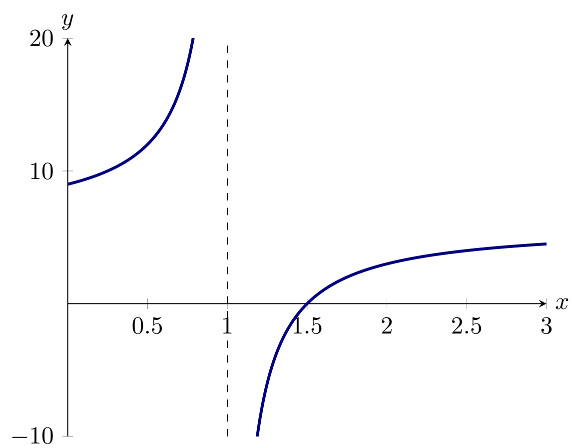
Dig-In:

Horizontal asymptotes

We explore functions that behave like horizontal lines as the input grows without bound.

Consider the function:

$$f(x) = \frac{6x-9}{x-1}$$



As x approaches infinity, it seems like $f(x)$ approaches a specific value. Such a limit is called a *limit at infinity*.

Definition 1. *If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large, we write*

$$\lim_{x\to\infty} f(x) = L$$

and we say, the ***limit at infinity*** *of $f(x)$ is L.*

If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large and negative, we write

$$\lim_{x\to-\infty} f(x) = L$$

and we say, the ***limit at negative infinity*** *of $f(x)$ is L.*

Learning outcomes: Find horizontal asymptotes using limits. Recognize that a curve can cross a horizontal asymptote. Calculate the limit as x approaches $\pm\infty$ of common functions algebraically. Find the limit as x approaches $\pm\infty$ from a graph. Define a horizontal asymptote. Compute limits at infinity of famous functions. Identify horizontal asymptotes by looking at a graph.

Example 1. *Compute*

$$\lim_{x\to\infty} \frac{6x-9}{x-1}.$$

Explanation. *Write*

$$\begin{aligned}\lim_{x\to\infty} \frac{6x-9}{x-1} &= \lim_{x\to\infty} \frac{6x-9}{x-1} \cdot \frac{1/x}{1/x} \\ &= \lim_{x\to\infty} \frac{\frac{6x}{x} - \frac{9}{x}}{\frac{x}{x} - \frac{1}{x}} \\ &= \lim_{x\to\infty} \frac{6}{1} \\ &= 6.\end{aligned}$$

Sometimes one must be careful, consider this example.

Example 2. *Compute*

$$\lim_{x\to-\infty} \frac{x^3+1}{\sqrt{x^6+5}}$$

Explanation. *In this case we multiply the numerator and denominator by* $-1/x^3$*, which is a positive number as since* $x \to -\infty$*,* x^3 *is a negative number.*

$$\begin{aligned}\lim_{x\to-\infty} \frac{x^3+1}{\sqrt{x^6+5}} &= \lim_{x\to-\infty} \frac{x^3+1}{\sqrt{x^6+5}} \cdot \frac{-1/x^3}{-1/x^3} \\ &= \lim_{x\to-\infty} \frac{-1-1/x^3}{\sqrt{x^6/x^6+5/x^6}} \\ &= -1.\end{aligned}$$

Note, since

$$\lim_{x\to\infty} f(x) = \lim_{x\to0^+} f\left(\frac{1}{x}\right)$$

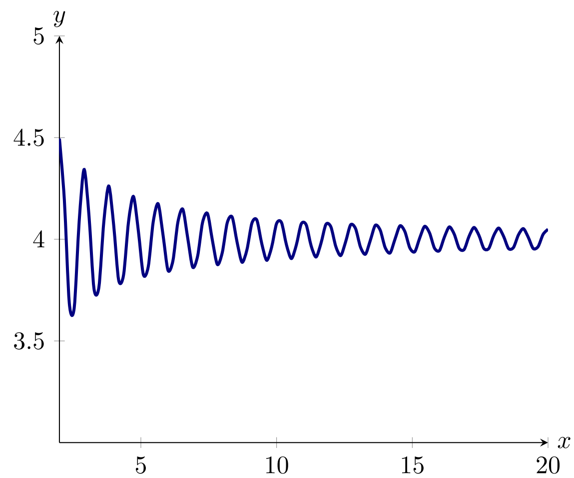
and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to0^-} f\left(\frac{1}{x}\right)$$

we can also apply the Squeeze Theorem when taking limits at infinity. Here is an example of a limit at infinity that uses the Squeeze Theorem, and shows that functions can, in fact, cross their horizontal asymptotes.

