Selected Exercises §18

Colton Kinstley

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Question 2

Suppose $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

Solution

Not necessarily. Consider any constant function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = a$ for some $a \in \mathbb{R}$. Then given any $A \neq \varnothing$ we have $f(A) = \{a\}$. But, this set has no limit points. For a counter example to the claim take $A = (0, 1)$ this set has limit points $A' = [0, 1]$. In particular $f(1) = a$, but any neighborhood of $f(1)$ intersects $\{a\}$ only at $f(1) = a$ and thus is not a limit point of $f(A)$.

Question 7a

Claim

Suppose $f : \mathbb{R}_\ell \to \mathbb{R}$. Then f is continuous if and only if

$$\lim_{x \to a^+} f(x) = f(a).$$

Proof

Suppose the above limit holds. Then by the definition of this limit we have

$$\forall \epsilon > 0 \text{ and } \forall a \in \mathbb{R},\ \exists \delta > 0 \text{ s.t.}$$
$$a < x < a + \delta \Rightarrow |f(x) - f(a)| < \epsilon.$$

Given any $a \in \mathbb{R}$ and neighborhood V of a there exists a basis element (c, d) such that $a \in (c, d) \subset V$. Take $\epsilon = \min\{f(a) - c, f(a) + d\}$, and set $V^* = (f(a) - \epsilon, f(a) + \epsilon) \subset V$. Then by hypothesis there exists $U = [a, a + \delta)$ such that $f(U) \subset f(V^*) \subset V$. But, U is a neighborhood containing a and since a and V were chosen arbitrarily we have satisfied equivalence (4) of theorem 18.1 [Munkres] and may conclude that f is continuous.

Conversely, suppose f is continuous. Given $\epsilon > 0$ and $a \in \mathbb{R}$ note that

$$V = (f(a) - \epsilon, f(a) + \epsilon)$$

is a neighborhood containing $f(a)$. Thus there exists a neighborhood U of a open in $\mathbb{R}_\ell$ such that $a \in U$ and $f(U) \subset V$. Let $[b, c) \subset U$ be a basis element containg a. Then take $\delta = c - a$ to get $[a, a + \delta) \subset [b, c) \subset U$. Since

$$f([a, a + \delta)) \subset f(U) \subset V = (f(a) - \epsilon, f(a) + \epsilon)$$

we have that for any $\epsilon > 0$ and $a \in \mathbb{R}$ there is a δ such that $a < x < a + \delta$ implies that $|f(x) - f(a)| < \epsilon$.

$\square$

Question 9b

Let $\{A_\alpha\}$ be a collection of subsets of X such that $X = \bigcup_\alpha A_\alpha$. Let $f : X \to Y$. Suppose $f\big|_{A_\alpha}$ is continuous for each α. Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but f is not continuous.

Solution

Take for example $X = \mathbb{Z}$ under the finite complement topology. Then U is open in $\mathbb{Z}$ if $\mathbb{Z} \setminus U$ is finite or all of $\mathbb{Z}$. Let $\{A_\alpha\} = \{n\}_{n \in \mathbb{Z}}$ then each $\{n\}$ is closed in $\mathbb{Z}$ as $\mathbb{Z} \setminus (\mathbb{Z} \setminus \{n\})$ is open and their union is clearly all of $\mathbb{Z}$. Let $f : \mathbb{Z} \to \mathbb{R}$ by $f(x) = x$ then the restriction $f\big|_{\{n\}}$ is continuous because given any open set V containing $f(n) = n$ we have that $U = f\big|_{\{n\}}^{-1}(n) = n$ and $\{n\} \setminus \{n\} = \varnothing$. So U is open.

On the other hand, given any open set in $\mathbb{R}$ of the form (a, b) where $-\infty < a < b < \infty$ we see that $U = f^{-1}((a, b)) = \{n \in \mathbb{Z} \mid a < n < b\}$. Clearly this set is finite, thus $\mathbb{Z} \setminus U$ is infinite and not open. We conclude that f is not continuous.

Extra Question

Prove that the interval $(0, 1)$ is homeomorphic to $\mathbb{R}$ but not to the interval $[0, 1)$.

Proof

To show that $(0,1)$ is homeomorphic to $\mathbb{R}$ we exhibit such a homeomorphism. Take $f : (0,1) \to \mathbb{R}$ by $f(x) = \frac{x-1/2}{x(x-1)}$. This is a rational polynomial with nonzero denominator on its domain and is thus continuous. Further $lim_{x\to\infty} f(x) = \infty$ and $lim_{x\to-\infty} f(x) = -\infty$, thus f onto. Since f is monotonically decreasing on $(0,1)$, (we have $f'(x) = \frac{-(2x^2-2x+1)}{2(x^2-x)^2} < 0$) we know that f is one to one and thus is a bijection. Computing the inverse,

$$f^{-1}(y) = \frac{1+y-\sqrt{1+y^2}}{2y} \text{ for } y \neq 0, \text{ and}$$
$$f^{-1}(y) = 1/2 \text{ for } y = 0$$

we see that it is indeed continuous.

To see that there is no homeomorphism from (0,1) to [0,1) we will show that there can be no continuous bijection from (0,1) to [0,1); this follows from continuity in real analysis. Indeed if there were such a function then there would a unique point $x \in (0,1)$ such that $f(x) = 0$ since f is continuous we have that closed/open intervals are mapped to closed/open intervals. Consider that there must be intervals $[x, x+\epsilon]$ and $[x-\epsilon, x]$ such that

$$f([x, x+\epsilon]) = [0, a)$$
$$f([x-\epsilon, x]) = [0, b)$$

since f is bijective this implies that at least one of the values in $[0, \min a, b)$ are taken twice by f. Which contradicts the infectivity of f.