Dig-In:

Working with substitution

We explore more difficult problems involving substitution.

We begin by restating the substitution formula.

Theorem 1 (Integral Substitution Formula)**.** *If g is differentiable on the interval $[a,b]$ and f is differentiable on the interval $[g(a), g(b)]$, then*

$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(g)\,dg.$$

We spend pretty much this entire section working out examples.

Example 1. *Compute:*

$$\int_2^3 \frac{1}{x\ln(x)}\,dx$$

Explanation. *Let*

$$g = \underbrace{\boxed{\ln(x)}}_{\text{given}},$$

computing dg, we find

$$dg = \underbrace{\boxed{\frac{1}{x}}}_{\text{given}}\,dx$$

and solving for dx we find

$$dx = \underbrace{\boxed{x}}_{\text{given}}\,dg.$$

Learning outcomes: To be able to use the method of substitution to solve more difficult types of integrals. To be able to both correctly identify what to substitute for and to be able to successfully carry out the process to correctly solve the problem.

Now

$$\begin{aligned}
\int_2^3 \frac{1}{x\ln(x)}\,dx &= \int_{g(2)}^{g(3)} \frac{1}{x\cdot g} x\,dg \\
&= \int_{\underbrace{\boxed{\ln(2)}}_{\text{given}}}^{\underbrace{\boxed{\ln(3)}}_{\text{given}}} \underbrace{\boxed{\frac{1}{g}}}_{\text{given}}\,dg \\
&= \left[\underbrace{\boxed{\ln(g)}}_{\text{given}}\right]_{\underbrace{\boxed{\ln(2)}}_{\text{given}}}^{\underbrace{\boxed{\ln(3)}}_{\text{given}}} \\
&= \ln(\ln(3)) - \ln(\ln(2)).
\end{aligned}$$

The next example requires a new technique.

Example 2. *Compute:*

$$\int x^3\sqrt{1-x^2}\,dx$$

Explanation. *Here it is not apparent that the chain rule is involved. However, if it was involved, perhaps a good guess for g would be*

$$g = \underbrace{\boxed{1-x^2}}_{\text{given}}$$

and then

$$\begin{aligned}
dg &= \underbrace{\boxed{-2x}}_{\text{given}}\,dx, \\
dx &= \underbrace{\boxed{\frac{-1}{2x}}}_{\text{given}}\,dg.
\end{aligned}$$

Now we consider the integral we are trying to compute

$$\int x^3\sqrt{1-x^2}\,dx$$

and we substitute using our work above. Write with me

$$\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx &= \int x^3 \underbrace{\boxed{\sqrt{g}}}_{\text{given}} \left(\underbrace{\boxed{\frac{-1}{2x}}}_{\text{given}}\right) dg \\
&= \int \frac{-x^2 \underbrace{\boxed{\sqrt{g}}}_{\text{given}}}{2}\,dg.
\end{aligned}$$

However, we cannot continue until each x is replaced. We know that

$$\begin{aligned} g &= 1 - x^2 \\ \Rightarrow \qquad g - 1 &= -x^2 \\ \Rightarrow \qquad \underbrace{\boxed{1 - g}}_{\text{given}} &= x^2 \end{aligned}$$

so now we may replace x^2

$$\int x^3 \sqrt{1 - x^2}\, dx = \int -\frac{\underbrace{\boxed{(1-g)}}_{\text{given}}\sqrt{g}}{2}\, dg.$$

At this point, we are close to being done. Write

$$\begin{aligned} \int -\frac{(1-g)\sqrt{g}}{2}\, dg &= \int \left(\frac{g\sqrt{g}}{2} - \frac{\sqrt{g}}{2} \right) dg \\ &= \int \frac{\underbrace{\boxed{g^{3/2}}}_{\text{given}}}{2}\, dg - \int \frac{\sqrt{g}}{2}\, dg \\ &= \underbrace{\boxed{\frac{g^{5/2}}{5}}}_{\text{given}} - \underbrace{\boxed{\frac{g^{3/2}}{3}}}_{\text{given}}. \end{aligned}$$

Now recall that $g = 1 - x^2$. Hence our final answer is

$$\int x^3 \sqrt{1 - x^2}\, dx = \underbrace{\boxed{\frac{(1-x^2)^{5/2}}{5}}}_{\text{given}} - \frac{(1-x^2)^{3/2}}{3} + C.$$

Sometimes it is not obvious how a fraction could have been obtained using the chain rule. A common trick though is to substitute for the *denominator* of a fraction. Like all tricks, this technique does not always work. Regardless the next two examples present how this technique can be used.

