Dig-In:

Applied related rates

We work related rates problems in context.

Now we are ready to work related rates problems in context. Just as before, we are going to follow essentially the same plan of attack in each problem.

Draw a picture. If possible, draw a schematic picture with all the relevant information.

Find equations. We want equations that relate all relevant functions.

Differentiate the equations. Here we will often use implicit differentiation.

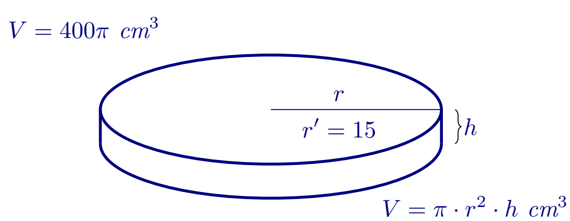
Evaluate and solve. Evaluate each equation at all known desired values and solve for the relevant rate.

Formulas

Example 1. *A hand-tossed pizza crust starts off as a ball of dough with a volume of 400π cm^3. First, the cook stretches the dough to the shape of a cylinder of radius 12 cm. Next the cook tosses the dough.*

If during tossing, the dough maintains the shape of a cylinder and the radius is increasing at a rate of 15 cm/min, how fast is its thickness changing when the radius is 20 cm?

Explanation. *To start,* ***draw a picture****. Here we see a cylinder that represents our pizza dough.*



Learning outcomes: Identify word problems as related rates problems. Solve related rates word problems. Translate word problems into mathematical expressions.

*Next we need to **find equations**. We see that we have*

$$400\pi = \underbrace{\boxed{\pi \cdot r^2 \cdot h}}_{\text{given}},$$

which immediately simplifies to

$$400 = r^2 \cdot h.$$

Imagining that r and h are functions of time, we now may write

$$400 = r(t)^2 \cdot h(t)$$

*and so we may now **differentiate the equation** using implicit differentiation, treating all functions as functions of t,*

$$0 = 2 \cdot r(t) \cdot r'(t) \cdot h(t) + r(t)^2 \cdot h'(t).$$

*Now we'll **evaluate and solve**. We know that $r(t) = \underbrace{\boxed{20}}_{\text{given}}$ cm and that $r'(t) = \underbrace{\boxed{15}}_{\text{given}}$ cm/min. Moreover, we can now find $h(t)$ as we have*

$$400 = r(t)^2 \cdot h(t) \qquad \textit{meaning} \qquad h(t) = \frac{400}{r(t)^2}.$$

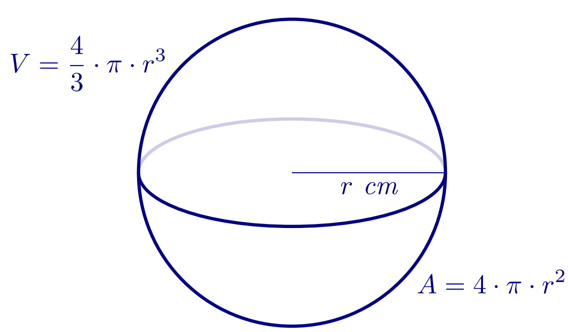
Since $20^2 = 400$, we see that $h(t) = 1$. Substituting in, we see

$$\begin{aligned} 0 &= 2 \cdot 20 \cdot 15 \cdot 1 + 20^2 \cdot h'(t), \\ -2 \cdot 20 \cdot 15 &= 20^2 \cdot h'(t), \\ \frac{-2 \cdot 20 \cdot 15}{20^2} &= h'(t) \\ -1.5 &= h'(t). \end{aligned}$$

Hence the thickness of the dough is changing at a rate of $\underbrace{\boxed{-1.5}}_{\text{given}}$ cm/min.

