Dig-In:

The idea of substitution

We learn a new technique, called substitution, to help us solve problems involving integration.

Computing antiderivatives is not as easy as computing derivatives. One issue is that the chain rule can be difficult to "undo." We have a general method called "integration by substitution" that will somewhat help with this difficulty. The idea is this, we know from the chain rule that

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

so if we conisder

$$\begin{aligned}\int_a^b f'(g(x))g'(x)\,dx &= \Big[f(g(x))\Big]_a^b\\ &= f(g(b)) - f(g(a))\\ &= \Big[f(g)\Big]_{g(a)}^{g(b)}\\ &= \int_{g(a)}^{g(b)} f'(g)\,dg.\end{aligned}$$

This "transformation" is worth stating explicity:

Theorem 1 (Integral Substitution Formula)**.** *If g is differentiable on the interval $[a,b]$ and f is differentiable on the interval $[g(a),g(b)]$, then*

$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(g)\,dg.$$

Three similar techinques

There are several different ways to think about substitution. The first is directly using the formula

$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(g)\,dg.$$

Learning outcomes: To be able to use the method of substitution to solve some "simple" integrals, with an emphasis on being able to correctly identify what to substitute for. Undo the Chain Rule. Calculate indefinite integrals (antiderivatives) using basic substitution. Calculate definite integrals using basic substitution.

Example 1. *Compute:*

$$\int_1^3 x\cos(x^2)\,dx$$

Explanation. *A little thought reveals that if $x\cos(x^2)$ is the derivative of some function, then it must have come from an application of the chain rule.*

$$\int \underbrace{x}_{\textit{derivative of inside}} \cos(\underbrace{x^2}_{\textit{inside}})\,dx.$$

Set $g(x) = x^2$, so $g'(x) = 2x$, and now it must be that $f'(g) = \dfrac{\cos(g)}{2}$. Now we see

$$\begin{aligned}\int_1^3 x\cos(x^2)\,dx &= \int_1^9 \frac{\cos(g)}{2}\,dg\\ &= \left[\frac{\sin(g)}{2}\right]_1^9\\ &= \frac{\sin(9)-\sin(1)}{2}.\end{aligned}$$

Notice the change of endpoints in the first equality! We obtained the new integrands by the computations

$$\begin{aligned}g(1) &= 1^2 = 1\\ g(3) &= 3^2 = 9.\end{aligned}$$

We will usually solve these problems in a slightly different way. Let's do the same example again, this time we will think in terms of differentials.

Example 2. *Compute:*

$$\int_1^3 x\cos(x^2)\,dx$$

Explanation. *Here we will set $g(x) = x^2$. Then $dg = \underbrace{\boxed{2x}}_{\text{given}}\,dx$, where we are thinking in terms of differentials. So we can solve for dx to get $dx = \dfrac{dg}{\underbrace{\boxed{2x}}_{\text{given}}}$. We then see that*

$$\begin{aligned}\int_1^3 x\cos(x^2)\,dx &= \int_{g(1)}^{g(3)} x\cos(g)\frac{dg}{2x}\\ &= \int_1^9 \frac{\cos(g)}{2}\,dg.\end{aligned}$$

At this point, we can continue as we did before and write

$$\int_1^3 x\cos(x^2)\,dx = \frac{\sin(9)-\sin(1)}{2}.$$

Finally, sometimes we simply want to deal with the antiderivative on its own, we'll repeat the example one more time demonstrating this.

Example 3. *Compute:*

$$\int_1^3 x\cos(x^2)\,dx$$

Explanation. *Here we start as we did before, setting* $g(x) = x^2$*. Now* $dg = 2x\,dx$*, again thinking in terms of differentials. Now we see that*

$$\int x\cos(x^2)\,dx = \int x\cos(g)\frac{dg}{2x} = \int \frac{\cos(g)}{2}\,dg.$$

Hence

$$\int x\cos(x^2)\,dx = \frac{\sin(g)}{2} = \frac{\sin(x^2)}{2}.$$

So

$$\begin{aligned}\int_1^3 x\cos(x^2)\,dx &= \left[\frac{\sin(x^2)}{2}\right]_1^3\\ &= \frac{\sin(9)-\sin(1)}{2}.\end{aligned}$$

More examples

With some experience, it is (usually) not too hard to see what to substitute as g. We will work through the following examples in the same way that we did for Example 2. Let's see another example.

