Selected Exercises §27 & §30

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Question 2

Let X be a metric space with metric d; let $A \subset X$ be nonempty.

(a) Show that ...

(b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

(c) Define the ...

(d) Assume that A is compact; let U be and open set containing A. Show that some ϵ-neighborhood of A is contained in U.

(e) Show the result in (d) need not hold if A is closed but not compact.

Solution (b)

Since $d : X \times X \to \mathbb{R}$ is continuous its restriction to $X \times A$ is also continuous. Fixing $x \in X$ we minimize the continuous function $d\big|_A : A \to \mathbb{R}$ to obtain the value of $d(x, A)$. Because A is compact we can apply theorem 27.4 [Munkres] (the extreme value theorem) to obtain $c \in A$ such that $d(x, c) \leq d(x, a)$ for all $a \in A$. Hence the infimum in the definition of $d(x, A)$ is in fact obtained by $c \in A$ and $d(x, A) = d(x, c)$.

Solution (d)

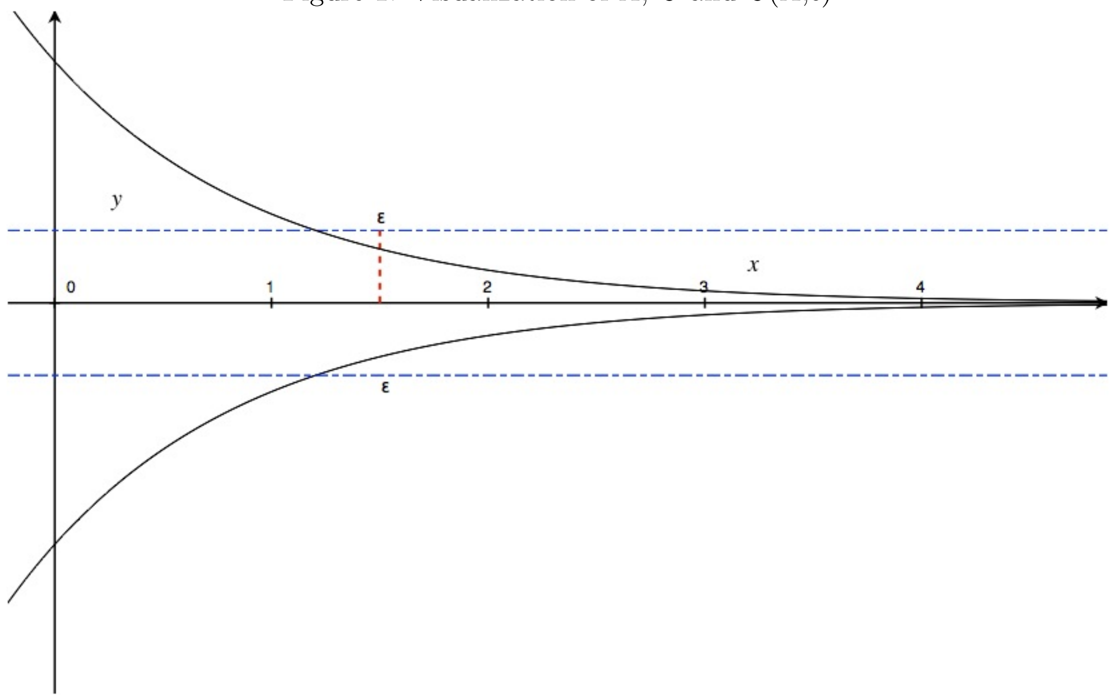
Since U is an open set in X we have that for each $a \in A \subset U$ there is a basis element $B_d(a, \epsilon)$ with ϵ depending on a that is a subset of U. This set of balls forms an open cover of A and because A is compact there exists $\{B_d(a_i, \epsilon_i)\}_{i=1}^n$ a finite subcover. Let $\epsilon = \min_{i=1,\ldots,n}\{\epsilon_i\}$ then we have $U(A, \epsilon) = \bigcup_i B_d(a_i, \epsilon_i) \subset U$, since for each $i = 1, \ldots, n$

$$B_d(a_i, \epsilon) \subset B_d(a_i, \epsilon_i) \subset U.$$

Solution (e)

Take $(X, d) = \mathbb{R} \times \mathbb{R}$ with the usual metric. Let $A = [1, \infty) \times \{0\}$. Then an open set containing A is $U = (0, \infty) \times \mathbb{R} \cap \{(x, y) \mid x > 0, -e^{-x} < y < e^{-x}\}$.

Figure 1: Visualization of A, U and U(A,ϵ)



Suppose that there were an $\epsilon > 0$ such that $U(A, \epsilon) \subset U$. Then the point $(1 - \ln \epsilon, \epsilon/2)$ would lie in $U(A, \epsilon)$ but if $U(A, \epsilon) \subset U$ this implies that

$$\epsilon/2 < e^{-(1-\ln \epsilon)} = \epsilon/e$$

a contradiction.

Question 10

Show that if X is a countable product of spaces having countable dense subsets, then X has a countable dense subset.

Solution

Let $X = \prod_{i=1}^{\infty} X_i$ be a countable product of spaces and suppose for each $i = 1, \ldots, n$ $A_i \subset X_i$ is a countable dense subset. We can show that X has a countable dense subset by constructing one. Let $A = \prod_{i=1}^{\infty} A_i$, we will show that A is a countable dense subset of X. That A is countable is clear as it is the countable product of countable sets. To see that A is dense in X we apply theorem 19.5 [Munkres] to see that

$$\bar{A} = \overline{\prod_i A_i} = \prod_i \bar{A}_i = \prod_i X_i = X.$$