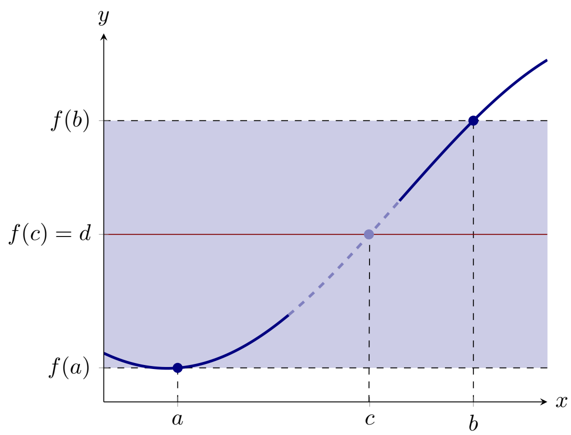
Dig-In:

The Intermediate Value Theorem

Here we see a consequence of a function being continuous.

The *Intermediate Value Theorem* should not be brushed off lightly. Once it is understood, it may seem "obvious," but mathematicians should not underestimate its power.

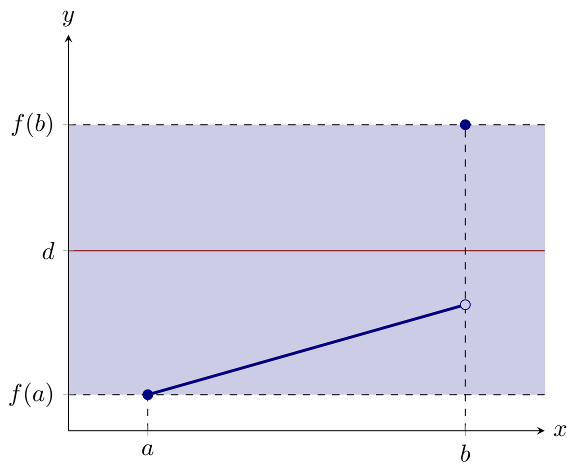
Theorem 1 (Intermediate Value Theorem)**.** *If f is a continuous function for all x in the closed interval $[a, b]$ and d is between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ such that $f(c) = d$.*



Now, let's contrast this with a time when the conclusion of the Intermediate Value Theorem does not hold.

Question 1 *Consider the following situation,*

Learning outcomes: State the Intermediate Value Theorem including hypotheses. Sketch pictures indicating why the Intermediate Value Theorem is true, and why all hypotheses are necessary. Explain why certain points exist using the Intermediate Value Theorem.



and select all that are true:

Select All Correct Answers:

(a) f *is continuous on* (a, b). ✓

(b) f *is continuous on* $[a, b)$. ✓

(c) f *is continuous on* $(a, b]$.

(d) f *is continuous on* $[a, b]$.

(e) *There is a point* c *in* $[a, b]$ *with* $f(c) = d$.

Building on the question above, it is not difficult to see that each of the hypothesis of the Intermediate Value Theorem are necessary.

Let's see the Intermediate Value Theorem in action.

Example 1. *Explain why the function* $f(x) = x^3 + 3x^2 + x - 2$ *has a root between* 0 *and* 1.

Explanation. *Since* f *is a polynomial, we see that* f *is continuous for all real numbers. Since* $f(0) = \underbrace{\boxed{-2}}_{\text{given}}$ *and* $f(1) = \underbrace{\boxed{3}}_{\text{given}}$, *and* 0 *is between* -2 *and* 3, *by the Intermediate Value Theorem, there is a point* c *in the interval* $[0, 1]$ *such that* $f(c) = \underbrace{\boxed{0}}_{\text{given}}$.

This example also points the way to a simple method for approximating roots.

Example 2. *Approximate a root of* $f(x) = x^3 + 3x^2 + x - 2$ *between* 0 *and* 1 *to within one decimal place.*

Explanation. *Again, since* f *is a polynomial, we see that* f *is continuous for all real numbers. Compute*

x	$f(x)$
0.1	$\underbrace{\boxed{-1.869}}_{\text{given}}$
0.2	$\underbrace{\boxed{-1.672}}_{\text{given}}$
0.3	$\underbrace{\boxed{-1.403}}_{\text{given}}$
0.4	$\underbrace{\boxed{-1.056}}_{\text{given}}$
0.5	$\underbrace{\boxed{-0.625}}_{\text{given}}$
0.6	$\underbrace{\boxed{-0.104}}_{\text{given}}$
0.7	$\underbrace{\boxed{0.513}}_{\text{given}}$

