**Break-Ground:**

# Volumes of aluminum cans

*Two young mathematicians discuss optimizing aluminum cans.*

Check out this dialogue between two calculus students (based on a true story):

**Devyn:** Riley, have you ever noticed aluminum cans?

**Riley:** So very recyclable!

**Devyn:** I know! But I've also noticed that there are some that are short and fat, and others that are tall and skinny, and yet they can still have the same volume!

**Riley:** So very observant!

**Devyn:** This got me wondering, if we want to make a can with volume $V$, what shape of can uses the least aluminum?

**Riley:** Ah! This sounds like a job for calculus! The volume of a cylindrical can is given by

$$V = \pi \cdot r^2 \cdot h$$

where $r$ is the radius of the cylinder and $h$ is the height of the cylinder. Also the surface area is given by

$$\begin{aligned} A &= \underbrace{\pi \cdot r^2}_{\text{bottom}} + \underbrace{2 \cdot \pi \cdot r \cdot h}_{\text{sides}} + \underbrace{\pi \cdot r^2}_{\text{top}} \\ &= 2 \cdot \pi \cdot r^2 + 2 \cdot \pi \cdot r \cdot h. \end{aligned}$$

Somehow we want to minimize the surface area, because that's the amount of aluminum used, but we also want to keep the volume constant.

**Devyn:** Whoa, we have way too many letters here.

**Riley:** Yeah, somehow we need only one variable. Yikes. Too many letters.

**Problem 1** *Suppose we wish to construct an aluminum can with volume $V$ that uses the least amount of aluminum. In the context above, what do we want to minimize?*

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Learning outcomes: Interpret an optimization problem as the procedure used to make a system or design as effective or functional as possible. Set up an optimization problem by identifying the objective function and appropriate constraints.

***Multiple Choice:***

(a) $A$ ✓

(b) $V$

(c) $h$

(d) $r$

**Problem 2** *In the context above, what should be considered a constant?*

***Select All Correct Answers:***

(a) $A$

(b) $V$ ✓

(c) $h$

(d) $r$

As Devyn and Riley noticed, when we work out this type of problem, we need to reduce the problem to a single variable.

**Problem 3** *Consider $r$ to be the variable, and express $A$ as a function of $r$.*

***Hint:*** *First, let's eliminate the variable $h$. We know that $V$ is a constant and that*

$$V = \pi \cdot r^2 \cdot h$$

*so* $h = \boxed{V/(\pi r^2)}$. *Substitute this expression for $h$ in the equation*

$$A = 2 \cdot \pi \cdot r^2 + 2 \cdot \pi \cdot r \cdot h.$$

$$A = \boxed{2 \cdot \pi \cdot r^2 + 2 \cdot \pi \cdot r \cdot V/(\pi r^2)}$$

**Problem 4** *Now consider $h$ to be the variable, and express $A$ as a function of $h$.*

***Hint:*** *First, let's eliminate the variable $r$. We know that $V$ is a constant and that*

$$V = \pi \cdot r^2 \cdot h$$

*so* $r = \boxed{\sqrt{V/(\pi h)}}$. *Substitute this expression for $h$ in the equation*

$$A = 2 \cdot \pi \cdot r^2 + 2 \cdot \pi \cdot r \cdot h.$$

$$A = \boxed{2 \cdot \pi \cdot V/(\pi h) + 2 \cdot \pi \cdot \sqrt{V/(\pi h)} \cdot h}$$

Notice that we've reduced (one way or another) this function of two variables to a function of one variable. This process will be a key step in nearly every problem in this next section.