Dig-In:

Differentiability implies continuity

We see that if a function is differentiable at a point, then it must be continuous at that point.

There are connections between continuity and differentiability.

Theorem 1 (Differentiability Implies Continuity). *If f is a differentiable function at $x = a$, then f is continuous at $x = a$.*

Explanation. *To explain why this is true, we are going to use the following definition of the derivative*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Assuming that $f'(a)$ exists, we want to show that $f(x)$ is continuous at $x = a$, hence we must show that

$$\lim_{x \to a} f(x) = f(a).$$

Starting with

$$\lim_{x \to a} (f(x) - f(a))$$

we multiply and divide by $(x - a)$ to get

$$\begin{aligned}
&= \lim_{x \to a} \left((x - a) \frac{f(x) - f(a)}{x - a} \right) \\
&= \left(\lim_{x \to a} (x - a) \right) \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) && \textit{Limit Law.} \\
&= \underbrace{\boxed{0}}_{\text{given}} \cdot f'(a) = \underbrace{\boxed{0}}_{\text{given}}.
\end{aligned}$$

Since

$$\lim_{x \to a} (f(x) - f(a)) = 0$$

we see that $\lim_{x \to a} f(x) = f(a)$, and so f is continuous at $x = a$.

This theorem is often written as its contrapositive:

> If $f(x)$ is not continuous at $x = a$, then $f(x)$ is not differentiable at $x = a$.

Learning outcomes: Explain the relationship between differentiability and continuity. Determine whether a piecewise function is differentiable.

Thus from the theorem above, we see that all differentiable functions on $\mathbb{R}$ are continuous on $\mathbb{R}$. Nevertheless there are continuous functions on $\mathbb{R}$ that are not differentiable on $\mathbb{R}$.

Question 1 *Which of the following functions are continuous but not differentiable on $\mathbb{R}$?*

Select All Correct Answers:

(a) x^2

(b) $\lfloor x \rfloor$

(c) $|x|$ ✓

(d) $\dfrac{\sin(x)}{x}$

Example 1. *Consider*

$$f(x) = \begin{cases} x^2 & \text{if } x < 3, \\ mx + b & \text{if } x \geq 3. \end{cases}$$

What values of m and b make f differentiable at $x = 3$?

Explanation. *To start, we know that we must make f both continuous and differentiable. Hence, we must ensure that the value of both pieces of f agree at $x = 3$. Write with me*

$$\begin{aligned} \Big[x^2\Big]_{x=3} &= \Big[mx+b\Big]_{x=3} \\ 9 &= m \cdot 3 + b. \end{aligned}$$

Now we must ensure that the derivatives of each piece of f agree at $x = 3$. Write with me

$$\begin{aligned} \frac{d}{dx}x^2 &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} (2x + h) \\ &= 2x. \end{aligned}$$

Moreover,

$$\frac{d}{dx}(mx+b) = m$$

by the definition of a tangent line. Hence we must have

$$\begin{aligned}\left[\frac{d}{dx}x^2\right]_{x=3} &= \left[\frac{d}{dx}(mx+b)\right]_{x=3}\\ \left[2x\right]_{x=3} &= \left[m\right]_{x=3}\\ 6 &= m.\end{aligned}$$

Ah! So now

$$\begin{aligned}9 &= m\cdot 3+b\\ 9 &= 6\cdot 3+b\\ 9 &= 18+b,\end{aligned}$$

so $b=-9$*. Thus setting* $m=6$ *and* $b=-9$ *will give us a continuous and differentiable piecewise function.*