Journal of Risk and Uncertainty, 20:5–44 (2000)

Utility Functions for Wealth

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Abstract

We specify all utility functions on wealth implied by four special conditions on preferences between risky prospects in four theories of utility, under the presumption that preference increases in wealth. The theories are von Neumann-Morgenstern expected utility (EU), rank dependent utility (RDU), weighted linear utility (WLU), and skew-symmetric bilinear utility (SSBU). The special conditions are a weak version of risk neutrality, Pfanzagl's consistency axiom, Bell's one-switch condition, and a contextual uncertainty condition. Previous research has identified the functional forms for utility of wealth for all four conditions under EU, and for risk neutrality and Pfanzagl's consistency axiom under WLU and SSBU. The functional forms for the other condition-theory combinations are derived in this paper.

Key words: expected utility, rank dependent utility, weighted utility, skew-symmetric bilinear utility, consistency axiom, one-switch condition, contextual uncertainty

JEL Classification: D81

There are two predominant interpretations for utility of wealth in theories of decision among risky prospects with monetary outcomes. The first, which dates from Bernoulli (1738), treats utility of wealth $u(w)$ as a measure of a person's intensity of preference for wealth w that is to be assessed in a strength-of-preference manner without reference to risk or outcome probabilities. The second, which originated in von Neumann and Morgenstern (1944), eschews the earlier intensity interpretation and views $u(w)$ as a consequence of simple preference comparisons between risky prospects that is assessed with the aid of outcome probabilities, largely through indifference judgments. In-depth analyses of the two interpretations are presented in Ellsberg (1954) and Fishburn (1989), but see also Allais (1953, 1979) for a dissenting opinion which maintains that von Neumann and Morgenstern intended a Bernoullian interpretation for their notion of utility.

The present paper lies squarely in the tradition of von Neumann and Morgenstern. Its aim is to specify all functional forms for $u(w)$ and related functions that occur in four types of utility representations when special conditions on preferences are imposed on their structures. The four utility theories are von Neumann-Morgenstern expected utility and three generalizations: rank dependent expected utility, weighted linear utility, and skew-symmetric bilinear utility. Our special conditions, also four in number, range from a restrictive condition of risk neutrality to Bell's (1988) more accommodating one-switch rule.

We assume throughout that more wealth is preferred to less. Although psychologically innocuous, this is important mathematically because it rules out non-monotonic solutions to functional equations that arise in our derivations.

Section 2 outlines the numerical representations of the four utility theories we consider and describes the four special conditions we apply to those theories. The most flexible special condition is the one-switch rule which says that if preference between two lotteries on increments to wealth reverses as wealth increases, then no further reversals occur beyond the initial switch point. Section 3 presents our results in theorems which show how the special conditions affect the numerical representations. The most prevalent special forms involved exponential terms such as e^{cw}, but other cases arise. We note which results have been proved earlier and which have not. The results are summarized in a 4×4 array which shows the effects of each (theory, special condition) combination.

Sections 4 through 7 cover the proofs of new results. Section 4 focuses on rank dependent utility, Sections 5 and 6 consider the one-switch rule for skew-symmetric bilinear utility and weighted linear utility, respectively, and Section 7 deals with a contextual uncertainty condition.

We have mentioned the Bernoullian tradition of intensive utility to clarify what our study is not about as well as what it is about. To explain this further, we first recall the expected utility model of both interpretations.

Throughout the paper, a *risky prospect* is defined as a random function $\tilde{w}$ with finite support $S(\tilde{w})$ on levels of wealth. When $\tilde{w}$ has probability $p(w)$ for $w \in S(\tilde{w})$, we have $\sum_{S(\tilde{w})} p(w) = 1$ and refer to p as the *lottery* associated with $\tilde{w}$. Under the obvious convention that $p(w) = 0$ for $w \notin S(\tilde{w})$, each lottery is a simple probability distribution on all wealth levels. With respect to a utility function u on levels of wealth, the *expected utility* of $\tilde{w}$ with associated lottery p is defined by

$$Eu(\tilde{w}) = \sum_{w \in S(\tilde{w})} p(w)u(w). \tag{1}$$

This form is used by Bernoulli and by von Neumann and Morgenstern, and in both cases risky prospect $\tilde{w}$ is regarded at least as good as $\tilde{w}'$ if and only if $Eu(\tilde{w}) \geq Eu(\tilde{w}')$. However, the two approaches differ radically in their interpretation and assessment of u.

As indicated earlier, Bernoulli's u is intended to reflect a person's intensities of preference for levels of wealth. It is riskless and logically precedes, or is at least independent of, probability considerations. On the other hand, von Neumann and Morgenstern's u reflects attitudes toward risk and is inextricably tied to preference and indifference judgments between lotteries. The difference between the two interpretations impacts our study in at least three ways.

First, the special forms for u and related functions specified later should not be viewed as intrinsic, intensive measures of utility. They are merely consequences of (1) in the expected utility case, or other representational models in generalized cases, and of our special conditions.

Second, the logarithmic form of u favored by Bernoulli, which can be written as

$$u(w) = \log(1 + cw) \quad \text{for } w \geq 0 \tag{2}$$

for some positive constant c, will not be found among our special forms. In particular, our special conditions do not permit this form even though it would be allowed by other conditions such as a relative-risk-constant condition in Harvey (1990, p. 1485). Bernoulli argued for (2) with an early expression of the law of diminishing marginal utility which says that the increment of utility for the next bit of wealth ought to be inversely proportional to the amount of wealth prior to the incremental increase. Support for (2) as an intensive measure was obtained by Allais in 1952 from questionnaire responses that are reported in Allais (1979). Although Savage was definitely not a Bernoullian in the sense used here, he says in Savage (1954, p. 94) that "To this day, no other function has been suggested as a better prototype for Everyman's utility function." Perhaps Savage would have reconsidered had he known about results in Bell (1988, 1995a, 1995b). The reason is that, within the von Neumann-Morgenstern interpretation that Savage favored,

$$u(w) = aw - be^{-cw} \tag{3}$$

for constants $a \geq 0$ and $b, c > 0$, is the *only* increasing ($u' > 0$) and risk averse ($u'' < 0$) form that satisfies the one-switch and contextual uncertainty conditions in the context of (1).

Our third point concerns the treatment of special conditions in (1) and its generalizations. In our approach, we begin with (1) or a generalization such as rank dependent utility or weighted linear utility on the basis of preference axioms which correspond to the theory under consideration. We then ask how the special conditions constrain the functions involved in the representation. Although there are also generalizations of (1) in the Bernoullian approach, as described for example in Allais (1953, 1979) and Hagen (1972, 1979), our approach is inadmissible there because u exists independently of risk or special conditions on preferences between risky prospects. It follows that special conditions in Bernoullian theory can only be accounted for by modifying the expectation rule of (1) for combining probabilities and utilities, not by modifying u. This is a perfectly legitimate route of inquiry, but it is not pursued here apart from alternatives to (1) that generalize von Neumann-Morgenstern expected utility prior to imposition of our special conditions.

1. Utility theories and special conditions

The numerical utility representations of our study apply to a *strict preference relation* $\succ$ on the set W^+ of all risky prospects $\tilde{w}, \tilde{x}, \tilde{y}, \ldots$ whose supports are subsets of the set $\mathbb{R}^+$ of nonnegative real numbers. We could also allow negative wealth with some fixed lower bound since this would not entail any substantial changes under the translation that treats the lower bound in the way we treat zero wealth in what follows. The *indifference relation* $\sim$ and *preference-or-indifference relation* $\succsim$ induced by $\succ$ are defined for $\tilde{x}, \tilde{y} \in W^+$ by

$$\tilde{x} \sim \tilde{y} \quad \text{if neither } \tilde{x} \succ \tilde{y} \text{ nor } \tilde{y} \succ \tilde{x},$$

$$\tilde{x} \succsim \tilde{y} \quad \text{if } \tilde{x} \succ \tilde{y} \text{ or } \tilde{x} \sim \tilde{y}.$$

We denote the set of simple probability distributions on $\mathbb{R}^+$ by P^+ and let $\tilde{w} \leftrightarrow p$ mean that $p \in P^+$ is the lottery associated with $\tilde{w} \in W^+$. Preference on P^+ is defined in the obvious way from preference on W^+ by

$$p \succ q \quad \text{if } \tilde{x} \succ \tilde{y} \quad \text{when } \tilde{x} \leftrightarrow p \quad \text{and} \quad \tilde{y} \leftrightarrow q.$$

The degenerate risky prospect that assigns probability 1 to wealth w is denoted simply by w, so the assumption that more wealth is preferred to less can be expressed as

$$x \succ y \quad \text{when } x > y \geq 0. \tag{4}$$

We assume (4) throughout.

Our presumption that wealth is *nonnegative* has a practical appeal, but it is also important in our derivations and a few results. For example, one special form for u that arises later is the quadratic form in which

$$u(w) = aw^2 + bw \quad \text{for } w \geq 0,$$

with $a \geq 0$, $b \geq 0$ and $a + b > 0$ to satisfy (4) for constants a and b. If wealth were unbounded below, with $x \succ y \Leftrightarrow x > y$, then the quadratic case would reduce to the linear case of $u(w) = w$ because any nonzero a would violate monotonicity.

On the other hand, factors such as borrowing and catastrophic events can render net wealth negative. To account for this, it is natural to replace $\mathbb{R}^+$, W^+, and P^+ in the preceding definitions by $\mathbb{R}$, the set W of all risky prospects whose supports are subsets of $\mathbb{R}$, and the set P of simple probability distributions on $\mathbb{R}$, respectively. Most of the results in the next section for $\succ$ on W^+ hold also for the unconstrained case of $\succ$ on W, and when they do not we will say so. But proofs will be given only for $\succ$ on W^+, which tends to be the more delicate case because of the $w \geq 0$ constraint.

Expectations

Our four utility representations for $\succ$ on W^+ are identified by EU (expected utility), RDU (rank dependent utility), WLU (weighted linear utility), and SSBU (skew-symmetric bilinear utility). Each is based on a notion of expected value, which is (1) for EU and WLU. For WLU, we work with a ratio $Eu_1(\tilde{w})/Eu_2(\tilde{w})$ with $u_2 > 0$. This reduces to the EU representation when the weighting function u_2 is constant.

For RDU, we define Π as the set of increasing functions from $[0,1]$ onto $[0,1]$. Members of Π are viewed as transformations of objective probabilities of lotteries into psychologically equivalent subjective probabilities. Each $\pi \in \Pi$ is continuous and increasing with $\pi(0) = 0$ and $\pi(1) = 1$. When $S(\tilde{w})$ has n members ordered as $w_1 < w_2 < \cdots < w_n$, and $\tilde{w} \leftrightarrow p$, the *rank dependent expectation* of u with respect to $\pi \in \Pi$ is defined by

$$E_\pi u(\tilde{w}) = \sum_{k=1}^{n} u(w_k)\left\{\pi\left[\sum_{j=1}^{k} p(w_j)\right] - \pi\left[\sum_{j=1}^{k-1} p(w_j)\right]\right\}. \tag{5}$$

This reduces to $Eu(\tilde{w})$ when π is the identity transform with $\pi(\lambda) = \lambda$.

For SSBU, let Φ denote the set of skew-symmetric functions ϕ from $\mathbb{R}^+ \times \mathbb{R}^+$ into $\mathbb{R}$ for which $\phi(x,y) > 0$ when $x > y \geq 0$. The latter constraint corresponds to (4) since the SSBU representation has $\phi(x,y) > 0 \Leftrightarrow x \succ y$. *Skew-symmetry* for $\phi \in \Phi$ means that

$$\phi(x,y) + \phi(y,x) = 0 \quad \text{for all } x, y \in \mathbb{R}^+. \tag{6}$$

In particular, $\phi(w,w) = 0$. The *bilinear expectation* of $\phi \in \Phi$ for the ordered pair $(\tilde{x},\tilde{y})$ of risky prospects $\tilde{x}$ and $\tilde{y}$ with $\tilde{x} \leftrightarrow p$ and $\tilde{y} \leftrightarrow q$ is defined by

$$E\phi(\tilde{x},\tilde{y}) = \sum_{x \in S(\tilde{x})} \sum_{y \in S(\tilde{y})} p(x)q(y)\phi(x,y). \tag{7}$$

This reduces to the EU form when $\phi(x,y)$ has the decomposition $\phi(x,y) = u(x) - u(y)$, for then $E\phi(\tilde{x},\tilde{y}) = Eu(\tilde{x}) - Eu(\tilde{y})$. More generally, given the representation $\tilde{x} \succ \tilde{y} \Leftrightarrow E\phi(\tilde{x},\tilde{y}) > 0$, it reduces to the WLU form when $\phi(x,y)$ can be written as $u_1(x)u_2(y) - u_1(y)u_2(x)$ with $u_2 > 0$, for then $E\phi(\tilde{x},\tilde{y}) > 0$ is equivalent to $Eu_1(\tilde{x})/Eu_2(\tilde{x}) > Eu_1(\tilde{y})/Eu_2(\tilde{y})$.

Four representations

We now specify our four numerical representations for $\succ$ on W^+ along with admissible transformations for their functions, which are used later to obtain the simplest forms for those functions under the special conditions described shortly.

Although there is no need here to go into the details of axioms which correspond to the representations, a few words about axioms may provide perspective.

Representations like those that follow are usually axiomatized in terms of the behavior of $\succ$ on a convex set P^* of lotteries or probability distributions on a set of outcomes, which in our particular setting are wealth levels. Convexity means that $\lambda p + (1-\lambda)q$ is in P^* whenever $p, q \in P^*$ and $0 \leq \lambda \leq 1$, where the convex combination assigns probability $\lambda p(x) + (1-\lambda)q(x)$ to outcome x.

Three axioms are typically used for EU: see, for example, Herstein and Milnor (1953), Jensen (1967) and Fishburn (1970, 1988). They are an ordering axiom, an independence condition, and an Archimedean axiom to ensure real-valued utilities. The independence condition, which in one version says that

$$p \succ q \Rightarrow \lambda p + (1-\lambda)r \succ \lambda q + (1-\lambda)r \tag{8}$$

whenever $p, q, r \in P^*$ and $0 < \lambda < 1$, is central to the derivation of representation by expected utilities.

