Selected Exercises §29, §34 & §38

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Question

Prove that $T_{3\frac{1}{2}}$ is a hereditary property.

Proof

Suppose X is completely regular and Y is a subspace of X. Let $B \subset Y$ be closed and $y_0 \in Y$ a point not in B. Then $B = A \cap Y$ for some A closed in X. Note that y_0 is not in A since $y_0 \in Y$ and $Y_0 \notin B$. Because X is completely regular there is a continuous function $f : X \to [0, 1]$ where $f(A) = \{0\}$ and $f(y_0) = 1$. The restriction of this function to Y is continuous and $f|_Y(B) = 0$ (since $B \subset A$) and $f|_Y(y_0) = 1$. Thus Y is completely regular.

$\square$

Question §34-4

Let X be a locally compact Hausdorff space. Is is true that if X has a countable basis, then X is metrizable? Is it true that if X is metrizable , then X has a countable basis?

Solution

The first claim is true, we will show that locally compact Hausdorff spaces are regular. Then the result follows from the Urysohn metrization theorem. Theorem 29.2 [Munkres] states that a space X is locally compact Hausdorff if and only if given any x and neighborhood U of x, there exists a neighborhood V of x such that $\bar{V} \subset U$ with $\overline{V}$ compact.

Given $x \in X$ and K closed such that $x \notin K$ take $U = X \setminus K$ as in the above theorem. Then there exist V and $\overline{V}$ disjoint from K. Letting $W = X \setminus \overline{V}$ we have $K \subset W$. Then V and W are disjoint open sets that sepreate x and K. So X is regular.

The second part is not true. We can take as counter example $\mathbb{R}$ with the discrete topology. It is clearly Hausdorff, it's metrizable by taking as metric the function $p(x, y) = 0$ for $x = y$ and $p(x, y) = 1$ for $x \neq y$. This space is locally compact as every subset is both open and closed so every open set equals its closure. But is not second countable since any basis $\mathcal{B}$ would have to contain as a subset $\{x \mid x \in \mathbb{R}\}$ which is uncountable.

Question §29-1

Show that the rationals $\mathbb{Q}$ are not locally compact.

Solution

Suppose that $\mathbb{Q}$ as a subspace of $\mathbb{R}$ is locally compact, we will use theorem 29.2 to show that there exists a compact interval $[a, b] \subset \mathbb{Q}$ then derive a contradiction. Take $x \in U$ any point and neighborhood. Then there exists a neighborhood of V of x such that $\overline{V}$ is compact. Let $(a, b) \subset V$ be a basic set containing x. Then it's closure $[a, b]$ is a subset of $\overline{V}$. Take this interval as a closed subspace of a compact metric space, $\overline{V}$, by theorem 26.2 [Munkres] $[a, b]$ is compact. Thus by Theorem 28.2 [Munkres] $[a, b]$ is sequentially compact. However, given a sequence of rational numbers in $[a, b]$ converging to an irrational number say $\alpha \in [a, b] \subset \mathbb{R}$ that same sequence will not converge in $[a, b] \subset \mathbb{Q}$ considering that all subsequences will also converge to α we conclude that $[a, b]$ is not sequentially compact. A contradiction.

Question §29-2

Let $X = \{X_\alpha\}$ e and indexed family of nonempty spaces.

(a) Show that if $X = \prod X_\alpha$ is locally compact, then each X_α is locally compact and X_α is compact for all but finitely many values of α.

(b) Prove the converse, assuming the Tychnoff theorem.

Proof (a)

Suppose $X = \prod X_\alpha$ then given any $x \in X$ there exists U open and K compact with $x \in U \subset K$. We have that $U = \prod U_\alpha$ for some U_α open in X_α and $U_\alpha = X_\alpha$ for all but finitely many α. Since $U \subset K$ we must have that if $U_\alpha = X_\alpha$, then $\pi_\alpha(K) = X_\alpha$. Each K_β is compact since given any cover $\{A_\lambda\}_{\lambda \in \Lambda}$ of K_β we can form a cover of K by taking $\mathcal{B} = \{B \subset X \mid \pi_\alpha(B) = X_\alpha \text{ for } \alpha \neq \beta \text{ and } \pi_\beta(B) = A_\lambda \text{ for } \lambda \in \Lambda\}$ which has a finite subcover (since K is compact), say $\{B_i\}_{i=1}^n$. The set $\{\pi_\beta(B_i)\}_{i=1}^n$ is a finite cover of K_β. We conclude that each X_β is locally compact since given any $x_\beta \in X_\beta$ there is an $x \in X$ such that $x_\beta = \pi_\alpha(x)$. But, $\pi_\beta(U) \subset \pi_\beta(K)$, $\pi_\beta(U)$ is open and $\pi_\beta(K)$ is compact. Further since for any K satisfying the definition of local compactness each $\pi_\alpha(K)$ is compact and only finitely man of the $\pi_\alpha(K)$ are not equal to all of X_α we conclude that all but finitely many of the X_α are compact.

$\square$

Proof (b)

To prove the converse let A be the finite set of α for which X_α is not compact. Then given any $x \in X$ there exists $U_a \in X_a$ open and $K_a \in X_a$ compact such that $U_a \subset K_a$ for each $a \in A$. Let U be the open set formed by taking the product of all U_a for $a \in A$ with X_α for $\alpha \notin A$ and K_a for $a \in A$ with X_α for $\alpha \notin A$. Then U is open in X and by Tychnoff's theorem K is compact in X.

$\square$

Question

Assume X is a locally compact space, denote by Y its one-point compactification; characterize the continuous functions $f : X \to \mathbb{R}$ that have an extension to Y.

Solution

Question §38-2

Show that the bounded continuous function $f : (0,1) \to \mathbb{R}$ defined by $g(x)\cos(1/x)$ cannot be extended to the compactification of Example 3. define an imbedding $h : (0,2) \to [0,1]^3$ such that the functions x, $\sin(1/x)$, and $\cos(1/x)$ are all extendable to the compactification induced by h.

Solution

In Example 3 the embedding is given by $Y \simeq \overline{h((0,1))}$ where $h(x) = (x, sin(\frac{1}{x}))$ is an imbedding of $(0,1)$ into $[0,1]^2$. To show that there is no continuous extension of $cos(1/x)$ to the compactification Y we consider the sequence in $(0,1)$ given by $x_n = \frac{1}{\pi n}$ it's image under h is $h(x_n) = \{(\frac{1}{\pi n}, sin(\pi n))\}$. As $k \to \infty$ we see that $h(x_n) \to (0,0)$. Since any continuous extension must also converge to a limit point with this sequence and agree with $\cos(1/x)$ on $(0,1)$ we arrive at a contradiction as $\lim_{n\to\infty} \cos(\pi n)$ does not exist.

For the second part of the question we follow the line from Example 4 but we let our compactification be induced by the function $h : (0,1) \to [0,1]^3$ by $h(x) = (x, \sin(1/x), \cos(1/x))$. So that if $Y \simeq \overline{h((0,1))}$, then the composition

$$Y \xrightarrow{H} \mathbb{R}^3 \xrightarrow{\pi_1} \mathbb{R}$$

is continuous since it is the composition of continuous functions and is equal to x on X. Changing the final composition to the projection function to the second and third coordinates provides the other two extensions for the same reason.