Selected Exercises §13

Colton Kinstley

June 20, 2018

Question 1

Let X be a topological space; let A be a subset of X. Suppose the for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Solution

To show that A is an open set in X we will show that it is the union of open sets of X. Given $x \in A$ let U_x be an open set containing X such that $U \subset A$. Then

$$\bigcup_{x \in A} U_x = A$$

thus A is a union of open sets.

Question 6

Show that the topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable.

Solution

A basis for $\mathbb{R}_l$ is the set of all intervals $[a, b)$ and a basis for $\mathbb{R}_K$ is the set of all open intervals (a, b) along with the intervals of the form $(a, b) \setminus K$ where $K = \{1/n \mid n \in \mathbb{Z}_+\}$. Let $[x, b)$ be a basis element of $\mathbb{R}_l$ where $1 < x < b$. Then there is no basis element B of $\mathbb{R}_K$ such that $x \in B$ and $B \subset [x, b)$. Thus by lemma 13.3 [Munkres] $\mathbb{R}_K \not\subset \mathbb{R}_l$.

Now take $D = (-1, 1) \setminus K$ as a basis element of $\mathbb{R}_K$. No basis element B of $\mathbb{R}_l$ both contains the point 0 and is contained within D. If so then $B = (a, b)$ where $a < 0 < b$ but there exists $n \in \mathbb{Z}$ such that $0 < 1/n < b$ so $B \not\subset D$. Thus by lemma 13.3 [Munkres] $\mathbb{R}_l \not\subset \mathbb{R}_K$. We conclude that $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable.

Question 8

(a)

ApplyLemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(ab,) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generated the standard topology on $\mathbb{R}$.

Proof

Let $\mathcal{T}$ be the topology generated by $\mathcal{B}$. We will show that $\mathcal{T} = \mathcal{T}_{\mathbb{R}}$. Each element of $\mathcal{B}$ is also a basis element of $\mathcal{T}_{\mathbb{R}}$ so $\mathcal{T} \subset \mathbb{R}$.

Now suppose $U = (\alpha, \beta)$ is a basis element of $\mathcal{T}_{\mathbb{R}}$. For any $x \in U$ there exists an open interval (a, b) with a and b rational such that $\alpha < a < x < b < \beta$ because $\mathbb{Q}$ is dense in $\mathbb{R}$. Thus by lemma 13.3 $\mathcal{T}_{\mathbb{R}} \subset \mathcal{T}$.

$\square$

(b)

Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Solution

Let $\mathcal{T}$ be the topology generated by $\mathcal{C}$. Take $[x, \beta)$ with x irrational as basis element of $\mathbb{R}_l$. There is no element of $\mathcal{C}$ that contains x and is contained within $[x, \beta)$. Thus by lemma 13.3 $\mathcal{T} \not\subset \mathbb{R}_l$. Thus $\mathcal{T} \neq \mathbb{R}_l$.

Extra Question

Determine the convergent sequences for the finite complement topology on $\mathbb{R}$.

Claim

If x_n is a sequence in $(\mathbb{R}, \mathcal{T}_F)$ then x_n converges if and only if x_n has 0 or 1 constant subsequence.

Proof

First we note that any open set U in $(\mathbb{R}, \mathcal{T}_F)$ can be written in the form

$$U = \mathbb{R} \setminus \{a_i\}_{i=1}^n.$$

Suppose a sequence x_n has no constant subsequences. We claim that this sequence will converge to any point l in $\mathbb{R}$. Take some neighborhood of l, $U = \mathbb{R} \setminus \{a_i\}_{i=1}^n$. Since x_n has no constant subsequences the sequence coincides with $\{a_i\}_{i=1}^n$ on finitely many terms. That is to say $x_{i_k} = a_k$ for only finitely many $i_k \in \mathbb{N}$. Otherwise, a_k would be the value of a constant subsequence. Thus we may take

$$N = \max_{\substack{k \in 1,\dots,n \\ i_k \in \mathbb{N}}} \{i_k\} + 1.$$

in the definition of convergence. So that for all $n \geq N$ we have that $x_n \notin \{a_i\}_{i=1}^n$ thus $x_n \in U$ for $n \geq N$.

Now suppose that x_n has only one constant subsequence, say $x_m = l$ for $m \in M \subset \mathbb{N}$ where M has infinite cardinality. Then x_n converges to l. To see this take any neighborhood U of l. Then $x_m = l \in U$ for all $m \in M$ and the remaining subsequence x_k where $k \in \mathbb{N} \setminus M$ is a sequence with no constant subsequence. From the above argument it will also converge to l, thus x_n as a whole converges to l.

For the converse we will prove its contrapositive; if x_n has two or more constant subsequences, then x_n does not converge to any point in $\mathbb{R}$. Suppose x_n is a series with two or more distinct constant subsequences, say

$$\begin{gathered} x_{m_1} = l_1, \\ x_{m_2} = l_2, \\ \vdots \\ \text{for } m_i \in M_i \subset \mathbb{N} \text{ and } M_i \text{ infinite.} \end{gathered}$$

Suppose that x_n does converge to some point $l \in \mathbb{R}$, then l is different from at least one of l_1, $l_2, \dots$ and the neighborhood of l, $U = \mathbb{R} \setminus \{l_k\}$ where $l \neq l_k$ is such that for all $m_k \in M_k$ we have that $x_{m_k} \notin U$. But, since M_k is infinite this contradicts the supposition that x_n converges to l.

$\square$