Common violations of (8) described in Allais (1953, 1979), MacCrimmon and Larsson (1979), and Kahneman and Tversky (1979), among others, motivated the other representations considered here, each of which weakens (8) to accommodate some of the observed violations. Axioms for RDU are discussed in Quiggin (1982, 1993), Quiggin and Wakker (1994), and Segal (1989, 1993); axioms for WLU appear in Chew and MacCrimmon (1979), Chew (1982), Fishburn (1983, 1988), and Nakamura (1984); SSBU is axiomatized in Fishburn (1982, 1988). The last of these also weakens the ordering axiom to allow preference cycles such as $p \succ q \succ r \succ p$. However, if transitivity of $\sim$ on P^* is added to the SSBU axioms, then SSBU reduces to WLU.

Our four representations are: for all $\tilde{x}, \tilde{y} \in W^+$,

EU: $\tilde{x} \succ \tilde{y} \Leftrightarrow Eu(\tilde{x}) > Eu(\tilde{y})$

RDU: $\tilde{x} \succ \tilde{y} \Leftrightarrow E_\pi u(\tilde{x}) > E_\pi u(\tilde{y})$

WLU: $\tilde{x} \succ \tilde{y} \Leftrightarrow Eu_1(\tilde{x})/Eu_2(\tilde{x}) > Eu_1(\tilde{y})/Eu_2(\tilde{y})$

SSBU: $\tilde{x} \succ \tilde{y} \Leftrightarrow E\phi(\tilde{x}, \tilde{y}) > 0,$

where u, u_1 and u_2 map $\mathbb{R}^+$ into $\mathbb{R}$, u is strictly increasing, $u_2 > 0$, $u_1(x)/u_2(x) > u_1(y)/u_2(y)$ when $x > y$, $\pi \in \Pi$ and $\phi \in \Phi$. The theorems in the next section invoke additional smoothness conditions for u, u_1, u_2 and ϕ because they are either implied by our special conditions or facilitate derivation of the special forms, for example by the use of differential equations. The following uniqueness properties, or admissible transformations, do not presume the smoothness conditions.

For EU, u is *unique up to a positive affine transformation*: v satisfies the representation in place of u if and only if there are constants $a > 0$ and b such that

$$v(w) = au(w) + b \quad \text{for all } w \geq 0. \tag{9}$$

If v does not satisfy (9) with $a > 0$, then $Ev(\tilde{x}) \leq Ev(\tilde{y})$ for some $\tilde{x}$ and $\tilde{y}$ for which the given $\succ$ has $\tilde{x} \succ \tilde{y}$.

For RDU, π is unique and u is unique up to a positive affine transformation. For WLU, u_1 and $u_2 > 0$ are unique up to the transformation pair

$$\begin{aligned} v_1(w) &= au_1(w) + bu_2(w) \quad \text{for all } w \geq 0 \\ v_2(w) &= cu_1(w) = du_2(w) \quad \text{for all } w \geq 0 \end{aligned} \tag{10}$$

with $ad > bc$ and $cu_1(w) + du_2(w) > 0$ for all $w \geq 0$: see Fishburn (1988, p. 132). Under smoothness assumptions (continuity suffices) we can have $v_1 > 0$ as well as $v_2 > 0$ for our $w \geq 0$ context by taking $a = 1$, b suitably large, $c = 0$ and $d = 1$.

For SSBU, ϕ is unique up to a *positive multiplicative transformation*: ψ satisfies the representation in place of ϕ if and only if there is an $a > 0$ such that

$$\psi(x, y) = a\phi(x, y) \quad \text{for all } x, y \in \mathbb{R}^+. \tag{11}$$

Every other transformation in Φ has $E\psi(\tilde{x}, \tilde{y}) \leq 0$ for some $\tilde{x}$ and $\tilde{y}$ for which $\tilde{x} \succ \tilde{y}$.

Four special conditions

A few new definitions are involved in our special conditions. When $\tilde{w} \in W^+$, $x \in \mathbb{R}^+$ and $\tilde{w} \sim x$, x is the *certainty equivalent* of $\tilde{w}$. For $x, y \in \mathbb{R}^+$, $(x, 1/2, y)$ is the *even-chance prospect* that assigns probability $\frac{1}{2}$ to each of x and y when $x \neq y$, and assigns probability 1 to x when $x = y$.

The next definition is stated in the unrestricted W mode. Given $\tilde{x}_j \in W$, $\alpha_j \in \mathbb{R}$, and $\tilde{x}_j \leftrightarrow p_j$ for $j = 1, \dots, m$, the *weighted convolution* $\sum \alpha_j \tilde{x}_j$ is defined as the risky prospect associated with $p \in P$ given by

$$p(y) = \sum_{(x_1, \dots, x_n)} \left\{ p_1(x_1) p_2(x_2) \cdots p_m(x_m) : \sum_{j=1}^{m} \alpha_j x_j = y \right\}, \tag{12}$$

for all $y \in \mathbb{R}$. Each $\tilde{x}_j$ in this definition, including identical $\tilde{x}_j$'s with different indices, is treated as an independent random function for the probability calculation. Some examples follow.

Suppose $\tilde{x}$ has probability $\frac{1}{3}$ for $x = 1$ and probability $\frac{2}{3}$ for $x = 3$. Then

$2\tilde{x}$ has pr. $\frac{1}{3}$ for 2 and pr. $\frac{2}{3}$ for 6;

$\tilde{x} + \tilde{x}$ has pr. $\frac{1}{9}$ for 2, pr. $\frac{4}{9}$ for 4 and pr. $\frac{4}{9}$ for 6.

Note in particular that $\alpha\tilde{x} + \beta\tilde{x}$ is not generally equal to $(\alpha + \beta)\tilde{x}$. In addition, if $\tilde{y}$ has probability $\frac{3}{4}$ for $y = 2$ and probability $\frac{1}{4}$ for $y = 4$, then

$\tilde{x} + \tilde{y}$ has pr. $\frac{3}{12}$ for 3, pr. $\frac{7}{12}$ for 5 and pr. $\frac{2}{12}$ for 7;

$\tilde{x} + \tilde{y} - 3$ has pr. $\frac{3}{12}$ for 0, pr. $\frac{7}{12}$ for 2 and pr. $\frac{2}{12}$ for 4.

In terms of (12), the latter case has $(\tilde{x}_1, \alpha_1) = (\tilde{x}, 1)$, $(\tilde{x}_2, \alpha_2) = (\tilde{y}, 1)$ and $(\tilde{x}_3, \alpha_3) = (3, -1)$.

We also let $\tilde{x}^+$ for $\tilde{x} \in W$ denote the set of all $w \in \mathbb{R}$ for which $\tilde{x} + w \in W^+$. Equivalently,

$$\tilde{x}^+ = \{w \in \mathbb{R}: \min\{x + w: x \in S(\tilde{x})\} \geq 0\}.$$

Then $\tilde{x}^+ \cap \tilde{y}^+$ is the set of $w \in \mathbb{R}$ for which both $\tilde{x} + w$ and $\tilde{y} + w$ are in W^+.

Our first special condition, C1, is a limited version of risk neutrality which says that some risky prospects with identical mean wealths are indifferent.

C1. For all $x, y, z \in \mathbb{R}^+$, $(x, \frac{1}{2}, y + z) \sim (x + y, \frac{1}{2}, z)$.

An even simpler condition, $(x + y)/2 \sim (x, \frac{1}{2}, y)$, is equivalent to C1 when $\sim$ is transitive, but not otherwise (Fishburn, 1998). C1 is very strong and seems plausible only when x and z are approximately equal or y is comparatively large. For EU and RDU, it implies $u(x) = ax + b$, $a > 0$, or $u(x) = x$ under affine rescaling. Its effects for WLU and SSBU are less restrictive. For example, under SSBU, it says that $\phi(x, y)$ depends only on the difference $x - y$, and it is consistent with the existence of preference cycles.

Our next two special conditions focus on how preference between $w + \tilde{x}$ and $w + \tilde{y}$ in W^+ changes as w varies with $\tilde{x}$ and $\tilde{y}$ fixed. The more restrictive of the two, C2, is tantamount to Pfanzagl's consistency axiom (Pfanzagl, 1959) when all risky prospects have certainty equivalents. Pfanzagl's axiom says that if x is the certainty equivalent of $\tilde{w} \in W^+$, then $x + y$ is the certainty equivalent of $\tilde{w} + y$ for $y \in \tilde{w}^+$. It is also similar to conditions used in Rothblum (1975) and Farquhar and Nakamura (1987). We refer to C2 as the *zero-switch condition* (Bell, 1988).

C2. For all $\tilde{x}, \tilde{y} \in W$, *if* $w + \tilde{x} \sim w + \tilde{y}$ *for some* $w \in \tilde{x}^+ \cap \tilde{y}^+$, *then* $w + \tilde{x} \sim w + \tilde{y}$ *for all* $w \in \tilde{x}^+ \cap \tilde{y}^+$.

This also is fairly strong. For example, it implies $u(x) = x$ or $u(x) = ae^{ax}$, $a \neq 0$, under EU or RDU. Its companion, Bell's *one-switch condition*, says that preference between $w + \tilde{x}$ and $w + \tilde{y}$ can reverse as w increases, but if this happens then it happens only once with no further changes beyond the point of reversal. A closely related condition that has a similar effect under EU is discussed in Farquhar and Nakamura (1987, 1988).

C3. For all $\tilde{x}, \tilde{y} \in W$, *if* $w_1 + \tilde{x} \succ w_1 + \tilde{y}$ *and* $w_2 + \tilde{y} \succsim w_2 + \tilde{x}$ *for some* $w_1, w_2 \in \tilde{x}^+ \cap \tilde{y}^+$, *then there is a unique* $w^* \in \tilde{x}^+ \cap \tilde{y}^+$ *such that either*

(i) $w + \tilde{x} \succ w + \tilde{y}$ *for all* $w > w^*$; $w + \tilde{y} \succ w + \tilde{x}$ *for all* $w < w^*$ *in* $\tilde{x}^+ \cap \tilde{y}^+$, *or*
(ii) $w + \tilde{y} \succ w + \tilde{x}$ *for all* $w > w^*$; $w + \tilde{x} \succ w + \tilde{y}$ *for all* $w < w^*$ *in* $\tilde{x}^+ \cap \tilde{y}^+$.

Let $\tilde{x} = \$3000$ and $\tilde{y} = (\$0, \frac{1}{2}, \$8000)$, and suppose that a person of modest wealth w_0 has $w_0 + \tilde{x} \succ w_0 + \tilde{y}$ and $w_0 + \$5{,}000{,}000 + \tilde{y} \succ w_0 + \$5{,}000{,}000 + \tilde{x}$. Then C3 asserts that, as t increases, preference for $w_0 + t + \tilde{x}$ over $w_0 + t + \tilde{y}$ switches to the opposite preference and stays the same thereafter. Apart from the issue of determining the change point precisely, we regard C3 as very appealing. It is the least restrictive of our four special conditions and allows the most variety for the functions in our four representations.

Even more liberal conditions than C3 would allow two or more reversals, as when there are $w_1 < w_2 < w_3$ with

$$w_1 + \tilde{x} \succ w_1 + \tilde{y}$$

$$w_2 + \tilde{y} \succ w_2 + \tilde{x}$$

$$w_3 + \tilde{x} \succ w_3 + \tilde{y}.$$

We do not consider two-switch and higher-switch conditions further, but note that the two-switch case is related to the doubly-inflected utility function in Friedman and Savage (1948), and that examples of double switches arise later in our proofs for C3. In addition, Bell (1988) demonstrates that the logarithmic function (2) violates C3; indeed it permits any finite number of switches (his Proposition 9).

Our final special condition, C4, is based on a version of what Bell (1995b) refers to as a contextual uncertainty condition (CUC) in the EU context. Its thrust is that if you can resolve one of two independent subprospects or "contextual uncertainties" that are scale multiples of each other prior to making a decision, then you would just as soon resolve the subprospect with the larger spread. Suppose for example that the subprospects are $\tilde{z} = (\$0, \frac{1}{2}, \$100)$ and $100\tilde{z} = (\$0, \frac{1}{2}, \$10000)$. Then, stated in the negative, CUC says that if resolution of $100\tilde{z}$ would *not* affect your choice between main prospects that include $100\tilde{z} + \tilde{z}$, resolution of $\tilde{z}$ rather than $100\tilde{z}$ also would not affect your choice. The version of CUC that we adopt here is:

C4. For all $k > 1$ *and all* $\tilde{x} + k\tilde{z} + \tilde{z}, \tilde{y} + k\tilde{z} + \tilde{z} \in W^+$, *if there is no* $a \in S(\tilde{z})$ *such that*

$$\tilde{x} + ka + \tilde{z} \succ \tilde{y} + ka + \tilde{z},$$

then there is no $a \in S(\tilde{z})$ *for which*

$$\tilde{x} + k\tilde{z} + a \succ \tilde{y} + k\tilde{z} + a.$$

It is easy to see that C4 and CUC are equivalent for EU when CUC is defined by the requirement that if $\tilde{x} \succsim \tilde{y}$ when neither contextual uncertainty is resolved, and if $\tilde{x} \succsim \tilde{y}$ for every possible resolution of the larger contextual uncertainty, then $\tilde{x} \succsim \tilde{y}$ for every possible resolution of the smaller contextual uncertainty. Then, by definition, CUC implies C4. Moreover, if $\tilde{x} \succsim \tilde{y}$ for all resolutions of the larger contextual uncertainty, then it follows by taking expectations that $\tilde{x} \succsim \tilde{y}$ if neither contextual uncertainty is resolved. Hence C4 implies CUC.

The same argument shows that C4 and CUC are equivalent for SSBU because its expectation mode is similar to that of EU. We shall see, however, that C4 and CUC are not equivalent for RDU and WLU.

