Comănac Dragoș-Mihail - group 912 - 9.05.2020

$with(linalg) : with(DEtools) : with(VectorCalculus) : with(plots) :$

Exercise I.2

a)

$$solve\left(\left\{x - 2\,x\cdot y = 0,\ \frac{x^2}{2} - y = 0\right\}\right)$$

$$\{x=0, y=0\},\ \left\{x=1, y=\frac{1}{2}\right\},\ \left\{x=-1, y=\frac{1}{2}\right\} \quad \textbf{(1)}$$

Three equilibria: (0,0), (1,1/2), (-1,1/2)

b)

$$Jm := Jacobian\left(\left[x - 2\,x\cdot y,\ \frac{x^2}{2} - y\right], [x, y]\right)$$

$$\begin{bmatrix} 1 - 2\,y & -2\,x \\ x & -1 \end{bmatrix} \quad \textbf{(2)}$$

$$A1 := subs([x=0, y=0], Jm)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \textbf{(3)}$$

$$eigenvalues(A1)$$

$$1, -1 \quad \textbf{(4)}$$

Both eigenvalues are real different from zero. Therefore (0, 0) is hyperbolic and we can apply the Linearization method. Because the eigenvalues have opposite signs, the system X'=A1*X has a saddle point, so the equilibrium point (0,0) is unstable.

$$A2 := subs\left(\left[x=1, y=\frac{1}{2}\right], Jm\right)$$

$$\begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix} \quad \textbf{(5)}$$

$$eigenvalues(A2);$$

$$-\frac{1}{2} + \frac{1}{2}\,\mathrm{I}\sqrt{7},\ -\frac{1}{2} - \frac{1}{2}\,\mathrm{I}\sqrt{7} \quad \textbf{(6)}$$

Both eigenvalues are complex conjugate, whose real part are different from zero. Therefore (1,1/2) is hyperbolic and we can apply the Linearization method. Because the real part of the eigenvalues are negative, the system X'=A2*X has an attracting focus, so the equilibrium (1, 1/2) of the nonlinear sysyem is also an attractor, and stable.

