Correlation of Boolean Functions

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Presentation Outline

- Sequences, Correlation, Cryptographic Properties, Cryptanalysis, and Their Relation to Transforms for Signals
- Indication Functions: A Bridge to Connect Resiliency (Cross Correlation) and Propagation (Additive Autocorrelation)
- Constructions of Boolean Functions with 2-Level (Multiplicative) AC and Three-valued Additive AC, and more
- Discussions

Applications of Pseudo-random Sequences

In communications:

- Orthogonal codes, cyclic codes
- CDMA (code division multiple access) applications
- Synchronization codes
- Radar, and deep water distance range
- Testing vectors of hardware design

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In cryptography:

- Key Stream Generators in Stream Cipher Models
- Functions in Block Ciphers
- Session Key Generators
- Pseudo-random Number Generators in Digital Signature Standard (DSS), etc.
- Digital Water-mark

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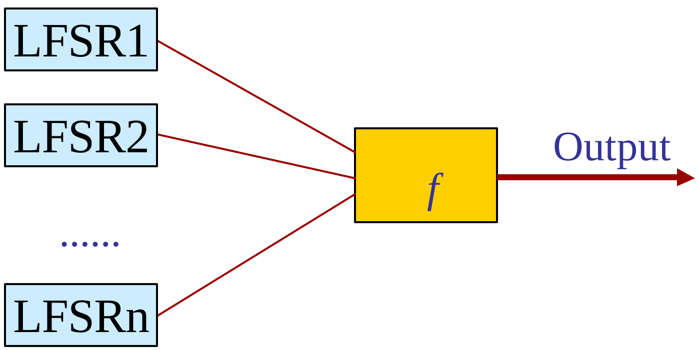
Design of Pseudo-random Sequence Generators