It was proved in Bell (1995b) that for EU, CUC is equivalent to the function V, defined by

$$V(w) = Eu(w + \tilde{x}) - Eu(w + \tilde{y}),$$

being strictly monotonic whenever $\tilde{x}$ and $\tilde{y}$ switch in w. An equivalent result holds for SSBU, namely that C4 holds if and only if $V(w) = E\phi(w + \tilde{x}, w + \tilde{y})$ is strictly monotone in w whenever $\tilde{x}$ and $\tilde{y}$ switch. For RDU the implications are more stark. The only cases for RDU that satisfy C4 require $\pi(p) = p$, which reduces RDU to EU. In other words, rank-dependent utility is identical to expected utility when C4 is imposed. Finally, for WLU we show that C4 entails C3, and that two of the four functions that satisfy C3 also satisfy C4.

2. Theorems

We organize our results into four theorems, one for each basic utility representation. Each theorem notes the effects of C1 through C4 in terms of the functional form or forms allowed by the special conditions. Constants are denoted by a, b, c, d and f, and restrictions on their values that are implied by (4) are noted in brackets, e.g. $[a \neq 0]$ and $[a \geq 0,\ b \geq 0,\ a + b > 0]$. We say that a function is *smooth* if it has derivatives of all orders. For convenience, continuity or smoothness assumptions are stated at the beginning of a theorem and apply to each special condition, but in many cases such properties are implied by (4) and the special condition. For example, in Theorems 1 and 2, C1 or C2 and (4) imply continuity for u, and C3 or C4 and (4) imply that $u(w)$ is continuous for all $w > 0$. For Theorems 3 and 4, the smoothness hypothesis is used in our proofs with C3 and C4, but continuity suffices for the proofs with C1 and C2 as noted in Fishburn (1998).

The functional forms stated in each theorem have important implications for risk attitudes and risky choice behavior in many applications, but we do not discuss these here. They are described for the EU representation in some detail in Bell (1988, 1995a, 1995b), and to a lesser extent for WLU and SSBU in Fishburn (1988, Chapter 6) and other references given there. See also Quiggin (1993) for further analyses of RDU.

The functional forms in the theorems are unique up to the admissible transformations noted in (9)–(11).

Theorem 1. *Suppose* $\succ$ *on* W^+ *satisfies* **EU** *and* u *is continuous. Then*:

C1 $\Rightarrow u(w) = w$;
C2 $\Rightarrow$ (i) $u(w) = w$, *or*
(ii) $u(w) = ae^{bw}$ $[ab > 0]$;
C3 $\Rightarrow$ (i) $u(w) = aw^2 + bw$ $[a \geq 0,\ b \geq 0,\ a + b > 0]$, *or*
(ii) $u(w) = ae^{bw} + ce^{dw}$ $[ab > 0,\ ab + cd \geq 0,\ d \leq b$ *if* $cd < 0$, $b(a + c) > 0$ *if* $b = d]$, *or*
(iii) $u(w) = aw + be^{cw}$ $[a + bc \geq 0,\ a > 0$ *if* $bc = 0,\ b \geq 0$ *if* $c > 0]$, *or*
(iv) $u(w) = (aw + b)e^{cw}$ $[ac \geq 0,\ a + bc \geq 0,\ a + bc > 0$ *if* $ac = 0]$;
C4 $\Rightarrow$ C3(i), *or* C3(ii) *with* $bd \leq 0$, *or* C3(iii).

Proofs for C1 and C2 are included in Bell (1995a) and Fishburn (1998), C3 is proved in Bell (1988, 1995a), and C4 in Bell (1995b). Precisely the same forms hold for the unrestricted case of $\succ$ on W, except that a must then be set at 0 in the quadratic form of C3(i). Generalizations of C3(ii) and C3(iii) are characterized in Nakamura (1996), and Harvey (1981, 1990) axiomatizes families of special u functions that are closely related to those in Theorem 1.

It is easily seen for $\succ$ on W^+ that (4) and C3 (one-switch) or C4 imply that u is continuous on $(0, \infty)$. If continuity at $w = 0$ were not presumed, two more EU forms that are discontinuous at the origin arise for C3 as follows:

(v) $u(0) = 0$, $u(w) = a + w$ for $w > 0$ $[a > 0]$,
(vi) $u(0) = 0$, $u(w) = 1 - a + ae^{bw}$ for $w > 0$ $[ab > 0]$.

The parts of these for $w > 0$ are similar to C2(i) and C2(ii), respectively, so (v) and (vi) satisfy the zero-switch condition on the strictly positive domain.

Theorem 2. *Suppose* $\succ$ *on* W^+ *satisfies* **RDU** *and* u *is continuous. Then* $C1$ *through* $C4$ *have precisely the same implications for* u *as in Theorem* 1. *In addition*, $C1$ *implies* $\pi(\frac{1}{2}) = \frac{1}{2}$, $C4$ *requires* $\pi(p) = p$ *for all* $p \in [0,1]$, *but* $C2$ *and* $C3$ *place no restrictions on* $\pi \in \Pi$.

The initial axiomatization for RDU in Quiggin (1982) presumed $\pi(\frac{1}{2}) = \frac{1}{2}$, but this was removed in subsequent axiomatizations. Wu and Gonzalez (1996) and Prelec (1998) provide nice discussions about shapes for π.

Our proof of Theorem 2 is presented in the next section.

Theorem 3. *Suppose* $\succ$ *on* W^+ *satisfies* **WLU** *and* u_1 *and* u_2 *are smooth. Then*:

C1 $\Rightarrow$ (i) $u_1(w) = w$, $u_2(w) = 1$, *or*
(ii) $u_1(w) = ae^{aw}$, $u_2(w) = e^{-aw}$ $[a \neq 0]$;
C2 $\Rightarrow$ (i) $u_1(w) = we^{aw}$, $u_2(w) = e^{aw}$, *or*
(ii) $u_1(w) = ae^{bw}$, $u_2(w) = e^{cw}$ $[a(b-c) > 0]$;
C3 $\Rightarrow$ (i) $u_1(w) = (aw^2 + bw)e^{cw}$, $u_2(w) = e^{cw}$ $[a \geq 0,\ b \geq 0,\ a + b > 0]$, *or*
(ii) $u_1(w) = ae^{bw} + ce^{dw}$, $u_2(w) = e^{fw}$ $[a(b-f) > 0,\ a(b-f) + c(d-f) \geq 0,\ d \leq b$ *if* $c(d-f) < 0]$, *or*
(iii) $u_1(w) = awe^{bw} + ce^{dw}$, $u_2(w) = e^{bw}$ $[a^2 + c^2 > 0,\ a + c(d-b) \geq 0,\ a > 0$ *if* $b = d$, $c > 0$ *if* $d > b]$, *or*
(iv) $u_1(w) = (aw + b)e^{cw}$, $u_2(w) = e^{dw}$ $[a + b(c-d) \geq 0,\ a(c-d) \geq 0,\ a + b(c-d) > 0$ *if* $a(c-d) = 0]$;
C4 $\Rightarrow$ C3(i) *with* $ac = 0$, C3(ii) *with* $bd \leq 0$, C3(iii) *with* $bd \leq 0$, *or* C3(iv) *with* $c = d$.

Parts C1 and C2 of Theorem 3 are proved in Fishburn (1998), but because those proofs leave u_2 implicit we sketch their (u_1, u_2) proofs here in Section 6. Part C3 is also proved in Section 6, and part C4 is proved in Section 7. When $\mathbb{R}^+$ is replaced by $\mathbb{R}$, we need to take $a = 0$ in C3(i), but no changes are required for C3(ii)–(iv). It is worth noting that in applying a special form of the theorem, expectations are taken separately for u_1 and u_2. For example, in C2(i), $u_1(x)/u_2(x) = x$, but $Eu_1(\tilde{x})/Eu_2(\tilde{x})$ is not generally equal to $E\tilde{x}$ unless $\tilde{x}$ is degenerate or $a = 0$, in which case C2(i) reduces to C1(i).

For our final theorem, which involves functions of $\Delta = x - y$, we define h: $\mathbb{R} \to \mathbb{R}$ as

even if $h(\Delta) = h(-\Delta)$ for all $\Delta \in \mathbb{R}$, and

odd if $h(\Delta) = -h(-\Delta)$ for all $\Delta \in \mathbb{R}$.

In Theorem 4, α *and* β *denote even functions*, *and g denotes an odd function*. These functions are unrestricted except for the presumption of smoothness and the requirement of monotonicity in (4). We emphasize that expressions like $\alpha(x-y)$ and $g(x-y)$ denote the value of the function α or g at argument $\Delta = x - y$.

Theorem 4. *Suppose* $\succ$ *on* W^+ *satisfies* **SSBU** *and* ϕ *is smooth. Then*:

C1 $\Rightarrow$ $\phi(x, y) = g(x-y)$ $[g(\Delta) > 0$ *if* $\Delta > 0]$;
C2 $\Rightarrow$ (i) $\phi(x, y) = g(x-y)$ $[g(\Delta) > 0$ *if* $\Delta > 0]$, *or*
(ii) $\phi(x, y) = \alpha(x-y)(e^{ax} - e^{ay})$ $[a\alpha(\Delta) > 0$ *if* $\Delta \neq 0]$;
C3 $\Rightarrow$ (i) $\phi(x, y) = \alpha(x-y)(x^2 - y^2) + \beta(x-y)(x-y)$ $[\alpha(\Delta) \geq 0$ *and* $\Delta\alpha(\Delta) + \beta(\Delta) > 0$ *for* $\Delta > 0]$, *or*
(ii) $\phi(x, y) = \alpha(x-y)(e^{ax} - e^{ay}) + \beta(x-y)(e^{bx} - e^{by})$ $[\alpha(\Delta)(e^{a\Delta} - 1)e^{ay} + \beta(\Delta)(e^{b\Delta} - 1)e^{by} > 0$ *for all* $\Delta > 0,\ y \geq 0]$, *or*

(iii) $\phi(x, y) = \alpha(x-y)(x-y) + \beta(x-y)(e^{ax} - e^{ay})$ $[\Delta\alpha(\Delta) + \beta(\Delta)(e^{a\Delta} - 1) > 0$ *for* $\Delta > 0$; $\alpha(\Delta) \geq 0$ *if* $a < 0$, $\beta(\Delta) \geq 0$ *if* $a > 0$ *for* $\Delta > 0]$, *or*

(iv) $\phi(x, y) = \alpha(x-y)(xe^{ax} - ye^{ay}) + \beta(x-y)(e^{ax} - e^{ay})$ $[a\alpha(\Delta) \geq 0$ *and* $\Delta\alpha(\Delta)e^{a\Delta} + \beta(\Delta)(e^{a\Delta} - 1) > 0$ *for* $\Delta > 0]$;

C4 $\Rightarrow$ C3(i), *or* C3(ii) *with* $ab < 0$, *or* C3(iii).

Parts C1 and C2 of Theorem 4 are proved in Fishburn (1998), part C3 is proved in Section 5, and part C4 is proved in Section 7. None of the special forms of the theorem precludes preference cycles. Examples for cycles under C1 are given in Fishburn (1984; 1988, pp. 74–75).

Table 1 provides a quick reference to the results of Theorems 1 through 4. It omits the restrictions needed to satisfy (4), but includes the additional constraints imposed by C4.

3. RDU proofs

We assume **RDU** with u continuous and increasing in w and $\pi \in \Pi$. Let $\lambda = \pi(\frac{1}{2})$.

C1 proof. It is easily seen that C1 implies $(x+y)/2 \sim (x, \frac{1}{2}, y)$ for $x, y \geq 0$, so

$$x < y \Rightarrow u\left(\frac{x+y}{2}\right) = \lambda u(x) + (1-\lambda)u(y).$$

Hence

$$u(x) = \lambda u(x-d) + (1-\lambda)u(x+d) \quad \text{for all } x \geq d \geq 0.$$

We have $\lambda[u(x) - u(x-d)] = (1-\lambda)[u(x+d) - u(x)]$ for $x \geq d \geq 0$. Suppose $\lambda \neq \frac{1}{2}$. Let $\tau = \lambda/(1-\lambda)$, so $\tau \neq 1$. Then for every $x \geq 0$ and $n \geq 1$,

$$\begin{aligned} u(x+1) - u(x) &= \sum_{i=0}^{n-1} [u(x + (i+1)/n) - u(x + i/n)] \\ &= \sum_{i=0}^{n-1} \left(\frac{\lambda}{1-\lambda}\right)^i [u(x + 1/n) - u(x)] \\ &= \left(\frac{\tau^n - 1}{\tau - 1}\right)[u(x + 1/n) - u(x)]. \end{aligned}$$

Then

$$\frac{u(x + 1/n) - u(x)}{1/n} = \frac{[u(x+1) - u(x)]n(\tau - 1)}{\tau^n - 1}.$$

Table 1.

	EU $u(w)$	RDU $u(w)$	WLU $u_1(w)$	WLU $u_2(w)$	SSBU* $\phi(x, y)$
C1	w	w	w	1	$g(x-y)$
		$\pi(\frac{1}{2}) = \frac{1}{2}$	or ae^{aw}	e^{-aw}	
C2	w	w	we^{aw}	e^{aw}	$(x-y)$
	or ae^{aw}	or ae^{aw}	or ae^{bw}	e^{cw}	or $\alpha(x-y)(e^{ax}-e^{ay})$
C3	$aw^2 + bw$	$aw^2 + bw$	$(aw^2 + bw)e^{cw}$	e^{cw}	$\alpha(x-y)(x^2-y^2) + \beta(x-y)(x-y)$
	or $ae^{bw} + ce^{dw}$	or $ae^{bw} + ce^{dw}$	or $ae^{bw} + ce^{dw}$	e^{fw}	or $\alpha(x-y)(e^{ax}-e^{ay}) + \beta(x-y)(e^{bx}-e^{by})$
	or $aw + be^{cw}$	or $ae^{bw} + ce^{dw}$	or $awe^{bw} + ce^{dw}$	e^{bw}	or $\alpha(x-y)(x-y) + \beta(x-y)(e^{ax}-e^{ay})$
	or $(aw+b)e^{cw}$	or $(aw+b)e^{cw}$	or $(aw+b)e^{cw}$	e^{dw}	or $\alpha(x-y)(xe^{ax}-ye^{ay}) + \beta(x-y)(e^{ax}-e^{ay})$
C4	$aw^2 + bw$	$\pi(p) \equiv p$,	$ae^{bw} + ce^{dw}$	e^{fw}	$\alpha(x-y)(x^2-y^2) + \beta(x-y)(x-y)$
	or $ae^{bw} + ce^{dw}$	so	$bd \leq 0$		or $\alpha(x-y)(e^{ax}-e^{ay}) + \beta(x-y)(e^{bx}-e^{by})$
	$bd < 0$	identical	or $awe^{bw} + ce^{dw}$	e^{bw}	$ab < 0$
	or $aw + be^{cw}$	to EU	$bd \leq 0$		or $\alpha(x-y)(x-y) + \beta(x-y)(e^{ax}-e^{ay})$

*g odd, α and β even.

