```
import numpy as np
from scipy.constants import G
import matplotlib.pyplot as plt
```

```
from google.colab import drive
drive.mount('/content/drive')
```

```
Mounted at /content/drive
```

Sympletic Integrator

$$x_{i+1} = q_i + c_i \frac{p_{i+1}}{m} t$$
$$v_{i+1} = v_i + d_i F(q_i) t$$

Update the position i of the particle by adding to it its (previously updated) velocity i multiplied by c_i

Update the velocity i of the particle by adding to it its acceleration (at updated position) multiplied by d_i

First Order

$k = 1$

$c_1 = d_1 = 1$

Second Order

$k = 2$

$c_1 = 0,\, c_2 = 1,\, d_1 = d_2 = \frac{1}{2}$

Third Order

$k = 3$

$c_1 = 1,\, c_2 = -\frac{2}{3},\, c_3 = \frac{2}{3},\, d_1 = -\frac{1}{24},\, d_2 = \frac{3}{4},\, d_3 = \frac{7}{24}$

Fourth Order

$k = 4$

$c_1 = c_4 = \frac{1}{2(2-2^{1/3})},\, c_2 = c_3 = \frac{1-2^{1/3}}{2(2-2^{1/3})},\, d_1 = d_3 = \frac{1}{2(2-2^{1/3})},\, d_2 = -\frac{2^{1/3}}{2(2-2^{1/3})},\, d_4 = 0$

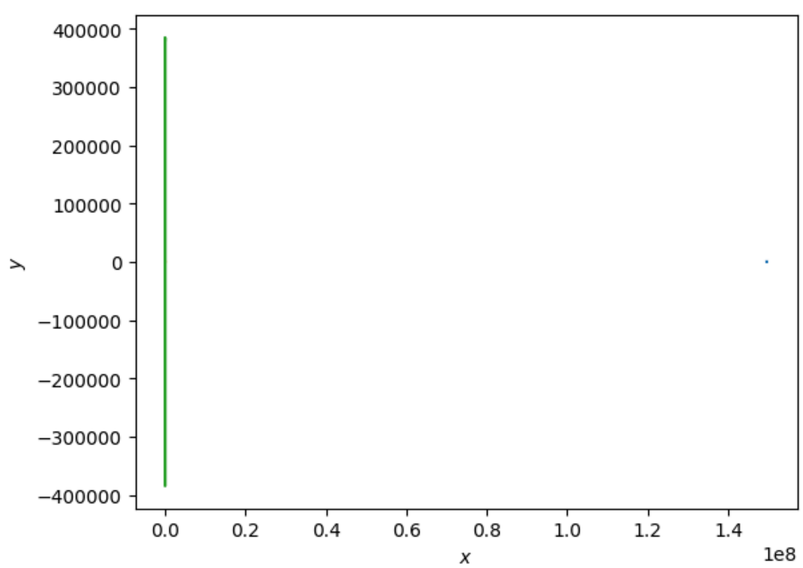
```
# 3-body: Earth, Sun 1, & Sun 2

# An array of the position of both x-axis and y-axis of the 3-body
px = np.array([0.00e00, 0.00e00, 0.00e00])
py = np.array([0.00e00, 4.50e10, -4.50e10])

# An array of the velocity of both x-axis and y-axis of the 3-body
vx = np.array([0.05e04, 3.00e04, -3.00e04])
vy = np.array([0.00e00, 0.00e00, 0.00e00])

#An array of the mass of the 3-body
m = np.array([5.974e24, 1.989e30, 1.989e30])
```

Solve with SciPy

$$r' = v$$
$$v' = -f(r) r^{-1} \mathbf{r}$$

```
from scipy.integrate import solve_ivp

diff_eq = [lambda x, v, t: x, lambda x, v, t: v]
```

```
m = np.array([1.9891e30,5.97219e24,7.34767309e22])
r=np.array([[149600000.,0.],[1.,0.],[0.,384400.]])
v=np.array([[0.,0.],[0.,1.],[1.,0.]])


dt=.001
t_max=1000

def calc_forces(r,m,G):
  N=len(m)
  forces = np.zeros((N,2))
  for i in range(N):
    for j in range(i+1,N):
      distance = np.linalg.norm(r[i]-r[j])
      force = G* m[i]*m[j]/ distance**2
      direction = (r[j]-r[i]/distance)
      forces[i]+= force*direction
      forces[j]-= force*direction
    return forces

t=0

body_0 = []
body_1 = []
body_2 = []

while t<t_max:
  forces = calc_forces(r,m,G)
  a= forces / m.reshape(-1,1)
  v+=a*dt
  r+=v*dt
  t+= dt
  body_0.append([t, r[0, 0], r[0, 1]])
  body_1.append([t, r[1, 0], r[1, 1]])
  body_2.append([t, r[2, 0], r[2, 1]])

body_0 = np.array(body_0)
body_1 = np.array(body_1)
body_2 = np.array(body_2)


plt.plot(body_0[:, 1], body_0[:, 2])
plt.plot(body_1[:, 1], body_1[:, 2])
plt.plot(body_2[:, 1], body_2[:, 2])
plt.xlabel("$x$")
plt.ylabel("$y$")
plt.show()
```



RK4

```
import scipy as sci
```

```
from mpl_toolkits.mplot3d import Axes3D
from matplotlib import animation


m_Earth = 5.9722e24 # kg
m_Sun = 1.989e+30 # kg
r_AU = 149597870700 # m
v_Earth = 29.78e3 # m/s
t_Earth = 60*60*24*365 # s


def plot_orbits(orbits):
  #Create figure
  fig=plt.figure(figsize=(15,15))

  #Create 3D axes
  ax=fig.add_subplot(111, projection="3d")

  #Plot the orbits
  for orbit in orbits:
    ax.plot(orbit[:, 0], orbit[:, 1], orbit[:, 2])

  #Plot the final positions of the stars
  for orbit in orbits:
    ax.scatter(orbit[-1,0], orbit[-1,1], orbit[-1,2], marker="o", s=100)

  #Add a few more bells and whistles
  ax.set_xlabel("x-coordinate", fontsize=14)
  ax.set_ylabel("y-coordinate", fontsize=14)
  ax.set_zlabel("z-coordinate", fontsize=14)
  ax.set_title("Visualization of orbits of stars in a n-body system\n", fontsize=14)

  return fig
```