(a) Towards 2-Level Auto-Correlation and Low Correlation

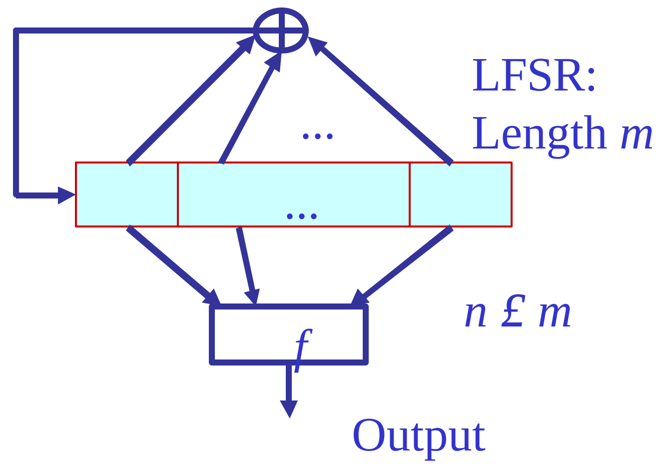
(b) Towards Large Linear Span

LFSR as Basic Blocks

Stream Cipher Applications



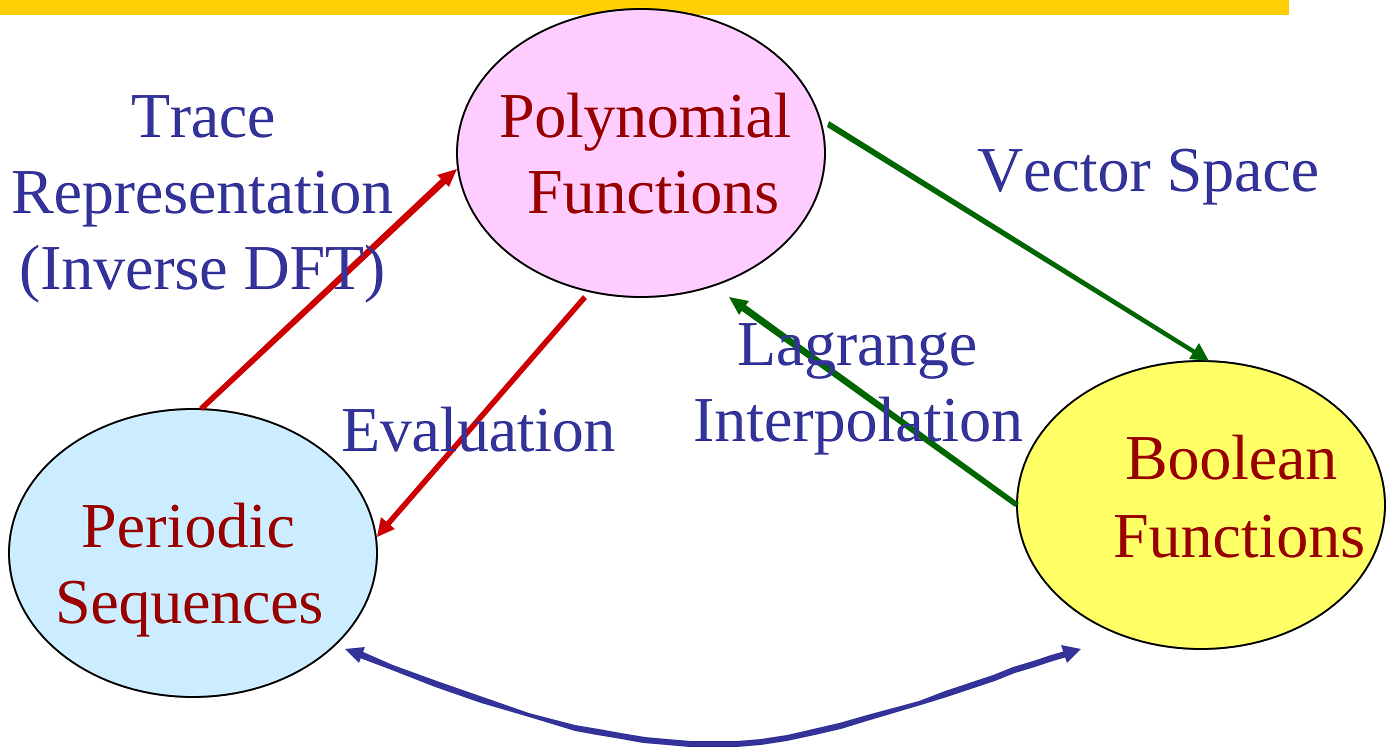
A Combinatorial Function Generator



A Filtering Generator

f is a boolean function in *n* variables .

1-1 Correspondences Between Sequences, Polynomial Functions and Boolean Functions



Notation

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- F = GF(2^n), a finite field, α is a primitive element of F
  F_2 = GF(2), binary field.

- $\mathbf{a} = \{a_i\}$, a binary sequence with period $N \mid 2^n - 1$; $f(x)$, the trace representation of **a**, i.e.,

$$a_i = f(\alpha^i), \quad i = 0, 1, \ldots.$$

Note. $f(x)$ is a polynomial function from GF(2^n) to GF(2) which can be represented by

$$f(x) = \sum_k Tr_1^{n_k}(A_k x^k), \; A_k \in GF(2^{n_k})$$

where the k's are different coset leaders modulo $2^n - 1$, and n_k is the size of the coset containing k.

- $x = x_0 + x_1\alpha + \cdots + x_{n-1}\alpha^{n-1} = (x_1, \cdots, x_n)$, an element in finite field GF(2^n) or an element in the vector space F_2^n.

