Dig-In:

Relating velocity and position, antiderivatives and areas

We see a connection between approximating antiderivatives and approximating areas.

A central theme of this course has been that we can often gain a better understanding of a function by looking at its derivative, and then working backwards. This has been our approach to max/min problems, curve sketching, linear approximation, and so on. So antiderivatives have really been important to us all along. We have a geometric interpretation of the derivative as the slope of a tangent line at a point. We have not yet found a geometric interpretation of antiderivatives.

More than one perspective

We'll start with a question:

Question 1 *Suppose you are in slow traffic moving at 4 mph from 2pm to 5pm. How far have you traveled?*

$\boxed{12}$ *miles.*

Hint: *A fancy way of saying this is that f is a line of slope 4, and we are looking for the rise corresponding to a run of $5 - 2 = 3$.*

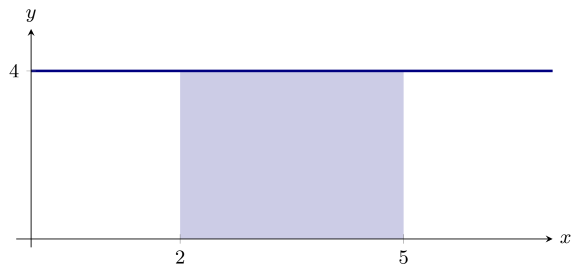
Now we move to a seemingly unrelated question:

Question 2 *What is the area of the region bounded by the graph of $f(x) = 4$, the horizontal axis, and the vertical lines $x = 2$ and $x = 5$?*

$\boxed{12}$

Learning outcomes: Interpert the product of rate and time as area. Approximate position from velocity. Recognize Riemann sums.

Hint:



Hint: *This is a rectangle with height* 4 *and width* 3*, so the area is* 12*.*

The fact that these two answers are the same is the germ of one of the most "fundamental" ideas in all of calculus. However, before we can step ahead, we might first look back to our even younger days of being mathematicians.

Recall that there are (at least!) two basic models of multiplication: A "rate times time" perspective and an "area" perspective. For instance, we could interpret

$$4 \times 3$$

as an answer to the question:

> If I am going 4mph for 3 hours, how far have I traveled?

or as the answer to the question

> What is the area of a rectangle with height 4 and width 3?

In what follows below, we will leverage these two different notions to gain insight into our study of functions.

From position to area

Suppose you have a continuous function $v(t)$ representing the velocity of some object at time t. We know that velocity is the change in position over time. If we wanted to recover the position of the object, we could approximate a graph by taking values of $v(t)$ and projecting the position via a small amount of time. Let's say this twice more:

$$\text{change in position} = \text{velocity} \times \text{change in time}$$

letting $s(t)$ be position, we see this translates **directly** into the language of differentials, since $s'(t) = v(t)$,

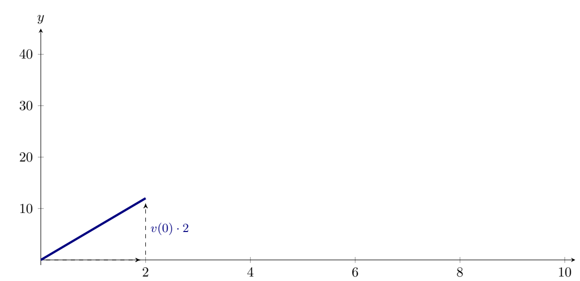
$$ds = v(t)\,dt.$$

Suppose we want to know how far we have traveled over the time interval $[0, 10]$. One way to proceed would be to cut the interval $[0, 10]$ into equal sized sections, we'll say five sections to keep things easy. Now we need to know the velocity of our object at those times. We'll describe $v(t)$ with the following table:

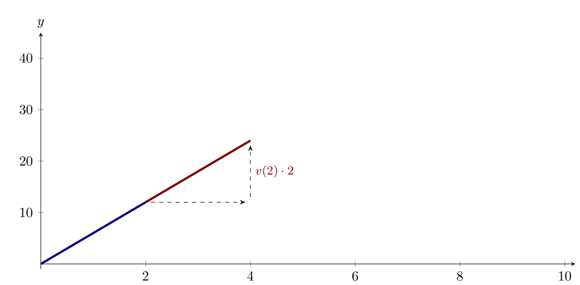
t	$v(t)$
0	6
2	6
4	4
6	0
8	-6

Example 1. *Assuming $s(0) = 0$, make an approximate graph of $y = s(t)$ with a piecewise linear function using the idea of differentials with $dt = 2$.*

