Dig-In:

Concepts of graphing functions

We use the language of calculus to describe graphs of functions.

In this section, we review the graphical implications of limits, and the sign of the first and second derivative. You already know all this stuff: it is just important enough to hit it more than once, and put it all together.

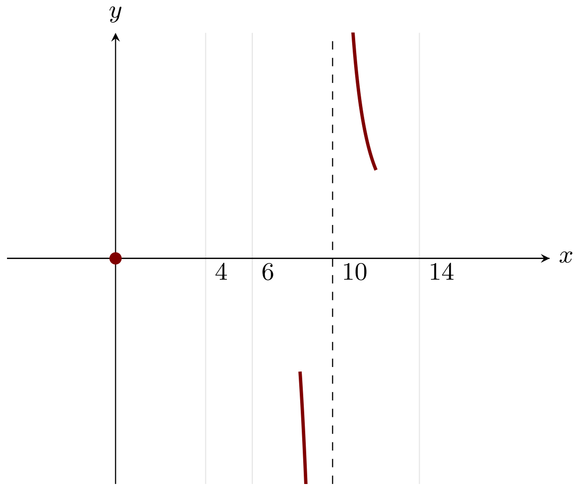
Example 1. *Sketch the graph of a function f which has the following properties:*

- $f(0) = 0$
- $\lim_{x\to 10^+} f(x) = +\infty$
- $\lim_{x\to 10^-} f(x) = -\infty$
- $f'(x) < 0$ *on* $(-\infty, 0) \cup (6, 10) \cup (10, 14)$
- $f'(x) > 0$ *on* $(0, 6) \cup (6, 10) \cup (14, \infty)$
- $f''(x) < 0$ *on* $(4, 10)$
- $f''(x) > 0$ *on* $(-\infty, 4) \cup (10, \infty)$

Explanation. *Try this on your own first, then either check with a friend or check the online version.*

Hint: *The first thing we will do is to plot the point $(0, 0)$ and indicate the appropriate vertical asympote due to the limit conditions. We also mark all of the places where f' or f'' change sign.*

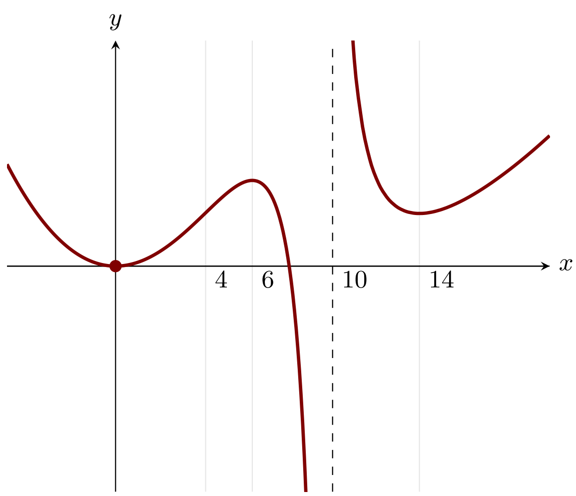
Learning outcomes:



Hint: *Now we classify the behaviour on each of the intervals:*

- *On* $(-\infty, 0)$, f *is* (increasing/decreasing ✓) *and concave* (up ✓ /down)
- *On* $(0, 4)$, f *is* (increasing ✓ /decreasing) *and concave* (up ✓ /down)
- *On* $(4, 6)$, f *is* (increasing ✓ /decreasing) *and concave* (up/down ✓)
- *On* $(6, 10)$, f *is* (increasing/decreasing ✓) *and concave* (up/down ✓)
- *On* $(10, 14)$, f *is* (increasing/decreasing ✓) *and concave* (up ✓ /down)
- *On* $(14, \infty)$, f *is* (increasing ✓ /decreasing) *and concave* (up ✓ /down)

Hint: *Utilizing all of this information, we are forced to sketch something like the following:*

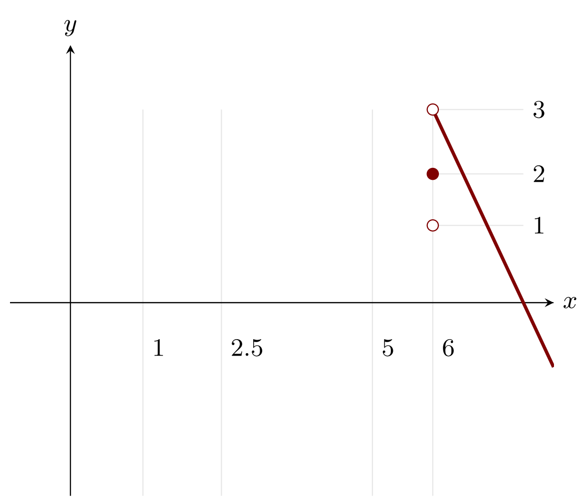


Example 2. *Sketch the graph of a function f which has the following properties:*

- $f(0) = 1$
- $f(6) = 2$
- $\lim_{x\to 6^+} f(x) = 3$
- $\lim_{x\to 6^-} f(x) = 1$
- $f'(x) < 0$ *on* $(-\infty, 1)$
- $f'(x) > 0$ *on* $(1, 6)$
- $f'(x) = -2$ *on* $(6, \infty)$
- $f''(x) < 0$ *on* $(2.5, 5)$
- $f''(x) > 0$ *on* $(-\infty, 2.5) \cup (5, 6)$

