Comparison of Introductory Zero-Knowledge Proof Examples

By Navid Roux orcid.org/0000-0002-8348-2441, 2020-06-10.

	Sudoku	3-COL	Hamiltonian Cycle	Any "hard" Graph Property	Discrete Log (variant)	Discrete Log (Schnorr variant)
Statement public or sent from P to V	Partial sudoku Ψ is solvable	Graph G is 3-colorable	Graph G is Hamiltonian[1]	Let $L \in \mathsf{NP}$ be any graph-isomorphism-invariant graph property believed to be hard.[3] Graph $G \in L$	Let $\mathbb{G}$ of order q and $y \in \mathbb{Z}_q$ be fixed. I know $x \in \mathbb{Z}_q$ such that $[x] = y$	Let $\mathbb{G}$ of order q and $y \in \mathbb{G}$ be fixed. I know $x \in \mathbb{Z}_q$ such that $[x] = y$
Witness only known to P	Solution $\overline{\Psi}$	3-coloring w	Hamiltonian cycle w	Certificate w	x	x
Iteration						
1. Rerandomization by P of 1. Problem Statement 2. Solution	Pick set isomorphism $i\colon \{1,\ldots,9\} \to \{1,\ldots,9\}$ 1. $\Psi' := i[\Psi]$ is solvable 2. $\overline{\Psi}' := i[\overline{\Psi}]$ is solution to Ψ'	Pick color permutation $i\colon \{1,2,3\} \to \{1,2,3\}$ 1. -/-(choose same graph G) 2. $w' := i[w]$ is alternative 3-coloring for G	Pick graph isomorphism $i\colon G \to G'$ (just relabel vertices) 1. $G' := i[G]$ is Hamiltonian 2. $w' := i[w]$ is Hamiltonian cycle for G'[2]	Pick graph isomorphism $i\colon G \to G'$ (just relabel vertices) 1. $G' := i[G] \in L$ 2. $w' := i[w]$ is certificate for G'	Pick $r \leftarrow \mathbb{Z}_q$ uniformly at random	Pick $r \leftarrow \mathbb{Z}_q$ uniformly at random
2. Commitment by P	Send all of • $(\mathrm{com}(\overline{\Psi}'_{j,k}))_{1\leq j,k\leq 9}$ • $\mathrm{com}(\Psi')$	Send all of • G • $(\mathrm{com}(w'_v))_{v\in V(G)}$ – coloring of each vertex	Send all of • G • G' • $\mathrm{com}(i)$ • $\mathrm{com}(w')$	Send all of • G • G' • $\mathrm{com}(i)$ • $\mathrm{com}(w')$	Send all of • $\hat{g} := [r]$ • $\mathrm{com}(r)$	Send $[r]$
3. Pose Challenge by V	Ask for one of • the nine permuted rows • the nine permuted columns • the nine permuted squares • permuted statement In total, this gives $9+9+9+1 = 28$ challenge types	Pick edge $(u,v) \leftarrow E(G)$ uniformly at random and ask for coloring of u and v	Ask for one of • isomorphism $i\colon G \to G'$ • Hamiltonian cycle w' in G'	Ask for one of • isomorphism $i\colon G \to G'$ • certificate w' for $G' \in L$	Ask for one of • r • $x + r$ and denote response by resp.	Pick $c \leftarrow \mathbb{Z}_q$ uniformly at random. Ask for $cx + r$ and denote response by resp.
4. Respond to challenge by P	canonical: respond exactly with what was asked					
5. Verify response by V	Check • that no numbers occur twice in row, column, or square, • or that the permuted statement is in fact a permutation	Check that coloring of u and v are distinct	Check • conditions on isomorphism, • or check that cycle is indeed Hamiltonian	Check • conditions on isomorphism, • or check that certificate is valid	Check • that indeed $\hat{g} = [\text{resp.}]$ • that $[\text{resp.}] = y + \hat{g}$ namely if indeed $[x] = y$, then $y + \hat{g} = [x] + [r] = [x + r] = [\text{resp.}]$	Check that $[\text{resp.}] = cy + [r]$
Completeness P can convince V in case P actually had a solution	Since step 4 above is canonical, provers can convince with prob. of 1					
Soundness P cannot convince V for statements not in the language. Shown are the success prob. of still trying to do so	$\left(\frac{27}{28}\right)^{\#\text{iter}}$	$\left(\frac{\lvert E(G)\rvert - 1}{\lvert E(G)\rvert}\right)^{\#\text{iter}}$	$\left(\frac{1}{2}\right)^{\#\text{iter}}$	$\left(\frac{1}{2}\right)^{\#\text{iter}}$	-/-	-/-
Soundness of Knowledge P cannot convince V without having a witness. Shown are the success prob. of still trying to do so	-/-	-/-	-/-	-/-	$\left(\frac{1}{2}\right)^{\#\text{iter}}$	$\left(\frac{1}{2}\right)^{\#\text{iter}}$
Zero Knowledge V doesn't learn anything about the witness	In each round, V learns *either* a Sudoku "building block" (row, column, square) of the permuted solution *or* the permuted solution statement. The second case is obviously useless. In the first case, V only learns what the round's permutation i does on the numbers that the original puzzle Ψ had already prefilled in the corresponding row, column, or square. In particular, nothing is learned about the solution entries, i.e. $\overline{\Psi}' \setminus \Psi'$, due to the Sudoku property of every number (mapping) occurring exactly once in such a building block.	In each round, V just learns the *round-dependent* colorings of two nodes: w'_u and w'_v. This is useless information as such and can furthermore – due to the rerandomization – not be correlated with colorings learnt in other rounds. Note that if V asked instead for colorings of three vertices, then the learned colorings could very well possess more information content. Namely, it isn't granted anymore that all three vertices must have pairwise distinct colors attached to them.	In each round, V learns *either* a useless isomorphism *or* a Hamiltonian cycle in $G' \cong G$. Since the graph isomorphism problem is believed to be hard, learning about such a cycle in G' without learning the isomorphism is useless as well.	Same argument as in the cell to the left.	In each round, V learns *either* a useless random r *or* $x + r$. In the latter case, however, since $r \sim \mathcal{U}(\mathbb{Z}_q)$, we also have $(x + r) \sim \mathcal{U}(\mathbb{Z}_q)$[4]	In each round, V only learns $[r]$ and $cx + r$ for a c chosen by them. Due to DLOG assumed to be hard in $\mathbb{G}$, in the eyes of V we have $r \sim \mathcal{U}(\mathbb{Z}_q)$ and hence $(cx + r) \sim \mathcal{U}(\mathbb{Z}_q)$.[4]