By the Intermediate Value Theorem, f *has a root between* 0.6 *and* 0.7*. Repeating the process*

x	$f(x)$
0.61	$\underbrace{\boxed{-0.046719}}_{\text{given}}$
0.62	$\underbrace{\boxed{0.011528}}_{\text{given}}$

so by the Intermediate Value Theorem, f *has a root between* 0.61 *and* 0.62*, and the root is* 0.6 *rounded to one decimal place.*

The Intermediate Value Theorem can be use to show that curves cross:

Example 3. *Explain why the functions*

$$\begin{aligned} f(x) &= x^2 \ln(x) \\ g(x) &= 2x \cos(\ln(x)) \end{aligned}$$

intersect on the interval $[1, e]$.

To start, note that both f *and* g *are continuous functions, and hence* $h = f - g$ *is also a continuous function. Now*

$$\begin{aligned} h(1) &= f(1) - g(1) \\ &= (\underbrace{\boxed{1}}_{\text{given}})^2 \cdot \ln(\underbrace{\boxed{1}}_{\text{given}}) - 2 \cdot \underbrace{\boxed{1}}_{\text{given}} \cdot \cos(\ln(\underbrace{\boxed{1}}_{\text{given}})) \\ &= \underbrace{\boxed{-2}}_{\text{given}}. \end{aligned}$$

and in a similar fashion

$$\begin{aligned} h(e) &= f(e) - g(e) \\ &= e^2 \cdot \underbrace{\boxed{1}}_{\text{given}} - 2 \cdot e \cos(\underbrace{\boxed{1}}_{\text{given}}) \end{aligned}$$

Since $e > 2$ *and* $0 < \cos(1) < 1$ *we see that the expression above is positive. Hence by the Intermediate Value Theorem,* f *and* g *intersect on the interval* $[1, e]$.

Now we move on to a more subtle example:

Example 4. *Suppose you have two cats, Roxy and Yuri. Is there a time when Roxy and Yuri have the same amount of water in their bowls assuming:*

- *They start and finish drinking at the same times.*
- *Roxy starts with more water than Yuri, and leaves less water left in her bowl than Yuri.*

Explanation. *To solve this problem, consider two functions:*

- $W_{\text{Roxy}}(t) =$ *the amount of water in Roxy's bowl at time* t.
- $W_{\text{Yuri}}(t) =$ *the amount of water in Yuri's bowl at time* t.

Now if t_{start} *is the time the cats start drinking and* t_{finish} *is the time the cats finish drinking. Then we have*

$$W_{\text{Roxy}}(t_{\text{start}}) - W_{\text{Yuri}}(t_{\text{start}}) > 0$$

and

$$W_{\text{Roxy}}(t_{\text{finish}}) - W_{\text{Yuri}}(t_{\text{finish}}) < 0.$$

Since the amount of water in a bowl at time t *is a continuous function, as water is "lapped" up in continuous amounts,*

$$W_{\text{Roxy}} - W_{\text{Yuri}}$$

is a continuous function, and hence the Intermediate Value Theorem applies. Since $W_{\text{Roxy}} - W_{\text{Yuri}}$ *is positive when at* t_{start} *and negative at* t_{finish}, *there is some time* t_{equal} *when the value is zero, meaning*

$$W_{\text{Roxy}}(t_{\text{equal}}) - W_{\text{Yuri}}(t_{\text{equal}}) = 0$$

meaning there is the same amount of water in each of their bowls.

And finally, an example when the Intermediate Value Theorem *does not* apply.

Example 5. *Suppose you have two cats, Roxy and Yuri. Is there a time when Roxy and Yuri have the same amount of dry cat food in their bowls assuming:*

- *They start and finish eating at the same times.*
- *Roxy starts with more food than Yuri, and leaves less food uneaten than Yuri.*

Explanation. *Here we could try the same approach as before, setting:*

- $F_{\text{Roxy}}(t) =$ *the amount of dry cat food in Roxy's bowl at time* t.
- $F_{\text{Yuri}}(t) =$ *the amount of dry cat food in Yuri's bowl at time* t.

However in this case, the amount of food in a bowl at time t is ***not*** *a continuous function! This is because dry cat food consists of discrete kibbles, and is not eaten in a continuous fashion. Hence the Intermediate Value Theorem* ***does not apply****, and we can make no definitive statements concerning the question above.*