{2}}_{\text{given}}.$$

Learning outcomes: Recognize a composition of functions. Take derivatives of compositions of functions using the chain rule. Take derivatives that require the use of multiple derivative rules. Use the chain rule to calculate derivatives from a table of values. Understand rate of change when quantities are dependent upon each other. Use order of operations in situations requiring multiple derivative rules. Apply chain rule to relate quantities expressed with different units.

Hence

$$\begin{aligned}\frac{d}{dx}\sin(1+2x) &= \frac{d}{dx}f(g(x))\\ &= f'(g(x))g'(x)\\ &= \cos(\underbrace{\boxed{1+2x}}_{\text{given}})\cdot\underbrace{\boxed{2}}_{\text{given}}\\ &= 2\cos(1+2x).\end{aligned}$$

Let's see a more complicated chain of compositions.

Example 2. *Compute:*

$$\frac{d}{dx}\sqrt{1+\sqrt{x}}$$

Explanation. *Set* $f(x)=\sqrt{x}$ *and* $g(x)=1+x$. *Hence,*

$$\sqrt{1+\sqrt{x}} = f(g(\underbrace{\boxed{f}}_{\text{given}}(x)))$$

and by the chain rule we know

$$\frac{d}{dx}f(g(f(x))) = f'(g(f(x)))g'(f(x))f'(x).$$

Since

$$f'(x) = \underbrace{\boxed{\frac{1}{2\sqrt{x}}}}_{\text{given}} \qquad \text{and} \qquad g'(x) = \underbrace{\boxed{1}}_{\text{given}}$$

We have that

$$\frac{d}{dx}\sqrt{1+\sqrt{x}} = \frac{1}{2\sqrt{1+\sqrt{x}}}\cdot 1\cdot\underbrace{\boxed{\frac{1}{2\sqrt{x}}}}_{\text{given}}.$$

The chain rule allows to differentiate compositions of functions that would otherwise be difficult to get our hands on.

Example 3. *Compute:*

$$\frac{d}{dx}e^{\sin(x^2)}$$

Explanation. *set* $f(x) = \underbrace{\boxed{e^x}}_{\text{given}}$, $g(x) = \underbrace{\boxed{\sin(x)}}_{\text{given}}$, *and* $h(x) = \underbrace{\boxed{x^2}}_{\text{given}}$ *so that* $f(g(x)) = e^{\sin(x^2)}$. *Now*

$$\begin{aligned}\frac{d}{dx}e^{\sin(x^2)} &= \frac{d}{dx}f(g(h(x)))\\ &= f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)\\ &= e^{\underbrace{\boxed{\sin(x^2)}}_{\text{given}}}\cdot\cos(\underbrace{\boxed{x^2}}_{\text{given}})\cdot\underbrace{\boxed{2x}}_{\text{given}}.\end{aligned}$$

Using the chain rule, the power rule, and the product rule it is possible to avoid using the quotient rule entirely.

Example 4. *Compute:*

$$\frac{d}{dx}\frac{x^3}{x^2+1}$$

Explanation. *Rewriting this as*

$$\frac{d}{dx}x^3(x^2+1)^{-1},$$

set $f(x) = \underbrace{\boxed{x^{-1}}}_{\text{given}}$ *and* $g(x) = \underbrace{\boxed{x^2+1}}_{\text{given}}$ *so that* $f(g(x)) = (x^2+1)^{-1}$. *Now*

$$x^3(x^2+1)^{-1} = x^3 f(g(x)),$$

and by the product and chain rules

$$\frac{d}{dx}x^3 f(g(x)) = \underbrace{\boxed{3x^2}}_{\text{given}} \cdot f(g(x)) + \underbrace{\boxed{x^3}}_{\text{given}} \cdot f'(g(x))g'(x).$$

Since $f'(x) = \underbrace{\boxed{\frac{-1}{x^2}}}_{\text{given}}$ *and* $g'(x) = \underbrace{\boxed{2x}}_{\text{given}}$, *write*

$$\frac{d}{dx}\frac{x^3}{x^2+1} = \frac{3x^2}{x^2+1} - \frac{2x^4}{(x^2+1)^2}.$$