1 This means it contains a so-called Hamiltonian cycle that is a path visiting every node exactly once. The problem of finding such a cycle is NP-complete.
2 Here, v is effectively a sequence of edges, on which the isomorphism is applied elementwise.
3 Take for example HAMILTONIAN, 3-COL, or CLIQUE. From $L \in \mathsf{NP}$ it follows that for every $G \in L$ there is a certificate w for membership of length $\mathrm{poly}(\lvert G\rvert)$ that can be verified in $\mathrm{poly}(\lvert G\rvert)$ time.
By graph-isomorphism invariance we demand that for $G \cong G'$ witnessed by an isomorphim $i\colon G \to G'$, certificates w for $G \in L$ can be transformed to certificates w' for $G' \in L$. We denote the latter by $i[w]$.
4 This is a simple lemma holding for arbitrary groups. The security of the OTP is based on this, usually phrased in the language of the group $(\{0,1\}^n, \oplus)$.

Useful Links

- Sudoku (slightly different challenges are given, though)
  - https://manishearth.github.io/blog/2016/08/10/interactive-sudoku-zero-knowledge-proof/
  - https://manishearth.github.io/sudoku-zkp/zkp.html
- 3-COL: [GMW91]
- Hamiltonian Cycle: [Wik20b], originally due to [Blu86]
- Any "hard" Graph Property: sketched on my own; [Blu86] describes this, too
- Discrete Log (variant): [Wik20a]
- Discrete Log (Schnott variant)
  - Lecture Notes by Prof. Schröder on "Privacy-Preserving Cryptocurrencies" (currently non-public; only accessible to students enrolled in their course)
  - [Sch90]

References

[Blu86] Manuel Blum. "How to Prove a Theorem So No One Else Can Claim It". In: *Proceedings of the International Congress of Mathematicians*. Berkeley, California, USA: Almquist & Wiksell, 1986, pp. 1444–1451. URL: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.469.9048&rep=rep1&type=pdf.

[GMW91] Oded Goldreich, Silvio Micali, and Avi Wigderson. "Proofs That Yield Nothing but Their Validity or All Languages in NP Have Zero-Knowledge Proof Systems". In: *J. ACM* 38.3 (July 1991), pp. 690–728. ISSN: 0004-5411. DOI: 10.1145/116825.116852. URL: https://doi.org/10.1145/116825.116852.

[Sch90] C. P. Schnorr. "Efficient Identification and Signatures for Smart Cards". In: *Advances in Cryptology — CRYPTO' 89 Proceedings*. Ed. by Gilles Brassard. New York, NY: Springer New York, 1990, pp. 239–252. ISBN: 978-0-387-34805-6.

[Wik20a] Wikipedia contributors. *Zero-knowledge proofs (Discrete log of a given value) — Wikipedia, The Free Encyclopedia*. [Online; accessed 2020-05-21]. 2020. URL: https://en.wikipedia.org/w/index.php?title=Zero-knowledge_proof&oldid=957331895#Discrete_log_of_a_given_value.

[Wik20b] Wikipedia contributors. *Zero-knowledge proofs (Hamiltonian cycle for a large graph)— Wikipedia, The Free Encyclopedia*. [Online; accessed 2020-05-21]. 2020. URL: https://en.wikipedia.org/w/index.php?title=Zero-knowledge_proof&oldid=957331895#Hamiltonian_cycle_for_a_large_graph.