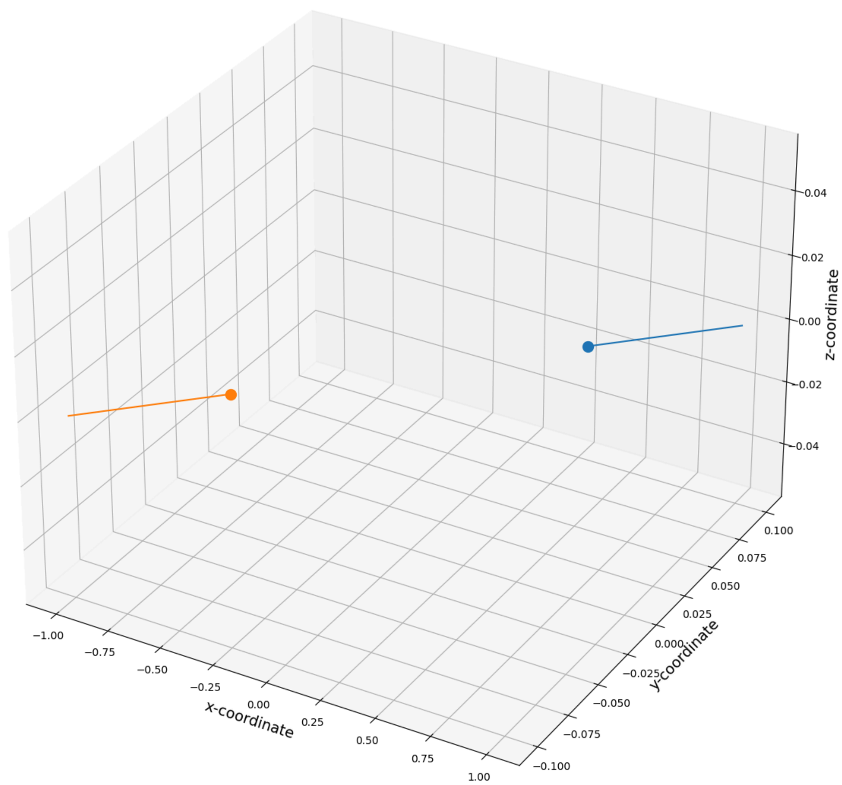
```
from matplotlib.animation import FuncAnimation
import IPython.display as dsp
from base64 import b64encode

def play_video(name):
  # use this fix to display mp4 since Colab can't do it with display.Video
  # https://stackoverflow.com/questions/57377185/how-play-mp4-video-in-google-colab
  mp4 = open(name,'rb').read()
  data_url = "data:video/mp4;base64," + b64encode(mp4).decode()

  return dsp.HTML("""
  <video width=400 controls>
        <source src="%s" type="video/mp4">
  </video>
  """ % data_url)

def create_animation(name, orbits, times):
  """
  display interactive animation of orbits.

  name: filename of the saved animation
  orbits: array-like of n time series, each time series a t_list * 3 size matrix
  times: array-like of t_list * 1 of time
  """

  #Create figure
  fig=plt.figure(figsize=(7,7))

  #Create 3D axes
  ax=fig.add_subplot(111, projection="3d")

  def animate(frame):
    ax.clear()

    ax.text(0.7, 0.7, 0.7, f"t={times[frame]:.3f}", transform=ax.transLimits)

    # Plot the orbits
    for i, orbit in enumerate(orbits):
      ax.plot(orbit[:frame, 0], orbit[:frame, 1], orbit[:frame, 2])

    # Plot the final positions of the stars
    for orbit in orbits:
      ax.scatter(orbit[frame-1,0], orbit[frame-1,1], orbit[frame-1,2], marker="o", s=100)


  desired_num = 250
  frame_num = len(orbits[0])
  scaling = int(np.ceil(frame_num / desired_num))
  # print(f"range(0, {frame_num}, {scaling})")
  anim = FuncAnimation(fig, animate, frames=range(0, frame_num, scaling))

  anim.save(name, codec=None)
```

```
def nbody(r, v, m, time_span, m_norm=1.989e+30, r_norm=1.495978707e11, v_norm=29784.8, t_norm=60*60*24*365):
  init_params = np.array([r, v]).flatten()

  # Net constants
  K1=G*t_norm*m_norm/(r_norm**2*v_norm)
  K2=v_norm*t_norm/r_norm

  def NBodyEquations(w, t, m):
    n = len(m)

    r = [w[3*i:3*i+3] for i in range(n)]
    v = [w[3*i+3*n:3*i+3+3*n] for i in range(n)]

    r_mn = np.array([[np.linalg.norm(r[i] - r[j]) for i in range(n)] for j in range(n)])

    # print("r_Mn")
    # print(r_mn)
    #r_str = [f"K2*v_{i}" for i in range(len(v))]
    #v_str = [" + ".join([f"K1*m[{j}]*(r[{j}]-r[{i}])/r_mn[{i}][{j}]**3" for j in range(n)  if i != j]) for i in range(n)]
    #print(np.concatenate((r_str, v_str), axis=0).flatten())

    # print(r)
    # print(v)

    drbydt = np.array([K2*v_i for v_i in v])

    #print("dr/dt")
    #print(drbydt)

    dvbydt = np.array([np.sum([K1*m[j]*(r[j]-r[i])/r_mn[i][j]**3 for j in range(n)  if i != j], axis=0) for i in range(n)])

    #print("dv/dt")
    #print(dvbydt)

    derivs = np.concatenate((drbydt, dvbydt), axis=0).flatten()

    #print(derivs)

    return derivs

  # Run the ODE solver
  n_body_sol=sci.integrate.odeint(NBodyEquations, init_params, time_span, args=((m,)))
  orbits = [n_body_sol[:, 3*i:3*i+3] for i in range(len(m))]

  return orbits
```

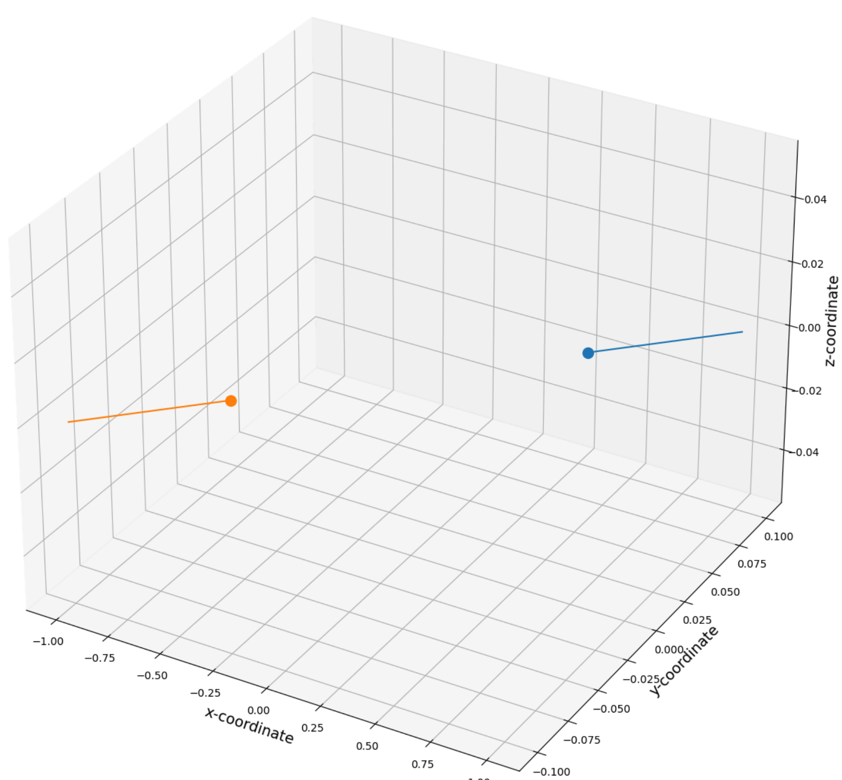
Simulation: Two Bodies

```
r = np.array([[1, 0.1, 0], [-1, -0.1, 0]], dtype="float64")

v = np.array([[0, 0, 0], [0, 0, 0]], dtype="float64")

m = np.array([10, 10], dtype="float64")

time_span = np.linspace(0, 0.09, 2500)

orbits = nbody(r, v, m, time_span)
plot_orbits(orbits)
#create_animation("two_body.mp4", orbits, time_span)
#play_video("two_body.mp4")
```