$$A3 := subs\left(\left[x=-1, y=\frac{1}{2}\right], Jm\right)$$

$$\begin{bmatrix} 0 & 2 \\ -1 & -1 \end{bmatrix} \tag{7}$$

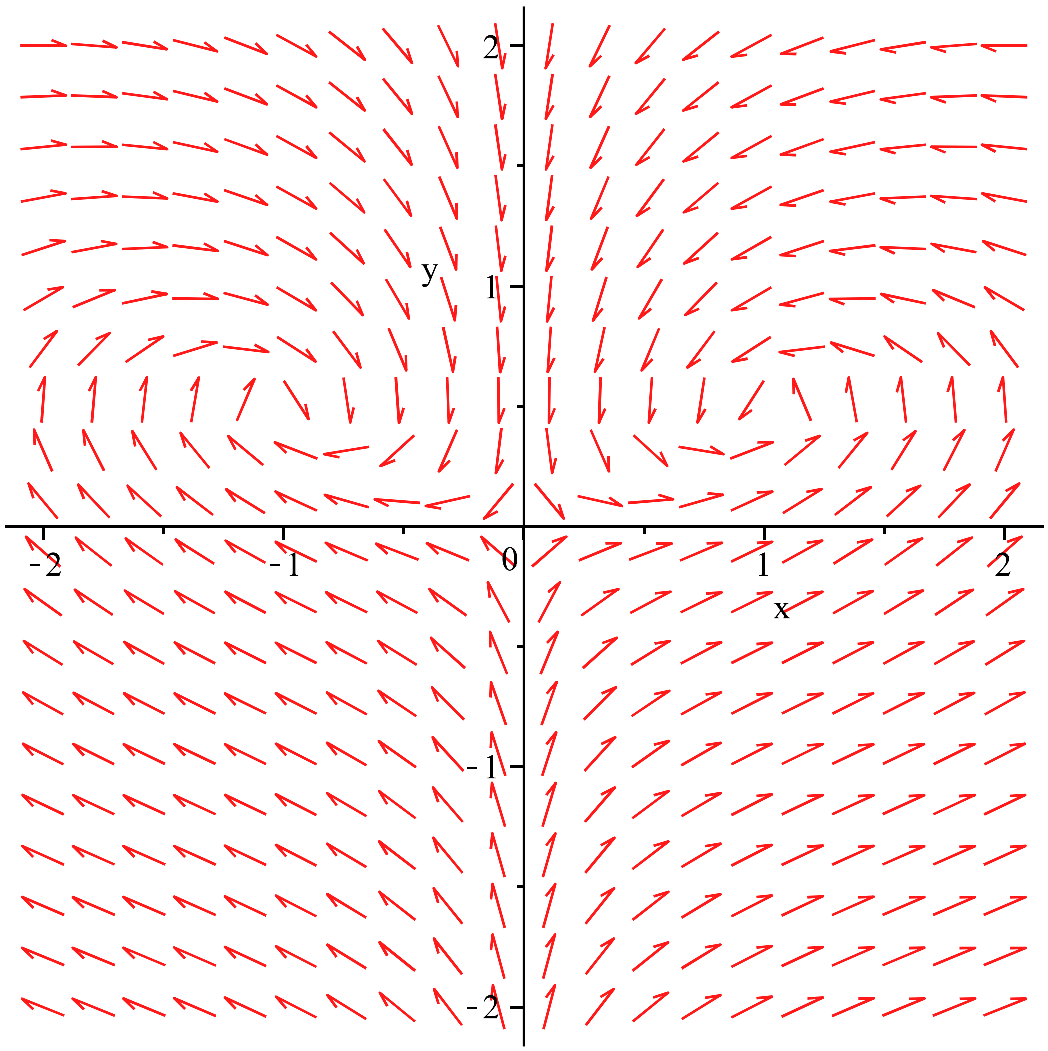
eigenvalues$(A3)$

$$-\frac{1}{2}+\frac{1}{2}\,\mathrm{I}\sqrt{7},\ -\frac{1}{2}-\frac{1}{2}\,\mathrm{I}\sqrt{7} \tag{8}$$

These are the same eigenvalues as the last ones therefore (-1,1/2) is also an attractor of the nonlinear system and stable.

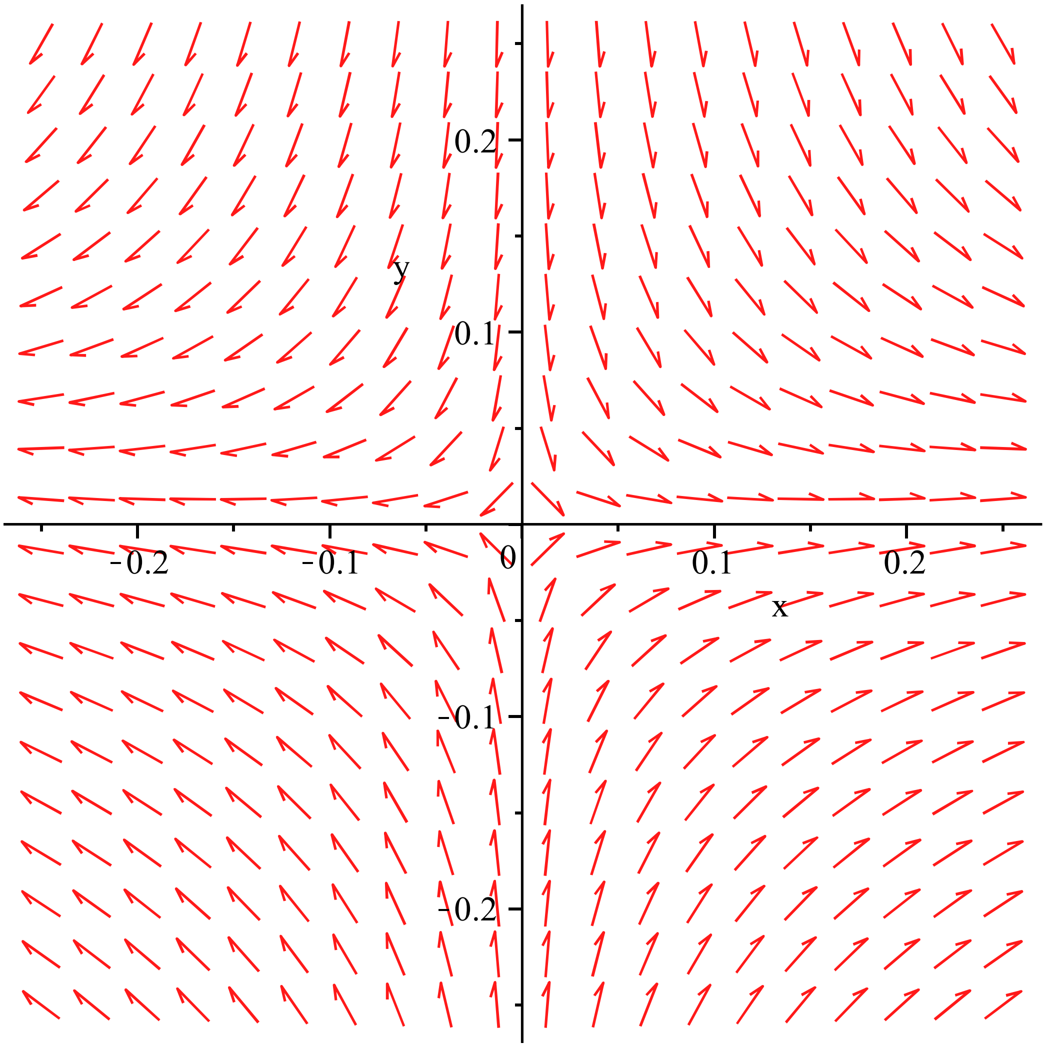
c)

