Dig-In:

The limit laws

We give basic laws for working with limits.

In this section, we present a handful of rules called the *Limit Laws* that allow us to find limits of various combinations of functions.

Theorem 1 (Limit Laws)**.** *Suppose that* $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$.

Sum/Difference Law $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$.

Product Law $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = LM$.

Quotient Law $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$, *if* $M \neq 0$.

Question 1 *True or false: If f and g are continuous functions on an interval I, then $f \pm g$ is continuous on I.*

Multiple Choice:

(a) *True* ✓

(b) *False*

Feedback (attempt): *This follows from the Sum/Difference Law.*

Question 2 *True or false: If f and g are continuous functions on an interval I, then f/g is continuous on I.*

Multiple Choice:

(a) *True*

(b) *False* ✓

Feedback (attempt): *In this case, f/g will not be continuous for x where $g(x) = 0$.*

Learning outcomes: Calculate limits using the limit laws.

Example 1. *Compute the following limit using limit laws:*

$$\lim_{x\to 1}(5x^2+3x-2)$$

Explanation. *Well, get out your pencil and write with me:*

$$\lim_{x\to 1}(5x^2+3x-2) = \lim_{x\to 1} 5x^2 + \lim_{x\to 1} \underbrace{\boxed{3x}}_{\text{given}} - \lim_{x\to 1} 2$$

by the Sum/Difference Law. So now

$$= 5\lim_{x\to 1} x^2 + 3\lim_{x\to 1} x - \lim_{x\to 1} \underbrace{\boxed{2}}_{\text{given}}$$

by the Product Law. Finally by continuity of x^k and k,

$$= 5(1)^2 + 3(1) - 2 = \underbrace{\boxed{6}}_{\text{given}}.$$

We can check our answer by looking at the graph of $y = f(x)$:

Graph of $5x^2+3x-2$

We can generalize the example above to get the following theorems.

Theorem 2 (Continuity of Polynomial Functions)**.** *All polynomial functions, meaning functions of the form*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a whole number and each a_i is a real number, are continuous for all real numbers.

Theorem 3 (Continuity of Rational Functions)**.** *Let f and g be polynomials. Then a rational function, meaning an expression of the form*

$$h = \frac{f}{g}$$

is continuous for all real numbers except where $g(x) = 0$. That is, rational functions are continuous wherever they are defined.

Explanation. *Let a be a real number such that $g(a) \neq 0$. Then, since $g(x)$ is continuous at a, $\lim_{x\to a} g(x) \neq 0$. Therefore, write with me,*

$$\lim_{x\to a} h(x) = \lim_{x\to a} \frac{f(x)}{g(x)}$$

and now by the Quotient Law,

$$\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

and by the continuity of polynomials we may now set $x = a$

$$\frac{f(a)}{g(a)} = h(a).$$

Since we have shown that $\lim_{x\to a} h(x) = h(a)$, we have shown that h is continuous at $x = a$.

Question 3 *Where is $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$ continuous?*

Multiple Choice:

(a) *for all real numbers*

(b) *at $x = 2$*

(c) *for all real numbers, except $x = 2$* ✓

(d) *impossible to say*

Now, we give basic rules for how limits interact with composition of functions.

Theorem 4 (Composition Limit Law)**.** *If $f(x)$ is continuous at $x = \lim_{x\to a} g(x)$, then*

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)).$$

Because the limit of a continuous function is the same as the function value, we can now pass limits inside continuous functions.

Corollary 1 (Continuity of Composite Functions)**.** *If g is continuous at $x = a$, then $f(g(x))$ is continuous at $x = a$.*

Example 2. *Compute the following limit using limit laws:*

$$\lim_{x\to 0} \sqrt{\cos(x)}$$

Explanation. *By continuity of x^k, assuming $\lim_{x\to 0} \cos(x) > 0$,*

$$\lim_{x\to 0} \sqrt{\cos(x)} = \sqrt{\lim_{x\to 0} \cos(x)},$$

and now since cosine is continuous for all real numbers,

$$\sqrt{\cos(0)} = \sqrt{1} = 1.$$

Many of the Limit Laws and theorems about continuity in this section might seem like they should be obvious. You may be wondering why we spent an entire section on these theorems. The answer is that these theorems will tell you exactly when it is easy to find the value of a limit, and exactly what to do in those cases.