(Multiplicative) Autocorrelation

The (multiplicative) autocorrelation of function $f(x)$ is defined as the autocorrelation of the sequence **a**, which is given by

$$C_f(t) = 1 + C(t) = 1 + \sum_{i=0}^{N-1} (-1)^{a_{i+t} + a_i}, \; t = 0, 1, \ldots$$

The sequence **a** has an (ideal) 2-level autocorrelation if

$$C(t) = \begin{cases} N & \text{if } t \equiv 0 \pmod{N} \\ -1 & \text{otherwise} \end{cases}$$

Convolution or Additive Autocorrelation

The additive autocorrelation of *f* (or the additive autocorrelation of the sequence is defined through its trace representation *f*) is defined as the convolution of $f(x)$:

$$A_f(w) = \sum_{x \in F} (-1)^{f(x)+f(x+w)}$$

Known Constructions of 2-Level Autocorrelation Sequences (or Orthogonal Codes, or Hadamard Difference Sets)

- Number theory approach (N is a prime): quadratic residue sequences (with $N \equiv 3 \bmod 4$), Hall sextic residue sequences, and the twin prime sequences.

$N = 2^n - 1$:

- PN-sequences = m-sequences (1931, Singer, 1958, Golomb)
- GMW sequences (1961, Golden-Miller-Welch, 1984, Welch-Scholtz)
- Conjectured Sequences (Gong-Gaal-Golomb, 1997, No-Golomb-Gong-Lee-Gaal, 1998)
- Hyper-oval Construction: (Maschietti, 1998)
- Kasami Power Function Construction (Dobbertin, Dillon, 1998)

2-level Additive Autocorrelation

f(*x*) is a bent function if and only if

$$\hat{f}(l) = \pm\sqrt{2^n}, \quad \forall l \in F$$

Note. Bent functions only exists for *n* even.

f(*x*) has 2-level additive autocorrelation if and only if *f*(*x*) is bent. There are two general constructions for bent functions (compared with the constructions of the binary sequences with 2-level (multi.) autocorrelation, this is relatively easy).

Question: What is the best additive autocorrelation for *n* odd?

Transforms for Signal (Sequence) Design (Engineering Perspective)

Hadamard (Walsh) Transform of *f*:

$$\hat{f}(l) = \sum_{x \in F} (-1)^{Tr(lx) + f(x)}$$

Time domain	Frequency domain
$f(x)$	$\hat{f}(l)$

Convolution or Additive autocorrelation of *f*:

$$A_f(w) = \sum_{x \in F} (-1)^{f(x) + f(x+w)}$$

They are related by the Convolution Law.

In other words, the square of the Hadamard transform of *f* is equal to the Hadamard transform of the convolution of *f* with itself or additive autocorrelation of *f*. Conversely,

$$A_f(w) = \frac{1}{2^n} \sum_{l \in F} (-1)^{Tr(wl)} \hat{f}^2(l)$$

which is a fundamental relation through this representation.

Desired Cryptographic Properties of Boolean Functions

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Definition 1 (Siegenthaler, 1984)

A Boolean function $f(x)$ in n variables is ***k*th-order correlation immune** if for each k-subset K of $\{0, \ldots, n-1\}$, $Z = f(x)$, considered as a random variable over F_2, is independent of all x_i for $i \in K$. Furthermore, if $f(x)$ is balanced and kth-order correlation immune, then $f(x)$ is said to be ***k*-order resilient**.

Nonlinearity of f is defined as the minimum distance of $f(x)$ with all affine functions, or equivalently,

$$N_f = 2^{n-1} - \frac{1}{2}\max_l |\hat{f}(l)|$$

Property (Xiao and Massey, 1988). $f(x)$ is ***k*th-order correlation immune** if and only if

$$\hat{f}(l) = 0,\ 1 \le H(l) \le k$$

where $H(x)$ is the Hamming weight of x.

A historical remark. Golomb studied these concepts under the terminology of **invariants** of boolean functions in 1959, and he is the first to compute them using Hadamard transform.