Example 3. *Compute:*

$$\lim_{x\to\infty} \frac{\sin(7x)+4x}{x}$$



Explanation. *We can bound our function*

$$\frac{-1+4x}{x} \leq \frac{\sin(7x)+4x}{x} \leq \frac{1+4x}{x}.$$

Now write with me

$$\begin{aligned}\lim_{x\to\infty} \frac{-1+4x}{x} \cdot \frac{1/x}{1/x} &= \lim_{x\to\infty} \frac{-1/x+4}{1} \\ &= 4.\end{aligned}$$

And we also have

$$\begin{aligned}\lim_{x\to\infty} \frac{1+4x}{x} \cdot \frac{1/x}{1/x} &= \lim_{x\to\infty} \frac{1/x+4}{1} \\ &= 4.\end{aligned}$$

Since

$$\lim_{x\to\infty} \frac{-1+4x}{x} = 4 = \lim_{x\to\infty} \frac{1+4x}{x}$$

we conclude by the Squeeze Theorem, $\lim_{x\to\infty} \frac{\sin(7x)}{x} + 4 = 4.$

Definition 2. *If*

$$\lim_{x\to\infty} f(x) = L \qquad \textit{or} \qquad \lim_{x\to-\infty} f(x) = L,$$

then the line $y = L$ is a ***horizontal asymptote*** *of $f(x)$.*

Example 4. *Give the horizontal asymptotes of*

$$f(x) = \frac{6x-9}{x-1}$$

Explanation. *From our previous work, we see that* $\lim_{x\to\infty} f(x) = 6$, *and upon further inspection, we see that* $\lim_{x\to-\infty} f(x) = 6$. *Hence the horizontal asymptote of* $f(x)$ *is the line* $y = 6$.

It is a common misconception that a function cannot cross an asymptote. As the next example shows, a function can cross a horizontal asymptote, and in the example this occurs an infinite number of times!

Example 5. *Give a horizontal asymptote of*

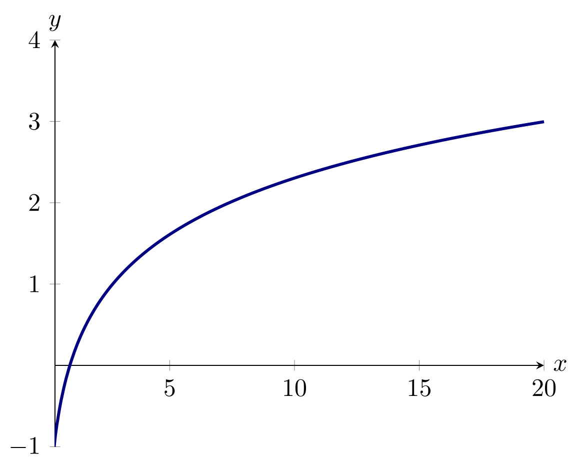
$$f(x) = \frac{\sin(7x) + 4x}{x}.$$

Explanation. *Again from previous work, we see that* $\lim_{x\to\infty} f(x) = \underbrace{\boxed{4}}_{\text{given}}$. *Hence* $y = \underbrace{\boxed{4}}_{\text{given}}$ *is a horizontal asymptote of* $f(x)$.

We conclude with an infinite limit at infinity.

Example 6. *Compute*

$$\lim_{x\to\infty} \ln(x)$$



Explanation. *The function* $\ln(x)$ *grows very slowly, and seems like it may have a horizontal asymptote, see the graph above. However, if we consider the definition of the natural log as the inverse of the exponential function*

$\ln(x) = y$ *means that* $e^y = x$ *and that* x *is positive.*

We see that we may raise e to higher and higher values to obtain larger numbers. This means that $\ln(x)$ *is unbounded, and hence* $\lim_{x\to\infty} \ln(x) = \infty$.