Dig-In:

Computations for graphing functions

We will give some general guidelines for sketching the plot of a function.

Let's get to the point. Here we use all of the tools we know to sketch the graph of $y = f(x)$:

- Compute f' and f''.
- Find the y-intercept, this is the point $(0, f(0))$. Place this point on your graph.
- Find any vertical asymptotes, these are points $x = a$ where $f(x)$ goes to infinity as x goes to a (from the right, left, or both).
- If possible, find the x-intercepts, the points where $f(x) = 0$. Place these points on your graph.
- Analyze end behavior: as $x \to \pm\infty$, what happens to the graph of f? Does it have horizontal asymptotes, increase or decrease without bound, or have some other kind of behavior?
- Find the critical points (the points where $f'(x) = 0$ or $f'(x)$ is undefined).
- Use either the first or second derivative test to identify local extrema and/or find the intervals where your function is increasing/decreasing.
- Find the candidates for inflection points, the points where $f''(x) = 0$ or $f''(x)$ is undefined.
- Identify inflection points and concavity.
- Determine an interval that shows all relevant behavior.

At this point you should be able to sketch the plot of your function.

Example 1. *Sketch the plot of* $2x^3 - 3x^2 - 12x$.

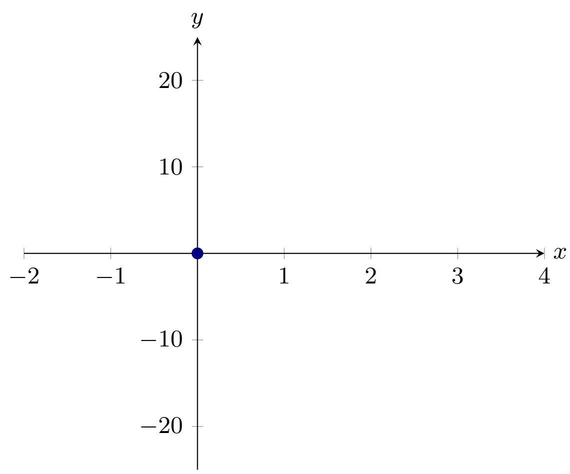
Explanation. *Try this on your own first, then either check with a friend, a graphing calculator (like www.desmos.com) or check the online version.*

Hint: *Compute* $f'(x)$ *and* $f''(x)$,

$$f'(x) = \underbrace{\boxed{6x^2 - 6x - 12}}_{\text{given}} \qquad \text{and} \qquad f''(x) = \underbrace{\boxed{12x - 6}}_{\text{given}}.$$

Learning outcomes: Determine how the graph of a function looks without using a calculator.

Hint: *The y-intercept is $(0, \underbrace{\boxed{0}}_{\text{given}})$. Place this point on your plot.*



Hint: *Which of the following are vertical asymptotes? Select all that apply.*

Select All Correct Answers:

(a) $x = 0$
(b) $x = 1$
(c) $x = -1$
(d) $x = \sqrt{2}$
(e) *There are no vertical asymptotes* ✓

Hint: *In this case, $f(x) = 2x^3 - 3x^2 - 12x$, we can find the x-intercepts. There are three x intercepts. Call them a, b, and c, and order them such that $a < b < c$. Then*

$$a = \underbrace{\boxed{\frac{3 - \sqrt{105}}{4}}}_{\text{given}},$$
$$b = \underbrace{\boxed{0}}_{\text{given}},$$
$$c = \underbrace{\boxed{\frac{3 + \sqrt{105}}{4}}}_{\text{given}}.$$

Hint: *Which of the following best describes the end behavior of f as $x \to \infty$?*

Multiple Choice:

(a) *f increases without bound.* ✓
(b) *f decreases without bound.*
(c) *f has a horizontal asymptote.*
(d) *f has some other behavior at ∞.*

Which of the following best describes the end behavior of f as $x \to -\infty$?