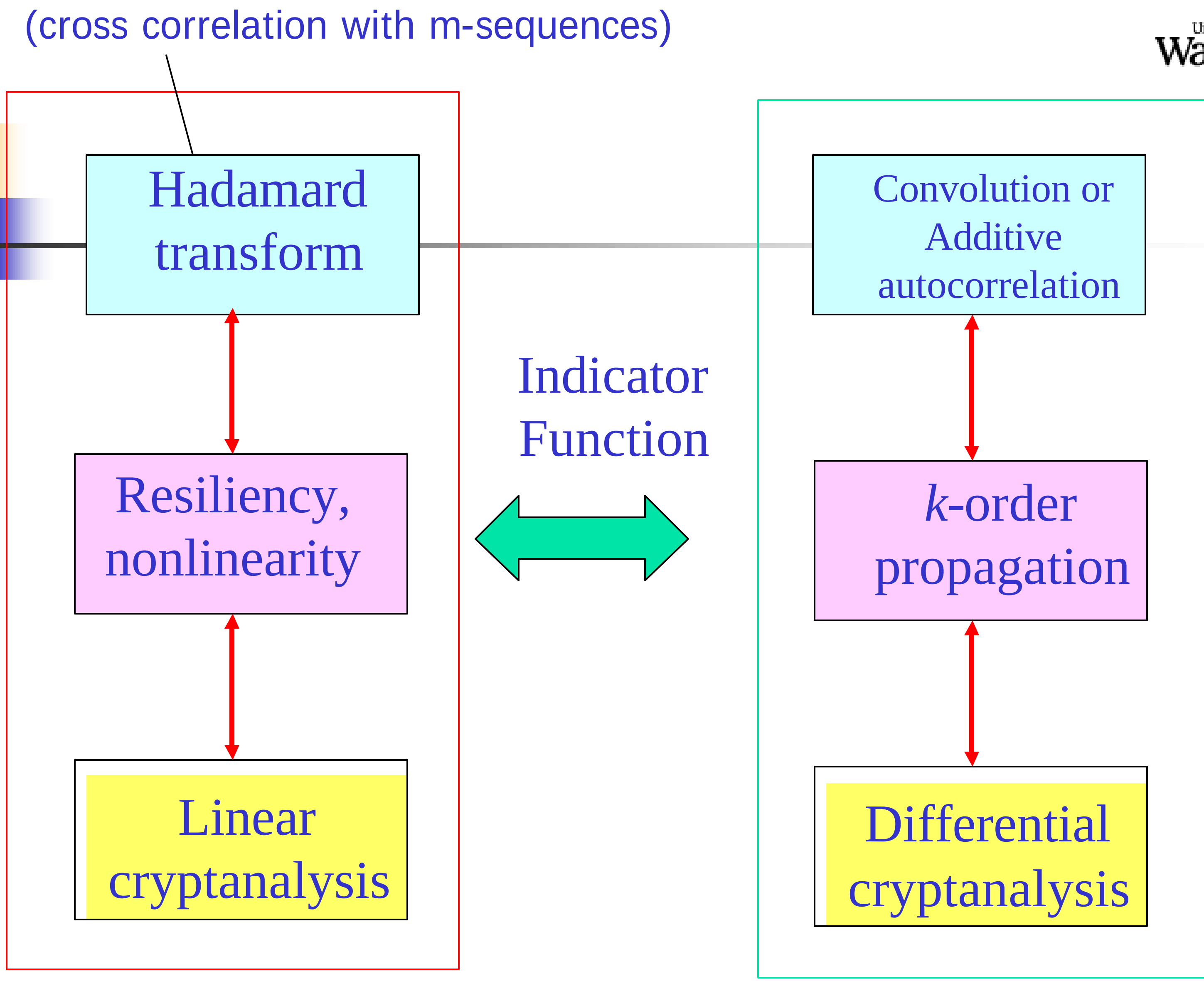
Definition 2.

