Exercises §13

Colton Kinstley

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Question 1

Let X be a topological space; let A be a subset of X. Suppose the for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Solution

To show that A is an open set in X we will show that it is the union of open sets of X. Given $x \in A$ let U_x be an open set containing X such that $U \subset A$. Then

$$\bigcup_{x \in A} U_x = A$$

thus A is a union of open sets.

Question 2

Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.

Solution

We shall label the topologies as elements in a matrix and give a sample of comparisons.

1. $\mathcal{T}_{11}$ is comparable to and coarser than all others.
2. $\mathcal{T}_{33}$ is comparable to and finer than all others.
3. $\mathcal{T}_{12}$, $\mathcal{T}_{31}$ and $\mathcal{T}_{32}$ are comparable with $\mathcal{T}_{31} \subset \mathcal{T}_{12} \subset \mathcal{T}_{32}$

Question 3

Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{U | X \setminus U \text{ is infinite or empty or all of } X\}$$

a topology on X?

Solution (part 1)

Let $\mathcal{T}_c$ be the collection of subsets, U of X such that $X \setminus U = X$ or $X \setminus U$ is countable. First we see that $X \setminus X = \varnothing$ which is countable and $X \setminus \varnothing = X$ so $X, \varnothing \in \mathcal{T}_c$.

Next we will show that $\mathcal{T}_c$ is closed under unions. Given any union of open sets $\bigcup U_\alpha$ we have that

$$X \setminus \bigcup U_\alpha = \bigcap (X \setminus U_\alpha).$$

Because $X \setminus U_\alpha$ is countable for all α and the intersection of countable sets is countable we have that $\bigcup U_\alpha \in \mathcal{T}_c$.

Finally, given any finite intersection $\bigcup_{i=1}^n U_i$ of open sets of X we have that

$$X \setminus \bigcup_{i=1}^{n} U_i = \bigcap_{i=1}^{n} (U_i \setminus X).$$

For each i, $X \setminus U_i$ is countable and the finite union of countable sets is countable.

Solution (part 2)

$\mathcal{T}_\infty$ is not a topology as it is not closed under finite intersections. For a counterexample consider $X = \mathbb{Z}$ and the two subsets $U_{-1} = \{-1, 0, \dots\}$ and $U_1 = \{\dots, 0, 1\}$. Though both U_1 and U_{-1} are clearly in $\mathcal{T}_\infty$ their intersection,

$$U_{-1} \bigcap U_1 = \{-1, 0, 1\}$$

is finite.

Question 4a

If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?

Solution (part 1)

We have that $\varnothing$ and X are in $\{\mathcal{T}_\alpha\}$ for all α so $\varnothing, X \in \bigcap \mathcal{T}_\alpha$. Let $\{U_i\}_{i\in I}$ be a collection of sets in $\bigcap \mathcal{T}_\alpha$. Since each U_i is an element of $\mathcal{T}_\alpha$ for all α and $\mathcal{T}_\alpha$ is closed under unions for each α we must have that

$$\bigcup_{i\in I} U_i \in \bigcap \mathcal{T}_\alpha.$$

Likewise if $\{U_i\}$ is a finite collection

$$\bigcap_{i=1}^{n} U_i \in \mathcal{T}_\alpha$$

as each $U_i \in \mathcal{T}_\alpha$ for some α and $\mathcal{T}_\alpha$ is closed under finite unions.

Solution (part 2)

No, $\bigcup \mathcal{T}_\alpha$ is not a topology in general. For a counterexample examine the topologies in Example 1 of §12. Observe that $\mathcal{T}_{12} \bigcup \mathcal{T}_{21}$ is not a topology; for one $\{b\}$ is not in the union but can be obtained from intersection of elements in $\mathcal{T}_{12} \bigcup \mathcal{T}_{21}$.

Question 4b

Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.

Solution

Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X and let $\mathcal{A}$ be the collection of all unions and finite intersections of elements in $\bigcup \mathcal{T}_\alpha$. By definition $\mathcal{A}$ is closed under unions and finite intersections and because each topology $\mathcal{T}_\alpha$ contains X and $\varnothing$ so will $\mathcal{A}$; thus $\mathcal{A}$ is a topology.

Clearly for each α, we have $\mathcal{T}_\alpha \subset \mathcal{A}$. Suppose there is some other topoogy $\mathcal{A}'$ for which $\mathcal{T}_\alpha \subset \mathcal{A}'$ for all α. Given any $U \in \mathcal{A}$ either $U = \bigcup U_\alpha$ or $\bigcap_{i=1}^{n} U_i$ with $U_\alpha, U_i \in \bigcup \mathcal{T}_\alpha$. However, because $\mathcal{A}'$ is a topology that contains each $\mathcal{T}_\alpha$, we must then have that $U \in \mathcal{A}'$. So $\mathcal{A} \subset \mathcal{A}'$. Thus $\mathcal{A}$ is the unique smallest topology on X containing all $\mathcal{T}_\alpha$.

Now, let $\mathcal{T} = \bigcap \mathcal{T}_\alpha$. We claim that $\mathcal{T}$ is the unique largest topology contained in all $\mathcal{T}_\alpha$. By 4a $\mathcal{T}$ is a topology. Suppose there is some other topology $\mathcal{T}'$ such that $\mathcal{T}' \subset \mathcal{T}_\alpha$ for all α. Given $U \in \mathcal{T}'$ we have that $U \in \mathcal{T}_\alpha$ for all alpha. Thus $\mathcal{T}' \subset \mathcal{T}$.

Question 4c

If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\varnothing, X, \{a\}, \{a, b\}\} \text{ and } \mathcal{T}_2 = \{\varnothing, X, \{a\}, \{b, c\}\}.$$

Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

Solution

The smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ is

$$\mathcal{T}_3 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}.$$

The largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$ is

$$\mathcal{T}_4 = \{\varnothing, X, \{a\}\}.$$

Question 5

Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

Proof