Dig-In:

Explanation of the product and chain rules

We give explanation for the product rule and chain rule.

Now that we know about differentials, let's use them to give some intuition as to why the product and chain rules are true.

Explanation of the product rule

Linear approximations can help us explain why the product rule works.

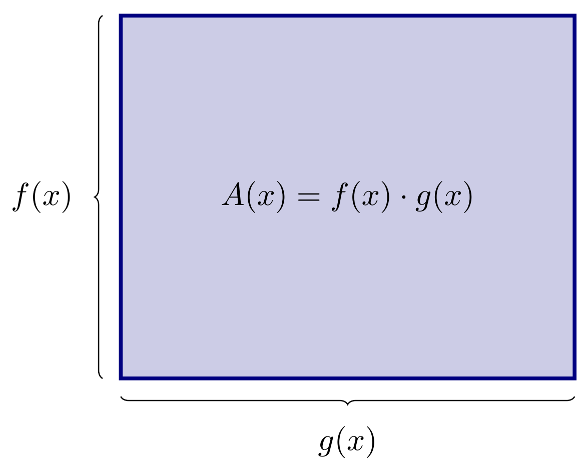
Theorem 1 (The product rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + f'(x)g(x).$$

Explanation. *To start, we need some way to understand the function*

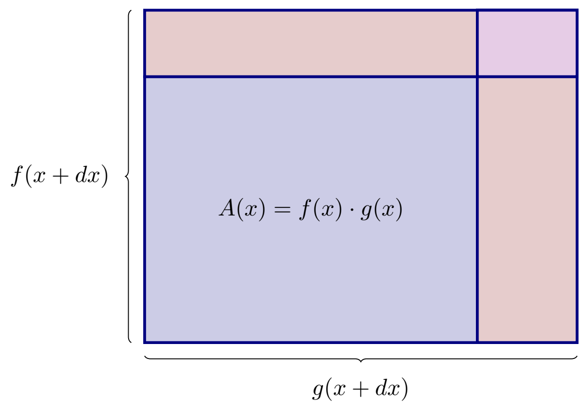
$$A(x) = f(x) \cdot g(x).$$

One interpretation of multiplication is it is the area of a $f(x) \times g(x)$ rectangle:



Learning outcomes:

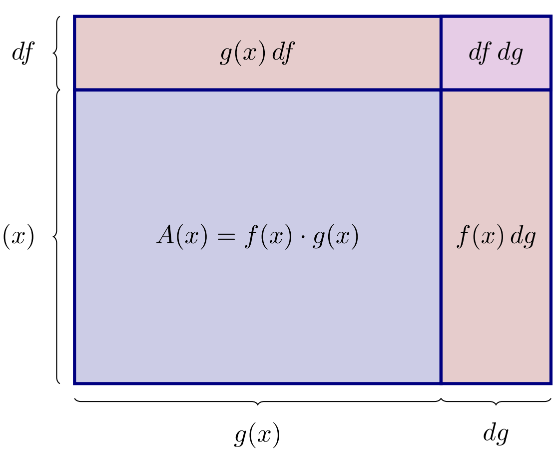
To understand the derivative of the product, we must understand how the area, A, changes as x changes. If we change the inputs of f and g by dx, then the size of the rectangle changes:



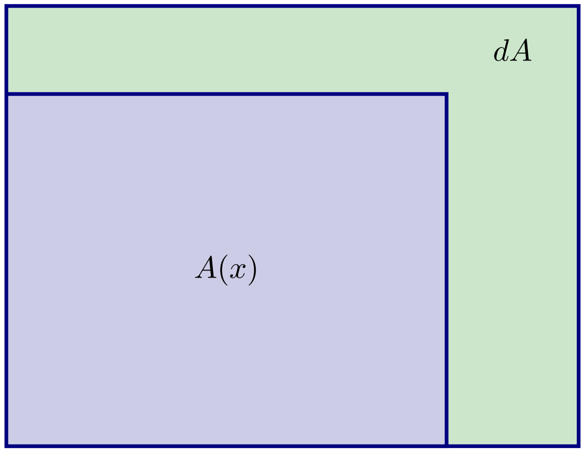
However, we know from our previous work that

$$
\begin{aligned}
f(x+dx) &\approx f(x) + df,\\
g(x+dx) &\approx g(x) + dg,
\end{aligned}
$$

so now our picture becomes:



Note, if we think of $A(x) = f(x) \cdot g(x)$, then we can also label our picture as follows:



Finally, from the pictures above and recalling that

$$df = f'(x)\,dx$$
$$dg = g'(x)\,dx,$$

we see that

$$\begin{aligned} dA &= f(x)\,dg + g(x)\,df + df\,dg \\ &= f(x)g'(x)\,dx + g(x)f'(x)\,dx + f'(x)g'(x)\,(dx)^2\,. \end{aligned}$$

Dividing both sides by dx we see

$$\frac{dA}{dx} = f(x)g'(x) + g(x)f'(x) + f'(x)g'(x)\,dx$$

and letting dx go to zero we see

$$A'(x) = f(x)g'(x) + g(x)f'(x).$$

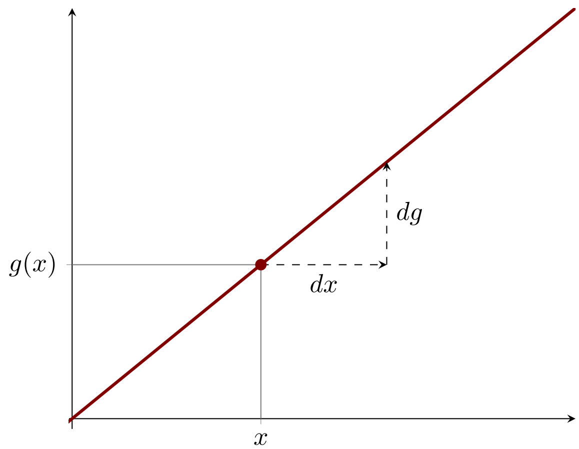
Explanation of the chain rule

Now we'll use linear approximations to help explain why the chain rule is true.

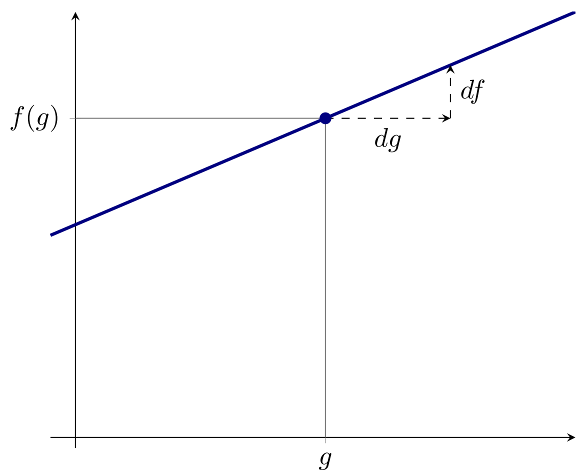
Theorem 2 (Chain Rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

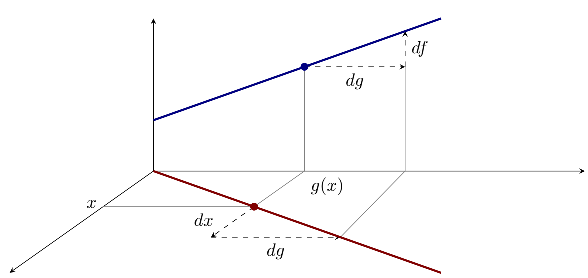
Explanation. *We'll try to understand this geometrically. In what follows, the functions f and g look like lines; however, the young mathematician should realize that we are **not** looking a true lines, instead we are looking at f and g sufficiently "zoomed-in" so that they appear to be lines. First consider a graph of g with respect to x:*



Now consider a graph of f with respect to g:



If we combine these graphs, by laying the graph of g on its side, we obtain:



Ah! From this we see that

$$\begin{aligned} df &= f'(g)\,dg \\ &= f'(g(x))g'(x)\,dx, \end{aligned}$$

so

$$\frac{df}{dx} = f'(g(x))g'(x).$$

These "explanations" are not meant to be the end of the story for the product rule and chain rule, rather they are hopefully the beginning. As you learn more mathematics, these explanations will be refined and made precise.