The most important thing to learn from this section is whether the limit laws can be applied for a certain problem, and when we need to do something more interesting. We will begin discussing those more interesting cases in the next section. For now, we end this section with a question:

A list of questions

Let's try this out.

Question 4 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 2} \frac{x^2+3x+2}{x+2}$$

Multiple Choice:

(a) *yes* ✓

(b) *no*

Question 5 *Compute:*

$$\lim_{x\to 2} \frac{x^2+3x+2}{x+2} = \boxed{3}$$

Feedback (attempt): *Since* $f(x) = \dfrac{x^2+3x+2}{x+2}$ *is a rational function, and the denominator does not equal* 0*, we see that* $f(x)$ *is continuous at* $x = 2$*. Thus, to find this limit, it suffices to plug* 2 *into* $f(x)$*.*

Question 6 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 2} \frac{x^2-3x+2}{x-2}$$

Multiple Choice:

(a) *yes*

(b) *no* ✓

Feedback (attempt): $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$ *is a rational function, but the denominator* $x - 2$ *equals* 0 *when* $x = 2$*. None of our current theorems address the situation when the denominator of a fraction approaches* 0*.*

Question 7 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 0} x\sin(1/x)$$

Multiple Choice:

(a) *yes*

(b) *no* ✓

Feedback (attempt): *If we are trying to use limit laws to compute this limit, we would first have to use the Product Law to say that*

$$\lim_{x\to 0} x\sin(1/x) = \lim_{x\to 0} x \cdot \lim_{x\to 0} \sin(1/x).$$

We are only allowed to use this law if both limits exist, so we must check this first. We know from continuity that

$$\lim_{x\to 0} x = 0.$$

However, we also know that $\sin(1/x)$ *oscillates "wildly" as* x *approaches* 0*, and so the limit*

$$\lim_{x\to 0} \sin(1/x)$$

does not exist. Therefore, we cannot use the Product Law.

Question 8 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 0} \cot(x^3)$$

Multiple Choice:

(a) *yes*

(b) *no* ✓

Feedback (attempt): *Notice that*

$$\cot(x^3) = \frac{\cos(x^3)}{\sin(x^3)}.$$

If we are trying to use limit laws to compute this limit, we would like to use the Quotient Law to say that

$$\lim_{x\to 0} \frac{\cos(x^3)}{\sin(x^3)} = \frac{\lim\limits_{x\to 0} \cos(x^3)}{\lim\limits_{x\to 0} \sin(x^3)}.$$

We are only allowed to use this law if both limits exist and the denominator is not 0*. We suspect that the limit on on the denominator might equal* 0*, so we check this limit.*

$$\begin{aligned}\lim_{x\to 0} \sin(x^3) &= \sin(\lim_{x\to 0} x^3)\\ &= \sin(0)\\ &= 0.\end{aligned}$$

This means that the denominator is zero and hence we cannot use the Quotient Law.

Question 9 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 0} \sec^2(e^x - 1)$$

Multiple Choice:

(a) *yes* ✓

(b) *no*

Question 10 *Compute:*

$$\lim_{x\to 0} \sec^2(e^x - 1) = \boxed{1}$$

Feedback (attempt): *Notice that*

$$\lim_{x\to 0} \sec^2(e^x - 1) = \lim_{x\to 0} \frac{1}{\cos^2(e^x - 1)}.$$

If we are trying to use Limit Laws to compute this limit, we would now have to use the Quotient Law to say that

$$\lim_{x\to 0} \frac{1}{\cos^2(e^x - 1)} = \frac{\lim\limits_{x\to 0} 1}{\lim\limits_{x\to 0} \cos^2(e^x - 1)}.$$

We are only allowed to use this law if both limits exist and the denominator is not 0. *Let's check the denominator and numerator separately. First we'll compute the limit of the denominator:*

$$\begin{aligned}\lim_{x\to 0}\cos^2(e^x-1) &= \cos^2(\lim_{x\to 0}(e^x-1))\\ &= \cos^2(\lim_{x\to 0}(e^x)-\lim_{x\to 0}(1))\\ &= \cos^2(1-1)\\ &= \cos^2(0)\\ &= 1\end{aligned}$$