Visualization of orbits of stars in a n-body system

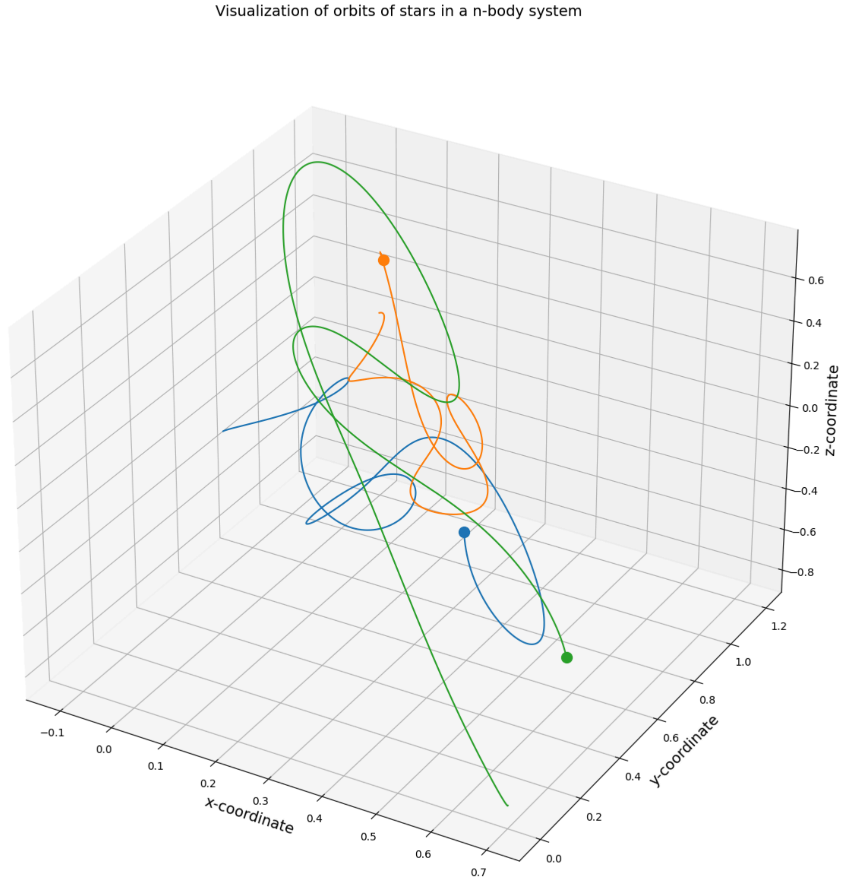


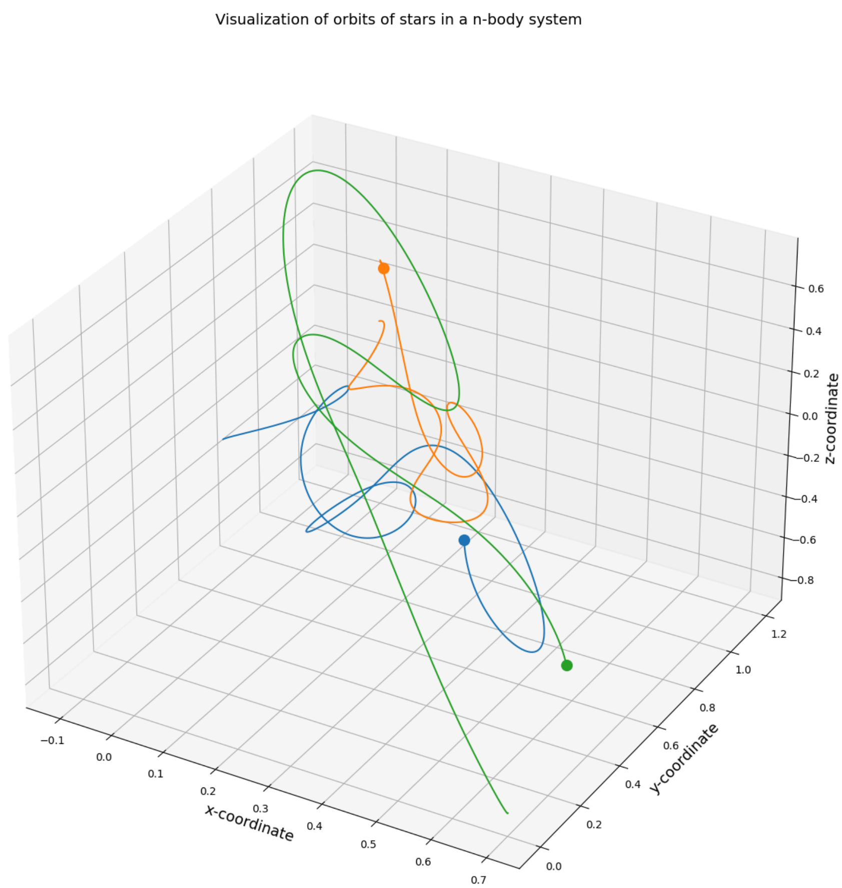
Visualization of orbits of stars in a n-body system



Simulation: Three Body Chaos

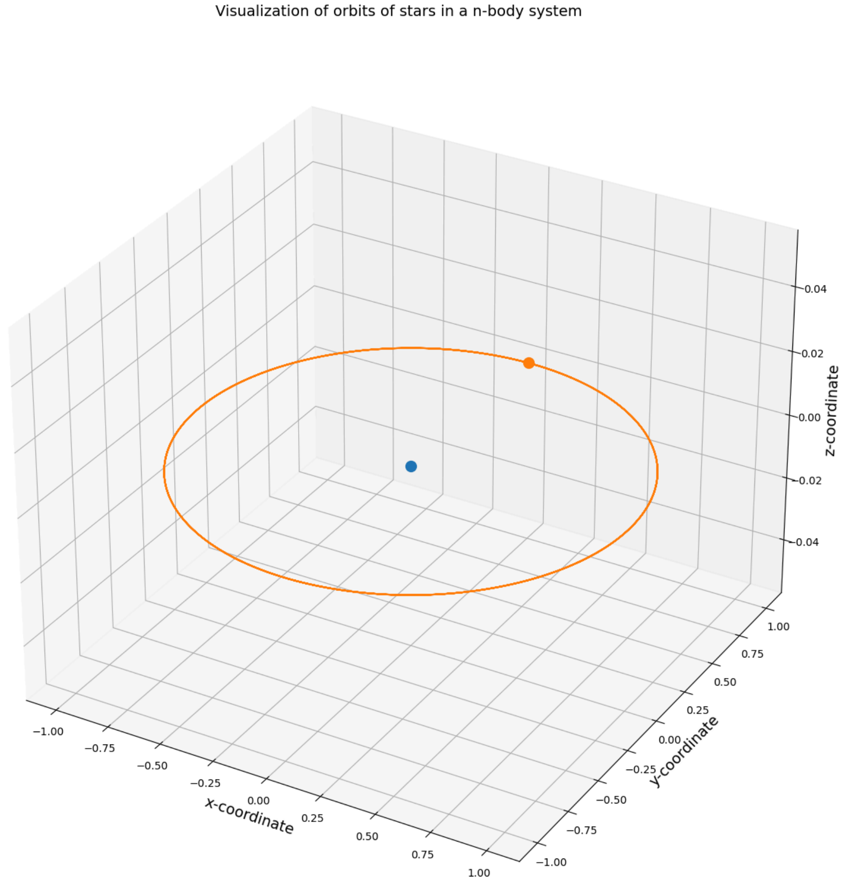
```
r = np.array([[1, 2, 3],
              [0, 12, 0],
              [7, 0, -8]])*0.1

v = np.array([[1, 2, 2],
              [4, 2, 4],
              [1, -3, -1]])*0.01

m = np.array([0.5, 0.7, 0.3])

time_span=np.linspace(0, 1, 5000)

orbits = nbody(r, v, m, time_span)
plot_orbits(orbits)
```