Example 2. *Consider a melting snowball. We will assume that the rate that the snowball is melting is proportional to its surface area. Show that the radius of the snowball is changing at a constant rate.*

Explanation. *To start, **draw a picture**.*



Next we need to ***find equations****. The equations we'll use are*

$$V = \underbrace{\boxed{(4/3)\cdot\pi\cdot r^3}}_{\text{given}} \qquad \textit{and} \qquad A = \underbrace{\boxed{4\cdot\pi\cdot r^2}}_{\text{given}}.$$

Now the key words are "the rate that the snowball is melting is proportional to its surface area." From this we have the following equation:

$$\underbrace{V'}_{\textit{rate the snowball is melting}} = \overbrace{k}^{\textit{is proportional to}} \cdot \underbrace{A}_{\textit{its surface area}}$$

So we need to know V'*. We know* $V = \frac{4}{3}\cdot\pi\cdot r^3$*. If we imagine* r *as a function of* t*, we can write volume as a function of* t*:*

$$V(t) = \frac{4}{3}\cdot\pi\cdot r(t)^3$$

so

$$V'(t) = 4\cdot\pi\cdot r(t)^2\cdot r'(t).$$

Now we'll ***evaluate and solve****. We now know*

$$V'(t) = 4\cdot\pi\cdot r(t)^2\cdot r'(t) = k\cdot 4\cdot\pi\cdot r(t)^2 = k\cdot A(t).$$

So

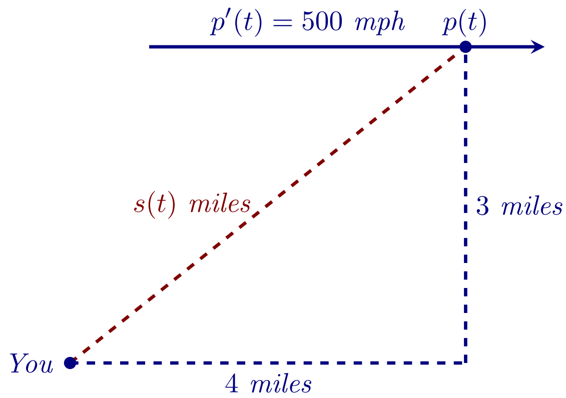
$$\begin{aligned} 4\cdot\pi\cdot r(t)^2\cdot r'(t) &= k\cdot 4\cdot\pi\cdot r(t)^2 \\ r'(t) &= k. \end{aligned}$$

Hence the radius is changing at a constant rate.

Right triangles

Example 3. *A plane is flying directly away from you at* 500 *mph at an altitude of* 3 *miles. How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground* 4 *miles from you?*

Explanation. *To start,* ***draw a picture****.*



Next we need to ***find equations****. By the Pythagorean Theorem we know that*

$$p^2 + 3^2 = s^2.$$

Imagining that p and s are functions of time, we now ***differentiate the equation****. Write*

$$2 \cdot p(t) \cdot p'(t) = 2 \cdot s(t) \cdot s'(t).$$

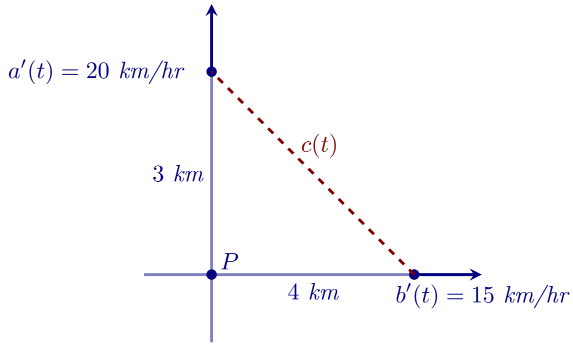
Now we'll ***evaluate and solve****. We are interested in the time at which* $p(t) = \underbrace{\boxed{4}}_{\text{given}}$ *and* $p'(t) = \underbrace{\boxed{500}}_{\text{given}}$*. Additionally, at this time we know that* $4^2 + 9 = s^2$*, so* $s(t) = \underbrace{\boxed{5}}_{\text{given}}$*. Putting together all the information we get*

$$2(4)(500) = 2(5)s'(t),$$

thus $s'(t) = \underbrace{\boxed{400}}_{\text{given}}$ *mph.*

Example 4. *A road running north to south crosses a road going east to west at the point P. Cyclist A is riding north along the first road, and cyclist B is riding east along the second road. At a particular time, cyclist A is* 3 *kilometers to the north of P and traveling at* 20 *km/hr, while cyclist B is* 4 *kilometers to the east of P and traveling at* 15 *km/hr. How fast is the distance between the two cyclists changing?*

Explanation. *We start the same way we always do, we* ***draw a picture****.*



Here $a(t)$ is the distance of cyclist A north of P at time t, and $b(t)$ the distance of cyclist B east of P at time t, and $c(t)$ is the distance from cyclist A to cyclist B at time t.