Since $u(x+1) - u(x) > 0$, it follows that if $\tau > 1$ then the right derivative of u at x equals 0, and if $\tau < 1$ then the right derivative of u at x equals ∞. Since this is impossible for every $x \geq 0$, we conclude that $\tau = 1$, or $\lambda = \frac{1}{2}$. Hence $u(x) - u(x - d) = u(x+d) - u(x)$ for all $x \geq d \geq 0$, so $u'(w) = a > 0$ for all $w \geq 0$. This implies that $u(w) = aw + b$, $a > 0$, or $u(w) = w$ with affine rescaling. Moreover, $\pi \in \Pi$ is unrestricted except for $\pi(\frac{1}{2}) = \frac{1}{2}$ since C1 holds for any such π.

C2 and C3 sufficiency. It is easily checked that the EU forms for C2 and C3 also satisfy C2 and C3 for RDU without restrictions on $\pi \in \Pi$. We illustrate this for the quadratic form of C3(i): see also Bell (1988, p. 1418). Suppose $u(w) = aw^2 + bw$ for $w \geq 0$. Then for $w + \tilde{x}$, $x + \tilde{y} \in W^+$,

$$E_\pi u(w + \tilde{x}) = aw^2 + 2awE_\pi \tilde{x} + aE_\pi \tilde{x}^2 + bw + bE_\pi \tilde{x}$$

$$E_\pi u(w + \tilde{y}) = aw^2 + 2awE_\pi \tilde{y} + aE_\pi \tilde{y}^2 + bw + bE_\pi \tilde{y}.$$

Therefore

$$E_\pi u(w + \tilde{x}) - E_\pi u(w + \tilde{y}) = w2a\left[E_\pi \tilde{x} - E_\pi \tilde{y}\right] + K,$$

where K does not involve w. We vary w with $\tilde{x}$ and $\tilde{y}$ fixed. If $E_\pi \tilde{x} = E_\pi \tilde{y}$, there are no switches; otherwise $E_\pi u(w + \tilde{x}) = E_\pi u(w + \tilde{y})$ for at most one w, so the quadratic form is one-switch (C3) for RDU.

The restrictions of (4) on the utility functions for these cases are also easily verified. We illustrate this for C3(ii), where

$$u'(w) = abe^{bw} + cde^{dw}.$$

$u'(0) \geq 0$ requires $ab + cd \geq 0$, and $u(w) > 0$ for $w > 0$ requires at least either $ab > 0$ or $cd > 0$. Assume $ab > 0$ without loss of generality. If $cd < 0$, then $ab/(-cd) > e^{(d-b)w}$ for all $w > 0$, and this requires $d \leq b$. If $b = d$ then $u'(w) = (a + c)be^{bw}$, and $u'(w) > 0$ requires $b(a + c) > 0$.

Derivation for C2. Assume C2. Let λ satisfy $\pi(\lambda) = \frac{1}{2}$, and let (x, λ, y) denote the risky prospect with probability λ for x and probability $1 - \lambda$ for y, $x \neq y$. Then (x, λ, y) with $0 \leq x < y$ has a unique certainty equivalent $c(x, y)$ strictly between x and y:

$$c(x, y) \sim (x, \lambda, y) \quad \text{and} \quad c(x, y) = u^{-1}\left(\frac{u(x) + u(y)}{2}\right).$$

By the zero-switch condition C2, for every w with $\min\{w + x, w + y\} \geq 0$ we have $c(w + x, w + y) \sim (w + x, \lambda, w + y)$ and

$$c(w+x, w+y) = w + c(x,y) = u^{-1}\left(\frac{u(w+x)+u(w+y)}{2}\right).$$

Therefore

$$u^{-1}\left(\frac{u(w+x)+u(w+y)}{2}\right) = w + u^{-1}\left(\frac{u(x)+u(y)}{2}\right).$$

It then follows from Aczél (1966, p. 153) that u has one of the two forms for C2 in Theorem 1.

Derivation for C3. Assume C3. The proof of Result 2 in Bell (1995a) for C3 under EU also implies that C3(i)–(iv) of Theorem 1 are the only u solutions for C3 under RDU, given (4). Note in particular that p and q in Bell's proof can be replaced by π values under our continuity assumption for π, so the EU proof holds also for RDU.

Derivation for C4. The comprehensive impact of C4 on RDU stems from the re-ranking of outcomes that occur for the particular contextual uncertainties involved.

We will demonstrate a class of counterexamples to C4 in which u is linear. Each specific counterexample thus applies to all utility functions that are differentiable in a closed interval. Since all utility functions that satisfy C4 for EU are differentiable everywhere, and since all other functions violate C4 for RDU even when π is the identity function, we are done.

Let $\tilde{x} = (x, p, 0)$ and $\tilde{y} = (y, q, 0)$. The contextual uncertainties for our counterexamples are $(2, \frac{1}{2}, -2)$ and $(1, \frac{1}{2}, -1)$. We use specific numbers for clarity only. Without loss of generality, we take $2 < x < 4$, $2 < y < 4$, $p < 1/2$ and $q < 1/2$.

Suppose $(2, \frac{1}{2}, -2)$ has been resolved with payoff outcome 2. The evaluation of $\tilde{x}$, combined with $(1, \frac{1}{2}, -1)$, proceeds as follows:

payoffs, highest to lowest:
$3+x, 1+x, 3, 1$
probabilities, respectively:
$p/2, p/2, (1-p)/2, (1-p)/2$
probability weights, respectively:
$1-\pi(1-p/2), \pi(1-p/2)-\pi(1-p), \pi(1-p)-\pi(\frac{1}{2}-p/2), \pi(\frac{1}{2}-p/2)$.

The RDU utility is therefore

$$\begin{aligned}
&(3+x)[1-\pi(1-p/2)] + (1+x)[\pi(1-p/2)-\pi(1-p)]\\
&\quad +3\left[\pi(1-p) - \pi\left(\frac{1}{2}-p/2\right)\right] + \pi\left(\frac{1}{2}-p/2\right)\\
&\qquad = x[1-\pi(1-p)] + 3\\
&\qquad\quad - 2\left[\pi(1-p/2) - \pi(1-p) - \pi\left(\frac{1}{2}-p/2\right)\right].
\end{aligned}$$

Thus $\tilde{x} \succ \tilde{y}$ if and only if

$$x[1 - \pi(1-p)] - y[1 - \pi(1-q)]$$
$$> 2\left[\pi(1-p/2) - \pi(1-q/2) - \pi(1-p) + \pi(1-q) + \pi\left(\frac{1}{2} - p/2\right) - \pi\left(\frac{1}{2} - q/2\right)\right].$$

The evaluation of $\tilde{x}$ and $\tilde{y}$ when $(2, \frac{1}{2}, -2)$ is resolved at -2 proceeds similarly and leads to exactly the same inequality because u is linear.

Consider now the resolution of $(1, \frac{1}{2}, -1)$. When this smaller contextual uncertainty is resolved at 1, the evaluation of $\tilde{x}$ goes as follows:

payoffs, highest to lowest:
$3 + x, 3, x - 1, -1$
probabilities:
$p/2, \frac{1}{2} - p/2, p/2, \frac{1}{2} - p/2$
probability weights:
$1 - \pi(1-p/2), \pi(1-p/2) - \pi(\frac{1}{2}), \pi(\frac{1}{2}) - \pi(\frac{1}{2} - p/2), \pi(\frac{1}{2} - p/2)$.

In this case, we find that $\tilde{x} \succ \tilde{y}$ if and only if

$$x\left[1 - \pi(1-p/2) + \pi\left(\frac{1}{2}\right) - \pi\left(\frac{1}{2} - p/2\right)\right]$$
$$> y\left[1 - \pi(1-q/2) + \pi\left(\frac{1}{2}\right) - \pi\left(\frac{1}{2} - q/2\right)\right].$$

Again, because u is linear, resolution of $(1, \frac{1}{2}, -1)$ at -1 leads to exactly the same inequality for $\tilde{x} \succ \tilde{y}$.

There are many ways to construct a counterexample to C4 from the preceding inequalities for $\tilde{x} \succ \tilde{y}$. For instance, assume that $\pi(p) = p$ except locally near $p = \frac{1}{2}$, where $\pi(\frac{1}{2}) = \frac{1}{2} + \epsilon$. Then the preceding two inequalities are

$$xp > yq$$

and

$$xp + (x - y)\epsilon > yq.$$

We need only choose x, p, y, q and ϵ so that

$$xp > yq > xp + (x - y)\epsilon$$

to contradict C4. A specific example is $(x, p, y, q, \epsilon) = (3, 1/3, 7/2, 1/(7/2 + \delta), 3\delta/(7/2 + \delta))$ with δ small and positive.

4. One-switch for SSBU

Assume that $\succ$ on W^+ satisfies **SSBU** with ϕ smooth, and let

$$\phi(w + \tilde{x}, w + \tilde{y}) = E\phi(w + \tilde{x}, w + \tilde{y})$$

as defined in (7). We first prove that the forms for ϕ in C3(i)–(iv) of Theorem 4 satisfy C3.

For fixed $\tilde{x}$ and $\tilde{y}$, we obtain

$$\begin{aligned}\phi(w + \tilde{x}, w + \tilde{y}) &= A + Bw \quad \text{for C3(i)}\\ &= Ae^{aw} + Be^{bw} \quad \text{for C3(ii)}\\ &= A + Be^{aw} \quad \text{for C3(iii)}\\ &= [A + Bw]e^{aw} \quad \text{for C3(iv)},\end{aligned}$$

where A and B are expectations with respect to $\tilde{x}$ and $\tilde{y}$ that do not involve w. To verify C3 (one-switch), it suffices to note that $\phi(w + \tilde{x}, w + \tilde{y}) = 0$ for no w, or exactly one w, or for all w. This is obvious for C3(i), C3(iii), and C3(iv), where $e^{aw} > 0$ for all w. Suppose C3(ii) obtains. If $AB \geq 0$, then $Ae^{aw} + Be^{bw}$ vanishes for no w or for all w. If $AB < 0$, equality to 0 requires $e^{(b-a)w} = -A/B$, which is true for no w or all w if $a = b$, and for no w or exactly one w if $a \neq b$.

We show next that (4), i.e., $\phi(x, y) > 0$ when $x > y \geq 0$, corresponds to the restrictions in brackets for the four ϕ functions in C3 of Theorem 4. Let $\Delta = x - y > 0$. For C3(i),

$$\begin{aligned}\phi(y + \Delta, y) &= \alpha(\Delta)\left[(y + \Delta)^2 - y^2\right] + \beta(\Delta)\Delta\\ &= \Delta[\Delta\alpha(\Delta) + \{2y\alpha(\Delta) + \beta(\Delta)\}].\end{aligned}$$

This is positive for all $y \geq 0$ and $\Delta > 0$ only if $\alpha(\Delta) \geq 0$, for if $\alpha(\Delta) < 0$ then large y will make the entire expression negative. Given $\alpha(\Delta) \geq 0$, at $y = 0$ we require $\Delta\alpha(\Delta) + \beta(\Delta) > 0$. Conversely, if $\alpha(\Delta) \geq 0$ and $\Delta\alpha(\Delta) + \beta(\Delta) > 0$ for all $\Delta > 0$, it is easily seen that $\phi(y + \Delta, y) > 0$ for all $y \geq 0$ and all $\Delta > 0$. For C3(ii),

$$\phi(y + \Delta, y) = e^{ay}\left[\alpha(\Delta)(e^{a\Delta} - 1)\right] + e^{by}\left[\beta(\Delta)(e^{b\Delta} - 1)\right],$$

so this must be positive for all $y \geq 0$ and $\Delta > 0$. For C3(iii),

$$\phi(y+\Delta, y) = e^{ay}\left[\alpha(\Delta)(e^{a\Delta}-1)\right] + e^{by}\left[\beta(\Delta)(e^{b\Delta}-1)\right],$$

so this must be positive for all $y \geq 0$ and $\Delta > 0$. For C3(iii),

$$\phi(y+\Delta, y) = \Delta\alpha(\Delta) + e^{ay}\left[\beta(\Delta)(e^{a\Delta}-1)\right].$$

At $y = 0$, this requires $\Delta\alpha(\Delta) + \beta(\Delta)(e^{a\Delta} - 1) > 0$ for all $\Delta > 0$. If $a < 0$ then the second term goes to 0 as $y \to \infty$, so we need $\alpha(\Delta) \geq 0$ in this case. We have already required $\alpha(\Delta) > 0$ when $a = 0$. If $a > 0$, we also need $\beta(\Delta) \geq 0$. For C3(iv),

$$\phi(y+\Delta, y) = e^{ay}\left\{\alpha(\Delta)\left[(y+\Delta)e^{a\Delta} - y\right] + \beta(\Delta)\left[e^{a\Delta}-1\right]\right\}.$$

For $y = 0$, we require $\Delta\alpha(\Delta)e^{a\Delta} + \beta(\Delta)(e^{a\Delta} - 1) > 0$ for all $\Delta > 0$. The other part within braces is $y\alpha(\Delta)(e^{a\Delta} - 1)$, so we need $\alpha(\Delta)(e^{a\Delta} - 1) \geq 0$ for all $\Delta > 0$. This is true if $a = 0$, or if $\alpha(\Delta) \geq 0$ if $a > 0$, or if $\alpha(\Delta) \leq 0$ if $a < 0$, so it holds if and only if $a\alpha(\Delta) \geq 0$ for all $\Delta > 0$.

