Merging precomputation and scalar multiplication

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Outline

1. Introduction

2. Classic Algorithms

3. The art of recycling

4. Finding better sets of digits

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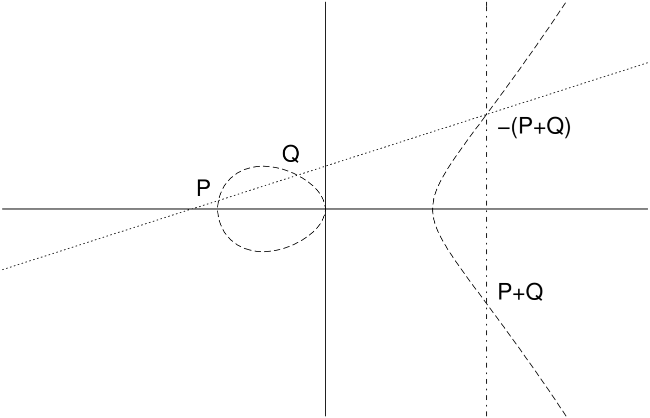
If you please ... draw me an elliptic curve (over $\mathbb{R}$)

group law of an elliptic curve over $\mathbb{R}$



Elliptic curve defined over a finite field

Definition

- $\mathbb{K}$ a characteristic p field (with $p > 3$)
- $P = (x, y) \in \mathbb{K}^2$ a point satisfaying $y^2 = x^3 + ax + b$, with $a, b \in \mathbb{K}$

Addition formulae

$P_1 = (x_1, y_1), P_2 = (x_2, y_2), P_3 = P_1 + P_2$

- $x_3 = (\frac{y_2 - y_1}{x_2 - x_1})^2 - x_1 - x_2$
- $y_3 = (\frac{y_2 - y_1}{x_2 - x_1})(x_1 - x_3) - y_1$

Point scalar multiplication

Point multiplication

- $k \in \mathbb{N}, P \in E(K)$
- $[k]P = \underbrace{P + \cdots + P}_{k \text{ fois}}$

Main issue

- Performing this operation as fast as possible

Three levels of arithmetic

- Finite field
- Point addition
- Scalar point multiplication

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Fast scalar multiplication

Double-and-add

- $k = \sum_{i=0}^{n-1} k_i 2^i$
- $n-1$ doublings and $n/2$ additions (on average)

Computation of 21P

- $k = 21 = 2^4 + 2^2 + 2^0 = 10101_2 = k_4 k_3 k_2 k_1 k_0$

Double-and-add

- Initialization: $Q \leftarrow P$
- $b_3 = 0 : Q \leftarrow 2Q \qquad (Q = 2P)$
- $b_2 = 1 : Q \leftarrow 2Q + P \qquad (Q = 5P)$
- $b_1 = 0 : Q \leftarrow 2Q \qquad (Q = 10P)$
- $b_0 = 1 : Q \leftarrow 2Q + P \qquad (Q = 21P)$

Non Adjacent Form (NAF)

- $k_i \in \{-1, 0, 1\}$
- $k = 31 = 11111_2 = 10000\overline{1}_{NAF}$
- at most n doublings and $\frac{n}{3}$ additions (on average)

Windows method: wNAF

- $|k_i| < 2^{w-1}$
- $k = 267 = 1\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1$
- 2-NAF: $1\ 0\ 0\ 0\ 1\ 0\ \overline{1}\ 0\ \overline{1}$
- 3-NAF: $1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 3$
- 4-NAF: $1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ \overline{5}$
- 5-NAF: $1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 11$

Method

- Compute and store $P, 3P, \ldots, 2^{w-1}P$
- Perform a Double-and-Add like scheme
- # of additions: $n/(w+1)$
- # of doublings: $n-w+1$

Going from wNAF to $(w+1)$NAF

- Twice as much precomputed points
- # of additions decreased by a factor $w/(w+1)$
- # of doublings decreased by 1!

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Simple example: $k = 10011k_{n-5}\dots k_0$

Using dictionary $\{1, 3, 5\}$

- recoding $k = 1003k_{n-5}\dots k_0$
- compute $2P, 3P, 5P$
- $(100)_2P = 8P, (1000)_2P = 16P, (1003)_2P = 19P$
- finish with double-and-add

Recycling approach

- compute $2P, 4P, 8P, 9P, 18P$ and $19P$ using double-and-add
- store $P, 9P, 19P$
- use dicionary $\{1, 9, 19\}$ during the recoding of k
- finish with double-and-add

Comparisons

Using dictionary $\{1, 3, 5\}$

- three point doublings and three point additions
- average density 2/9

Using dictionary $\{1, 9, 19\}$

- four point doublings and two point additions
- average density 2/9

We have traded a point addition for a doubling!

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Finding better sets of digits

Using Double-and-Add

- $\{1, 2, 4, 8, 9, 18, 19\}$ does not provide a *good* set because of too many doublings
- $2{=}(10)_2$, $4{=}(100)_2$ and $8{=}(1000)_2$ are not useful

Idea

- Use chains with more additions
- Choice: euclidean addition chains
- $\{1, 2, 3, 5, 7, 12, 19\}$ lead to the dictionary $\{1, 3, 5, 7, 19\}$
- average density $4/21(\simeq 0.19) < 2/9(\simeq 0.22)$

New Scalar Multiplication Scheme

$k = (k_{t-1} \dots k_0)_2$

- $k_H = \lfloor k/2^{t-l} \rfloor$ for some l
- $k_L = k - k_H * 2^{t-l}$
- find short Euclidean chain $S = (s_1, \dots s_m)$ computing k_H
- use S as a dictionary for the recoding of
- compute $n_H P$ using chain S and store the values $P, s_1 P, s_2 P, \dots, s_m P$
- compute $(k_H * 2^{t-l} + k_L)P$ using the double-and-add alg. starting from $k_H P$.

Example

$k = 1626832774 = 193 \times 2^{23} + 7831430$

- $k_H = 193$
- $k_L = 7831430$
- $S = 1, 2, 3, 5, 7, 12, 19, 31, 50, 81, 112, 193$
- $k_L = (5\ 0\ 0\ 0\ 0\ 0\ \text{-}81\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \text{-}1\ 0\ 0\ 0\ 0\ 0\ 3\ 0)_2$
- compute $193P$ using chain S and store $P, 3P, 5P$ and$81P$
- compute $193 \times 2^{23}P + 7831430P$ using the double-and-add alg. starting from $193P$.

The end

Any questions?