*We must **find equations**. By the Pythagorean Theorem,*

$$c(t)^2 = a(t)^2 + b(t)^2.$$

*Now we can **differentiate the equation**. Taking derivatives we get*

$$2c(t)c'(t) = 2a(t)a'(t) + 2b(t)b'(t).$$

*Now we can **evaluate and solve**. We know that* $a(t) = \underset{\text{given}}{\boxed{3}}$, $a'(t) = \underset{\text{given}}{\boxed{20}}$, $b(t) = \underset{\text{given}}{\boxed{4}}$ *and* $b'(t) = \underset{\text{given}}{\boxed{15}}$. *Hence by the Pythagorean Theorem,* $c(t) = \underset{\text{given}}{\boxed{5}}$. *So*

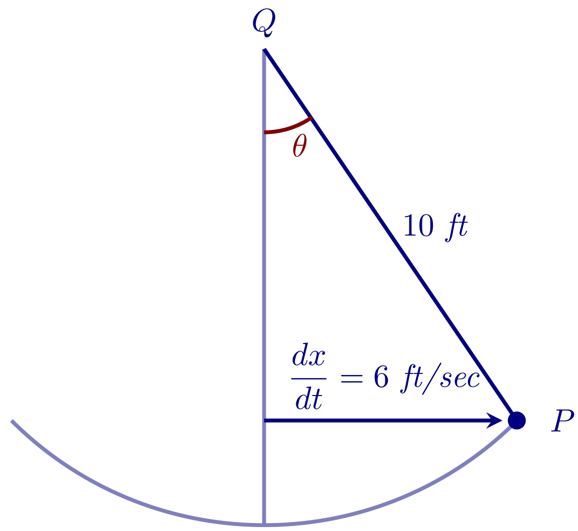
$$2 \cdot 5 \cdot c'(t) = 2 \cdot 3 \cdot 20 + 2 \cdot 4 \cdot 15$$

solving for $c'(t)$ we find $c'(t) = \underset{\text{given}}{\boxed{24}}$ *km/hr.*

Angular rates

Example 5. *A swing consists of a board at the end of a 10 ft long rope. Think of the board as a point P at the end of the rope, and let Q be the point of attachment at the other end. Suppose that the swing is directly below Q at time $t = 0$, and is being pushed by someone who walks at 6 ft/sec from left to right. What is the angular speed of the rope in rad/sec after 1 sec?*

Explanation. *To start, **draw a picture**.*



*Now we must **find equations**. From the right triangle in our picture, we see*

$$\sin(\theta) = \underbrace{\boxed{x/10}}_{\text{given}}.$$

*We can now **differentiate the equation**. Taking derivatives we obtain*

$$\cos(\theta) \cdot \theta'(t) = 0.1x'(t).$$

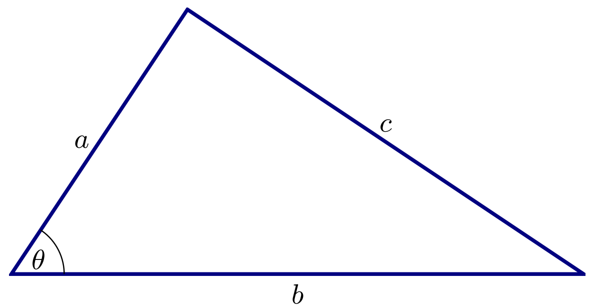
*Now we can **evaluate and solve**. When $t = 1$ sec, the person was pushed by someone who walks 6 ft/sec. Hence we have a $6 - 8 - 10$ right triangle, with $x'(t) = \underbrace{\boxed{6}}_{\text{given}}$, and $\cos\theta = 8/10$. Thus*

$$(8/10)\theta'(t) = \underbrace{\boxed{6/10}}_{\text{given}},$$

and so $\theta'(t) = \underbrace{\boxed{3/4}}_{\text{given}}$ rad/sec.