We now assume C3 and derive the forms for C3 in Theorem 4. Let $\phi_{1(i),2(j)}(w, w)$ or $\phi_{1\cdots 12\cdots 2}$ [i $1's$, j $2's$] denote the derivative of $\phi(w+x, w+y)$ i times with respect to x and j times with respect to y, evaluated at $x = y = 0$. Then a Taylor series expansion for a function of two variables (Taylor, 1955, p. 228) yields

$$\phi(w+x, w+y) = \phi(w,w) + \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^{i}\phi_{1(n-i),2(i)}(w,w).$$

Skew-symmetry gives $\phi(w,w) = 0$ and $\phi(w+x, w+y) + \phi(w+y, w+x) = 0$. The preceding equation then yields

$$\begin{aligned}
0 = {} & (x+y)(\phi_1+\phi_2) + \frac{1}{2}(x^2+y^2)(\phi_{11}+\phi_{22}) + 2xy\phi_{12} \\
& + \frac{1}{6}(x^3+y^3)(\phi_{111}+\phi_{222}) + \frac{1}{2}xy(x+y)(\phi_{112}+\phi_{122}) \\
& + \frac{1}{24}(x^4+y^4)(\phi_{1111}+\phi_{2222}) + \frac{1}{6}xy(x^2+y^2)(\phi_{1112}+\phi_{1222}) \\
& + \frac{1}{2}x^2y^2\phi_{1122} + \cdots .
\end{aligned}$$

We divide this by sums of powers of x and y and let x and y go to zero to conclude that

$$\phi_1 + \phi_2 = 0 \quad (\text{set } x = y, \text{divide by } x, \text{let } x \to 0)$$

$$\phi_{11} + \phi_{22} = 0 \quad (\text{set } y = 0, \text{divide by } x^2, \text{let } x \to 0)$$

$$\phi_{12} = 0 \quad (\text{set } x = y, \text{divide by } x^2, \text{let } x \to 0)$$

$$\phi_{111} + \phi_{222} = 0$$

$$\phi_{112} + \phi_{122} = 0$$

and so forth. It follows that

$$\begin{aligned}\phi(w+x, w+y) &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{i} x^i y^i (x^{n-2i} - y^{n-2i}) \phi_{1(n-i),2(i)}(w,w) \\ &= \sum_{i=0}^{\infty} (xy)^i \sum_{n=2i+1}^{\infty} \frac{1}{n!} \binom{n}{i} (x^{n-2i} - y^{n-2i}) \phi_{1(n-i),2(i)}(w,w). \end{aligned} \tag{13}$$

Equivalently,

$$\phi(w+x, w+y) = (x-y)\phi_1 + \frac{1}{2!}(x^2 - y^2)\phi_{11} + \frac{1}{3!}(x^3 - y^3)\phi_{111} + \frac{1}{2}(x^2 y - xy^2)\phi_{112} + \cdots ,$$

so, for lotteries $\tilde{x}$ and $\tilde{y}$, bilinear expansion gives

$$\begin{aligned}\phi(w+\tilde{x}, w+\tilde{y}) &= \sum_{i=0}^{\infty} \sum_{n=2i+1}^{\infty} \frac{1}{n!} \binom{n}{i} \left[E\tilde{x}^{n-i} E\tilde{y}^i - E\tilde{y}^{n-i} E\tilde{x}^i \right] \phi_{1(n-i),2(i)}(w,w) \\ &= \left[E\tilde{x} - E\tilde{y} \right] \phi_1 + \frac{1}{2} \left[E\tilde{x}^2 - E\tilde{y}^2 \right] \phi_{11} \\ &\quad + \frac{1}{6} \left[E\tilde{x}^3 - E\tilde{y}^3 \right] \phi_{111} + \frac{1}{2} \left[E\tilde{x}^2 E\tilde{y} - E\tilde{y}^2 E\tilde{x} \right] \phi_{112} + \cdots . \end{aligned} \tag{14}$$

If any ϕ term here after ϕ_1 and ϕ_{11} is not linearly dependent on ϕ_1 and ϕ_{11} as a function of w, then for suitably small (near 0) outcomes it is is possible to construct $\tilde{x}$ and $\tilde{y}$ to contradict C3 by a two-switch example. Thus, for $n \geq 3$ and $0 \leq i \leq \lfloor (n-1)/2 \rfloor$, there are constants $c_{1(n-i),2(i)}$ and $d_{1(n-i),2(i)}$ such that

$$\phi_{1(n-i),2(i)}(w,w) = c_{1(n-i),2(i)}\phi_1(w,w) + d_{1(n-i),2(i)}\phi_{11}(w,w)$$

for all w, i.e., for all $w \geq 0$. Let

$$a(w) = \phi_1(w,w), \quad b(w) = \phi_{11}(w,w).$$

Then, by (13),

$$\begin{aligned}
&\phi(w+x, w+y) \\
&\quad = (x-y)a(w) + \frac{1}{2}(x^2-y^2)b(w) + \sum_{i=0}^{\infty}(xy)^i \underset{\text{(exclude } n=1,2 \text{ for } i=0)}{\sum_{n=2i+1}^{\infty}} \frac{1}{n!}\binom{n}{i} \\
&\qquad \times \left(x^{n-2i} - y^{n-2i}\left[c_{1(n-i),2(i)}a(w) + d_{1(n-i),2(i)}b(w)\right]\right. \\
&\quad = \left\{x - y + \sum_{i=0}^{\infty}(xy)^i\left[a_i(x) - a_i(y)\right]\right\}a(w) \\
&\qquad + \left\{\frac{1}{2}(x^2-y^2) + \sum_{i=0}^{\infty}(xy)^i\left[b_i(x) - b_i(y)\right]\right\}b(w) \\
&\quad = A(x,y)a(w) + B(x,y)b(w),
\end{aligned} \tag{15}$$

where the a_i, b_i and A, B are defined in context, with A and B skew-symmetric. When x and y are suitably small with $x > y$, the leading terms $x - y$ in A and $\frac{1}{2}(x^2 - y^2)$ in B dominate to give $A(x,y) > 0$ and $B(x,y) > 0$. For lotteries $\tilde{x}$ and $\tilde{y}$ we have

$$\phi(w+\tilde{x}, w+\tilde{y}) = E\left[A(\tilde{x},\tilde{y})\right]a(w) + E\left[B(\tilde{x},\tilde{y})\right]b(w),$$

and it follows from (14) that there is a positive-radius disk with center $(0,0)$ such that every point in the disk has an $\tilde{x}, \tilde{y}$ at which that point equals $(E[A(\tilde{x},\tilde{y})], E[B(\tilde{x},\tilde{y})])$.

We assume without loss of generality that neither $a(w)$ nor $b(w)$ in (15) is uniformly zero. For example, if $\phi_{11} \equiv 0$, then $x > y$ implies $A(x,y)a(w) > 0$ for all w; hence a has constant sign and ϕ is zero-switch.

To determine forms for $a(w)$ and $b(w)$, let $u = \phi(w+x, w+y)$, let u_1 (u_2) denote the first derivative of u with respect to x (y)—with similar notation for higher-order partial derivatives, and series-expand $\phi(w + x + z_1, w + y + z_2)$ to obtain

$$\begin{aligned}
&\phi(w + x + z_1, w + y + z_2)\\
&\quad = u + z_1 u_1 + z_2 u_2\\
&\qquad + \frac{1}{2}\left(z_1^2 u_{11} + 2 z_1 z_2 u_{12} + z_2^2 u_{22}\right)\\
&\qquad + \frac{1}{3!}\left(z_1^3 u_{111} + 3 z_1^2 z_2 u_{112} + 3 z_1 z_2^2 u_{122} + z_2^3 u_{222}\right) + \cdots . \qquad (16)
\end{aligned}$$

Now let $\tilde{z}_1$ and $\tilde{z}_2$ be lotteries with the same first two moments:

$$\begin{aligned}
m_1 &= E\tilde{z}_1 = E\tilde{z}_2\\
m_2 &= E\tilde{z}_1^2 = E\tilde{z}_2^2.
\end{aligned}$$

Bilinear expansion and (16) yield

$$\begin{aligned}
&\phi(w + x + \tilde{z}_1, w + y + \tilde{z}_2)\\
&\quad = u + m_1(u_1 + u_2) + \frac{1}{2} m_2 (u_{11} + u_{22})\\
&\qquad + m_1^2 u_{12} + \frac{1}{3!}\left[E\tilde{z}_1^3 u_{111} + 3 m_1 m_2 (u_{112} + u_{122}) + E\tilde{z}_2^3 u_{222}\right] + \cdots .
\end{aligned}$$

Consider $x > y$ with $x - y$ small. If $u_{11} + u_{22}$ and u_{12} are not both linearly dependent on $u_1 + u_2$ and u, then it will be possible to construct $\tilde{z}_1$ and $\tilde{z}_2$ with outcomes near zero which satisfy the noted moment equalities and contradict C3. It follows for the given x and y that there are constants α and β such that

$$u_{11} + 2u_{12} + u_{22} = \alpha(u_1 + u_2) + \beta u$$

for all w. In terms of the total differentials ϕ' and ϕ'' of $\phi(w + x, w + y)$ with respect to w, this is precisely

$$\phi''(w + x, w + y) = \alpha\phi'(w + x, w + y) + \beta\phi(w + x, w + y). \qquad (17)$$

For convenience let $f(w) = \phi(w + x, w + y)$, and write the preceding equation as the linear differential equation

$$f''(w) - \alpha f'(w) - \beta f(w) = 0$$

with auxiliary equation $r^2 - \alpha r - \beta = 0$. Let r_1 and r_2 be the roots of the auxiliary equation. Then (Phillips, 1951) we have

$$f(w) = c_1 e^{r_1 w} + c_2 e^{r_2 w} \quad \text{if } r_1 \neq r_2$$

$$f(w) = (c_1 + c_2 w) e^{rw} \quad \text{if } r_1 = r_2 = r,$$

where c_1 and c_2 are constants. Actually, because α and β in (17) might depend on $x - y$, r_1 and r_2 can be written as continuous functions of $x - y$. Similarly, c_1 and c_2 can depend on (x, y), so we write them as functions of (x, y). The preceding f solutions for variable x and y with $x - y$ small then take the forms

$$\phi(w + x, w + y) = c_1(x, y) e^{wr_1(x-y)} + c_2(x, y) e^{wr_2(x-y)}$$

and

$$\phi(w + x, w + y) = [c_1(x, y) + wc_2(x, y)] e^{wr(x-y)}.$$

We compare these to (15), i.e., to

$$\phi(w + x, w + y) = A(x, y)a(w) + B(x, y)b(w).$$

Taking account of the comment after (15) for a positive-radius disk, it follows that $r_1(x - y)$ and $r_2(x - y)$, or $r(x - y)$, can be presumed (or must be, when the coefficients are nonzero) to be constant for small x and y with $|x - y|$ small and nonzero. Hence, for such x and y, either

$$\begin{aligned} \phi(w + x, w + y) &= c_1(x, y) e^{r_1 w} + c_2(x, y) e^{r_2 w} \\ &= A(x, y)a(w) + B(x, y)b(w) \quad \text{for all } w, \end{aligned} \tag{18}$$

or

$$\begin{aligned} \phi(w + x, w + y) &= [c_1(x, y) + wc_2(x, y)] e^{rw} \\ &= A(x, y)a(w) + B(x, y)b(w) \quad \text{for all } w, \end{aligned} \tag{19}$$

where c_1 and c_2, like A and B, are easily seen to be skew-symmetric.

Suppose (18) applies. We can then suppose that

$$a(w) = \alpha_1 e^{r_1 w} + \alpha_2 e^{r_2 w} + \alpha_3 U(w), \quad \alpha_3 \neq 0,$$

$$b(w) = \beta_1 e^{r_1 w} + \beta_2 e^{r_2 w} + \beta_3 V(w), \quad \beta_3 \neq 0,$$

where U and V have no additive term like $e^{r_i w}$ and each is either uniformly zero or linear independent of $e^{r_1 w}$ and $e^{r_2 w}$. When these are substituted into the AB

version of (18) and compared to its c_1c_2 version, it follows from the remarks following (15) that $U = V \equiv 0$. Then

$$\begin{aligned}\phi&(w+x, w+y)\\ &= [\alpha_1 A(x,y) + \beta_1 B(x,y)]e^{r_1 w} + [\alpha_2 A(x,y) + \beta_2 B(x,y)]e^{r_2 w}\end{aligned}$$

for all x, y and w. Let $C(x, y) = \alpha_1 A(x, y) + \beta_1 B(x, y)$ and $D(x, y) = \alpha_2 A(x, y) + \beta_2 B(x, y)$. Then

$$\phi(w+x, w+y) = C(x,y)e^{r_1 w} + D(x,y)e^{r_2 w}, \tag{20}$$

where C and D are skew-symmetric because A and B are skew-symmetric.