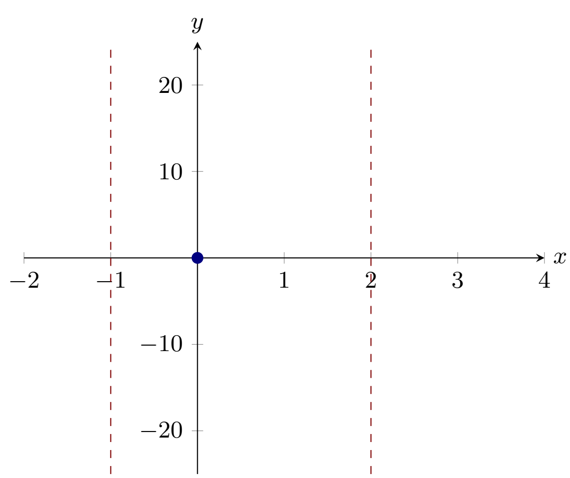
Multiple Choice:

(a) *f increases without bound.*
(b) *f decreases without bound.* ✓
(c) *f has a horizontal asymptote.*
(d) *f has some other behavior at ∞.*

Hint: *The critical points are where $f'(x) = 0$, thus we need to solve $6x^2 - 6x - 12 = 0$ for x. This equation has two solutions. If we call them a and b, with $a < b$, then what are a and b?*

$$a = \underbrace{\boxed{-1}}_{\text{given}} \qquad \text{and} \qquad b = \underbrace{\boxed{2}}_{\text{given}}.$$

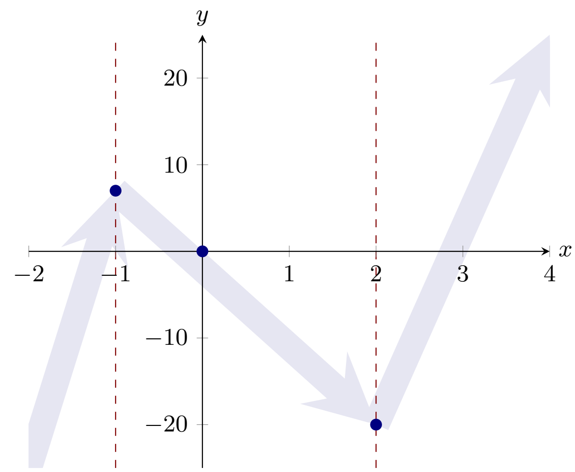
Hint: *Mark the critical points $x = 2$ and $x = -1$ on your plot.*



Hint: *Check the second derivative evaluated at the critical points. In this case,*

$$f''(-1) = \underbrace{\boxed{-18}}_{\text{given}} \qquad \text{and} \qquad f''(2) = \underbrace{\boxed{18}}_{\text{given}},$$

hence $x = -1$, corresponding to the point $(-1, 7)$ is a local (maximum ✓ / minimum) *and $x = 2$, corresponding to the point $(2, -20)$ is local* (maximum/ minimum ✓) *of $f(x)$. Moreover, this tells us that our function is* (increasing ✓ / decreasing) *on $[-2, -1)$,* (increasing/ decreasing ✓) *on $(-1, 2)$, and* (increasing ✓ / decreasing) *on $(2, 4]$. Identify this on your plot.*



Hint:

Hint: *The candidates for the inflection points are where* $f''(x) = 0$*, thus we need to solve* $12x - 6 = 0$ *for* x*.*

The solution to this is $x = \boxed{\frac{1}{2}}$.

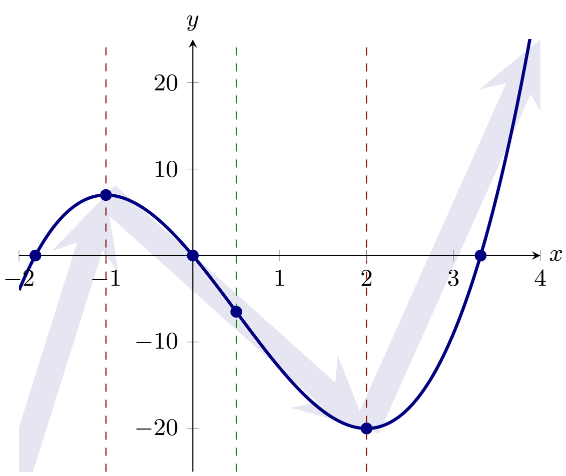
This is only a ***possible*** *inflection point, since the concavity needs to change to make it a true inflection point.*

f *is concave* (up/down ✓) *to the left of this point*

f *is concave* (up ✓ /down) *to the right of this point*

So this point (is ✓ /is not) *a point of inflection.*

Hint: *Since all of this behavior as described above occurs on the interval* $[-2, 4]$*, we now have a complete sketch of* $f(x)$ *on this interval, see the figure below.*



Example 2. *Sketch the plot of*

$$f(x) = \begin{cases} xe^x + 2 & \text{if } x < 0 \\ x^4 - x^2 + 3 & \text{if } x \geq 0. \end{cases}$$