Example 3. *Compute:*

$$\int \frac{\sec(y)\tan(y) + \sec^2(y)}{\sec(y) + \tan(y)}\, dy$$

Explanation. *We substitute*

$$g = \sec(y) + \tan(y)$$

and we immediately see that

$$dg = \underbrace{\boxed{(\sec(y)\tan(y) + \sec^2(y))}}_{\text{given}} dy,$$

$$dy = \frac{1}{\underbrace{\boxed{\sec(y)\tan(y) + \sec^2(y)}}_{\text{given}}} dg.$$

But this cancels perfectly with the numerator! So we have that

$$\begin{aligned}\int \frac{\sec(y)\tan(y) + \sec^2(y)}{\sec(y) + \tan(y)}\,dy &= \int \frac{1}{g}\,dg \\ &= \ln|g| + C \\ &= \ln|\sec(y) + \tan(y)| + C.\end{aligned}$$

Notice that

$$\frac{\sec(x)\tan(x) + \sec^2(x)}{\sec(x) + \tan(x)} = \frac{\sec(x)(\tan(x) + \sec(x))}{\sec(x) + \tan(x)} = \sec(x)$$

when $\sec(x) \neq -\tan(x)$. So in a very contrived way, we have just proved

Theorem 2.

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C.$$

Notice the variable in this next example.

Example 4. *Compute:*

$$\int \frac{g}{1 - g^2}\,dg$$

Explanation. *We want to substitute for* $1 - g^2$*. But the variable "g" has already been used... OH NO! Never fear! We can substitute with whatever variable that we want. In particular, let us use "h" for this problem. So we let*

$$h = 1 - g^2$$

and then

$$dh = \underbrace{\boxed{-2g}}_{\text{given}} dg,$$

$$dg = \underbrace{\boxed{\frac{-1}{2g}}}_{\text{given}} dh.$$

Thus

$$\begin{aligned}\int \frac{g}{1-g^2}\,dg &= \int \frac{g}{h}\left(\underbrace{\boxed{\frac{-1}{2g}}}_{\text{given}}\right)dh \\ &= -\frac{1}{2}\int \frac{1}{h}\,dh \\ &= -\frac{1}{2}\ln|h| + C \\ &= -\frac{1}{2}\ln|1-g^2| + C.\end{aligned}$$

Example 5. *Compute:*

$$\int \tan(x)\,dx$$

Explanation. *We begin by writing*

$$\begin{aligned}\tan(x) &= \frac{\sin(x)}{\cos(x)} \\ &= \frac{\sin(x)}{\cos(x)}\cdot\frac{\cos(x)}{\cos(x)} \\ &= \frac{\sin(x)\cos(x)}{1-\sin^2(x)}.\end{aligned}$$

We then make the substitution

$$g = \sin(x)$$

and so

$$\begin{aligned}dg &= \underbrace{\boxed{\cos(x)}}_{\text{given}}\,dx, \\ dx &= \frac{1}{\underbrace{\boxed{\cos(x)}}_{\text{given}}}\,dg.\end{aligned}$$

Then

$$\begin{aligned}\int \tan(x)\,dx &= \int \frac{\sin(x)\cos(x)}{1-\sin^2(x)}\,dx \\ &= \int \frac{g\cos(x)}{1-g^2}\cdot\frac{1}{\underbrace{\boxed{\cos(x)}}_{\text{given}}}\,dg \\ &= \int \frac{g}{1-g^2}\,dg.\end{aligned}$$

But this is the same problem as Example 4! And so we know that

$$\begin{aligned}\int \tan(x)\,dx &= -\frac{1}{2}\ln|1-g^2| + C\\ &= -\frac{1}{2}\ln|1-\sin^2(x)| + C\\ &= -\frac{1}{2}\ln|\cos^2(x)| + C\\ &= \ln|\cos^2(x)|^{-\frac{1}{2}} + C\\ &= \ln|\sec(x)| + C.\end{aligned}$$

We have just proved

Theorem 3.

$$\int \tan(x)\,dx = \ln|\sec(x)| + C.$$

Note that in Example 5, we could have instead made the substitution

$$g = 1 - \sin^2(x).$$

This would have gotten us to the answer quicker and without using Example 4. You are encouraged to work this out on your own right now!

