1 About cardinalities in Type Theories

The usual definition of "cardinality order" in set-theory is the following: For two sets A, B we write $A \leq_c B$ iff there is an injective function $f : A \to B$ and we write $A =_c B$ iff there is a bijective function $f : A \to B$.

In type theory there are two things to which we'd like to compare by cardinality: types and ensembles[1]. As is usual for such situations, we'd like to define concepts or prove theorems only for one case and use some well-behaved transfer principles between the two.

One apprach would be to define cardinality-order and -equality for types completely analogously to the definitions in set-theory. Then define the cardinality of an ensemble $P : A \to \mathsf{Type}$ as the cardinality of its sigma-type $\{a : A \mid Pa\}$.

This approach only "works well" if we have "proof irrelevance" for P. I.e. we need to have for all $a : A$ and proofs $\phi, \psi : Pa$ that $\phi = \psi$. Otherwise the sigma-type might even have a bigger cardinality than A had. Proof-irrelevance very often doesn't hold in type theories but can be added using axioms.

An axiom-free definition, but with a bit more bookkeeping is the following: Let A, B be types and $P : A \to \mathsf{Type}$, $Q : B \to \mathsf{Type}$ be ensembles on A resp. on B. Let $f : \{a : A \mid Pa\} \to \{b : B \mid Qb\}$ be a function between the sigma-types of P and Q. Let π_A be the canonical projection from $\{a : A \mid Pa\}$ to A and π_B be the canonical projection from $\{b : B \mid Qb\}$ to B. We'll call f *relatively injective* if for all $a_0, a_1 : \{a : A \mid Pa\}$, $\pi_B(f(a_0)) = \pi_B(f(a_1)) \Rightarrow \pi_A a_0 = \pi_A a_1$. We'll call f *relatively surjective* if for all $b_0 : \{b : B \mid Qb\}$ there exists some $a_0 : \{a : A \mid Pa\}$ such that $\pi_B b_0 = \pi_B(f(a_0))$. We'll call f *relatively bijective* if it is both relatively injective and relatively surjective.

We now write $P \leq_c Q$ if there exists a relatively injective function $f : \{a : A \mid Pa\} \to \{b : B \mid Qa\}$. We write $P =_c Q$ if there exists a relatively bijective function $f : \{a : A \mid Pa\} \to \{b : B \mid Qa\}$.

Now we can pull these relations back to types. For any type A let $F_A : A \to Type$ be the "full ensemble" i.e. function mapping $a \mapsto \top$ where $\top$ is some inhabited type (corresponding to the truth value "true"). Then for types A, B define $A \leq_c B$ iff $F_A \leq_c F_B$.

Using the natural map $A \hookrightarrow \{a : A \mid F_A a\}$ we can prove helper-lemmas and recover the "classical" definitions given initially. This way we avoided proof irrelevance for arguing about cardinalities.

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[1]Following the nomenclature of the Coq std. library. They are sometimes called "sets" (for example in the math-comp library) or "predicates". For a type A, ensembles of A have type $A \to \mathsf{Type}$ or $A \to \mathsf{Prop}$ depending on the formalism.