Suppose (19) applies. We can then presume that

$$a(w) = \alpha_1 e^{rw} + \alpha_2 we^{rw} + \alpha_3 U(w), \quad \alpha_3 \neq 0,$$

$$b(w) = \beta_1 e^{rw} + \beta_2 we^{rw} + \beta_3 V(w), \quad \beta_3 \neq 0,$$

with stipulations on U and V similar to those above. We again get $U = V \equiv 0$, so

$$\begin{aligned}\phi(w+x, w+y) &= [\alpha_1 A(x,y) + \beta_1 B(x,y)]e^{rw}\\ &\quad + [\alpha_2 A(x,y) + \beta_2 B(x,y)]we^{rw}\\ &= C(x,y)e^{rw} + D(x,y)we^{rw}.\end{aligned} \tag{21}$$

We now proceed from (20) and (21) to our final forms for $\phi(x, y)$. There are three cases for (20). One of these, which is the case of complex conjugate roots, is inapplicable because it cannot satisfy the basic monotonicity property of $x > y \Rightarrow x \succ y$. The other two are the real (r_1, r_2) cases, one for $r_1 = 0$ and the other for $r_1 r_2 \neq 0$:

$$\text{I.} \quad \phi(w+x, w+y) = C(x,y) + D(x,y)e^{r_2 w} \quad r_2 \neq 0$$

$$\text{II.} \quad \phi(w+x, w+y) = C(x,y)e^{r_1 w} + D(x,y)e^{r_2 w} \quad r_1 r_2 \neq 0.$$

We also consider the two cases for (21), for $r = 0$ and $r \neq 0$:

$$\text{III.} \quad \phi(w+x, w+y) = C(x,y) + wD(x,y)$$

$$\text{IV.} \quad \phi(w+x, w+y) = C(x,y)e^{rw} + D(x,y)we^{rw} \quad r \neq 0.$$

Each case has the form $C(x, y)c(w) + D(x, y)d(w)$:

	$c(w)$	$d(w)$	
I.	1	e^{bw}	$b \neq 0$
II.	e^{aw}	e^{bw}	$ab \neq 0$
III.	1	w	
IV.	e^{bw}	we^{bw}	$b \neq 0.$

Given $\phi(w + x, w + y) = c(w)C(x, y) + d(w)D(x, y)$, we replace w, x and y by $w + (x + y)/2$, $(x - y)/2$ and $(y - x)/2$, respectively, to obtain

$$\phi(w + x, w + y) = c\left(w + \frac{x + y}{2}\right)s(x - y) + d\left(w + \frac{x + y}{2}\right)t(x - y), \tag{22}$$

where $s(x - y) = C((x - y)/2, (y - x)/2)$, $t(x - y) = D((x - y)/2, (y - x)/2)$, and s and t are odd because C and D are skew-symmetric.

I. With $c(w) = 1$ and $d(w) = e^{bw}$, $b \neq 0$, (22) gives

$$\begin{aligned}\phi(w + x, w + y) &= s(x - y) + e^{bw}e^{b(x+y)/2}t(x - y)\\ &= s(x - y) + e^{bw}\left(\frac{e^{bx} - e^{by}}{e^{b(x-y)/2} - e^{b(y-x)/2}}\right)t(x - y)\\ &= s(x - y) + [e^{b(w+x)} - e^{b(w+y)}]t_1(x - y),\end{aligned}$$

where t_1 is even. We have assumed here that $x \neq y$, and for $x = y$ can let $t_1(0)$ be any finite value, e.g. $t_1(0) = 0$. It follows that

$$\phi(x, y) = s(x - y) + (e^{bx} - e^{by})t_1(x - y),$$

with s odd and t_1 even. This matches the form of C3(iii) in Theorem 4.

II. With $c(w) = e^{aw}$ and $d(w) = e^{bw}$, $ab \neq 0$, a calculation like that for t in case I in (22) gives

$$\phi(x, y) = (e^{ax} - e^{ay})s_1(x - y) + (e^{bx} - e^{by})t_1(x - y),$$

with s_1 and t_1 even. This is C3(ii) in Theorem 4.

III. With $c(w) = 1$ and $d(w) = w$, (22) gives (for $x \neq y$)

$$\begin{aligned}
&\phi(w+x, w+y) \\
&\quad = s(x-y) + \left(w + \frac{x+y}{2}\right) t(x-y) \\
&\quad = [(w+x) - (w+y)] s_1(x-y) \\
&\qquad + \left[(w+x)^2 - (w+y)^2\right] \frac{\left(w + \frac{x+y}{2}\right) t(x-y)}{\left(2w(x-y) + x^2 - y^2\right)} \\
&\quad = [(w+x) - (x+y)] s_1(x-y) + \left[(w+x)^2 - (w+y)^2\right] \frac{t(x-y)}{2(x-y)} \\
&\quad = [(w+x) - (w+y)] s_1(x-y) + \left[(w+x)^2 - (w+y)^2\right] t_1(x-y)
\end{aligned}$$

where s_1 and t_1 are even. Therefore

$$\phi(x, y) = (x-y) s_1(x-y) + (x^2 - y^2) t_1(x-y),$$

which is C3(i) in Theorem 4.

IV. With $c(w) = e^{bw}$ and $d(w) = we^{bw}$, $b \neq 0$, (22) gives

$$\begin{aligned}
&\phi(w+x, w+y) \\
&\quad = e^{bw} e^{b(x+y)/2} s(x-y) + \left(w + \frac{x+y}{2}\right) e^{bw} e^{b(x+y)/2} t(x-y) \\
&\quad = e^{bw} e^{b(x+y)/2} \left(\frac{t(x-y)}{e^{b(x-y)/2} - e^{b(y-x)/2}}\right) \Big\{ w\left(e^{b(x-y)/2} - e^{b(y-x)/2}\right) \\
&\qquad\qquad + \left(\frac{x+y}{2}\right)\left(e^{b(x-y)/2} - e^{b(y-x)/2}\right) \\
&\qquad\qquad + \left(\frac{x-y}{2}\right) e^{b(x-y)/2} + e^{b(y-x)/2} \Big\} \\
&\qquad - e^{bw} e^{b(x+y)/2} \left\{ \frac{t(x-y)\left(\frac{x-y}{2}\right)\left(e^{b(x-y)/2} + e^{b(y-x)/2}\right)}{e^{b(x-y)/2} - e^{b(y-x)/2}} - s(x-y) \right\} \\
&\quad = t_1(x-y) e^{bw} \left\{ w(e^{bx} - e^{by}) + xe^{bx} - ye^{by} \right\} \\
&\qquad + e^{bw} e^{b(x+y)/2} s_1(x-y) \; [s_1 \text{ odd}, t_1 \text{ even}]
\end{aligned}$$

$$= \left[w(e^{b(w+x)} - e^{b(w+y)}) + xe^{b(w+x)} - ye^{b(w+y)}\right] t_1(x-y)$$

$$+ [e^{b(w+x)} - e^{b(w+y)}] s_2(x-y) \; [s_2 \text{ even}]$$

$$= \left[(w+x)e^{b(w+x)} - (w+y)e^{b(w+y)}\right] t_1(x-y)$$

$$+ [e^{b(w+x)} - e^{b(w+y)}] s_2(x-y).$$

Therefore

$$\phi(x,y) = (xe^{bx} - ye^{by}) t_1(x-y) + (e^{bx} - e^{by}) s_2(x-y),$$

which matches C3(iv) in Theorem 4.

5. Derivations for WLU

Assume that $\succ$ on W^+ satisfies **WLU** with u_1 and u_2 smooth. We focus here on C1−C3 in Theorem 3, beginning with outlines of the derivations of the forms for C1 and C2. We omit the straightforward proofs that the forms given in Theorem 3 for C1, C2, and C3 imply these conditions. Verification of this for C3 is essentially the same as for C3 in the second paragraph of preceding section because $\phi(x, y)$ defined by

$$\phi(x,y) = u_1(x)u_2(y) - u_1(y)u_2(x) \tag{23}$$

gives a case of SSBU. We also omit verification of the conditions in brackets that ensure (4).

C1 proof. Given (23), Fishburn (1998) proves that C1 implies for all $x \geq y \geq 0$ that, up to a positive multiplicative transformation, either

$$\text{(i)} \quad \phi(x,y) = \phi(x-y,0) = x-y$$

or

$$\text{(ii)} \quad \phi(x,y) = \phi(x-y,0) = a\left[e^{a(x-y)} - e^{-a(x-y)}\right], \quad a \neq 0.$$

Alternative (i) gives $\phi(\tilde{x}, \tilde{y}) = E\tilde{x} - E\tilde{y}$, and it suffices to take $u_1(w) = w$ and $u_2(w) = 1$ for all $w \geq 0$. This gives C1(i) in Theorem 3.

When (ii) holds, we have

$$\phi(\tilde{x}, \tilde{y}) = aEe^{a\tilde{x}}Ee^{-a\tilde{y}} - aEe^{a\tilde{y}}Ee^{-a\tilde{x}},$$

so for $\tilde{x} \succ \tilde{y} \Leftrightarrow u_1(\tilde{x})u_2(\tilde{y}) > u_1(\tilde{y})u_2(\tilde{x})$ it suffices to take $u_1(w) = ae^{aw}$ and $u_2(w) = e^{-aw}$, $a \neq 0$, which matches C1(ii) in Theorem 3.

C2 proof. The two WLU forms implied by C2 in Fishburn (1998) can be written, for all $x \geq y \geq 0$, as follows:

$$\text{(i)} \quad \phi(x, y) = (x - y)\beta^y \beta^{(x-y)/2} = (x - y)e^{a(x+y)}$$

$$\text{(ii)} \quad \phi(x, y) = u(x - y)\beta^y = a\left[e^{d(x-y)} - e^{-d(x-y)}\right]\beta^y \beta^{(x-y)/2}$$

$$= a\left[e^{bx+cy} - e^{by+cx}\right].$$

Skew-symmetry shows that both forms hold for $x < y$ as well as $x \geq y$. For (i),

$$\phi(\tilde{x}, \tilde{y}) = E(\tilde{x}e^{a\tilde{x}})Ee^{a\tilde{y}} - E(\tilde{y}e^{a\tilde{y}})Ee^{a\tilde{x}},$$

so for $\phi(\tilde{x}, \tilde{y}) = u_1(\tilde{x})u_2(\tilde{y}) - u_1(\tilde{y})u_2(\tilde{x})$ it suffices to take $u_1(w) = we^{aw}$ and $u_2(w) = e^{aw}$, which are the forms for C2(i) in Theorem 3.

For (ii),

$$\phi(\tilde{x}, \tilde{y}) = aEe^{b\tilde{x}}Ee^{c\tilde{y}} - aEe^{b\tilde{y}}Ee^{c\tilde{x}},$$

so we can take $u_1(w) = ae^{bw}$ and $u_2(w) = e^{cw}$, as in C2(ii) of Theorem 3.

Derivation for C3. We now assume C3 and derive the forms for C3 shown in Theorem 3. As remarked after (10), we assume without loss of generality that $u_1 > 0$ and $u_2 > 0$. We shall also let

$$v(w) = u_1'(w)u_2(w) - u_1(w)u_2'(w) \quad \text{for all } w \geq 0.$$

Because $x > y \Rightarrow u_1(x)/u_2(x) > u_1(y)/u_2(y)$, $u_1(w)/u_2(w)$ increases in w, and therefore

$$\frac{v(x)}{u_2(w)^2} = \frac{d(u_1(w)/u_2(w))}{dw} > 0,$$

so $v(w) > 0$ for all w. We note also that

$$v'(w) = u_1''(w)u_2(w) - u_1(w)u_2''(w),$$

$$v''(w) = \left[u_1'''(w)u_2(w) - u_1(w)u_2'''(w)\right] + \left[u_1''(w)u_2'(w) - u_1'(w)u_2''(w)\right].$$

Several steps in the ensuing proof are similar to steps in the SSBU proof for C3 in the preceding section.

We begin our derivation for C3 with a Taylor-series expansion of $u_i(w+x)$ around w:

$$u_i(w+x) = u_i(w) + \sum_{n=1}^{\infty} \frac{1}{n!} x^n u_i^{(n)}(w),$$

where $u_i^{(n)}(w) = d^n u_i(w+x)/dx^n|_{x=0}$. With $u_i^{(0)} = u_i(w)$ and $u_i^{(n)} = u_i^{(n)}(w)$, we have

$$u_1(w+x)u_2(w+y) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_1^{(j)} u_2^{(k)} x^j y^k / (j!k!).$$

Therefore

$$\begin{aligned}
&u_1(w+x)u_2(w+y) - u_1(w+y)u_2(w+x) \\
&\quad = \sum_{0 \le j < k} (x^k y^j - x^j y^k)\left(u_1^{(k)} u_2^{(j)} - u_1^{(j)} u_2^{(k)}\right) \Big/ (j!k!) \\
&\quad = (x-y)(u_1' u_2 - u_1 u_2') + \frac{1}{2}(x^2 - y^2)(u_1'' u_2 - u_1 u_2'') \\
&\qquad + \frac{1}{6}(x^3 - y^3)(u_1''' u_2 - u_1 u_2''') + \cdots \\
&\qquad + \frac{1}{2}(x^2 y - x y^2)(u_1'' u_2' - u_1' u_2'') \\
&\qquad + \frac{1}{6}(x^3 y - x y^3)(u_1''' u_2' - u_1' u_2''') + \cdots .
\end{aligned}$$

Let $u_1(\tilde{x}) = Eu_1(\tilde{x})$ and $u_2(\tilde{x}) = Eu_2(\tilde{x})$. Taking expectations for $\tilde{x}$ and $\tilde{y}$, we have

$$\begin{aligned}
&u_1(w+\tilde{x})u_2(w+\tilde{y}) - u_1(w+\tilde{y})u_2(w+\tilde{x}) \\
&\quad = \left[E\tilde{x} - E\tilde{y}\right](u_1' u_2 - u_1 u_2') + \frac{1}{2}\left[E\tilde{x}^2 - E\tilde{y}^2\right](u_1'' u_2 - u_1 u_2'') \\
&\qquad + \frac{1}{6}\left[E\tilde{x}^3 - E\tilde{y}^3\right](u_1''' u_2 - u_1 u_2''') + \cdots \\
&\qquad + \frac{1}{2}\left[E\tilde{x}^2 E\tilde{y} - E\tilde{x} E\tilde{y}^2\right](u_1'' u_2' - u_1' u_2'') + \cdots .
\end{aligned}$$

C3 implies that every parenthetical u term beyond $v(w) = u_1'u_2 - u_1u_2'$ and $v'(w) = u_1''u_2 - u_1u_2''$ must be linearly dependent on those two to avoid a two-switch situation. One of the consequences of this is $v''(w) = av'(w) + bv(w)$ for real a and b because

$$\begin{aligned} v'' &= (u_1'''u_2 - u_1u_2''') + (u_1''u_2' - u_1'u_2'') \\ &= (a_1v' + b_1v) + (a_2v' + b_2v) \\ &= (a_1 + a_2)v' + (b_1 + b_2)v. \end{aligned}$$

The solutions of the linear differential equation $v'' - av' - bv = 0$ for real roots of the auxiliary equation $r^2 - ar - b = 0$ are

$$v(w) = ae^{bw} + ce^{dw}, \quad b \neq d \tag{24}$$

$$v(w) = (a + bw)e^{cw}. \tag{25}$$

We do not consider the case of complex conjugate roots because it does not lead to viable monotonic one-switch functions u_1 and u_2. We return to (24) and (25) after we develop general solution forms for (24) and (25) based on the linear dependence implied by C3.