Example 4. *Compute:*

$$\int x^4(x^5+1)^9\,dx$$

Explanation. *Here we set* $g(x) = \underbrace{\boxed{x^5+1}}_{\text{given}}$*, so* $dg = \underbrace{\boxed{5x^4}}_{\text{given}}dx$*. Then*

$$\begin{aligned}\int x^4(x^5+1)^9\,dx &= \frac{1}{5}\int 5x^4(x^5+1)^9\,dx\\ &= \frac{1}{5}\int g^9\,dg\\ &= \underbrace{\boxed{\frac{g^{10}}{50}}}_{\text{given}}.\end{aligned}$$

Notice that this example is an indefinite integral and not a definite integral, meaning that there are no limits of integration. So we do not need to worry *about changing the endpoints of the integral. However, we do need to back-substitute into our answer, so that our final answer is a function of x. Recalling that $g(x) = x^5 + 1$, we have our final answer*

$$\int x^4(x^5+1)^9\,dx = \underbrace{\boxed{\frac{(x^5+1)^{10}}{50}}}_{\text{given}} + C.$$

If substitution works to solve an integral (and that is not always the case!), a common trick to find what to substitute for is to locate the "ugly" part of the function being integrated. We then substitute for the "inside" of this ugly part. While this technique is certainly not rigorous, it can prove to be very helpful. This is especially true for students new to the technique of substitution. The next two problems are really good examples of this philosophy.

Example 5. *Compute:*

$$\int_{-1}^{0} 12x^3 e^{x^4}\,dx$$

Explanation. *The "ugly" part of the function being integrated is e^{x^4}. The "inside" of this term is then x^4. So a good possibility is to try*

$$g(x) = \underbrace{\boxed{x^4}}_{\text{given}}.$$

Then

$$dg = \underbrace{\boxed{4x^3}}_{\text{given}}\,dx \qquad \Rightarrow \qquad dx = \underbrace{\boxed{\frac{1}{4x^3}}}_{\text{given}}\,dg$$

and so

$$\begin{aligned}
\int_{-1}^{0} 12x^3 e^{x^4}\,dx &= \int_{g(-1)}^{g(0)} 12x^3 e^{g} \underbrace{\boxed{\frac{1}{4x^3}}}_{\text{given}}\,dg \\
&= \int_{\underbrace{\boxed{1}}_{\text{given}}}^{\underbrace{\boxed{0}}_{\text{given}}} \underbrace{\boxed{3e^g}}_{\text{given}}\,dg \\
&= \left[\underbrace{\boxed{3e^g}}_{\text{given}}\right]_{\underbrace{\boxed{1}}_{\text{given}}}^{\underbrace{\boxed{0}}_{\text{given}}} \\
&= \underbrace{\boxed{3(1-e)}}_{\text{given}}.
\end{aligned}$$

Example 6. *Compute:*

$$\int_{1}^{e^{\frac{\pi}{4}}} \frac{\cos(\ln x)}{x}\,dx$$

Explanation. *Here the "ugly" part here is* $\cos(\ln x)$. *So we substitute for the inside:*

$$g(x) = \underbrace{\boxed{\ln x}}_{\text{given}}.$$

Then

$$dg = \underbrace{\boxed{\frac{1}{x}}}_{\text{given}} dx \qquad \Rightarrow \qquad dx = \underbrace{\boxed{x}}_{\text{given}} dg.$$

Notice that

$$g(1) = \ln(1) = \underbrace{\boxed{0}}_{\text{given}}$$

$$g\left(e^{\frac{\pi}{4}}\right) = \ln\left(e^{\frac{\pi}{4}}\right) = \underbrace{\boxed{\frac{\pi}{4}}}_{\text{given}}.$$

Then we substitute back into the original integral and solve:

$$\begin{aligned}
\int_1^{e^{\frac{\pi}{4}}} \frac{\cos(\ln x)}{x}\, dx &= \int_0^{\frac{\pi}{4}} \frac{\cos(g)}{x} x\, dg \\
&= \int_0^{\frac{\pi}{4}} \cos(g)\, dg \\
&= \left[\underbrace{\boxed{\sin(g)}}_{\text{given}}\right]_0^{\frac{\pi}{4}} \\
&= \underbrace{\boxed{\frac{\sqrt{2}}{2}}}_{\text{given}} - \underbrace{\boxed{0}}_{\text{given}} = \underbrace{\boxed{\frac{\sqrt{2}}{2}}}_{\text{given}}.
\end{aligned}$$

To summarize, if we suspect that a given function is the derivative of another via the chain rule, we let g denote a likely candidate for the inner function, then translate the given function so that it is written entirely in terms of g, with no x remaining in the expression. If we can integrate this new function of g, then the antiderivative of the original function is obtained by replacing g by the equivalent expression in x.