Linear Stability Analysis

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December 9, 2023

$$\frac{a_j^{n+1} - a_j^n}{\Delta t} = -u\frac{a_j^{n+1} - a_{j-1}^{n+1}}{\Delta x} \qquad (1)$$
$$a_j^n = A^n e^{ijk\Delta x}$$

$$[A^{n+1} - A^n]e^{ijk\Delta x} = \frac{-u\Delta t}{\Delta x}[A^{n+1}e^{ijk\Delta x} - A^{n+1}e^{i(j-1)k\Delta x}] \qquad (2)$$
$$\frac{u\Delta t}{\Delta x} = C;\ \text{divide both sides by } e^{ijk\Delta x}$$

$$A^{n+1} - A^n = -C[A^{n+1} - A^{n+1}e^{-ik\Delta x}] \qquad (3)$$

$$A^{n+1} - A^n = -CA^{n+1} + CA^{n+1}e^{-ik\Delta x} \qquad (4)$$

$$A^{n+1} + CA^{n+1} - CA^{n+1}e^{-ik\Delta x} = A^n \qquad (5)$$

$$A^{n+1}[1 + C - Ce^{-ik\Delta x}] = A^n \qquad (6)$$

$$\frac{A^{n+1}}{A^n} = \frac{1}{1 + C - Ce^{-ik\Delta x}} \qquad (7)$$
$$e^{-ik\Delta x} = \cos(k\Delta x) - i\sin(k\Delta x)$$

$$\frac{A^{n+1}}{A^n} = \frac{1}{1 + C - C[\cos(k\Delta x) - i\sin(k\Delta x)]} \qquad (8)$$

$$\frac{A^{n+1}}{A^n} = \frac{1}{1 + C - C\cos(k\Delta x) + iC\sin(k\Delta x)} \qquad (9)$$
$$\cos(k\Delta x) = K; \qquad \sin(k\Delta x) = S$$

$$\frac{A^{n+1}}{A^n} = \frac{1}{1 + C - CK + iCS} \qquad (10)$$
$$(1 - K) = B$$

$$\frac{A^{n+1}}{A^n} = \frac{1}{1 + CB + iCS} \qquad (11)$$

$$
\begin{aligned}
\|\frac{A^{n+1}}{A^n}\|^2 &= \frac{1}{(1+CB)^2 + C^2S^2} \\
(1+CB)^2 &= 1 + 2C(1-K) + C^2(1-K)^2 \\
&= 1 + 2C(1-K) + C^2 - 2C^2K + C^2K^2
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\|\frac{A^{n+1}}{A^n}\|^2 &= \frac{1}{1 + 2C(1-K) + C^2 - 2C^2K + C^2K^2 + C^2S^2} \\
C^2K^2 + C^2S^2 &= C^2(\cos^2(K\Delta x) + \sin^2(K\Delta x)) = C^2
\end{aligned} \tag{13}
$$

$$\|\frac{A^{n+1}}{A^n}\|^2 = \frac{1}{1 + 2C(1-K) + 2C^2 - 2C^2K} \tag{14}$$

$$\|\frac{A^{n+1}}{A^n}\|^2 = \frac{1}{1 + 2C(1-K) + 2C(C-CK)} \tag{15}$$

$$\|\frac{A^{n+1}}{A^n}\|^2 = \frac{1}{1 + 2C[(1-K) + (C-CK)]} \tag{16}$$

$$\|\frac{A^{n+1}}{A^n}\|^2 = \frac{1}{1 + 2C[(1+C)(1-K)]} \tag{17}$$

$$\|\frac{A^{n+1}}{A^n}\|^2 = \frac{1}{1 + 2C(1+C)(1-\cos(k\Delta x))} \tag{18}$$

$$1 + 2C(1+C)(1-\cos(k\Delta x)) \geq 1 \tag{19}$$

$$2C(1+C)(1-\cos(k\Delta x)) \geq 0 \tag{20}$$

$$1 - \cos(k\Delta x)) \geq 0 \rightarrow 2C(1+C) \geq 0 \tag{21}$$

$$C \geq 0 \tag{22}$$