By linear dependence, there are real λ_1 and λ_2 such that

$$u_1'''u_2 - u_1u_2''' = \lambda_1(u_1''u_2 - u_1u_2'') + \lambda_2(u_1'u_2 - u_1u_2').$$

Therefore, with $u_1 > 0$ and $u_2 > 0$ as noted earlier, we have

$$\frac{u_1''' - \lambda_1u_1'' - \lambda_2u_1'}{u_1} = \frac{u_2''' - \lambda_1u_2'' - \lambda_2u_2'}{u_2} = \theta(w),$$

where θ is defined by the common ratio. Thus

$$u_i''' = \lambda_1u_i'' + \lambda_2u_i' + \theta u_i \quad \text{for } i = 1, 2. \tag{26}$$

We will prove that $\theta(w)$ is a constant, independent of w, so the preceding equations for $i = 1, 2$ become order-3 linear differential equations that yield general solution forms for u_1 and u_2 subject to (24) and (25).

When the preceding expression is differentiated with respect to w, we obtain

$$u_i'''' = \lambda_1u_i''' + \lambda_2u_i'' + \theta u_i' + \theta'u_i \quad \text{for } i = 1, 2.$$

By linear dependence, there are real μ_1 and μ_2 such that

$$u_1''''u_2 - u_1u_2'''' = \mu_1(u_1''u_2 - u_1u_2'') + \mu_2(u_1'u_2 - u_1u_2').$$

Substitutions in this from the preceding two expressions give

$$\left(\lambda_1^2 + \lambda_2 - \mu_1\right)v' + (\theta + \lambda_1\lambda_2 - \mu_2)v = 0,$$

so that there are constants a_1 and b_1 for which

$$\theta(w) = a_1 + b_1\left(\frac{v'(w)}{v(w)}\right).$$

Linear dependence also gives π_1 and π_2 such that $u_1''''u_2' - u_1'u_2'''' = \pi_1 v' + \pi_2 v$, and by substitutions we obtain

$$\lambda_1(\alpha_1 v' + \beta_1 v) + \lambda_2(\alpha_2 v' + \beta_2 v) - \theta' v = \pi_1 v' + \pi_2 v,$$

so that there are constants a_2 and b_2 such that

$$\theta'(w) = a_2 + b_2\left(\frac{v'(w)}{v(w)}\right).$$

We therefore have the following equation pair for $\theta(w)$ and $\theta'(w)$:

$$\theta = a_1 + b_1(v'/v)$$

$$\theta' = a_2 + b_2(v'/v).$$

One solution to this pair is $\theta = a_1$ with $b_1 = a_2 = b_2 = 0$, in which case θ is constant and there are no implications for v'/v beyond what we already know from (24) and (25). There are also solutions when $b_1 \neq 0$. Given $b_1 \neq 0$, it follows without difficulty from (24) and (25) that the only solution to the (θ, θ') pair is $\theta =$ constant with v'/v also constant, in which case $v = ae^{bw}$. Because this v is a specialization of (24) and (25) (and in fact characterizes the zero-switch situation under C2), we proceed with the more general solution that leaves (24) and (25) intact.

Equation (26) with constant θ yields the following forms for u_1 and u_2 when the auxiliary equation $r^3 - \lambda_1 r^2 - \lambda_2 r - \theta = 0$ has three real roots:

I. All roots identical:

$$u_1(w) = (aw^2 + bw + c)e^{dw}$$

$$u_2(w) = (\alpha w^2 + \beta w + \gamma)e^{dw}$$

$$v(w) = \left[(a\beta - \alpha b)w^2 + 2(a\gamma - \alpha c)w + (b\gamma - \beta c)\right]e^{2dw};$$

II. Two roots identical:

$$u_1(w) = (aw + b)e^{cw} + de^{fw}, \quad c \neq f$$

$$u_2(w) = (\alpha w + \beta)e^{cw} + \delta e^{fw}$$

$$\begin{aligned} v(w) = (a\beta - \alpha b)e^{2cw} + e^{(c+f)w} \\ \times[(c-f)(a\delta - \alpha d)w \\ +(c-f)(b\delta - \beta d) + (a\delta - \alpha d)]; \end{aligned}$$

III. All three roots different:

$$u_1(w) = ae^{rw} + be^{sw} + ce^{tw}, \quad r \neq s \neq t \neq r$$

$$u_2(w) = \alpha e^{rw} + \beta e^{sw} + \gamma e^{tw}$$

$$\begin{aligned} v(w) &= e^{(r+s)w}(a\beta - \alpha b)(r-s) \\ &\quad + e^{(r+t)w}(a\gamma - \alpha c)(r-t) \\ &\quad + e^{(s+t)w}(b\gamma - \beta c)(s-t). \end{aligned}$$

We do not consider the auxiliary solution with complex conjugate roots because it does not yield admissible forms for u_1 and u_2.

The following restrictions are required for v in I, II and III so that it satisfies the necessary (24) or (25):

I. $a\beta - \alpha b = 0$
II. $a\beta - \alpha b = 0$, or $a\delta - \alpha d = 0$
III. $a\beta - \alpha b = 0$, or $a\gamma - \alpha c = 0$, or $b\gamma - \beta c = 0$.

We complete the proof of the C3 forms in Theorem 3 by showing how u_1 and u_2 of I–III reduce to C3(i)–(iv) under the preceding restrictions.

I. Given u_1 and u_2 for I and $a\beta = \alpha b$, we have

$$\begin{aligned} &u_1(\tilde{x})u_2(\tilde{y}) - u_1(\tilde{y})u_2(\tilde{x}) \\ &\quad = E(\tilde{x}^2 e^{d\tilde{x}})Ee^{d\tilde{y}}[a\gamma - \alpha c] \\ &\qquad - E(\tilde{y}^2 c^{d\tilde{y}})Ee^{d\tilde{x}}[a\gamma - \alpha c] \\ &\qquad + E(\tilde{x}e^{d\tilde{x}})Ee^{d\tilde{y}}[b\gamma - \beta c] \\ &\qquad - E(\tilde{y}e^{d\tilde{y}})Ee^{d\tilde{x}}[b\gamma - \beta c] \\ &\quad = \left[E(\tilde{x}^2 e^{d\tilde{x}})(a\gamma - \alpha c) + E(\tilde{x}e^{d\tilde{x}})(b\gamma - \beta c)\right]Ee^{d\tilde{y}} \\ &\qquad - \left[E(\tilde{y}^2 e^{d\tilde{y}})(a\gamma - \alpha c) + E(\tilde{y}e^{d\tilde{y}})(b\gamma - \beta c)\right]Ee^{d\tilde{x}}. \end{aligned}$$

With $\alpha_1 = a\gamma - \alpha c$ and $\alpha_2 = b\gamma - \beta c$, precisely the same conclusion obtains when we redefine u_1 and u_2 by

$$u_1(w) = \left(\alpha_1 w^2 + \alpha_2 w\right)e^{dw}$$

$$u_2(w) = e^{dw},$$

which give the forms in C3(i) of Theorem 3.

II. Given u_1 and u_2 for II, we have

$$\begin{aligned} u_1(\tilde{x})u_2(\tilde{y}) - u_1(\tilde{y})u_2(\tilde{x}) = {} & (a\beta - \alpha b)\left[E(\tilde{x}e^{c\tilde{x}})Ee^{c\tilde{y}} - E(\tilde{y}e^{c\tilde{y}})Ee^{c\tilde{x}}\right] \\ & +(a\delta - \alpha d)\left[E(\tilde{x}e^{c\tilde{x}})Ee^{f\tilde{y}} - E(\tilde{y}e^{e\tilde{y}})Ee^{f\tilde{x}}\right] \\ & +(b\delta - \beta d)\left[Ee^{c\tilde{x}}Ee^{f\tilde{y}} - Ee^{c\tilde{y}}Ee^{f\tilde{x}}\right]. \end{aligned}$$

when $a\beta = \alpha b$, this can be written as

$$\left[c_1 E(\tilde{x}e^{c\tilde{x}}) + c_2 Ee^{c\tilde{x}}\right]Ee^{f\tilde{y}} - \left[c_1 E(\tilde{y}e^{c\tilde{y}}) + c_2 Ee^{c\tilde{y}}\right]Ee^{f\tilde{x}},$$

which is the form for $u_1(\tilde{x})u_2(\tilde{y}) - u_1(\tilde{y})u_2(\tilde{x})$ obtained when we redefine u_1 and u_2 by

$$u_1(w) = (c_1 w + c_2)e^{cw}$$

$$u_2(w) = e^{fw},$$

as in C3(iv) of Theorem 3.

When $a\delta = \alpha d$, the initial expression can be written as

$$\left[c_0 E(\tilde{x}e^{c\tilde{x}}) - c_2 Ee^{f\tilde{x}}\right]Ee^{c\tilde{y}} - \left[c_0 E(\tilde{y}e^{c\tilde{y}}) - c_2 Ee^{f\tilde{y}}\right]Ee^{c\tilde{x}},$$

which is the form for $u_1(\tilde{x})u_2(\tilde{y}) - u_1(\tilde{y})u_2(\tilde{x})$ obtained when we redefine u_1 and u_2 by

$$u_1(w) = c_0 we^{cw} - c_2 e^{fw}$$

$$u_2(w) = e^{cw},$$

as in C3(iii) of Theorem 3.

III. Given u_1 and u_2 for III, it suffices from the symmetry of terms to suppose that $a\beta = \alpha b$. Then

$$\begin{aligned} u_1(\tilde{x})u_2(\tilde{y}) - u_1(\tilde{y})u_2(\tilde{x}) = {} & \left(\beta_1 Ee^{r\tilde{x}} + \beta_2 Ee^{s\tilde{x}}\right)Ee^{t\tilde{y}} \\ & -\left(\beta_1 Ee^{r\tilde{y}} + \beta_2 Ee^{s\tilde{y}}\right)Ee^{t\tilde{x}}, \end{aligned}$$

which is the same as the form obtained when we redefine u_1 and u_2 by

$$u_1(w) = \beta_1 e^{rw} + \beta_2 e^{sw}$$

$$u_2(w) = e^{tw}$$

as in C3(ii) of Theorem 3.

6. Contextual uncertainty under WLU and SSBU

We conclude our proofs with results for WLU and SSBU that involve C4. We begin under the conditions of Theorem 3.

Proof for WLU that C4 ⇒ C3. Let

$$V(w) = Eu_1(w+\tilde{x})Eu_2(w+\tilde{y}) - Eu_1(w+\tilde{y})Eu_2(w+\tilde{x}).$$

If w is itself uncertain, say $\tilde{w} = (w_1, p, w_2)$, then

$$\begin{aligned} V(\tilde{w}) = {} & pV(w_1) + (1-p)V(w_2) + p(1-p) \\ & \times\big[\big(Eu_1(w_1+\tilde{x}) - Eu_1(w_2+\tilde{x})\big)\big(Eu_2(w_2+\tilde{y}) - Eu_2(w_1+\tilde{y})\big) \\ & \quad + \big(Eu_1(w_1+\tilde{y}) - Eu_1(w_2+\tilde{y})\big)\big(Eu_2(w_1+\tilde{x}) - Eu_2(w_2+\tilde{x})\big)\big]. \end{aligned}$$

So long as $V(w_2) \neq 0$, $V(\tilde{w})$, $(1-p)V(w_2)$ and $pV(w_1) + (1-p)V(w_2)$ will have the same sign for sufficiently small p. This fact allows us to adapt the argument used to show that C4 ⇒ C3 for EU in Bell (1995b, p. 1149). That proof made use of the fact that, for EU at least, $V(\tilde{w}) = pV(w_1) + (1-p)V(w_2)$, but in fact the proof only requires that $V(\tilde{w})$ and $pV(w_1) + (1-p)V(w_2)$ have the same sign for small p. Hence Bell's argument also applies to WLU. Thus C4 ⇒ C3.

Proof of Theorem 3 for C4. Recall that four WLU families satisfy C3. The following table sets up an analogy between these families and closely related EU families.

	WLU Family	EU Family
(i)	$u_1 = ae^{bw} + ce^{dw},\ u_2 = e^{fw}$	$u = ae^{bw} + ce^{dw}$
(ii)	$u_1 = awe^{bw} + ce^{dw},\ u_2 = e^{bw}$	$u = ae^{bw} + ce^{dw}$
(iii)	$u_1 = (aw+b)e^{cw},\ u_2 = e^{dw}$	$u = (aw+b)e^{cw}$
(iv)	$u_1 = (aw^2+bw)e^{cw},\ u_2 = e^{cw}$	$u = aw^2 + bw.$

To begin, consider a WLU function with a general u_1 but specific u_2, namely $u_2 = e^{cw}$. To satisfy C4, we must show that the relative preference between $x + \tilde{z} + k\tilde{z} + \tilde{x}$ and $w + \tilde{z} + k\tilde{z} + \tilde{y}$ cannot depend on the particular resolution of $\tilde{z}$ if it does not depend on the particular resolution of $k\tilde{z}$. The difference in their expected utilities is

$$Eu_1(w + \tilde{z} + k\tilde{z} + \tilde{x})e^{cw}Ee^{c\tilde{y}}Ee^{c\tilde{z}}Ee^{ck\tilde{z}}$$
$$- Eu_1(w + \tilde{z} + k\tilde{z} + \tilde{y})e^{cw}Ee^{c\tilde{x}}Ee^{c\tilde{z}}Ee^{ck\tilde{z}},$$

which has the same sign as

$$Eu_1(w + \tilde{z} + k\tilde{z} + \tilde{x})Ee^{c\tilde{y}} - Eu_1(w + \tilde{z} + k\tilde{z} + \tilde{y})Ee^{c\tilde{x}}. \tag{27}$$

Since all the C3 families have u_2 exponential, it will suffice to think about the properties of (27) when $\tilde{z}$ or $k\tilde{z}$ is resolved.

