Characterization of the Finite Complement Topology on the Real Line

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Abstract

A proof that all open sets in the finitie complement topology on the real line can be written as a finite union $(-\infty, a_1) \cup (a_1, a_2) \cup \cdots \cup (a_{n-1}, a_n) \cup (a_n, \infty)$.

Definitions

Given a set X let $\mathcal{T}_f$ be the collection of all subsets U of X such that $X \setminus U$ is either all of X or is finite. That is,

$$\mathcal{T}_f = \{U \subset X : U \setminus X = X \text{ or } U \setminus X \text{ is finite}\}.$$

A Characterization of $\mathcal{T}_f$

Given any open set U in $(\mathbb{R}, \mathcal{T}_f)$, U is either all of $\mathbb{R}$ or can be written in the form

$$U = (-\infty, a_1) \cup (a_1, a_2) \cup \cdots \cup (a_{n-1}, a_n) \cup (a_n, \infty)$$

with $a_i \in \mathbb{R}$ and $i \in \mathbb{N}$.

Proof

By definition given any U open in $\mathbb{R}$ there must exist some $n \in \mathbb{N}$ such that $X \setminus U = \varnothing$ or

$$\mathbb{R} \setminus U = \{x_i\}_{i=1}^{n} \text{ with } x_i \in \mathbb{R}.$$

If $X \setminus U = \varnothing$ the hypotheses holds with $U = \mathbb{R} = (-\infty, \infty)$. In the second case the proof is by induction on n. The hypothesis holds for $n = 1$ with $U = (-\infty, x_1) \cup (x_1, \infty)$. Suppose the statement is true for all natural numbers less than n. Then we have

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