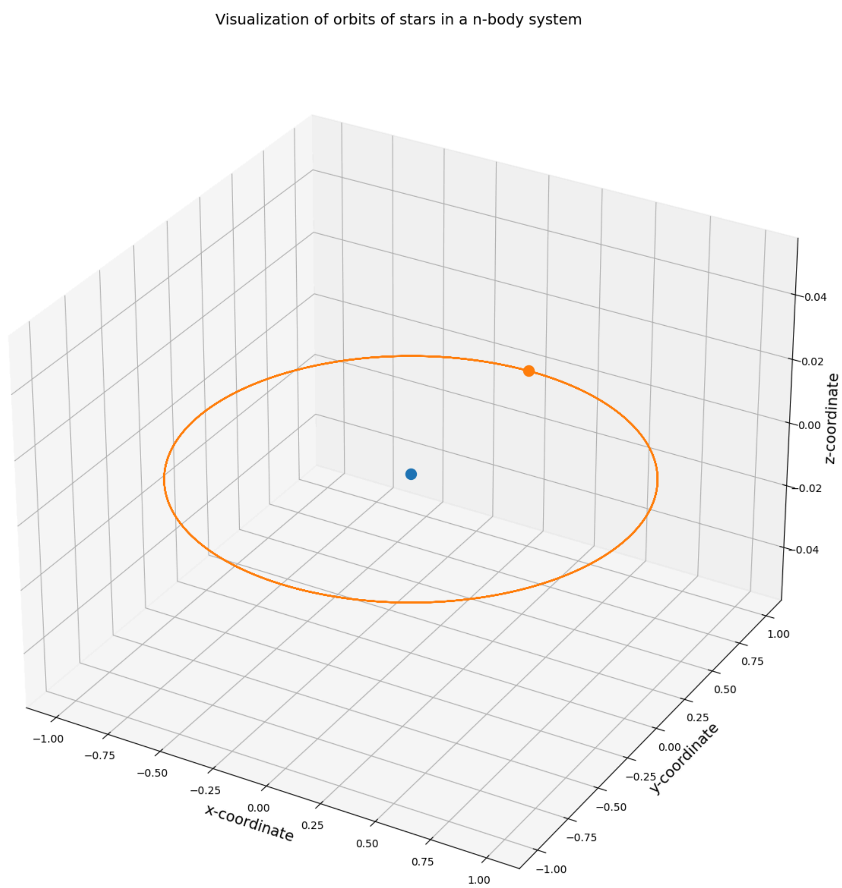




Simulation: Earth & Sun

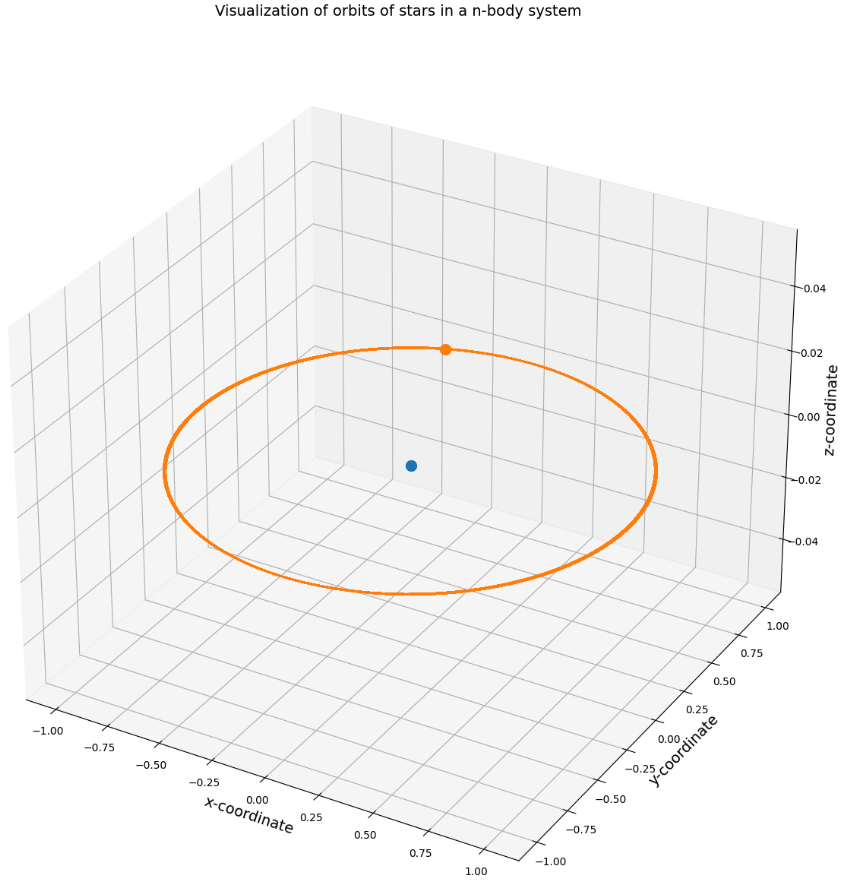
```
r = np.array([[0, 0, 0],
              [0, 1, 0]], dtype="float64")

v = np.array([[0, 0, 0],
              [1, 0, 0]], dtype="float64")

m = np.array([1, 5.97219e24/1.989e+30])

time_span=np.linspace(0, 5, 10000)

orbits = nbody(r, v, m, time_span)
plot_orbits(orbits)
```

