Dig-In

Extreme Values and Critical Points

We use derivatives to help locate extrema.

Whether we are interested in a function as a purely mathematical object or in connection with some application to the real world, it is often useful to know what the graph of the function looks like. We can obtain a good picture of the graph using certain crucial information provided by derivatives of the function.

Extrema

Local *extrema* on a function are points on the graph where the y-coordinate is larger (or smaller) than all other y-coordinates on the graph at points "close to" (x, y).

Definition 1.

(a) *A function f has a* ***local maximum*** *at $x = a$, if $f(a) \geq f(x)$ for every x near a.*

(b) *A function f has a* ***local minimum*** *at $x = a$, if $f(a) \leq f(x)$ for every x near a.*

A ***local extremum*** *is either a local maximum or a local minimum.*

Problem 1 *True or false: "All absolute extrema are also local extrema."*

Multiple Choice:

(a) *true* ✓

(b) *false*

Feedback (attempt): *All global extrema are local extrema.*

Learning outcomes: Define a critical point. Find critical points. Define absolute maximum and absolute minimum. Find the absolute max or min of a continuous function on a closed interval. Define local maximum and local minimum. Compare and contrast local and absolute maxima and minima. Identify situations in which an absolute maximum or minimum is guaranteed. Classify critical points. State the First Derivative Test. Apply the First Derivative Test. State the Second Derivative Test. Apply the Second Derivative Test. Define inflection points. Find inflection points.

Points where local maxima and minima occur are quite distinctive on the graph of a function, and are therefore useful in understanding the shape of the graph. In many applied problems we want to find the largest or smallest value that a function achieves (for example, we might want to find the minimum cost at which some task can be performed) and so identifying maxima and minima will be useful for applied problems as well.

Critical points

If $(x, f(x))$ is a point where f reaches a local maximum or minimum, and if the derivative of f exists at x, then the graph has a tangent line and the tangent line must be horizontal. This is important enough to state as a theorem, though we will not prove it.

Theorem 1 (Fermat's Theorem)**.** *If f has a local extremum at $x = a$ and f is differentiable at a, then $f'(a) = 0$.*

Problem 2 *Does Fermat's Theorem say that if $f'(a) = 0$, then f has a local extrema at $x = a$?*

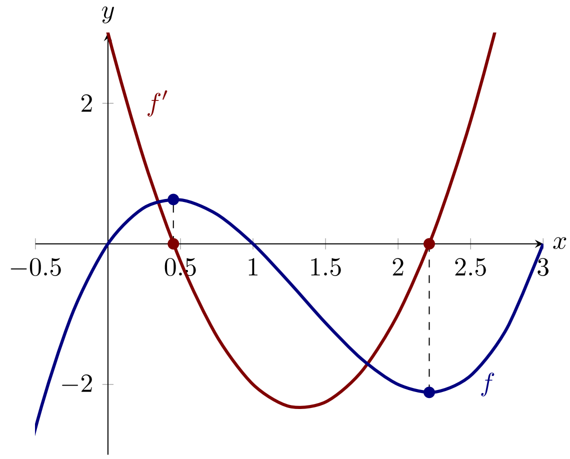
Multiple Choice:

(a) *yes*

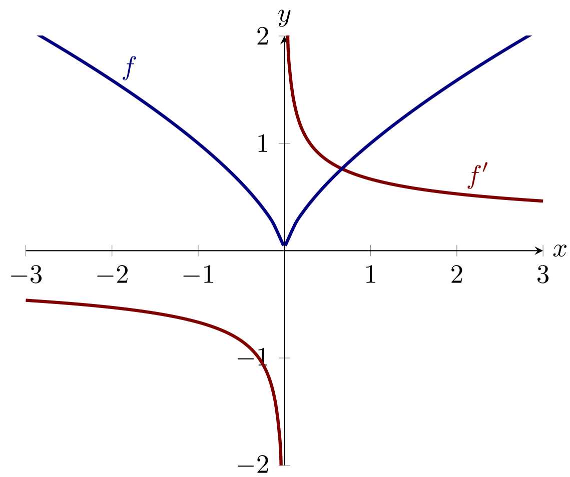
(b) *no* ✓

Feedback (attempt): *Consider $f(x) = x^3$, $f'(0) = 0$, but f does not have a local maximum or minimum at $x = 0$.*

Here is an example of a function and its derivative where Fermat's Theorem applies. Consider the plots of $f(x) = x^3 - 4x^2 + 3x$ and $f'(x) = 3x^2 - 8x + 3$, and look at the output of the derivative at the same x-values where local extrema occur.



