Logical Relations for MMT/LF

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Abstract. Logical relations are an established proof method for deriving meta-level theorems of formal systems. Common areas of application include proofs of *free theorems*, strong normalization, and subject reduction. Moreover, they have been used for reasoning about program equivalences and correctness of transformation algorithms (e.g., optimizations). Theories of logical relations have been stated for a wide range of formal systems including the simply typed lambda calculus, System F, F_ω (with various extensions), the Calculus of Constructions, and – subsuming the former – pure type systems.
In **RS:logrels:12**, Rabe and Sojakova give another very general definition of logical relations targetting formal systems that have been encoded via signatures and signature morphisms in a logical framework over a dependent type theory. They define an n-ary logical relation from signatures S to T to be a family of relations indexed by types in S over n interpretations given by signature morphisms $\mu_1, \ldots \mu_n : S \to T$.
In this paper, we work out the details for the Edinburgh LF Framework (LF) as realized in MMT in the special cases of $n = 1, 2$. While we do presuppose familarity with formalizations in LF, our presentation is mainly driven by concrete examples that are accessible without prior knowledge in the meta theory of type theories.

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1 Introduction

"It's a specific class of inductive arguments. Induction on terms with a different IH for every base type. Many meta theorems can be seen a LRs. So formalizing and internalizing them is a big deal." (source: Florian)

Motivation

Contribution

Related Work

2 Preliminaries

0-ary logical relations are morphisms!

MMT/LF

Prop. Logic in MMT/LF

3 Logical Relations over Identity Morphisms

3.1 Example I: Deriving $\vdash p \vee \neg p$ for all p in Prop. Logic

Definition 1 (Unary Logical Relation on Identity). *Let T be a theory and $(r_c)_{c \in T}$ a family of T-expressions. Define a mapping $r(-)$ on T-contexts and terms by:*

$$
\begin{aligned}
r(\cdot) &= \cdot \\
r(\Gamma,\ x\colon A) &= r(\Gamma),\ x\colon A,\ x^*\colon r(A)\ x \\[4pt]
r(\mathtt{type}) &= \lambda A\colon \mathtt{type}.\ A \to \mathtt{type} \\
r(\Pi x\colon A.\ B) &= \lambda f\colon (\Pi x\colon A.\ B).\ \Pi x\colon A.\ \Pi x^*\colon r(A)\ x.\ r(B)\ (f\ x) \\
r(A\ B) &= r(A)\ B\ r(B) \\
r(\lambda x\colon A.\ B) &= \lambda x\colon A.\ \lambda x^*\colon r(A)\ x.\ r(B) \\
r(c) &= r_c \\
r(x) &= x^*
\end{aligned}
$$

*We call r a **unary logical relation on** id_T, written $r\colon\ id_T\colon T \to T$, if $\vdash_T r_c\colon r(\tau)\ c$ for all constants $c\colon \tau$ in T.*

3.2 Example II: Showing $\Leftrightarrow$ is a Congruence in Prop. Logic

Definition 2 (Binary Logical Relation on Identity). *Let T be a theory and $(r_c)_{c \in T}$ a family of T-expressions. Define a mapping $r(-)$ on T-contexts and terms by:*

$$
\begin{aligned}
r(\cdot) &= \cdot \\
r(\Gamma,\ x\colon A) &= r(\Gamma),\ x\colon A,\ x'\colon A',\ x^*\colon r(A)\ x\ x' \\[4pt]
r(\mathtt{type}) &= \lambda A\colon \mathtt{type}.\ \lambda A'\colon \mathtt{type}.\ A \to A' \to \mathtt{type} \\
r(\Pi a\colon A.\ B) &= \lambda f\colon (\Pi a\colon A.\ B).\ \lambda f'\colon (\Pi a'\colon A'.\ B'). \\
&\qquad\quad \Pi x\colon A.\ \Pi x'\colon A'.\ \Pi x^*\colon r(A)\ x\ x'. \\
&\qquad\qquad r(B)\ (f\ x)\ (f'\ x') \\
r(A\ B) &= r(A)\ B\ B'\ r(B) \\
r(\lambda x\colon A.\ B) &= \lambda x\colon A.\ \lambda x'\colon A'.\ \lambda x^*\colon r(A)\ x\ x'.\ r(B) \\
r(c) &= r_c \\
r(x) &= x^*
\end{aligned}
$$

*where A' denotes systematic priming. We call r a **binary logical relation on** $id_T \times id_T$, written $r\colon\ id_T \times id_T\colon T \to T$, if $\vdash_T r_c\colon r(\tau)\ c\ c$ for all constants $c\colon \tau$ in T.*

4 Logical Relations over Non-Trivial Morphisms

4.1 Example III: Deriving Type Preservation during Type Erasure

Definition 3 (Unary Logical Relation Along Morphism). *Let S,T be theories, $\mu\colon S \to T$ a morphism, and $(r_c)_{c\in S}$ a family of T-expressions indexed by constants in S. Define a mapping $r(-)$ from S-contexts and terms to T-contexts and terms by:*

$$\begin{aligned}
r(\cdot) &= \cdot \\
r(\Gamma,\ x\colon A) &= r(\Gamma),\ x\colon \mu(A),\ x^*\colon r(A) \\
r(\texttt{type}) &= \lambda A\colon \texttt{type}.\ A \to \texttt{type} \\
r(\Pi a\colon A.\ B) &= \lambda f\colon \mu(\Pi a\colon A.\ B).\ \Pi x\colon \mu(A).\ \Pi x^*\colon r(A)\ x. \\
&\qquad r(B)\ (f\ x)\ (f'\ x') \\
r(A\ B) &= r(A)\ \mu(B)\ r(B) \\
r(\lambda x\colon A.\ B) &= \lambda x\colon \mu(A).\ \lambda x^*\colon r(A)\ x.\ r(B) \\
r(c) &= r_c \\
r(x) &= x^*
\end{aligned}$$

where A' denotes systematic priming. We call r a ***unary logical relation on*** *μ, written $r\colon\ \mu\colon\ \to$, if $\vdash_T r_c\colon r(\tau)\ \mu(c)$ for all constants $c\colon \tau$ in S.*

5 Evaluation

Adequacy as a Language Feature (Instead of Diag Op) versus just generating the interface theory via diag. op

Inaccessibility of the Basic Lemma from within the Formalization can we access it from within the formalization? probably not, right? then we cannot build upon the theorems we proved in the formalization itself

6 Conclusion

conclusion...

Future Work [**sharoda_operators:20**]