Case (i). When $u_1 = ae^{bw} + ce^{dw}$, equation (27) becomes

$$ae^{bw}Ee^{b\tilde{z}}Ee^{bk\tilde{z}}Ee^{b\tilde{x}}Ee^{f\tilde{y}} + ce^{dw}Ee^{d\tilde{z}}Ee^{dk\tilde{z}}Ee^{d\tilde{x}}Ee^{f\tilde{y}}$$
$$- ae^{bw}Ee^{b\tilde{z}}Ee^{bk\tilde{z}}Ee^{b\tilde{y}}Ee^{f\tilde{x}} - ce^{dw}Ee^{d\tilde{z}}Ee^{dk\tilde{z}}Ee^{d\tilde{y}}Ee^{f\tilde{x}}.$$

This may be written as

$$ae^{bw}Ee^{b\tilde{z}}Ee^{bk\tilde{z}}A + ce^{dw}Ee^{d\tilde{z}}Ee^{dk\tilde{z}}B, \tag{28}$$

where A and B are functions of $\tilde{x}$ and $\tilde{y}$ but not of $\tilde{z}$ or $k\tilde{z}$. Note what happens when we consider C4 for the EU family $ae^{bw} + ce^{dw}$. The difference in expected utility between $w + \tilde{z} + k\tilde{z} + \tilde{x}$ and $w + \tilde{z} + k\tilde{z} + \tilde{y}$ is

$$ae^{bw}Ee^{b\tilde{z}}Ee^{bk\tilde{z}}C + ce^{dw}Ee^{d\tilde{z}}Ee^{dk\tilde{z}}D, \tag{29}$$

where C and D are functions of $\tilde{x}$ and $\tilde{y}$ but not of $\tilde{z}$ or $k\tilde{z}$. Although $A \neq C$ and $B \neq D$, it is the case that if for some $\tilde{z}$ and $k\tilde{z}$ there is a counterexample to C4 for some choice of $\tilde{x}$ and $\tilde{y}$ in (28), then there will also exist a counterexample for the same $\tilde{z}$ and $k\tilde{z}$ in (29), albeit with a different $\tilde{x}$ and $\tilde{y}$, but chosen so that $AD = BC$.

Now $ae^{bw} + ce^{dw}$ satisfies C4 for EU if and only if $bd \leq 0$. Hence $u_1 = ae^{bw} + ce^{dw}$, $u_2 = e^{fw}$ satisfies C4 for WLU if and only if $bd \leq 0$.

Case (ii). When $u_1 = awe^{bw} + ce^{dw}$ and $u_2 = e^{bw}$, expression (27) is

$$E\left[a(w + \tilde{z} + k\tilde{z} + \tilde{x})e^{bw}e^{b\tilde{z}}e^{bk\tilde{z}}e^{b\tilde{x}}\right]Ee^{b\tilde{y}} + ce^{dw}Ee^{d\tilde{z}}Ee^{dk\tilde{z}}Ee^{d\tilde{x}}Ee^{b\tilde{y}}$$
$$- E\left[a(w + \tilde{z} + k\tilde{z} + \tilde{y})e^{bw}e^{b\tilde{z}}e^{bk\tilde{z}}e^{b\tilde{y}}\right]Ee^{b\tilde{x}}$$
$$- ce^{dw}Ee^{d\tilde{z}}Ee^{dk\tilde{z}}Ee^{d\tilde{y}}Ee^{b\tilde{x}},$$

which may be written as

$$ae^{bw}Ee^{b\tilde{z}}Ee^{bk\tilde{z}}A + ce^{dw}Ee^{d\tilde{z}}Ee^{dk\tilde{z}}B. \tag{30}$$

As before, A and B depend on $\tilde{x}$ and $\tilde{y}$ but not on $\tilde{z}$ or $k\tilde{z}$. Expression (30) is analogous to (29), which was derived from the EU family $ae^{bw} + ce^{dw}$. Thus $u_1 = awe^{bw} + ce^{dw}$, $u_2 = e^{bw}$ satisfies C4 if and only if $bd \le 0$.

Case (iii). When $u_1 = (aw + b)e^{cw}$ and $u_2 = e^{dw}$, (27) is

$$E\Big[\big[a(w + \tilde{z} + k\tilde{z} + \tilde{x}) + b\big]e^{cw}e^{c\tilde{z}}e^{ck\tilde{z}}e^{c\tilde{x}}\Big]Ee^{d\tilde{y}}$$

$$- E\Big[\big[a(w + \tilde{z} + k\tilde{z} + \tilde{y}) + b\big]e^{cw}e^{c\tilde{z}}e^{ck\tilde{z}}e^{c\tilde{y}}\Big]Ee^{d\tilde{x}},$$

or

$$E\Big[\big[a(w + \tilde{z} + k\tilde{z}) + b\big]e^{cw}e^{c\tilde{z}}e^{ck\tilde{z}}\Big]A + ae^{cw}Ee^{c\tilde{z}}Ee^{ck\tilde{z}}B, \tag{31}$$

where A and B depend on $\tilde{x}$ and $\tilde{y}$ but not on $\tilde{z}$ or $k\tilde{z}$. In fact, $A = Ee^{c\tilde{x}}Ee^{d\tilde{y}} - Ee^{c\tilde{y}}Ee^{d\tilde{x}}$ and $B = E(\tilde{x}e^{c\tilde{x}})Ee^{d\tilde{y}} - E(\tilde{y}e^{c\tilde{y}})E(e^{d\tilde{x}})$.

If $c = d$ then (31) has constant sign for all resolutions of $\tilde{z}$ or $k\tilde{z}$, and so this case satisfies C4. It is, however, a special case of case (ii). The EU family $(aw + b)e^{cw}$ leads to an expression identical to (31) except that A is replaced by $C = Ee^{c\tilde{x}} - Ee^{c\tilde{y}}$ and B by $D = E(\tilde{x}e^{c\tilde{x}}) - E(\tilde{y}e^{c\tilde{y}})$. By analogy with this case, no cases with $c \neq d$ satisfy C4.

Case (iv). When $u_1 = (aw^2 + bw)e^{cw}$ and $u_2 = e^{cw}$, expression (27) is

$$E\Big[\Big(a(w + \tilde{z} + k\tilde{z} + \tilde{x})^2 + b(w + \tilde{z} + k\tilde{z} + \tilde{x})\Big)e^{cw}e^{c\tilde{z}}e^{ck\tilde{z}}e^{c\tilde{x}}\Big]Ee^{c\tilde{y}}$$

$$- E\Big[\Big(a(w + \tilde{z} + k\tilde{z} + \tilde{y})^2 + b(w + \tilde{z} + k\tilde{z} + \tilde{y})\Big)e^{cw}e^{c\tilde{z}}e^{ck\tilde{z}}e^{c\tilde{y}}\Big]Ee^{c\tilde{x}},$$

which may be written (after dividing by e^{cw}) as

$$(2aw + b)Ee^{c\tilde{z}}Ee^{ck\tilde{z}}E\big((\tilde{x} - \tilde{y})e^{c\tilde{x}}e^{c\tilde{y}}\big) + 2aE\big((\tilde{z} + k\tilde{z})e^{c\tilde{z}}e^{ck\tilde{z}}\big)$$

$$\times E\big((\tilde{x} - \tilde{y})e^{c\tilde{x}}e^{c\tilde{y}}\big)$$

$$+aEe^{c\tilde{z}}Ee^{ck\tilde{z}}E\big((\tilde{x}^2 - \tilde{y}^2)e^{c\tilde{x}}e^{c\tilde{y}}\big)$$

$$= Ee^{c\tilde{z}}Ee^{ck\tilde{z}}((2aw + b)A + aB)$$

$$+2aE\big((\tilde{z} + k\tilde{z})e^{c\tilde{z}}e^{ck\tilde{z}}\big)A, \tag{32}$$

where $A = E((\tilde{x} - \tilde{y})e^{c\tilde{x}}e^{c\tilde{y}})$ and $B = E((\tilde{x}^2 - \tilde{y}^2)e^{c\tilde{x}}e^{c\tilde{y}})$. Comparing this to $aw^2 + bw$ in EU, we find that

$$V(w + \tilde{x} + \tilde{z} + k\tilde{z}) - V\big(w + \tilde{y} + \tilde{z} + k\tilde{z}\big)$$

$$= \big[(2aw + b)C + aD\big] + 2a(k + 1)E(\tilde{z})C, \tag{33}$$

where $C = E(\tilde{x} - \tilde{y})$ and $D = E(\tilde{x}^2 - \tilde{y}^2)$. Though the expressions are close, they are no longer identical but for A, B, C, D. While (33) is, in effect, independent of $\tilde{z}$ and $k\tilde{z}$ (there is no loss in assuming $E(\tilde{z}) = 0$) and so all quadratics satisfy C4 in EU, we can show that (32) fails C4 for all c (except $c = 0$, or $a = 0$). To see this, rewrite (32) as

$$(2aw + b)A + aB + 2aAZ, \tag{34}$$

where

$$Z = \frac{E\left[(\tilde{z} + k\tilde{z})e^{c\tilde{z}}e^{ck\tilde{z}}\right]}{Ee^{c\tilde{z}}Ee^{ck\tilde{z}}}.$$

If $a = 0$, then (34) is independent of the $\tilde{z}$'s and trivially satisfies C4. If $a \neq 0$, then the value of the terms $(2aw + b)A + aB$ may be selected at will by appropriate choice of w (when $A \neq 0$). There is no loss in assuming $aA > 0$ ($\tilde{x}$ and $\tilde{y}$ may be reversed at will). In order then to ensure that (34) is compatible with C4, it must be that the extremes of Z occur when $k\tilde{z}$ is resolved (as $kz_{\min}$ and $kz_{\max}$) rather than when $\tilde{z}$ is resolved (as $z_{\min}$ or $z_{\max}$). We require then, that

$$\frac{(E\tilde{z}e^{c\tilde{z}})e^{ckz_{\min}} + (Ee^{c\tilde{z}})kz_{\min}e^{ckz_{\min}}}{Ee^{c\tilde{z}}e^{ckz_{\min}}} < \frac{E(k\tilde{z}e^{ck\tilde{z}})e^{cz_{\min}} + z_{\min}e^{cz_{\min}}(Ee^{ck\tilde{z}})}{e^{cz_{\min}}Ee^{ck\tilde{z}}}$$

or

$$\frac{E(\tilde{z}e^{c\tilde{z}})}{Ee^{c\tilde{z}}} + kz_{\min} < \frac{kE(\tilde{z}e^{ck\tilde{z}})}{Ee^{ck\tilde{z}}} + z_{\min}.$$

Similarly, we require

$$\frac{E(\tilde{z}e^{c\tilde{z}})}{Ee^{c\tilde{z}}} + kz_{\max} > \frac{k(E\tilde{z}e^{ck\tilde{z}})}{Ee^{ck\tilde{z}}} + z_{\max}.$$

Substitute $\tilde{z} = (1, \frac{1}{2}, -1)$ to get the requirements

$$\frac{e^c - e^{-c}}{e^c + e^{-c}} - k < \frac{k(e^{ck} - e^{-ck})}{(e^{ck} + e^{-ck})} - 1$$

and

$$\frac{e^c - e^{-c}}{e^c + e^{-c}} + k > k\left(\frac{e^{ck} - e^{-ck}}{e^{ck} + e^{-ck}}\right) + 1,$$

so that

$$1 - k < \frac{k(e^{ck} - e^{-ck})}{(e^{ck} + e^{-ck})} - \frac{(e^{c} - e^{-c})}{(e^{c} + e^{-c})} < k - 1.$$

If $c > 0$, then as k gets large the right-hand inequality is violated. If $c < 0$, the left-hand inequality is violated. Thus $u_1 = (aw^2 + bw)e^{cw}$ and $u_2 = e^{cw}$ satisfies WLU only in trivial cases that are already covered.

This completes the proof of Theorem 3. We conclude with the completion of Theorem 4.

C4 for SSBU. Assume the hypotheses of Theorem 4 along with C3. The argument in Bell (1995b) for EU carries over precisely for SSBU, and so we omit the details here. In outline, we first show that if C3 is false then so is C4. One can also show that if $V(w) = E\phi(w + \tilde{x}, w + \tilde{y})$ is zero for exactly one value of w, but is not strictly monotonic (or equal to zero everywhere), then there exists a violation of C4. Thus C4 for SSBU is equivalent to the requirement that $V(w)$ is strictly monotonic (or constant) whenever it has at least one zero. (There are no restrictions on V if it never changes sign.)

It remains to establish which SSBU families have monotonic V functions. The forms for C3 in the theorem imply

(i) $V(w) = a_1 + b_1 w$

(ii) $V(w) = a_2 e^{aw} + b_2 e^{bw}$

(iii) $V(w) = a_3 + b_3 e^{aw}$

(iv) $V(w) = (a_4 w + b_4)e^{aw}$

where the a_j and b_j do not depend on w. Cases (i) and (iii) satisfy C4 without further restriction.

For (ii) we have $a_2 = E[\alpha(\tilde{x} - \tilde{y})(e^{a\tilde{x}} - e^{a\tilde{y}})]$ and $b_2 = E[\beta(\tilde{x} - \tilde{y})(e^{b\tilde{x}} - e^{b\tilde{y}})]$. If $\alpha \equiv 0$ or $\beta \equiv 0$ or $ab = 0$, (ii) reduces to (iii). Otherwise, (ii) satisfies C4 if and only if $ab < 0$.

For (iv) we have $a_4 = E[\alpha(\tilde{x} - \tilde{y})(e^{a\tilde{x}} - e^{a\tilde{y}})]$ and $b_4 = E[\alpha(\tilde{x} - \tilde{y})(\tilde{x}e^{a\tilde{x}} - \tilde{y}e^{a\tilde{y}})] + E[\beta(\tilde{x} - \tilde{y})(e^{a\tilde{x}} - e^{a\tilde{y}})]$. If either $\alpha \equiv 0$ or $a = 0$, (iv) reduces to (iii). Otherwise, it does not satisfy C4.

This completes the proof of Theorem 4.

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