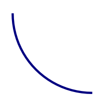
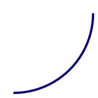
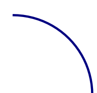
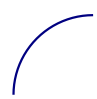
Dig-In:

Concavity

Here we examine what the second derivative tells us about the geometry of functions.

We know that the sign of the derivative tells us whether a function is increasing or decreasing at some point. Likewise, the sign of the second derivative $f''(x)$ tells us whether $f'(x)$ is increasing or decreasing at x. We summarize the consequences of this seemingly simple idea in the table below:

	$f'(x) < 0$	$0 < f'(x)$
$0 < f''(x)$	 Here $y = f(x)$ is decreasing, while the rate itself is increasing. In this case the curve is **concave up**.	 Here $y = f(x)$ is increasing, while the rate itself is increasing. In this case the curve is **concave up**.
$f''(x) < 0$	 Here $y = f(x)$ is decreasing, while the rate itself is decreasing. In this case the curve is **concave down**.	 Here $y = f(x)$ is increasing, while the rate itself is decreasing. In this case the curve is **concave down**.

If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. It is worth summarizing what we have seen already in to a single theorem.

Theorem 1 (Test for Concavity)**.** *Suppose that $f''(x)$ exists on an interval.*

(a) $f''(x) > 0$ *on that interval whenever* $y = f(x)$ *is concave up on that interval.*

(b) $f''(x) < 0$ *on that interval whenever* $y = f(x)$ *is concave down on that interval.*

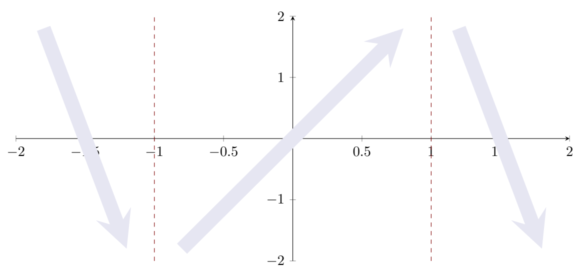
Learning outcomes: Use the first derivative to determine whether a function is increasing or decreasing. Identify the relationships between the function and its first and second derivatives. Sketch a graph of the second derivative, given the original function. Sketch a graph of the original function, given the graph of its first and second derivatives. State the relationship between concavity and the second derivative.

Example 1. *Let f be a continuous function and suppose that:*

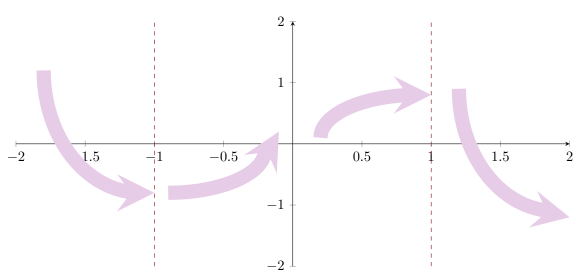
- $f'(x) > 0$ *for* $-1 < x < 1$.
- $f'(x) < 0$ *for* $-2 < x < -1$ *and* $1 < x < 2$.
- $f''(x) > 0$ *for* $-2 < x < 0$ *and* $1 < x < 2$.
- $f''(x) < 0$ *for* $0 < x < 1$.

Sketch a possible graph of f.

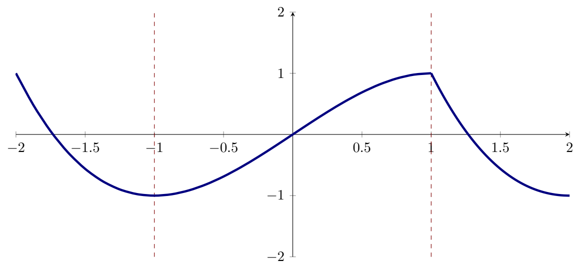
Explanation. *Start by marking where the derivative changes sign and indicate intervals where f is increasing and intervals f is decreasing. The function f has a negative derivative from -2 to $x =$* $\underbrace{\boxed{-1}}_{\text{given}}$. *This means that f is* (increasing/ decreasing ✓) *on this interval. The function f has a positive derivative from $x =$* $\underbrace{\boxed{-1}}_{\text{given}}$ *to $x =$* $\underbrace{\boxed{1}}_{\text{given}}$. *This means that f is* (increasing ✓ /decreasing) *on this interval. Finally, The function f has a negative derivative from $x =$* $\underbrace{\boxed{1}}_{\text{given}}$ *to 2. This means that f is* (increasing/decreasing ✓) *on this interval.*



Now we should sketch the concavity: (concave up ✓ /concave down) *when the second derivative is positive,* (concave up/ concave down ✓) *when the second derivative is negative.*



Finally, we can sketch our curve:


2
1
−1
−2
−2
−1.5
−1
−0.5
0.5
1
1.5
2