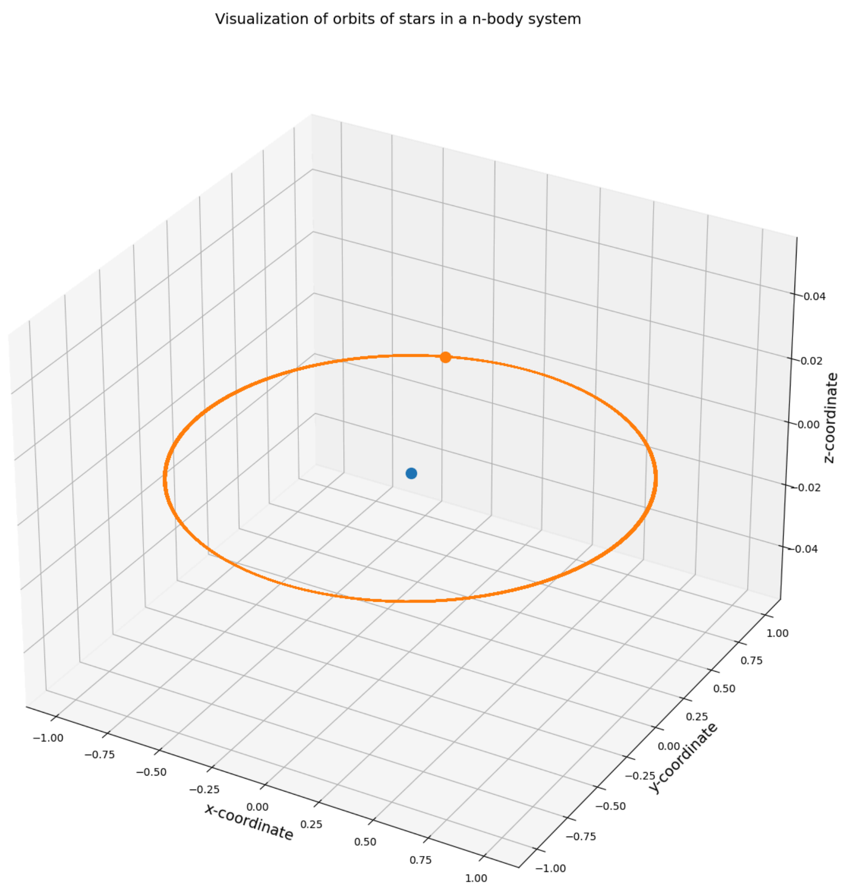




Simulation: Drift of RK$ - Normalized

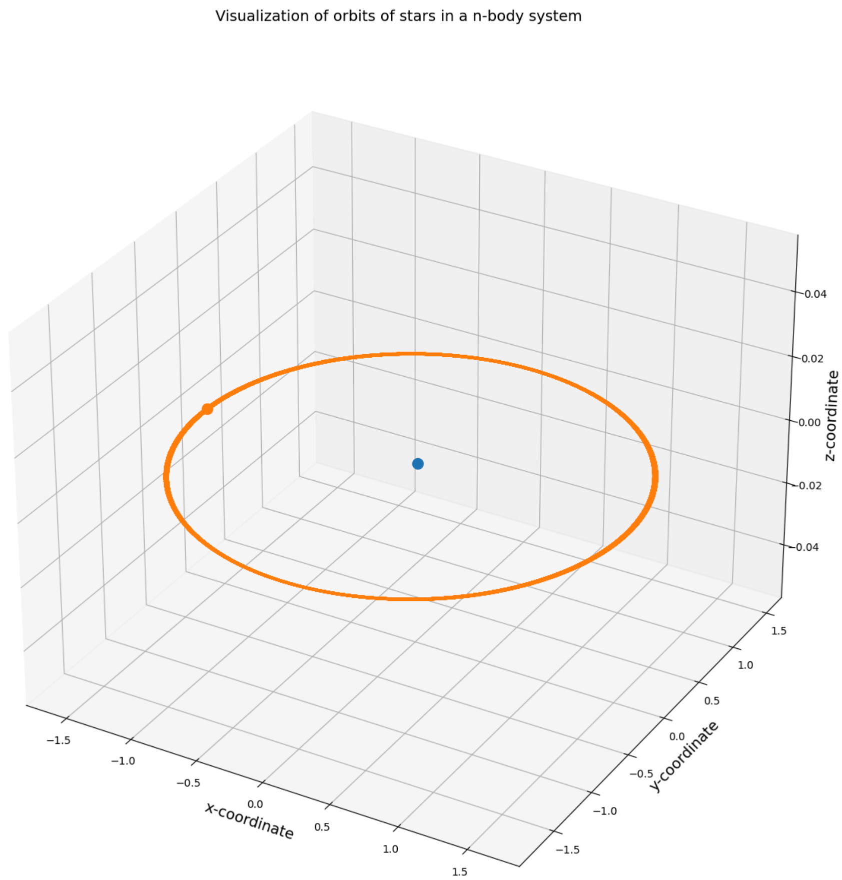
```
r = np.array([[0, 0, 0],
              [0, 1, 0]], dtype="float64")

v = np.array([[0, 0, 0],
              [1, 0, 0]], dtype="float64")

m = np.array([1, 5.97219e24/1.989e+30])

time_span=np.linspace(0, 500, 50000)

orbits = nbody(r, v, m, time_span)
plot_orbits(orbits)
```