Explanation. *Try this on your own first, then either check with a friend, a graphing calculator (like www.desmos.com), or check the online version.*

Hint: *Since this function is piecewise defined, we will analyze the cases $(-\infty, 0)$ and $[0, \infty)$ separately.*

Hint: *The derivative of f on $(-\infty, 0)$ is* $\underbrace{\boxed{xe^x + e^x}}_{\text{given}}$

The second derivative of f on $(-\infty, 0)$ is $\underbrace{\boxed{xe^x + 2e^x}}_{\text{given}}$

The derivative of f on $(0, \infty)$ is $\underbrace{\boxed{4x^3 - 2x}}_{\text{given}}$

The second derivative of f on $(0, \infty)$ is $\underbrace{\boxed{12x^2 - 2}}_{\text{given}}$

Hint: *Because f is piecewise defined, and potentially discontinuous at 0, it is important to understand the behavior of f near $x = 0$.*

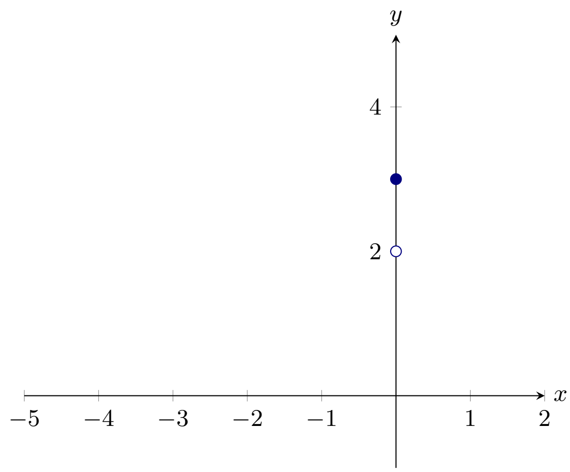
$$\lim_{x \to 0^-} f(x) = \underbrace{\boxed{2}}_{\text{given}}$$

$$\lim_{x \to 0^+} f(x) = \underbrace{\boxed{3}}_{\text{given}}$$

Moreover,

$$f(0) = \underbrace{\boxed{3}}_{\text{given}}$$

Record this information on our graph with filled and unfilled circles.



Hint:

Hint: *Which of the following are vertical asymptotes on $(-\infty, 0)$? Select all that apply.*

Select All Correct Answers:

(a) $x = 0$
(b) $x = 1$
(c) $x = -1$
(d) $x = \sqrt{2}$
(e) *There are no vertical asymptotes* ✓

Which of the following are vertical asymptotes on $(0, \infty)$? Select all that apply.

Select All Correct Answers:

(a) $x = 0$
(b) $x = 1$
(c) $x = -1$
(d) $x = \sqrt{2}$
(e) *There are no vertical asymptotes* ✓

Hint: *Which of the following best describes the end behavior of f as $x \to \infty$?*

Multiple Choice:

(a) *f increases without bound.* ✓
(b) *f decreases without bound.*
(c) *f has a horizontal asymptote of $y = 2$.*

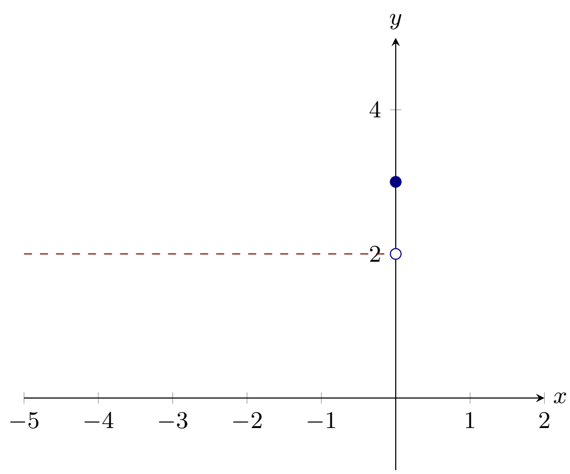
(d) *f has some other behavior at ∞.*

Which of the following best describes the end behavior of f as $x \to -\infty$?