A Boolean function $f(x)$ in n variables is said to satisfy the avalanche criterion (SAC) if

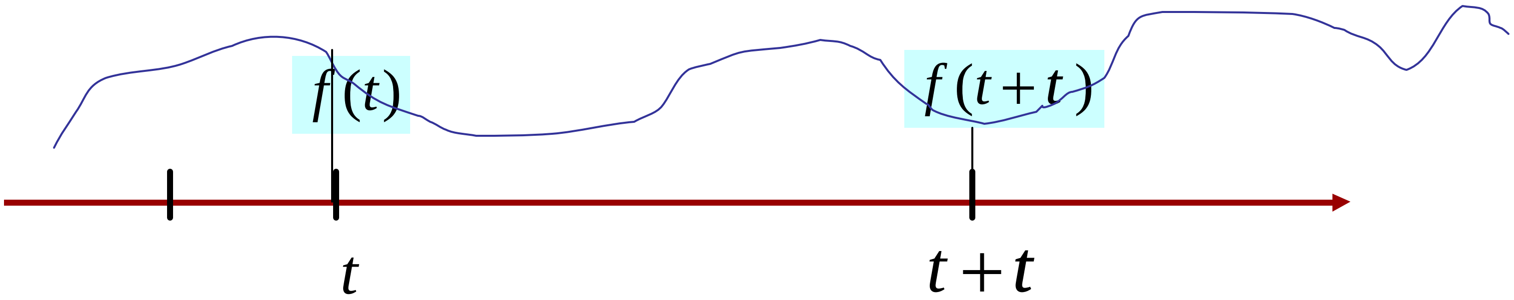
$$\boxed{A_f(w) = 0} \qquad \text{for all } w \text{ with } H(w) = 1$$

to have the k-order propagation if

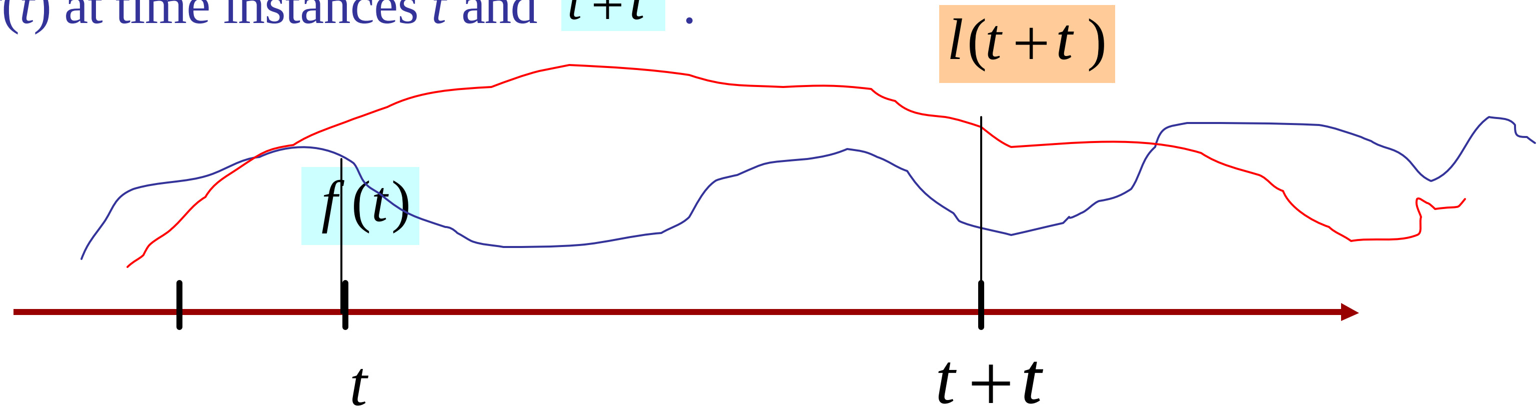
$$\boxed{A_f(w) = 0} \qquad \text{for all } w \text{ with } 1 \leq H(w) \leq k$$


(cross correlation with m-sequences)
Hadamard transform
Resiliency, nonlinearity
Linear cryptanalysis
Indicator Function
Convolution or Additive autocorrelation
k-order propagation
Differential cryptanalysis

Engineering Perspective of Differential Cryptanalysis and Linear Cryptanalysis Associated to Transforms for Signal (Sequence) Design



Differential cryptanalysis (or propagation) is to exploit the correlation of the signal $f(t)$ at time instances t and $t+t$.