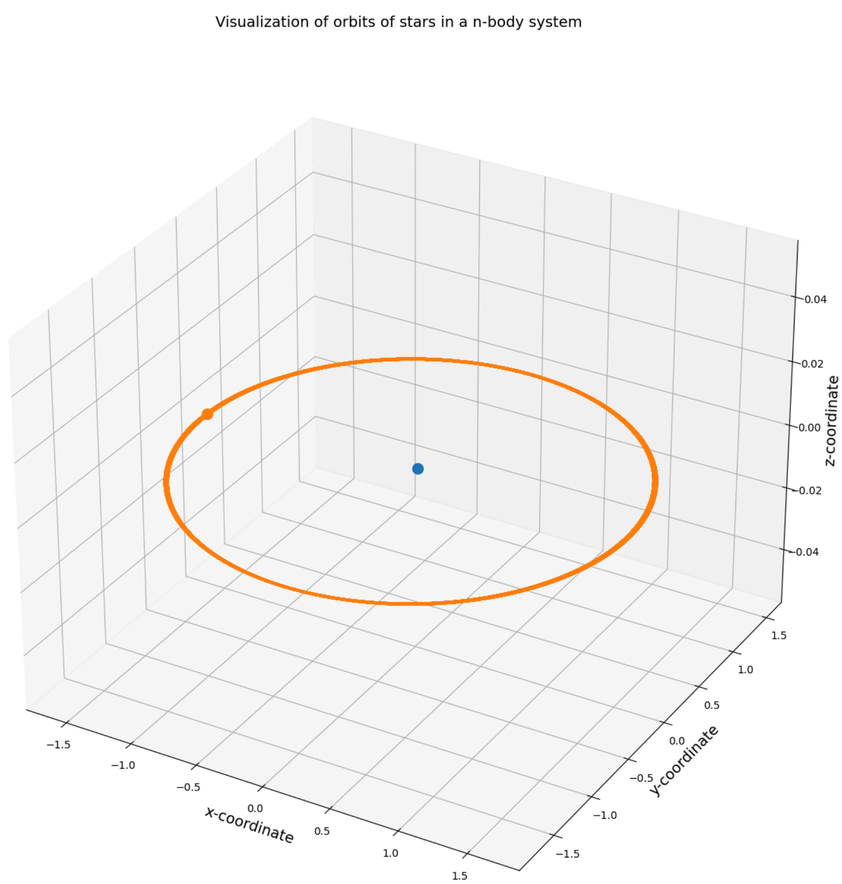




Simulation: Drift of RK4 - Unnormalized

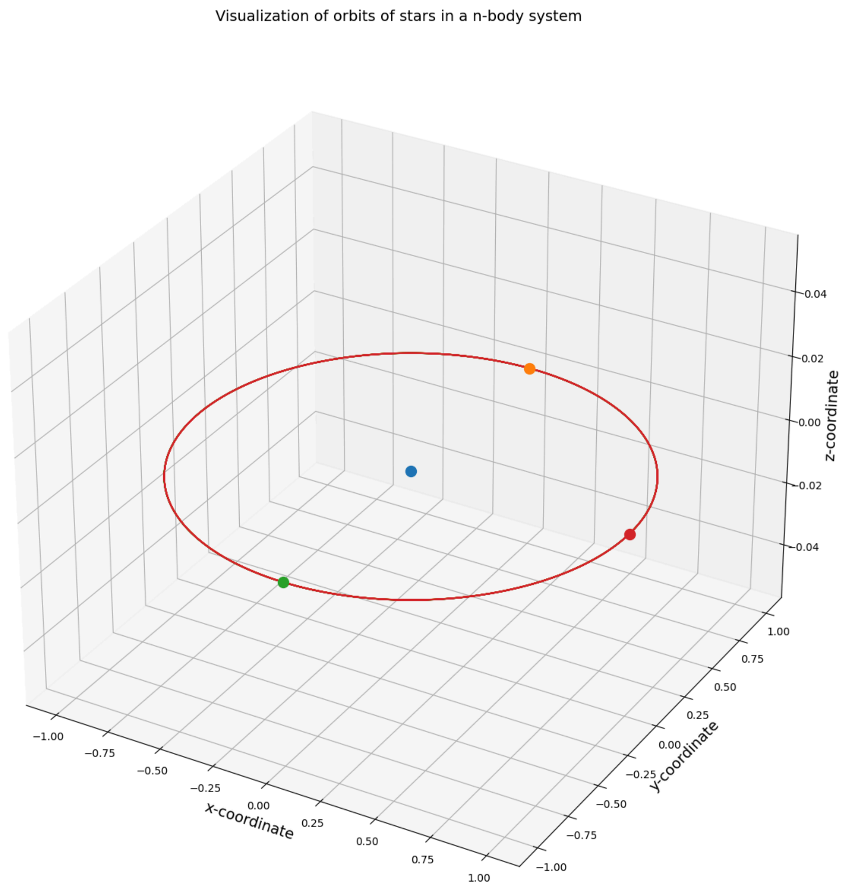
```
r = np.array([[0, 0, 0],
              [0, r_AU, 0]], dtype="float64")

v = np.array([[0, 0, 0],
              [v_Earth+1000, 390, 0]], dtype="float64")

m = np.array([m_Sun, m_Earth])

time_span = np.linspace(0, 500*t_Earth, 10000)

orbits = nbody(r, v, m, time_span, r_norm=1, m_norm=1, t_norm=1, v_norm=1)
plot_orbits(orbits)
```

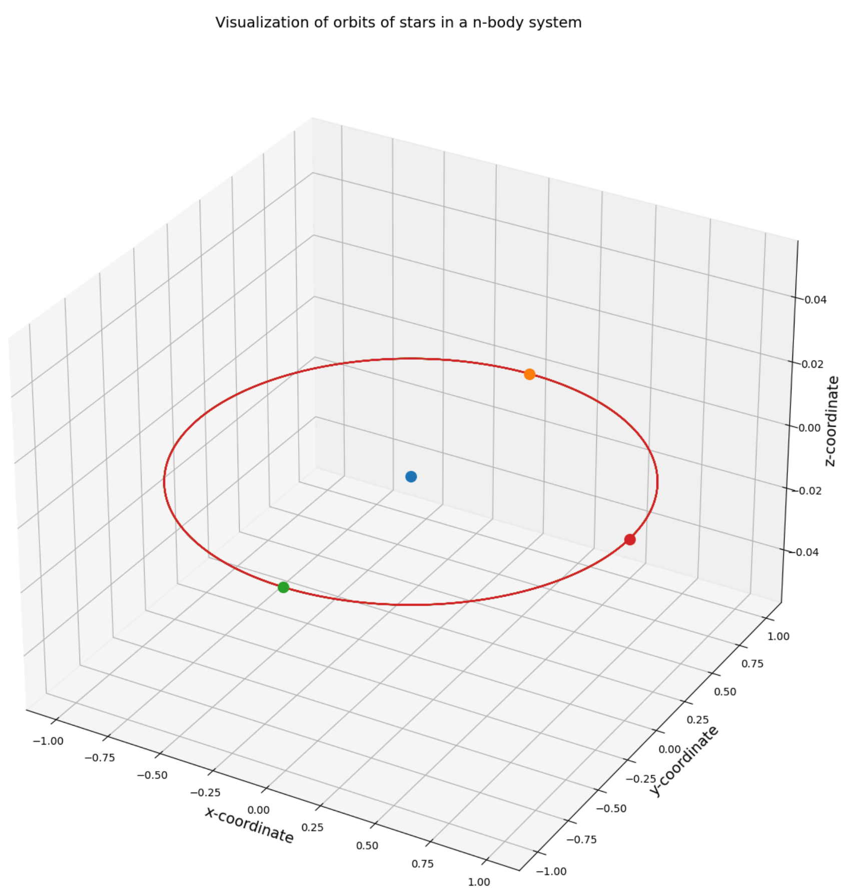

Visualization of orbits of stars in a n-body system
x-coordinate
y-coordinate
z-coordinate
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
0.04
0.02
0.00
−0.02
−0.04


Visualization of orbits of stars in a n-body system
x-coordinate
y-coordinate
z-coordinate
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
0.04
0.02
0.00
−0.02
−0.04

Simulation: Many Earths

```
r = np.array([[0, 0, 0],
              [0, 1, 0],
              [0, -1, 0],
              [1, 0, 0]], dtype="float64")

v = np.array([[0, 0, 0],
              [1, 0, 0],
              [-1, 0, 0],
              [0, -1, 0]], dtype="float64")

m = np.array([1, 5.97219e24/1.989e+30, 5.97219e24/1.989e+30, 5.97219e24/1.989e+30])

time_span=np.linspace(0, 3, 50000)

orbits = nbody(r, v, m, time_span)
plot_orbits(orbits)
```