Multiple Choice:

(a) *f increases without bound.*
(b) *f decreases without bound.*
(c) *f has a horizontal asymptote of $y = 2$.* ✓
(d) *f has some other behavior at ∞.*

Hint: *We mark the location of the horizontal asymptote:*

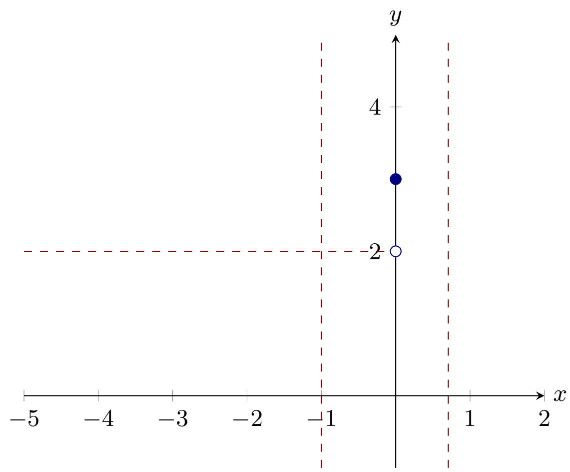


Hint: *The critical points are where $f'(x) = 0$ or does not exist. 0 is a critical point, since we have already seen it is a point of discontinuity for f, and thus $f'(0)$ does not exist there.*

On $(-\infty, 0)$, f has a critical point at $x = \underbrace{\boxed{-1}}_{\text{given}}$

On $(0, \infty)$, f has a critical point at $x = \underbrace{\boxed{\frac{1}{\sqrt{2}}}}_{\text{given}}$

Hint: *Mark the critical points $x = -1$ and $x = \frac{1}{\sqrt{2}}$ on your plot.*



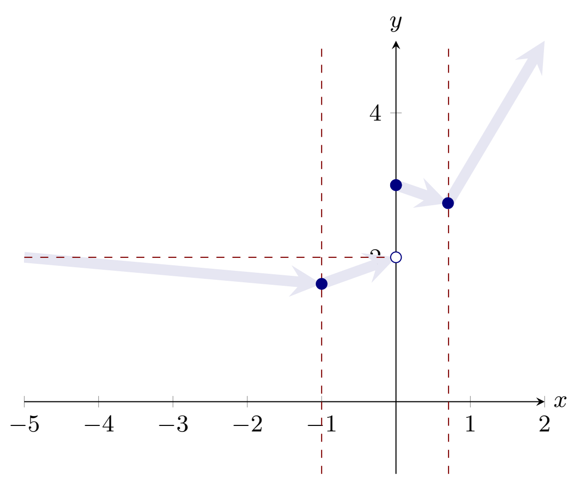
Hint: *Using the first derivative, we can see that*

On $(-\infty, -1)$, f *is* (increasing/decreasing ✓).

On $(-1, 0)$, f *is* (increasing ✓ /decreasing).

On $(0, \frac{1}{\sqrt{2}})$, f *is* (increasing/decreasing ✓).

On $(\frac{1}{\sqrt{2}}, \infty)$, f *is* (increasing ✓ /decreasing).

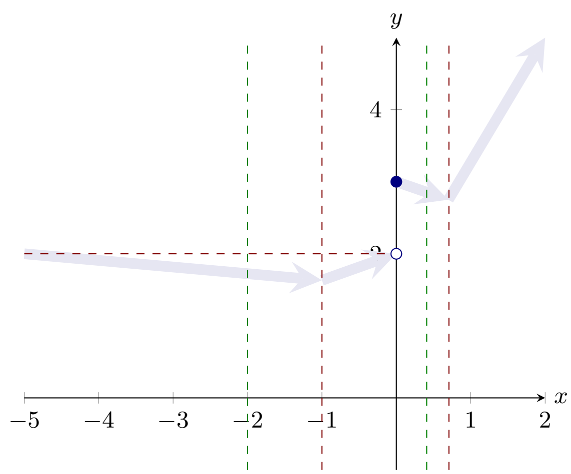


Hint:

Hint: *The candidates for the inflection points are where* $f''(x) = 0$.

On $(-\infty, 0)$, f'' *has one zero, namely* $x = \underbrace{\boxed{-2}}_{\text{given}}$. *The sign of* f'' *changes from* (positive to negative/negative to positive ✓]) *through this point.*

On $(0, \infty)$*,* f'' *has one zero, namely* $x = \underbrace{\boxed{\frac{1}{\sqrt{6}}}}_{\text{given}}$*. The sign of* f'' *changes from* (positive to negative/negative to positive ✓]) *through this point.*



Hint:

Hint: *Since all of this behavior as described above occurs on the interval* $[-5, 2]$*, we now have a complete sketch of* $y = f(x)$ *on this interval, see the figure below.*