However, this is only half of the story! Consider $f(x) = x^{2/3}$ and $f'(x) = \dfrac{2}{3x^{1/3}}$:



What do you notice about the value of the derivative at the place where the minimum occurs?

This brings us to our next definition.

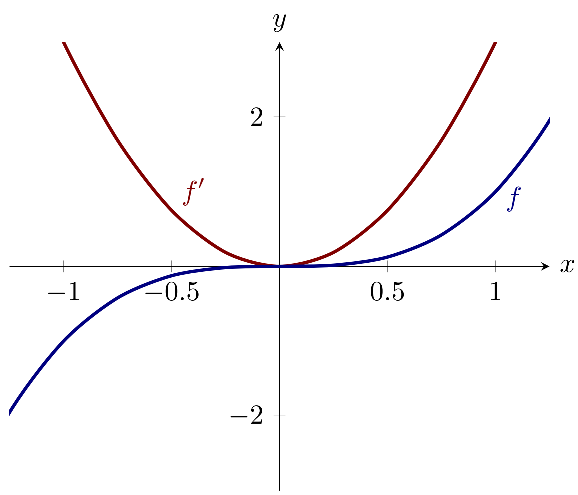
Definition 2. *A function has a **critical point** at an interior point in its domain $x = a$ if*

$$f'(a) = 0 \qquad \text{or} \qquad f'(a) \text{ does not exist.}$$

Warning 1. *When looking for local maximum and minimum points, you are likely to make two sorts of mistakes:*

- *You may forget that a maximum or minimum can occur where the derivative does not exist, so don't forget to check whether the derivative exists* *everywhere.*

- *You might assume that any place that the derivative is zero is a local maximum or minimum point, but this is not true, consider the plots of* $f(x) = x^3$ *and* $f'(x) = 3x^2$.



While $f'(0) = 0$*, there is neither a maximum nor minimum at* $(0, f(0))$.

Since the derivative is zero or undefined at both local maximum and local minimum points, we need a way to determine which, if either, actually occurs. The most elementary approach is to test directly whether the y coordinates near the potential maximum or minimum are above or below the y coordinate at the point of interest.

It is not always easy to compute the value of a function at a particular point. The task is made easier by the availability of calculators and computers, but they have their own drawbacks: they do not always allow us to distinguish between values that are very close together. Nevertheless, because this method is conceptually simple and sometimes easy to perform, you could consider it. However, be wary of using this approach as an exam answer- it is difficult to justify, and works only on nice enough functions. It is better if you use some of the methods you will learn over the next few cards.

Example 1. *Find all local maximum and minimum values for the function* $f(x) = x^3 - x$.

Explanation. *Write*

$$\frac{d}{dx} f(x) = \underbrace{\boxed{3x^2 - 1}}_{\text{given}}.$$

This is defined everywhere and is zero at $x = \pm\sqrt{3}/3$. *Looking first at* $x = \sqrt{3}/3$, *we see that*

$$f(\sqrt{3}/3) = \underset{\text{given}}{\boxed{-2\sqrt{3}/9}}.$$

Now we test two points on either side of $x = \sqrt{3}/3$, *making sure that neither is farther away than the nearest critical point; since* $\sqrt{3} < 3$, $\sqrt{3}/3 < 1$ *and we can use* $x = 0$ *and* $x = 1$. *Since*

$$f(0) = 0 > -2\sqrt{3}/9 \qquad \textit{and} \qquad f(1) = 0 > -2\sqrt{3}/9,$$

there must be a local minimum at $x = \underset{\text{given}}{\boxed{\sqrt{3}/3}}$.

For $x = -\sqrt{3}/3$, *we see that* $f(-\sqrt{3}/3) = 2\sqrt{3}/9$. *This time we can use* $x = 0$ *and* $x = -1$, *and we find that* $f(-1) = f(0) = 0 < 2\sqrt{3}/9$, *so there must be a local maximum at* $x = \underset{\text{given}}{\boxed{-\sqrt{3}/3}}$, *see the plot below:*