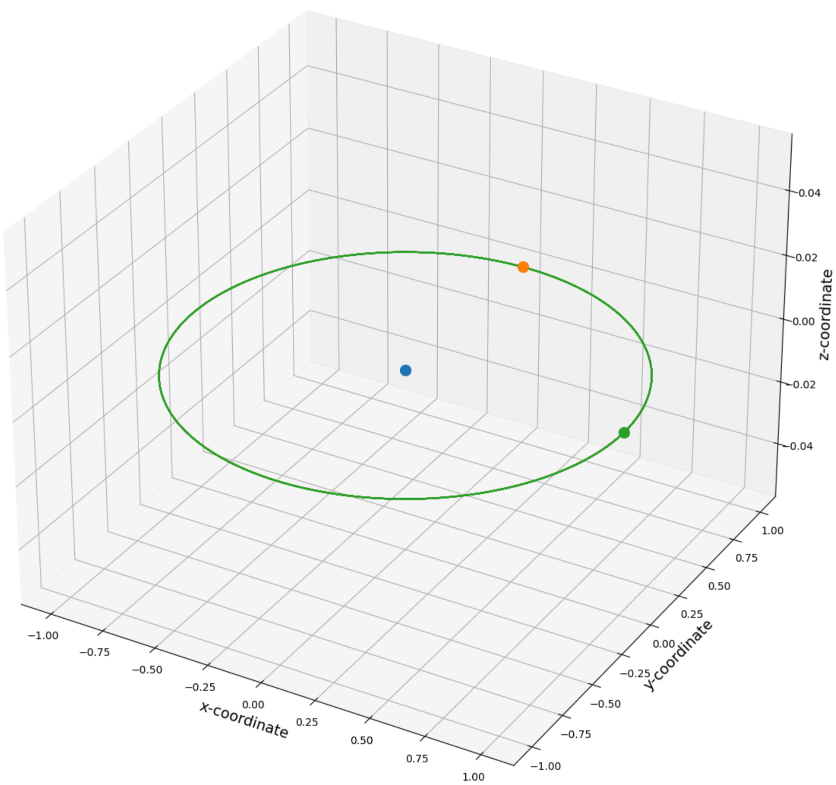

Visualization of orbits of stars in a n-body system
x-coordinate
y-coordinate
z-coordinate
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
0.04
0.02
0.00
−0.02
−0.04


Visualization of orbits of stars in a n-body system
x-coordinate
y-coordinate
z-coordinate
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
0.04
0.02
0.00
−0.02
−0.04

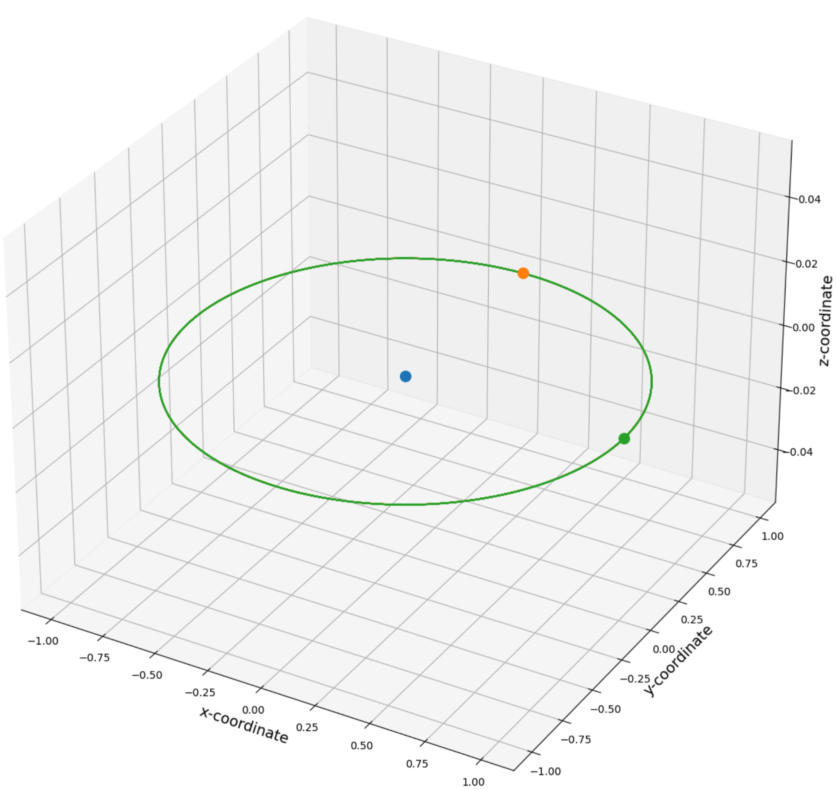
Simulation: Lagrange Points

```
r = np.array([[0, 0, 0],
              [0, 1, 0],
              [1, 0, 0]], dtype="float64")

v = np.array([[0, 0, 0],
              [1, 0, 0],
              [0, -1, 0]], dtype="float64")

m = np.array([m_Sun, m_Earth, 1000]) / m_Sun

time_span=np.linspace(0, 5, 500)

orbits = nbody(r, v, m, time_span)
plot_orbits(orbits)
```

