Dig-In:

A tale of three integrals

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Indefinite integrals

An indefinite integral, also called an **antiderivative** computes classes of functions:

$$\int f(x)\,dx = \text{"a class of functions whose derivative is } f\text{"}$$

Here there are no limits of integration, and your answer will have a "$+C$" at the end. Pay attention to the notation:

$$\int f(x)\,dx = F(x) + C$$

Where $F'(x) = f(x)$.

Explanation. *Indefinite integrals* (have/do not have ✓) *limits of integration, and they compute* (signed area/an antiderivative/a class of antiderivatives ✓).

Question 1 *Two students, say Devyn and Riley, are working with the following indefinite integral:*

$$\int \frac{2}{x\ln(x^2)}\,dx$$

Devyn computes the integral as

$$\int \frac{2}{x\ln(x^2)}\,dx = \ln|\ln|x^2|| + C$$

and Riley computes the integral as

$$\int \frac{2}{x\ln(x^2)}\,dx = \ln|\ln|x|| + C.$$

Which student is correct?

Learning outcomes:

Multiple Choice:

(a) *Devyn is correct*

(b) *Riley is correct*

(c) *Both students are correct* ✓

(d) *Neither student is correct*

Feedback (attempt): *Both students are correct! The seeming discrepancy arises from the fact that the "+C" in each case is different!*

Accumulation functions

An **accumulation function**, also called an **area function** computes accumulated area:

$$\int_a^x f(t)\,dt = \text{"a function } F \text{ whose derivative is } f\text{"}$$

This is a function of x whose derivative is f, with the additional property that $F(a) = 0$. Pay attention to the notation:

$$F(x) = \int_a^x f(t)\,dt$$

Where $F'(x) = f(x)$.

Explanation. *Accumulation functions* (have ✓ /do not have) *limits of integration, and they compute* (signed area/an antiderivative ✓ /a class of antiderivatives).

Question 2 *True or false: There exists a function f such that*

$$\int_0^x f(t)\,dt = e^x$$

Multiple Choice:

(a) *true*

(b) *false* ✓

Feedback (attempt): *Let*

$$F(x) = \int_0^x f(t)\,dt,$$

this is an accumulation function and $F(0) = 0$, since no area is accumulated yet. However, $e^0 = 1$. Hence there can be no such function f. On the other hand, there is a function g with

$$\int_0^x g(t)\,dt = e^x - 1$$

namely, $g(x) = e^x$. This subtlety arises from the fact that an accumulation function

$$F(x) = \int_a^x f(t)\,dt$$

*gives a **specific** antiderivative of f, the one that when evaluated at $x = a$ is zero.*

Definite integrals

A **definite integral** computes signed area:

$$\int_a^b f(x)\,dx = \text{"the signed area between the } x\text{-axis and } f\text{"}$$

Here we always have limits of integration, both of which are numbers. Moreover, definite integrals have definite values, the signed area between f and the x-axis. Pay attention to the notation:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Where $F'(x) = f(x)$.

Explanation. *Definite integrals* (have ✓ / do not have) *limits of integration, and they compute* (signed area ✓ / an antiderivative / a class of antiderivatives).

Question 3 *Consider*

$$f(x) = \begin{cases} -2 & \text{if } x < 1, \\ 2 & \text{if } x \geq 1. \end{cases}$$

If we compute an antiderivative of f, we find

$$F(x) = \begin{cases} -2x & \text{if } x < 1, \\ 2x & \text{if } x \geq 1. \end{cases}$$

Is it correct to say

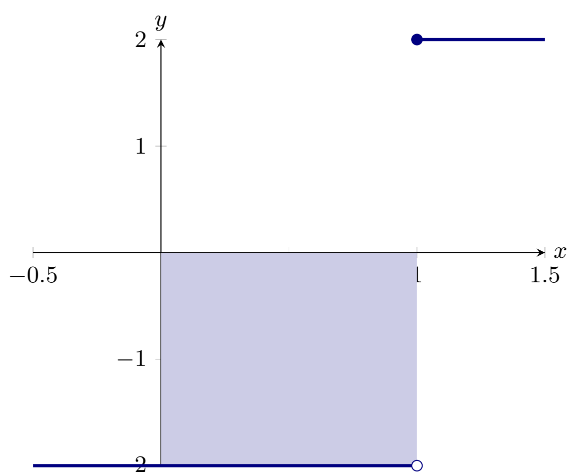
$$\begin{aligned} \int_0^1 f(x)\,dx &= \Big[F(x)\Big]_0^1 \\ &= F(1) - F(0) \\ &= 2? \end{aligned}$$

Multiple Choice:

(a) *yes*

(b) *no* ✓

Feedback (attempt): *Perhaps the first thing to do would be to attempt to analyze this geometrically. Here we see our function and the signed area computed by the integral:*



From the graph above, we can see that

$$\int_0^1 f(x)\,dx = -2.$$

So now the question is, "what went wrong" above? In this case our function f is **not** *continuous! For The Fundamental Theorem of Calculus to apply, the integrand* **must** *be continuous on the interval that one is integrating on. If this is not the case, the fundamental theorem may or may not yield valid results.*