Correlation immunitiy (or resiliency, nonlinearity, linear cryptalalysis) is to exploit the correlation between the signal $f(t)$ and the reference linear signal $l(t)$ at time instances t and $t+t$.

Indicator Function: A Bridge for Connecting Resiliency and Additive Autocorrelation

Definition. A indicator function of f, denoted by $s_f(x)$, is defined as

$$s_f(l) = \begin{cases} 0 \text{ if } & \hat{f}(l) = 0 \\ 1 \text{ if } & \hat{f}(l) \neq 0 \end{cases}$$

Example. For $n = 5$, $GF(2^5)$ defined by $a^5 + a^3 + 1 = 0$, and $f(x) = Tr(x^3)$.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\hat{f}(a^i)$	-8	0	0	0	0	-8	0	8	0	-8	-8	8	0	8	8	0	0	0	-8	8	-8	8	8	0	0	8	8	0	8	0	0
$s_f(a^i)$	1	0	0	0	0	1	0	1	0	1	1	1	0	1	1	0	0	0	1	1	1	1	1	0	0	1	1	0	1	0	0

Preferred set: For $n = 2m + 1$, f is said to be *preferred* if the Hadamard transform of f has the following three values:

$$P = \{0, \pm 2^{m+1}\}$$

Optimal Additive autocorrelation (AC): For $n = 2m + 1$, let f be balanced, the additive AC of f is said to be *optimal* if the maximal magnitude of the additive AC at nonzero, denoted as Δ_f, is 2^{m+1} and A_f has 2^{n-1} zeros in $GF(2^n)$.

Note.

1. According to the Parseval energy formula, 2^{m+1} is minimum among magnitudes of all 3-valued Hadamard spectra.

2. Zhang and Zheng (1995) conjectured that*

$$\Delta_f \geq 2^{m+1}$$

Observation 1: Indicator Function and Resiliency

Let f be preferred. Then f is 1-order resilient if and only if the dual of f is nonlinear.

Zhang and Zheng, 1999 under boolean forms, Gong and Yousself, 1999 under polynomial forms, Canteaut-Carlet-Charpin-Fontaine, 2000, for any three-valued Hadamard transform.

Observation 2: Indicator Function and Additive Autocorrelation

Let f be preferred. Then the additive autocorrelation functions of f at nonzero is equal to opposite of the Hadamard transform of f. In other words,

$$A_f(w) = -\hat{s}_f(w), \quad \forall\, 0 \neq w \in \mathrm{GF}(2^n)$$

Theorem. A Sufficient Condition for Preferred Additive Autocorrelation

If the Hadamard transforms of both *f* and its indicator functions are preferred, then the additive autocorrelation is preferred.

$$
\begin{tikzcd}
 & \hat{f}(l) \in P \arrow[d] & \hat{s}_f(l) \in P \arrow[d] \arrow[r, Rightarrow] & A_f(w) \in P \\
f \arrow[ur] \arrow[r] & s_f = g \arrow[ur] \arrow[r] \arrow[d, "\text{nonlinear}"] & s_g \arrow[d, "\text{nonlinear}"] & \\
 & \text{1-resiliency} & \text{1-propagation} &
\end{tikzcd}
$$

$$P = \{0, \pm 2^{(n+1)/2}\}$$

Constructions

All functions, listed in Tables 1-3, are from binary sequences with 2-level (multiplicative) autocorrelation.

Cryptographic Properties:

a) 2-level AC
b) Nonlinearity: $2^{n-1} - 2^{(n-1)/2}$
c) Preferred f
d) 1-order resiliency
e) Preferred additive AC, so optimal additive AC
f) 1-order propagation.