Visualization of orbits of stars in a n-body system

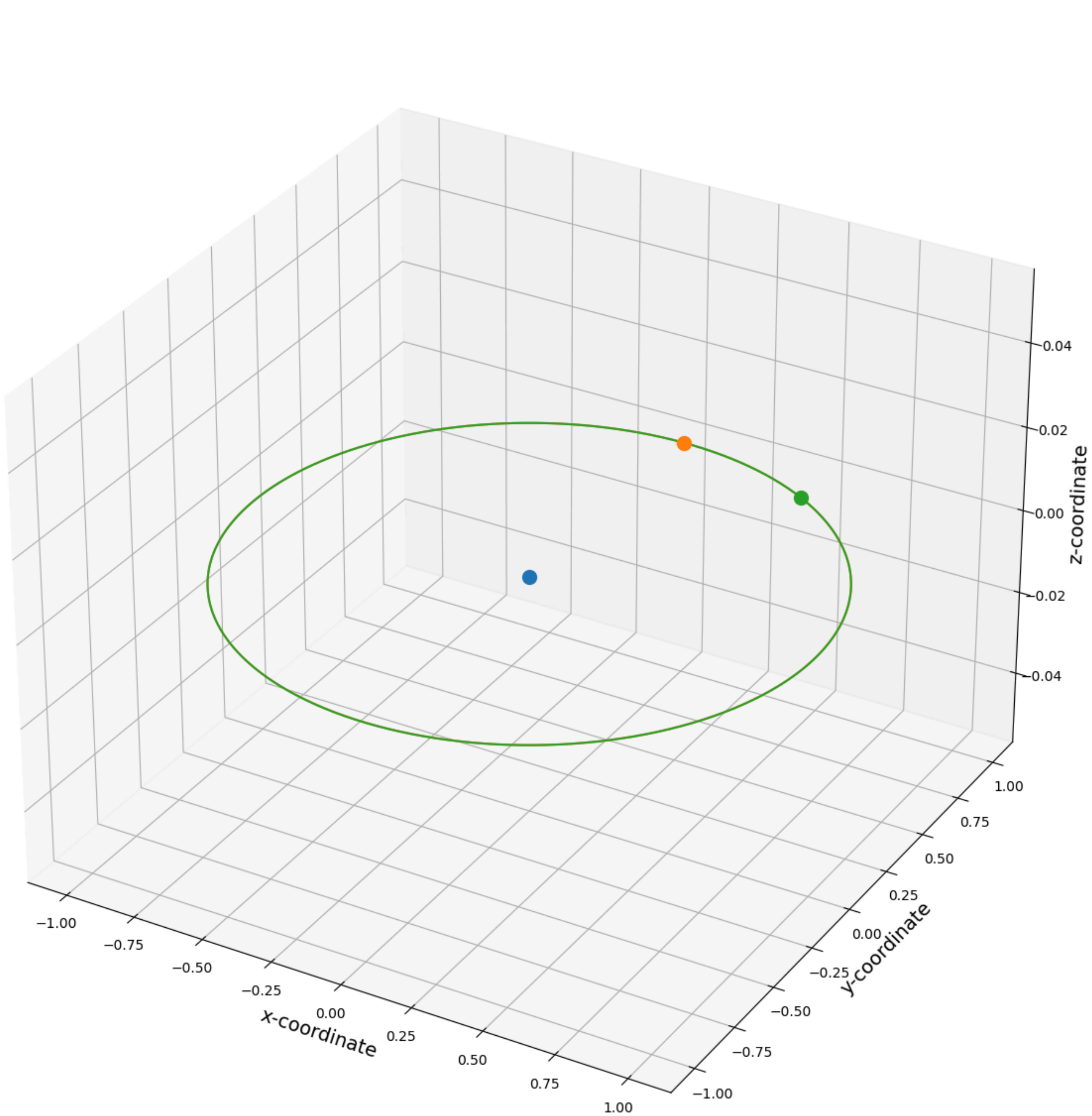


Visualization of orbits of stars in a n-body system

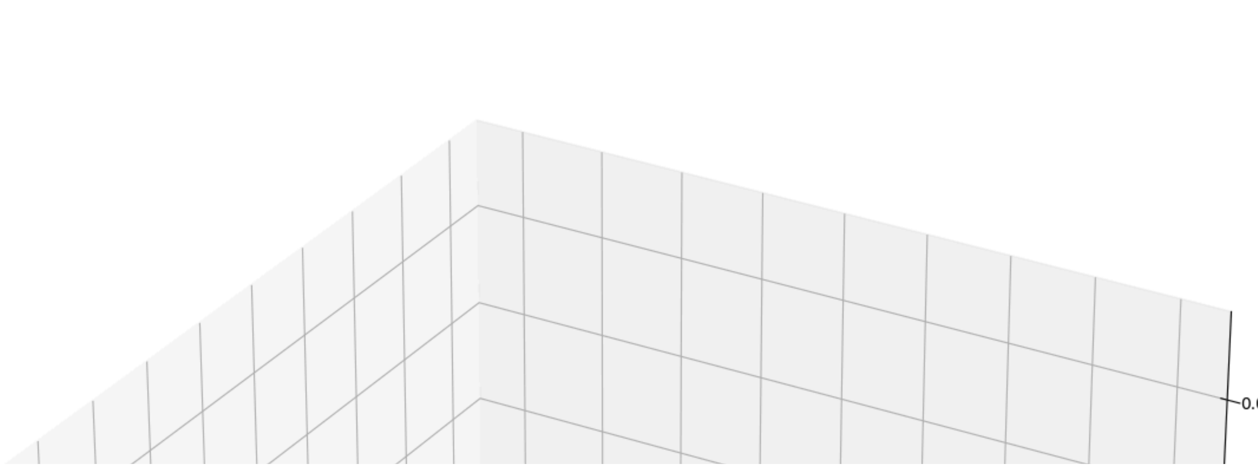


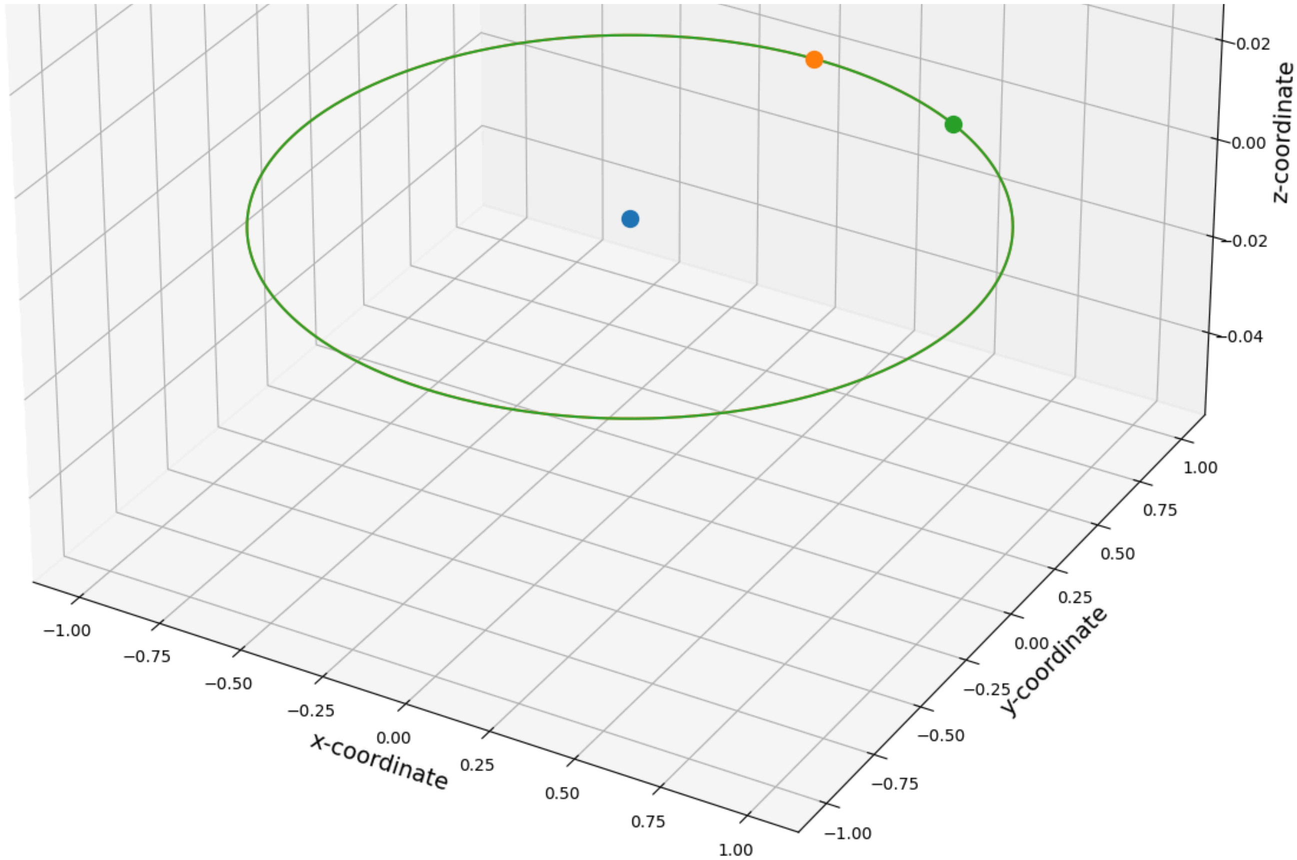
```
r = np.array([[0, 0, 0],
              [0, 1, 0],
              [np.cos(np.pi/3), np.sin(np.pi/3), 0]], dtype="float64")

v = np.array([[0, 0, 0],
              [1, 0, 0],
              [np.sin(2*np.pi/3)+0.0001, -np.cos(np.pi/3), 0]], dtype="float64")

m = np.array([m_Sun, m_Earth, 1000]) / m_Sun

time_span=np.linspace(0, 1, 500)

orbits = nbody(r, v, m, time_span)
plot_orbits(orbits)
```

Visualization of orbits of stars in a n-body system



Visualization of orbits of stars in a n-body system





Sympletic Leapfrog Integration

```
class Body:
    def __init__(self, mass, position, velocity):
        self.mass = mass
        self.position = np.array(position, dtype=float)
        self.velocity = np.array(velocity, dtype=float)

def calculate_gravitational_force(body1, body2):
```