Selected Exercises §17

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Question 6

Let A, B and A_α denote subsets of a space X. Prove the following:

(a) If $A \subset B$, then $\bar{A} \subset \bar{B}$

(b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

(c) $\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$

Proof (a)

By definition $\bar{B}$ is the intersection of all closed sets containing B. But, $A \subset B$ and $\bar{B}$ is closed; in particular $\bar{B}$ is a closed set containing A. Since $\bar{A}$ is equal to the intersection of all such sets we must have $\bar{A} \subset \bar{B}$.

□

Proof (b)

First we will show that $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$. Using (a) we have that the following implications hold

$$A \subset A \cup B \Rightarrow \bar{A} \subset \overline{A \cup B}$$
$$\text{and}$$
$$B \subset A \cup B \Rightarrow \bar{B} \subset \overline{A \cup B}.$$

Noting that both antecedents are tautologies our result follows.

Now we will show that $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$. Since $A \subset \bar{A}$ and $B \subset \bar{B}$ we have that $A \cup B \subset \bar{A} \cup \bar{B}$. The finite union of closed sets is closed, so $\bar{A} \cup \bar{B}$ is a closed set containing $A \cup B$ and thus contains $\overline{A \cup B}$.

□

Proof (c)

We wish to show that $\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$ but, for each α, $\overline{\bigcup A_\alpha}$ is a closed set containing A_α. Thus $\bar{A}_\alpha \subset \overline{\bigcup A_\alpha}$ for all α. The containment of their union follows.

□

Question 8

Let A, B and A_α denote subsets of a space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\subset$ or $\supset$ holds.

(a) $\overline{A \cap B} = \bar{A} \cap \bar{B}$

(b) $\overline{\bigcap A_\alpha} = \bigcap \bar{A}_\alpha$

(c) $\overline{A - B} = \bar{A} - \bar{B}$

Solution (a)

We have that $A \subset \bar{A}$ and $B \subset \bar{B}$. It follows that $A \cap B \subset \bar{A} \cap \bar{B}$. Since $\bar{A} \cap \bar{B}$ is closed we must have that $\overline{A \cup B} \subset \bar{A} \cap \bar{B}$ as $\overline{A \cup B}$ is the intersection of all closed sets containing $A \cap B$.

Solution (b)

Suppose $x \in \overline{\bigcap A_\alpha}$ then every neighborhood U of x intersects $\bigcap A_\alpha$. So given any neighborhood U, we have that for all α $U \cap A_\alpha \neq \varnothing$. Thus, $x \in \bar{A}_\alpha$ for all α and $\overline{\bigcap A_\alpha} \subset \bigcap \bar{A}_\alpha$.

Solution (c)

Question 13

Show that X is Hausdorff if and only if the *diagonal* $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Proof

Suppose that Δ is closed in $X \times X$. Then $W = X \times X \setminus \Delta$ is an open set. We wish to show that given $x_1, x_2 \in X$ there exist UV open in X such that $U \cap V = \varnothing$. Consider the point (x_1, x_2) in $X \times X$. Since $x_1 \neq x_2$ we have that $(x_1, x_2) \in W$. Then there exist $U, V \subset X$ open such that $x_1 \in U$ and $x_2 \in V$ and $U \times V \subset W$. We see that U and V do not intersect for if they did, say at $y \in X$ then $(y, y) \in W$ a contradiction.

Now suppose that X is Hausdorff. We have that for each pair $(x_i, x_j) \in X \times X$ where $x_i \neq x_j$ that there exist $U_{x_i}, V_{x_j} \in X$ such that $U_{x_i} \cap V_{x_j} = \varnothing$. Let W be the union of all $U_{x_i} \times V_{x_j}$. Then W is open and $X \times X \setminus W = \Delta$ is closed.

$\square$

Extra Question

Assume that (X, d) is a metric space, and $E \subset X$ is a nonempty subset. Denote by E' the limit (or accumulation) points of E.

(a) Prove that E' is closed.

(b) Prove that E and $\bar{E}$ have the same limit points.

(c) Do E and E' always have the same limit points?

Proof (a)

Take x a limit point of E'. Then for all neighborhoods V of x we have that $V \cap E' \neq \varnothing$, say V intersects E' at the point $\{y\}$. Then $y \in E'$ so for any neighborhood U of y, $U \cap E \neq \varnothing$. But, V is also a neighborhood of y. So $V \cap E \neq \varnothing$. Thus $x \in E'$; which shows that E' contains it's limit points and must be closed.

Proof (b)

First we will show that $E' \subset \bar{E}'$. Suppose $x \in E'$ then for every neighborhood V of x we have that $V \setminus \{x\} \cap E \neq \varnothing$. But, $E \subset \bar{E}$ so $V \setminus \{x\}$ also intersects $\bar{E}$. Thus $x \in \bar{E}'$.

For the reverse inclusion suppose $x \in \bar{E}'$. Then for all neighborhoods V of x we have $V \setminus \{x\} \cap \bar{E} \neq \varnothing$. Say V intersects $\bar{E}$ at the point $\{y\}$. Since $\bar{E} = E \cup E'$ either $y \in E$ or $y \in E'$. If $y \in E$ then $V \setminus \{x\}$ intersects E at y. So $x \in E'$. If $y \in E'$ then for any neighborhood U of y we have that U intersects E at some point other than y. But, $V \setminus \{x\}$ is also a neighborhood of y so $V \setminus \{x\}$ intersects E. Thus $x \in E'$.

$\square$

Part (c)

It is clear that the limit points of E are a subset of the limit points of $\bar{E}$ as $\bar{E} = E \cup E'$.