Therefore, the limit in the denominator exists and does not equal 0. *We can use the Quotient Law, so we will compute the limit of the numerator:*

$$\lim_{x\to 0} 1 = 1$$

Hence

$$\frac{\lim_{x\to 0} 1}{\lim_{x\to 0}\cos^2(e^x-1)} = \frac{1}{1} = 1$$

Question 11 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 1}(x-1)\cdot\csc(\ln(x))$$

Multiple Choice:

(a) *yes*

(b) *no* ✓

Feedback (attempt): *If we are trying to use limit laws to compute this limit, we would have to use the Product Law to say that*

$$\lim_{x\to 1}(x-1)\cdot\csc(\ln(x)) = \lim_{x\to 1}(x-1)\cdot\lim_{x\to 1}\csc(\ln(x)).$$

We are only allowed to use this law if both limits exist. Let's check each limit separately.

$$\begin{aligned}\lim_{x\to 1}(x-1) &= \lim_{x\to 1}(x)-\lim_{x\to 1}(1)\\ &= 1-1\\ &= 0.\end{aligned}$$

So this limit exists. Now we check the other factor. Notice that

$$\lim_{x\to 1}\csc(\ln(x)) = \frac{1}{\sin(\ln(x))}.$$

If we are trying to use limit laws to compute this limit, we would now have to use the Quotient Law to say that

$$\frac{1}{\sin(\ln(x))} = \frac{\lim\limits_{x\to 1} 1}{\lim\limits_{x\to 1} \sin(\ln(x))}.$$

We are only allowed to use this law if both limits exist and the denominator does not equal 0. *The limit in the numerator definitely exists, so lets check the limit in the denominator.*

$$\begin{aligned}\lim_{x\to 1} \sin(\ln(x)) &= \sin(\lim_{x\to 1} \ln(x)) \\ &= \sin(\ln(1)) \\ &= \sin(0) \\ &= 0\end{aligned}$$

Since the denominator is 0, *we cannot apply the Quotient Law.*

Question 12 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 0} x \ln x$$

Multiple Choice:

(a) *yes*

(b) *no* ✓

Feedback (attempt): *If we are trying to use limit laws to compute this limit, we would have to use the Product Law to say that*

$$\lim_{x\to 0} x \ln x = \lim_{x\to 0} x \cdot \lim_{x\to 0} \ln x.$$

We are only allowed to use this law if both limits exist. We know $\lim\limits_{x\to 0} x = 0$, *but what about* $\lim\limits_{x\to 0} \ln x$*? We do not know how to find* $\lim\limits_{x\to 0} \ln x$ *using limit laws because* 0 *is not in the domain of* $\ln x$.

Question 13 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 0} \frac{2^x - 1}{3^{x-1}}$$

Multiple Choice:

(a) *yes* ✓

(b) *no*

Question 14 *Compute:*

$$\lim_{x\to 0} \frac{2^x - 1}{3^{x-1}} = \boxed{0}$$

Feedback (attempt): *If we are trying to use limit laws to compute this limit, we would have to use the Quotient Law to say that*

$$\lim_{x\to 0} \frac{2^x - 1}{3^{x-1}} = \frac{\lim\limits_{x\to 0}(2^x - 1)}{\lim\limits_{x\to 0}(3^{x-1})}.$$

We are only allowed to use this law if both limits exist and the denominator does not equal 0*. Let's check each limit separately, starting with the denominator*

$$\begin{aligned}\lim_{x\to 0}(3^{x-1}) &= 3^{\lim\limits_{x\to 0}(x-1)} \\ &= 3^{-1} \\ &= \frac{1}{3}\end{aligned}$$

On the other hand the limit in the numerator is

$$\begin{aligned}\lim_{x\to 0}(2^x - 1) &= \lim_{x\to 0}(2^x) - \lim_{x\to 0}(1) \\ &= 1 - 1 \\ &= 0\end{aligned}$$

The limits in both the numerator and denominator exist and the limit in the denominator does not equal 0*, so we can use the Quotient Law. We find:*

$$\frac{\lim\limits_{x\to 0}(2^x - 1)}{\lim\limits_{x\to 0}(3^{x-1})} = \frac{0}{\frac{1}{3}} = 0.$$

Question 15 *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 0}(1+x)^{1/x}$$

Multiple Choice:

(a) *yes*

(b) *no* ✓

Feedback (attempt): *We do not have any limit laws for functions of the form* $f(x)^{g(x)}$*, so we cannot compute this limit.*