We end this section with two more difficult examples.

Example 6. *Compute:*

$$\int \frac{e^{2x}}{1-e^{2x}}\,dx$$

Explanation. *Maybe the biggest key to solving this problem is to recall that*

$$e^{2x} = (\underbrace{\boxed{e^x}}_{\text{given}})^2.$$

So we can rewrite the problem

$$\int \frac{e^{2x}}{1-e^{2x}}\,dx = \int \frac{(\underbrace{\boxed{e^x}}_{\text{given}})^2}{1-(\underbrace{\boxed{e^x}}_{\text{given}})^2}\,dx.$$

Now, if we make the substitution $g = e^x$, we have that

$$\begin{aligned} dg &= \underbrace{\boxed{e^x}}_{\text{given}}\,dx,\\ dx &= \frac{1}{\underbrace{\boxed{e^x}}_{\text{given}}}\,dg,\end{aligned}$$

and

$$\int \frac{e^{2x}}{1-e^{2x}}\,dx = \int \frac{(e^x)^2}{1-g^2} \cdot \frac{1}{\underbrace{\boxed{e^x}}_{\text{given}}}\,dg$$
$$= \int \frac{e^x}{1-g^2}\,dg$$
$$= \int \frac{g}{1-g^2}\,dg.$$

But now we are back to Example 4, and so we know that

$$\int \frac{e^{2x}}{1-e^{2x}}\,dx = -\frac{1}{2}\ln|1-g^2| + C$$
$$= -\frac{1}{2}\ln|1-e^{2x}| + C.$$

Again, in the previous example we could have instead made the substitution

$$g = 1 - e^{2x}$$

and avoided using Example 4. In general, any time that you make two successive substitutions in a problem, you could have instead just made one substitution. This one substitution is the *composition* of the two original substitutions. But sometimes it may not be obvious to make one clever substitution, and so two substitutions makes more sense. The next example helps to demonstrate this.

Example 7. *Compute:*

$$\int_0^{16} \sqrt{4-\sqrt{x}}\,dx$$

Explanation. *While it is not obvious at all, let us try the substitution*

$$g = \sqrt{x}.$$

Then

$$dg = \underbrace{\boxed{\frac{1}{2\sqrt{x}}}}_{\text{given}}\,dx,$$
$$dx = 2\sqrt{x}\,dg = 2g\,dg,$$

and so

$$\int_0^{16} \sqrt{4-\sqrt{x}}\,dx = \int_{g(0)}^{g(16)} \underbrace{\boxed{\sqrt{4-g}\cdot 2g}}_{\text{given}}\,dg$$
$$= \int_{\underbrace{\boxed{0}}_{\text{given}}}^{\underbrace{\boxed{4}}_{\text{given}}} \underbrace{\boxed{2g\sqrt{4-g}}}_{\text{given}}\,dg.$$

From here we now make the second (and more obvious) substitution

$$h = 4 - g.$$

Then $g = 4 - h$, *and*

$$\begin{aligned} dh &= -\,dg, \\ dg &= -\,dh. \end{aligned}$$

So

$$\begin{aligned}
\int_0^{16} \sqrt{4-\sqrt{x}}\,dx &= \int_0^4 2g\sqrt{4-g}\,dg \\
&= \int_{h(0)}^{h(4)} 2(4-h)\sqrt{h}(-1)\,dh \\
&= -\int_{\underset{\text{given}}{\boxed{4}}}^{\underset{\text{given}}{\boxed{0}}} (8h^{\frac{1}{2}} - 2h^{\frac{3}{2}})\,dh \\
&= \int_{\underset{\text{given}}{\boxed{0}}}^{\underset{\text{given}}{\boxed{4}}} (8h^{\frac{1}{2}} - 2h^{\frac{3}{2}})\,dh \\
&= \Bigg[\underset{\text{given}}{\boxed{8\left(\frac{2}{3}\right)h^{\frac{3}{2}} - 2\left(\frac{2}{5}\right)h^{\frac{5}{2}}}}\Bigg]_0^4 \\
&= \left(\frac{16}{3}(4)^{\frac{3}{2}} - \frac{4}{5}(4)^{\frac{5}{2}}\right) - (0-0) \\
&= \frac{128}{3} - \frac{128}{5} \\
&= \frac{256}{15}.
\end{aligned}$$