Dig-In:

Basic rules of differentiation

We derive the constant rule, power rule, and sum rule.

It is tedious to compute a limit every time we need to know the derivative of a function. Fortunately, we can develop a small collection of examples and rules that allow us to quickly compute the derivative of almost any function we are likely to encounter. We will start simply and build up to more complicated examples.

The constant rule

The simplest examples of functions, hence the best place to start our investigation, are constant functions. Recall that derivatives measure the rate of change of a function at a given point. This means the derivative of a constant function is zero. Here are some ways to think about this situation.

- The constant function plots a horizontal line—so the slope of the tangent line is 0.
- If $s(t)$ represents the position of an object with respect to time and $s(t)$ is constant, then the object is not moving, so its velocity is zero. Hence $\frac{d}{dt}s(t) = 0$.
- If $v(t)$ represents the velocity of an object with respect to time and $v(t)$ is constant, then the object's acceleration is zero. Hence $\frac{d}{dt}v(t) = 0$.

The examples above lead us to our next theorem. To gain intuition, you should compute the derivative of $f(x) = 6$ using the limit definition of the derivative.

Theorem 1 (The constant rule)**.** *Given a constant c,*

$$\frac{d}{dx}c = 0.$$

Learning outcomes: Use "shortcut" rules to find and use derivatives. Use the definition of the derivative to develop shortcut rules to find the derivatives of constants, constant multiples, powers of x, and sums of functions

Explanation. *From the limit definition of the derivative, write*

$$\begin{aligned}\frac{d}{dx}c &= \lim_{h\to 0}\frac{c-c}{\underbrace{\boxed{h}}_{\text{given}}}\\ &= \lim_{h\to 0}\frac{0}{h}\\ &= \lim_{h\to 0} 0 = 0.\end{aligned}$$

Question 1 *What is*

$$\frac{d}{dx}e$$

equal to?

Hint: *Remember,*

$$e = 2.718281828459045\ldots.$$

It equals $\underbrace{\boxed{0}}_{\text{given}}$.

The power rule

Next, let's examine derivatives of powers of a single variable. To gain intuition, you should compute the derivative of $f(x) = x^3$ using the limit definition of the derivative. Before computing this derivative, we should recall the *Binomial Theorem.*

Theorem 2 (Binomial Theorem)**.** *If n is a nonnegative integer, then*

$$(a+b)^n = a^n b^0 + \binom{n}{1} a^{n-1}b^1 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + a^0 b^n$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The Binomial Theorem tells us the pattern of the coefficients for the expanded form of

$$(a+b)^n,$$

which will help us prove our next derivative rule.

Example 1. *Expand $(x+h)^3$ using the Binomial Theorem.*

Explanation. *We apply the Binomial Theorem to the expression in question, then simplify.*

$$\begin{aligned}(x+h)^3 &= x^3h^0 + \binom{3}{1}x^{3-1}h^1 + \binom{3}{2}x^{3-2}h^2 + x^0h^3 \\ &= x^3 + \underbrace{\boxed{3}}_{\text{given}}x^2h + \underbrace{\boxed{3}}_{\text{given}}xh^2 + h^3\end{aligned}$$

Theorem 3 (The power rule)**.** *For any real number n,*

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Explanation. *At this point we will only prove this theorem for values of n which are positive integers. Later we will give the complete explanation. From the limit definition of the derivative, write with me*

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(\underbrace{\boxed{x+h}}_{\text{given}})^n - x^n}{h}.$$

Expand the term $(x+h)^n$:

$$= \lim_{h\to 0}\frac{x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n - x^n}{h}$$

Note that we are using the Binomial Theorem to write $\binom{n}{k}$ for the coefficients. Canceling the terms x^n and $-x^n$, and noting $\binom{n}{1} = n$, write with me

$$\begin{aligned}&= \lim_{h\to 0}\frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n}{h} \\ &= \lim_{h\to 0}\left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + \binom{n}{n-1}xh^{n-2} + h^{n-1}\right).\end{aligned}$$

Since every term but the first has a factor of h, we see

$$\frac{d}{dx}x^n = \underbrace{\boxed{nx^{n-1}}}_{\text{given}}.$$

Let's consider several examples. We begin with something basic.