Table 1. Properties (a)-(d)

Functions from the sequence sets	Indicator Functions	Comments
Kasami decimation: $Tr(x^d), d = 2^{2k} - 2^k + 1$	$Tr(x^{2^k+1})$, for $3k \equiv 1 \bmod n$	Kasami 1971, Dillon 1999
The other Kasami, Welch, Niho	nonlinear	
Subset of GMW sequences	Nonlinear	2-level AC (Goldon, Miller, Welch 1961) HT (Games (85), Klapper(96))
Welch-Gong sequences WG(x)	$Tr(x^{d^{-1}})$	2-level AC (No *et. al* 1998, Dillon *et. al*. 1999)
Glynn Type 1 hyperoval sequences	$Tr\left(x^{(k-1)/k}\right)$	2-level AC (Matchietti 1998), Hadamard transform (Xiang 1998, Dillon 1999)
Kasami power function sequences: $C_k(x)$	$Tr\left(x^{(2^k+1)/3}\right)$	2-level AC (No *et. al* 1998, Dillon *et. al*. 1999)

Table 2. Properties (a)-(e)

$$3k \equiv 1 \bmod n$$

(Boolean) Functions	Indicator Functions
Kasami sequences: $Tr(x^d), d = 2^{2k} - 2^k + 1$	$Tr(x^{2^k+1})$
Welch-Gong sequences $WG(x) = Tr(t(x+1)+1)$	$Tr(x^{d^{-1}})$
Kasami power function sequences: $C_3(x),\ k = 3$ $C_k(x) = Tr(t(x^{2^k+1}))$	$Tr\left(x^3\right)$ $Tr\left(x^{d^{-1}}\right)$

where

$$t(x) = x + x^{2^k+1} + x^{2^{2k}+2^k+1} + x^{2^{2k}-2^k+1} + x^{2^{2k}+2^k-1}$$

Table 3. Properties (a)-(f)

(Boolean) Functions	**Indicator Functions**
Welch-Gong sequences $WG(x) = Tr(t(x+1)+1)$	$Tr(x^{d^{-1}})$
Kasami power function sequences (5-term sequences): $C_k(x) = Tr(t(x^{2^k+1}))$	$Tr\left(x^{d^{-1}}\right)$

where

$$t(x) = x + x^{2^k+1} + x^{2^{2k}+2^k+1} + x^{2^{2k}-2^k+1} + x^{2^{2k}+2^k-1}, 3k \equiv 1 \bmod n$$

Example 11.7 Let $n = 7$. Then $k = 5 \Longrightarrow n - k = 2 \Longrightarrow 2^{n-k} + 1 = 5$, and $t(x) = x + x^5 + x^{21} + x^{13} + x^{29}$. Thus

$$
\begin{aligned}
C_5(x) &= Tr(t(x^{2^2+1})) = Tr(x^5 + x^{19} + x^{29} + x^3 + x^9) \\
WG(x) &= Tr(t(x+1)+1) = Tr(x + x^3 + x^7 + x^{19} + x^{29}).
\end{aligned}
$$

Both $C_5(x)$ and $WG(x)$ have the following properties:

(a) Orthogonal or 2-level autocorrelation.

(b) Nonlinearity $N_f = 56$.

(c) Hadamard transform is preferred, i.e., belongs to $\{0, \pm 16\}$.

(d) 1-resiliency under some basis.

(e) The additive autocorrelation function is preferred, i.e., belongs to $\{0, \pm 16\}$.

(f) 1-order propagation under some basis.

Discussions

- What are the additive autocorrelations of the rest functions with 2-level autocorrelation?

- The functions constructed from sequence design do not have linear structure for any fixed set of input variables (possible week leakage of Maiorana-McFarland like resilient functions).

- Experimental results show that there are many functions having preferred Hadamard transform, and preferred additive AC, so optimal additive AC, but not 2-level AC.