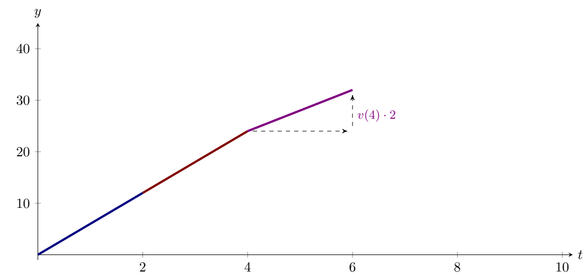
Explanation. *Our table from before gives us all the information we need! At $t = 0$, $s'(0) = v(0) = \underset{\text{given}}{\boxed{6}}$, so starting with the point $(0, 0)$ we attach a segment of slope $\underset{\text{given}}{\boxed{6}}$ over an interval of length 2:*



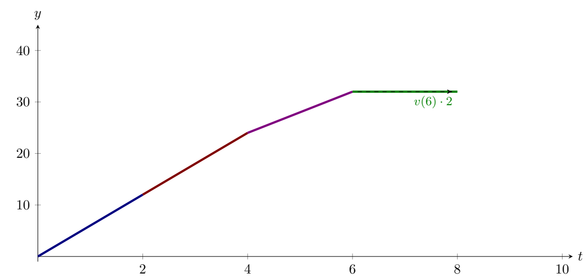
At $t = 2$, $s'(2) = v(2) = \underset{\text{given}}{\boxed{6}}$, so we attach a segment of slope $\underset{\text{given}}{\boxed{6}}$ over the next interval of length 2:



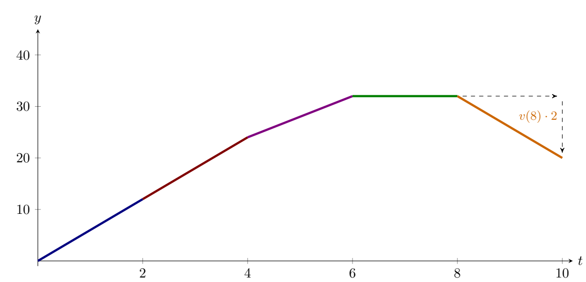
At $t = 4$, $s'(4) = v(4) = \underset{\text{given}}{\boxed{4}}$, so we attach a segment of slope $\underset{\text{given}}{\boxed{4}}$ over the next interval of length 2:



At $t = 6$, $s'(6) = v(6) = \underset{\text{given}}{\boxed{0}}$, *so we attach a segment of slope* $\underset{\text{given}}{\boxed{0}}$ *over the next interval of length* 2*:*



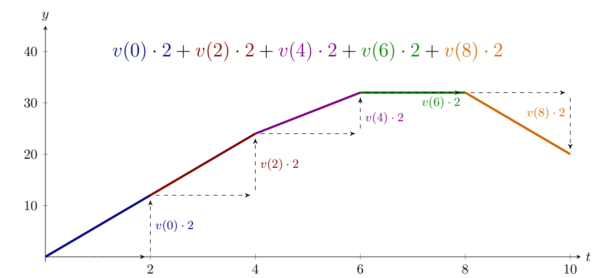
At $t = 8$, $s'(8) = v(8) = \underset{\text{given}}{\boxed{-6}}$, *so we attach a segment of slope* $\underset{\text{given}}{\boxed{-6}}$ *over the next interval of length* 2*:*



This gives us a plot of the approximate position of our object.

Example 2. *Use your approximation above with a step-size of* $dt = 2$ *to estimate* $s(10)$.

