Selected Exercises §16

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Question 1

Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Proof

Suppose $A \subset Y \subset X$ and let $\mathcal{T}_X$ be a topology on X and $\mathcal{T}_Y$ be the corresponding subspace topology on Y. Let $\mathcal{A}_X$ denote the subspace topology that A inherits from $(X, \mathcal{T}_X)$ and $\mathcal{A}_Y$ be the subspace topology A inherits from $(Y, \mathcal{T}_Y)$. We will show that $\mathcal{A}_X \subset \mathcal{A}_Y$. Let $U \in \mathcal{A}_X$ then $U = W \cap A$ for some W open in X. Since $Y \cap A = A$ we have that

$$U = W \cap (Y \cap A) = (W \cap Y) \cap A.$$

Where $W \cap Y$ is an open set in the subspace topology on Y. Thus $U \in \mathcal{A}_Y$.

On the other hand if $U \in \mathcal{A}_Y$, then $U = V \cap A$ for some V open in Y. But, since $V = W \cap Y$ for some W open in X we have that

$$U = (W \cap Y) \cap A = W \cap (Y \cap A) = W \cap A.$$

Thus $U \in \mathcal{A}_X$ and $\mathcal{A}_Y \subset \mathcal{A}_X$.

$\square$

Question 4

A map $f : X \to Y$ is said to be an *open map* if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

Proof

Suppose $u \times v$ is open in $X \times Y$ then

$$u \times v = \bigcup_{\alpha \in J} (u_\alpha \times v) = (\bigcup_{\alpha \in J} u_\alpha \times v)$$

for $\{u_\alpha\}$ some collection of open sets in X. Applying π_1 we have that

$$\pi_1(u \times v) = \pi_1(\bigcup_{\alpha \in J} (u_\alpha \times v)) = \bigcup_{\alpha \in J} u_\alpha$$

which is a union of open sets in X and thus itself is open in X. The proof that π_2 is an open map is nearly identical.

$\square$

Question 6

Show that the countable collection

$$\{(a, b) \times (c, d) \mid a < b, c < d \text{ and } a, b, c, d \in \mathbb{Q}\}$$

is a basis for $\mathbb{R}^2$.

Proof

The standard topology on $\mathbb{R}^2$ is formed by taking as basis the set

$$\mathcal{B} = \{u \times v \mid u, v \in \mathcal{T}_\mathbb{R}\}.$$

So by theorem 15.1 [Munkres] we need only show that

$$\mathcal{C} = \{(a, b) \mid a < b \text{ and } a, b \in \mathbb{Q}\}$$

is a basis for $\mathbb{R}$. But, this was done in Homework 1 question 8a. The proof of which is included as an appendix.

$\square$

Question 9

Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$ where $\mathbb{R}_d$ is the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.

Proof

We note that the set $\mathcal{B} = \{\{x\} \mid x \in \mathbb{R}\}$ is a basis for the discrete topology on $\mathbb{R}$. Each $x \in \mathbb{R}$ lies in an element of $\mathcal{B}$ and the intersection of basis elements B_1, B_2 is empty unless $B_1 = B_2$. Thus we have by theorem 15.1 [Munkres] that a basis element U of $\mathbb{R}_d \times \mathbb{R}$ is such that

$$U = \{\alpha\} \times (a, b) \text{ for } \alpha, a, b \in \mathbb{R}.$$

First we show that $(\mathbb{R}^2, \mathcal{T}_{dict}) \subset \mathbb{R}_d \times \mathbb{R}$. Using lemma 13.3 we take $(x, y) \in B$, $B = (a \times b, c \times d)$ a basis element of $(\mathbb{R}^2, \mathcal{T}_{dict})$. Since the dictionary order on $\mathbb{R}$ has no smallest or largest element we have only the open intervals as basis elements. We consider the following cases for (x, y).

Case 1:

If $a < x < c$ and $y \in \mathbb{R}$ then we may take $B' = \{x\} \times (y + \epsilon, y - \epsilon)$ any $\epsilon > 0$ as our basis element of $\mathbb{R}_d \times \mathbb{R}$. Then $x \in B' \subset B$.

Case 2:

If $x = a$ and $y > b$ we may take $B' = \{x\} \times (b, y)$.

Case 3:

If $x = a$ and $y < d$ then we take $B' = \{x\} \times (y, d)$, whence in all cases lemma 13.3 is satisfied.

Now we show that $\mathbb{R}_d \times \mathbb{R} \subset (\mathbb{R}^2, \mathcal{T}_{dict})$. We again apply lemma 13.3. Fixing $(x, y) \in B = \{x\} \times (a, b)$ we have,

$$(x, y) \in B' = (x \times a, x \times b) \subset \{x\} \times (a, b).$$

We can apply lemma 13.3 once more to see that when compared to the standard topology, $\mathbb{R}_d \times \mathbb{R}$ is finer than $\mathbb{R}^2$. For if $(x, y) \in B = (a, b) \times (c, d)$ then $(x, y) \in B' = \{x\} \times (c, d) \subset B$.

$\square$

Appendix

The countable collection

$$\mathcal{B} = \{(ab,) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

Proof

Let $\mathcal{T}$ be the topology generated by $\mathcal{B}$. We will show that $\mathcal{T} = \mathcal{T}_{\mathbb{R}}$. Each element of $\mathcal{B}$ is also a basis element of $\mathcal{T}_{\mathbb{R}}$ so $\mathcal{T} \subset \mathcal{T}_{\mathbb{R}}$.

Now suppose $U = (\alpha, \beta)$ is a basis element of $\mathcal{T}_{\mathbb{R}}$. For any $x \in U$ there exists an open interval (a, b) with a and b rational such that $\alpha < a < x < b < \beta$ because $\mathbb{Q}$ is dense in $\mathbb{R}$. Thus by lemma 13.3 $\mathcal{T}_{\mathbb{R}} \subset \mathcal{T}$.

$\square$