Selected Exercises §23 & §28

Colton Kinstley

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Question §23-7

Is the space $\mathbb{R}_\ell$ connected? Justify your answer.

Solution

No, $\mathbb{R}_\ell$ is not connected. To see this consider the open sets formed by fixing b in $\mathbb{R}_\ell$

$$U_1 = \bigcup_{x<b} [x, b) = (-\infty, b)$$
$$U_2 = \bigcup_{b<x} [b, x) = [b, \infty).$$

The sets U_1 and U_2 clearly form a separation of $\mathbb{R}_\ell$.

Question §23-11

Let $p : X \to Y$ be a quotient mp. Show that if each set $p^{-1}(\{y\})$ is connected, and if Y is connected, then X is connected.

Solution

Suppose X is not connected, let A and B be a separation of X. We will show that A and B are saturated with respect to p.

Given any $y \in Y$ the set $p^{-1}(\{y\})$ must lie entirely within A or B since $p^{-1}(\{y\})$ is connected. Thus A must contain any set $p^{-1}(\{y\})$ that it intersects. Likewise for B. Further, note that since p is surjective $p(A) \cup p(B) = Y$ and $p(A) \cup p(B) = \varnothing$, otherwise a point in the intersection would have pre-image lying in both A and B. Since p takes saturated open sets of X to open sets in Y, $p(A)$ and $p(B)$ are open and thus form a separation of Y, a contradiction.

Question §28-3

Let X be limit point compact.

(a) If $f : X \to Y$ is continuous, does it follow that $f(X)$ is limit point compact?

(b) If A is a closed subset of X, does it follow that A is limit point compact?

Solution (a)

Yes, $f(X)$ is limit point compact. Suppose $A \subset f(X)$ and A has infinitely man points. Then $f^{-1}(A)$ has infinitely many points and so has a limit point in X, say x. Then there exists a net in $f^{-1}(A)$, say $\{x_\lambda\}_{\lambda \in \Lambda}$ such that $x_\lambda \to x$. By continuity of f, $f(x_\lambda \to f(x)$, thus $f(x)$ is a limit point of A.

Solution (b)

Yes, A is limit point compact. If A is closed then A contains all of it's limit points. Given B, an infinite subset of A, B has a limit point in X, being an infinite subset of X. But since $B \subset A$ this limit point must lie within A. Thus A is limit point compact.