Example 2. *Compute:*

$$\frac{d}{dx}x^{13}$$

Explanation. *Applying the power rule, we write*

$$\frac{d}{dx}x^{13} = \underbrace{\boxed{13x^{12}}}_{\text{given}}.$$

Sometimes, it is not as obvious that one should apply the power rule.

Example 3. *Compute:*

$$\frac{d}{dx}\frac{1}{x^4}$$

Explanation. *Applying the power rule, we write*

$$\frac{d}{dx}\frac{1}{x^4} = \frac{d}{dx}x^{-4} = \underbrace{\boxed{-4x^{-5}}}_{\text{given}}.$$

The power rule also applies to radicals once we rewrite them as exponents.

Example 4. *Compute:*

$$\frac{d}{dx}\sqrt[5]{x}$$

Explanation. *Applying the power rule, we write*

$$\frac{d}{dx}\sqrt[5]{x} = \frac{d}{dx}x^{1/5} = \underbrace{\boxed{\frac{x^{-4/5}}{5}}}_{\text{given}}.$$

The sum rule

We want to be able to take derivatives of functions "one piece at a time." The *sum rule* allows us to do exactly this. The sum rule says that we can add the rates of change of two functions to obtain the rate of change of the sum of both functions. For example, viewing the derivative as the velocity of an object, the sum rule states that to find the velocity of a person walking on a moving bus, we add the velocity of the bus and the velocity of the walking person.

Theorem 4 (The sum rule)**.** *If $f(x)$ and $g(x)$ are differentiable and c is a constant, then*

(a) $\dfrac{d}{dx}\big(f(x) + g(x)\big) = f'(x) + g'(x)$,

(b) $\dfrac{d}{dx}\big(f(x) - g(x)\big) = f'(x) - g'(x)$,

(c) $\dfrac{d}{dx}\big(c \cdot f(x)\big) = c \cdot f'(x)$.

Explanation. *We will only prove part (a) above; the rest are similar. Write with me*

$$\begin{aligned}
\frac{d}{dx}\big(f(x)+g(x)\big) &= \lim_{h\to 0} \frac{f(\underbrace{\boxed{x+h}}_{\text{given}})+g(\underbrace{\boxed{x+h}}_{\text{given}})-(f(x)+g(x))}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} \left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right) \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= f'(x)+g'(x).
\end{aligned}$$

Question 2 *Using different notation for the derivative can be confusing. Which of the following are the same as part (a) of the Sum Rule?*

Select All Correct Answers:

(a) $(f(x)+g(x))' = f'(x)+g'(x)$ ✓

(b) $(f(x)+g(x))' = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)$ ✓

(c) $\frac{d}{dx}f(x)+\frac{d}{dx}g(x) = f'(x)+g'(x)$

(d) $(f'(x)+g(x))' = \frac{d}{dx}(f(x)+g(x))$

Example 5. *Suppose we have two functions, f, and g, and we know that $f'(2) = 6$ and $g'(2) = -3$. What is the slope of $f(x)+g(x)$ at $x = 2$?*

Explanation. *Using the sum rule, the slope of $f(x)+g(x)$ at $x = 2$ is the sum $f'(2)+g'(2)$. In this case, we have $f'(x)+g'(x) = \underbrace{\boxed{3}}_{\text{given}}$.*

We now have the tools to work some more complicated examples.

Example 6. *Compute:*

$$\frac{d}{dx}\left(x^5+\frac{1}{x}\right)$$

Explanation. *Write with me*

$$\begin{aligned}
\frac{d}{dx}\left(x^5+\frac{1}{x}\right) &= \frac{d}{dx}x^5+\frac{d}{dx}x^{-1} \\
&= \underbrace{\boxed{5x^4-x^{-2}}}_{\text{given}}.
\end{aligned}$$

Example 7. *Compute:*

$$\frac{d}{dx}\left(\frac{3}{\sqrt[3]{x}} - 2\sqrt{x} + \frac{1}{x^7}\right)$$

Explanation. *Write with me*

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{3}{\sqrt[3]{x}} - 2\sqrt{x} + \frac{1}{x^7}\right) \\
&\quad = 3\frac{d}{dx}x^{-1/3} - 2\frac{d}{dx}x^{1/2} + \frac{d}{dx}x^{-7} \\
&\quad = \underbrace{\boxed{-x^{-4/3} - x^{-1/2} - 7x^{-8}}}_{\text{given}}.
\end{aligned}$$