Explanation. *If we label our graph above, we will have a much easier time solving this problem:*



So

$$\begin{aligned} s(10) &\approx v(0)\cdot 2 + v(2)\cdot 2 + v(4)\cdot 2 + v(6)\cdot 2 + v(8)\cdot 2 \\ &= 6\cdot 2 + 6\cdot 2 + 4\cdot 2 + 0\cdot 2 + (-6)\cdot 2 \\ &= \underbrace{\boxed{20}}_{\text{given}}. \end{aligned}$$

Hmmm. Let's look at our answer from the previous question again:

$$v(0)\cdot 2 + v(2)\cdot 2 + v(4)\cdot 2 + v(6)\cdot 2 + v(8)\cdot 2$$

This is a Riemann sum!

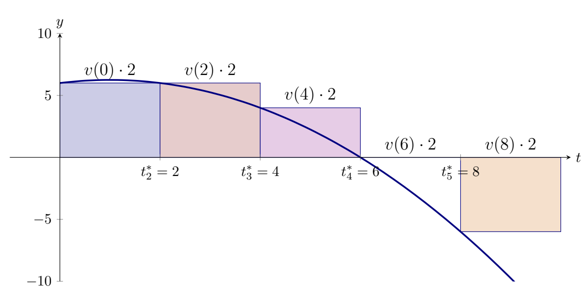
When we exclaim (with great excitement!) that "this is a Riemann sum," we are really saying that we are in a situation where we may view

$$\text{rate} \times \text{time}$$

as

$$\text{height} \times \text{width}$$

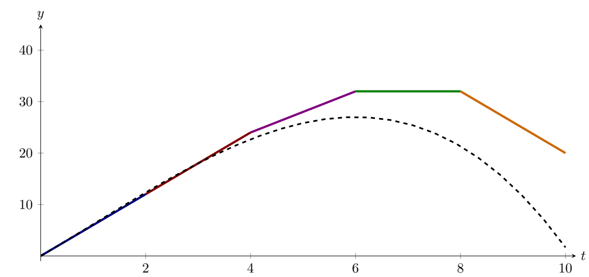
Check it out: here $n = 5$, $\Delta t = 2$, and we are looking at left-endpoints! Here is a suggestive graph of $y = v(t)$:



Here is the upshot: When we are computing values of antiderivatives, we are simultaneously computing areas between curves and the horizontal axis!

On the notation for antiderivatives

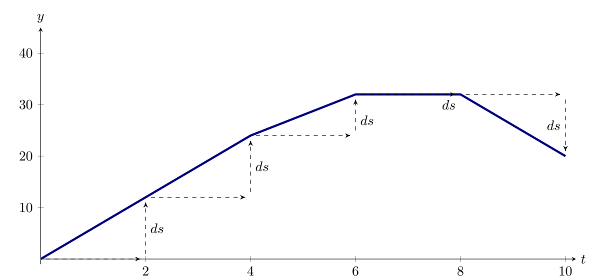
Let's look at our approximation of $s(t)$ we found above, we'll include the actual plot of $s(t)$ as well, shown as a dashed curve:



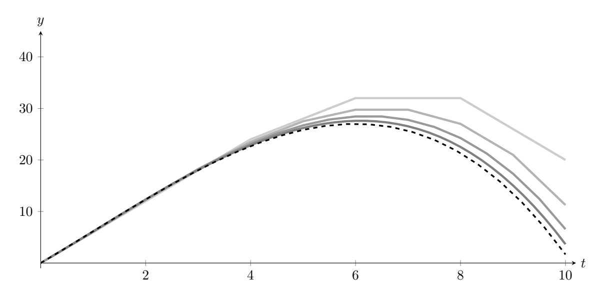
At each step we are computing

$$ds = v(t)\,dt$$

and adding it to the previous result:



If we wanted to, we could take a smaller step-size (our value for dt) and gain better, and better, approximations of $s(t)$:



At this point, we can explain the notation for antiderivatives. If we "sum" all of the "ds," we find the value of position. Hence

$$\begin{aligned}
ds &= v(t)\,dt \\
\text{“sum”}\;\; ds &= \text{“sum”}\;\; v(t)\,dt \\
\int ds &= \int v(t)\,dt \\
s(t) &= \int v(t)\,dt,
\end{aligned}$$

thus we see that an antiderivative is, essentially, a sum.