Example 6. *The Palace of Westminster in London has a large clock tower. The minute hand is 4.2 meters long and the hour hand is 2.7 meters long. At what rate is the distance between the tip of the hands is changing when the clock strikes 3 pm?*

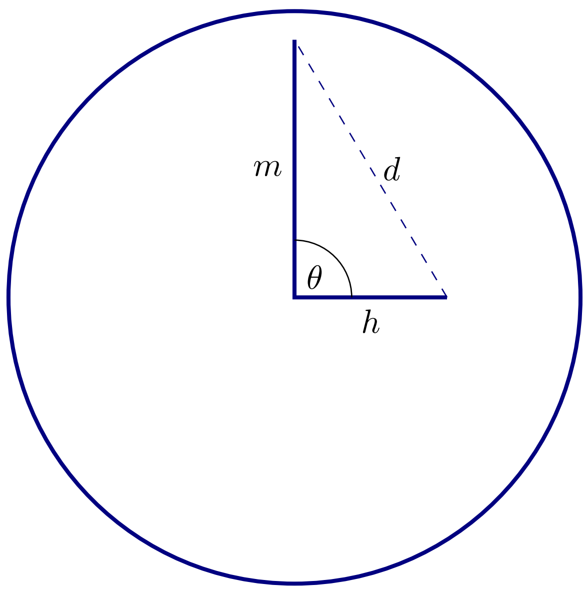
Hint: *Recall the Law of Cosines: Given a triangle with sides lengths a, b, and c,*



we then have

$$c^2 = a^2 + b^2 - 2ab\cos(\theta).$$

Explanation. *To start, we* ***draw a picture****.*



Now we must ***find equations*** *that combine relevant functions. Initially we might suppose that*

$$d^2 = m^2 + h^2;$$

however, here θ is function of time, so this relationship only holds for certain times. Hence we must use the Law of Cosines to write

$$d^2 = \underbrace{\boxed{m^2 + h^2}}_{\text{given}} - 2mh\cos(\theta).$$

To find θ, imagine we are measuring the angle starting at "twelve o'clock" with t being measured in hours. Then letting θ_m be the angle made by the minute *hand and θ_h be the angle made by the hour hand we have*

$$\theta_m(t) = \underbrace{\boxed{2\pi \cdot t}}_{\text{given}},$$
$$\theta_h(t) = \frac{2\pi}{12} \cdot t = \frac{\pi}{6} \cdot t.$$

Finally since θ is decreasing, as the minute hand is traveling faster than the hour hand,

$$\theta(t) = \theta_h(t) - \theta_m(t).$$

On the other hand, m and h are constants. We may now write

$$d(t)^2 = m^2 + h^2 - 2mh\cos(\theta(t)).$$

*If **differentiate the equations** using implicit differentiation we find*

$$\theta'(t) = \underbrace{\boxed{\frac{\pi}{6} - 2\pi}}_{\text{given}}$$

and

$$2 \cdot d(t) \cdot d'(t) = 2mh\sin(\theta(t)) \cdot \theta'(t).$$

*Now we **evaluate and solve**. We know that $m = 4.2$, $h = 2.7$, $\theta'(t) = \dfrac{-11\pi}{6}$, and since the time is 3 pm, $\theta(t) = \pi/2$. Thus*

$$d(t) \cdot d'(t) = \underbrace{\boxed{4.2}}_{\text{given}} \cdot 2.7 \cdot \frac{-11\pi}{6}.$$

on the other hand

$$d(t) = \sqrt{4.2^2 + 2.7^2}$$

and so

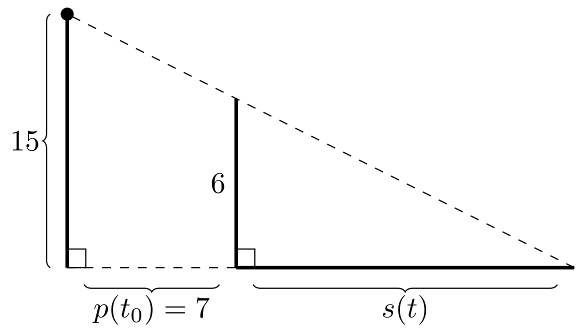
$$d'(t) = \frac{4.2 \cdot 2.7 \cdot \frac{-11\pi}{6}}{\sqrt{4.2^2 + 2.7^2}}.$$

This is the desired rate in units of meters per hour.