Explanation. *Try this on your own first, then either check with a friend or check the online version.*

Hint: *The first thing we will do is to plot the points $(0, 1)$ and $(6, 2)$, and the "holes" at $(6, 3)$ and $(6, 1)$ due to the limit conditions. We can immediately draw in what f looks like on $(6, \infty)$ since it is linear with slope 2, and must connect to the hole at $(6, 2)$. We also mark all of the places where f' or f'' change sign.*

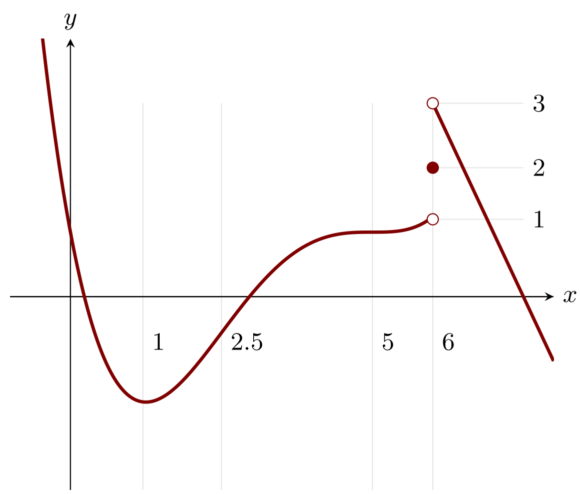


Hint: *Now we classify the behaviour on each of the intervals:*

- *On* $(-\infty, 1)$, f *is* (increasing/decreasing ✓) *and concave* (up ✓ /down)

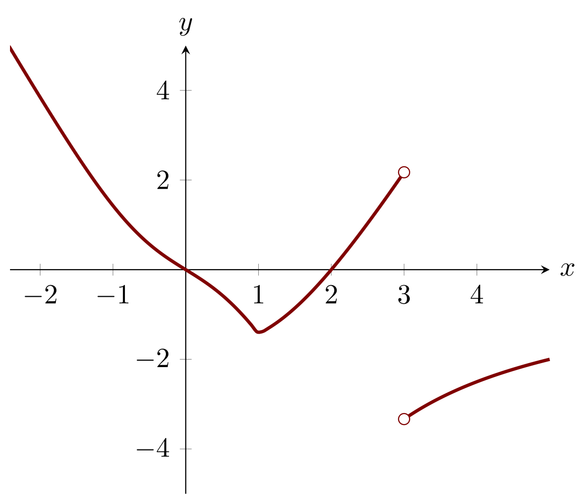
- *On* $(1, 2.5)$*, f is* (increasing ✓ /decreasing) *and concave* (up ✓ /down)
- *On* $(2.5, 5)$*, f is* (increasing ✓ /decreasing) *and concave* (up/down ✓)
- *On* $(5, 6)$*, f is* (increasing ✓ /decreasing) *and concave* (up ✓ /down)

Hint: *Utilizing all of this information, we are forced to draw something like the following:*



Example 3. *The graph of f' (the derivative of f) is shown below.*

Assume f is continuous for all real numbers.



Question 1 *On which of the following intervals is f increasing?*

Select All Correct Answers:

(a) $(-\infty, 0)$ ✓

(b) $(0, 1)$

(c) $(1, 2)$

(d) $(2, 3)$ ✓

(e) $(3, \infty)$

Hint: *f is increasing where $f'(x) > 0$, i.e. on the intervals $(-\infty, 0)$ and $(2, 3)$.*

Question 2 *Which of the following are critical points of f?*

Select All Correct Answers:

(a) $x = 0$ ✓

(b) $x = 1$

(c) $x = 2$ ✓

(d) $x = 3$ ✓

Hint: *f has a critical point at the zeros of f', and the places where f' does not exist. In this case, $x = 0$, $x = 2$, and $x = 3$.*

Question 3 *Where do the local maxima occur?*

Select All Correct Answers:

(a) $x = 0$ ✓

(b) $x = 1$

(c) $x = 2$

(d) $x = 3$ ✓

Hint: *A local maximum occurs at a critical point where the function transitions from increasing to decreasing, i.e. the derivative passes from positive to negative. In this case, we see that the local maxima occur at $x = 0$ and $x = 3$.*

Question 4 *Where does a point of inflection occur?*

Select All Correct Answers:

(a) $x = 0$

(b) $x = 1$ ✓

(c) $x = 2$

(d) $x = 3$

Hint: *A point of inflection occurs when the concavity of f changes. This is reflected in the sign of f'' changing. This only occurs at one point in this graph, namely $x = 1$.*

Question 5 *On which of the following intervals is f concave down?*

Select All Correct Answers:

(a) $(-\infty, 0)$ ✓

(b) $(0, 1)$ ✓

(c) $(1, 2)$

(d) $(2, 3)$

(e) $(3, \infty)$

Hint: *f is concave down when $f''(x) < 0$. This occurs for $x < 1$ on this graph. So the correct answer is to select both $(-\infty, 0)$ and $(0, 1)$.*