Similar triangles

Example 7. *It is night. Someone who is 6 feet tall is walking away from a street light at a rate of 3 feet per second. The street light is 15 feet tall. The person casts a shadow on the ground in front of them. How fast is the length of the shadow growing when the person is 7 feet from the street light?*

Explanation. *To start, **draw a picture**.*



Here t is the variable and t_0 is the specific time when $p(t_0) = 7$.

Now we need to ***find equations****. We use the fact that we have similar triangles to write:*

$$\begin{aligned}\frac{s(t)+p(t)}{\underbrace{\boxed{15}}_{\text{given}}} &= \frac{s(t)}{\underbrace{\boxed{6}}_{\text{given}}},\\ 6\cdot s(t) + 6\cdot p(t) &= 15\cdot s(t),\\ 6\cdot p(t) &= 9\cdot s(t),\\ 2\cdot p(t) &= 3\cdot s(t).\end{aligned}$$

Now we must ***differentiate the equation****. We should use implicit differentiation, and treat each of the variables as functions of t. Write*

$$2\cdot p'(t) = 3\cdot s'(t)$$

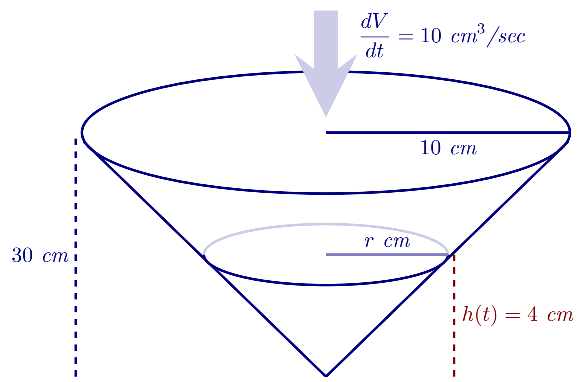
At this point we ***evaluate and solve****. Since the person is waling at a rate of 3 feet per second, we may write*

$$2\cdot 3 = 3\cdot s'(t),$$

and cancel to see that $s'(t) = \underbrace{\boxed{2}}_{\text{given}}$, meaning the shadow is growing at a rate of 2 feet per second.

Example 8. *Water is poured into a conical container at the rate of 10 cm^3/sec. The cone points directly down, and it has a height of 30 cm and a base radius of 10 cm. How fast is the water level rising when the water is 4 cm deep?*

Explanation. *To start,* ***draw a picture****.*



Note, no attempt was made to draw this picture to scale, rather we want all of the relevant information to be available to the mathematician.

Now we need to ***find equations****. The formula for the volume of a cone tells us that*

$$V = \frac{\pi}{3} r^2 h.$$

Also the dimensions of the cone of water must have the same proportions as those of the container. That is, because of similar triangles,

$$\frac{r}{h} = \frac{10}{30} \qquad \textit{so} \qquad r = \underbrace{\boxed{h/3}}_{\text{given}}.$$

Now we must ***differentiate the equation****. We should use implicit differentiation, and treat each of the variables as functions of t. Write*

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) \qquad \textit{and} \qquad \frac{dr}{dt} = \frac{1}{3}\cdot\frac{dh}{dt}.$$

At this point we ***evaluate and solve****. We plug in* $\frac{dV}{dt} = \underbrace{\boxed{10}}_{\text{given}}$, $r = \underbrace{\boxed{4/3}}_{\text{given}}$, $\frac{dr}{dt} = \frac{1}{3}\cdot\frac{dh}{dt}$ *and* $h = \underbrace{\boxed{4}}_{\text{given}}$. *Write*

$$\begin{aligned}
10 &= \frac{\pi}{3}\left(2\cdot\frac{4}{3}\cdot 4\cdot\frac{1}{3}\cdot\frac{dh}{dt} + \left(\frac{4}{3}\right)^2\frac{dh}{dt}\right)\\
10 &= \frac{\pi}{3}\left(\frac{32}{9}\frac{dh}{dt} + \frac{16}{9}\frac{dh}{dt}\right)\\
10 &= \frac{16\pi}{9}\frac{dh}{dt}\\
\frac{90}{16\pi} &= \frac{dh}{dt}.
\end{aligned}$$

Thus, $\dfrac{dh}{dt} = \underbrace{\boxed{\dfrac{90}{16\pi}}}_{\text{given}}$ *cm/sec.*