$$dfieldplot\left(\left[diff(x(t),t)=x(t)-2\,x(t)\cdot y(t),\ diff(y(t),t)=\frac{x(t)^2}{2}-y(t)\right],\ [x(t),y(t)],\ t=0..1,\ x=-2..2,\ y=-2..2\right)$$



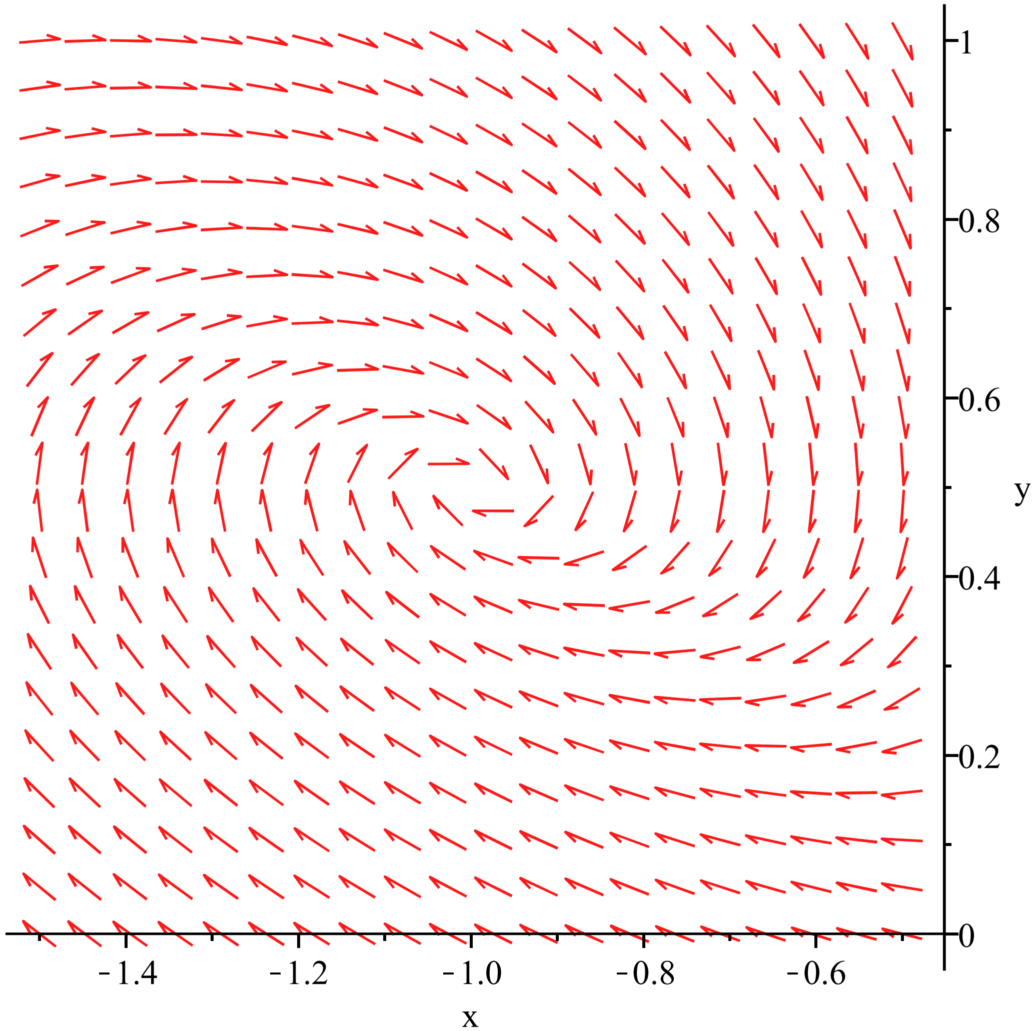
This is the direction field in the box [-2,2]x[-2,2]. Indeed around the equilibirum points (0,0), (1,1/2) (-1, 1/2) the direction field is not regular

$$dfieldplot\left(\left[diff(x(t), t) = x(t) - 2\,x(t)\cdot y(t), diff(y(t), t) = \frac{x(t)^2}{2} - y(t)\right], [x(t), y(t)], t = 0..1, x = -\frac{1}{4}..\frac{1}{4}, y = -\frac{1}{4}..\frac{1}{4}\right)$$



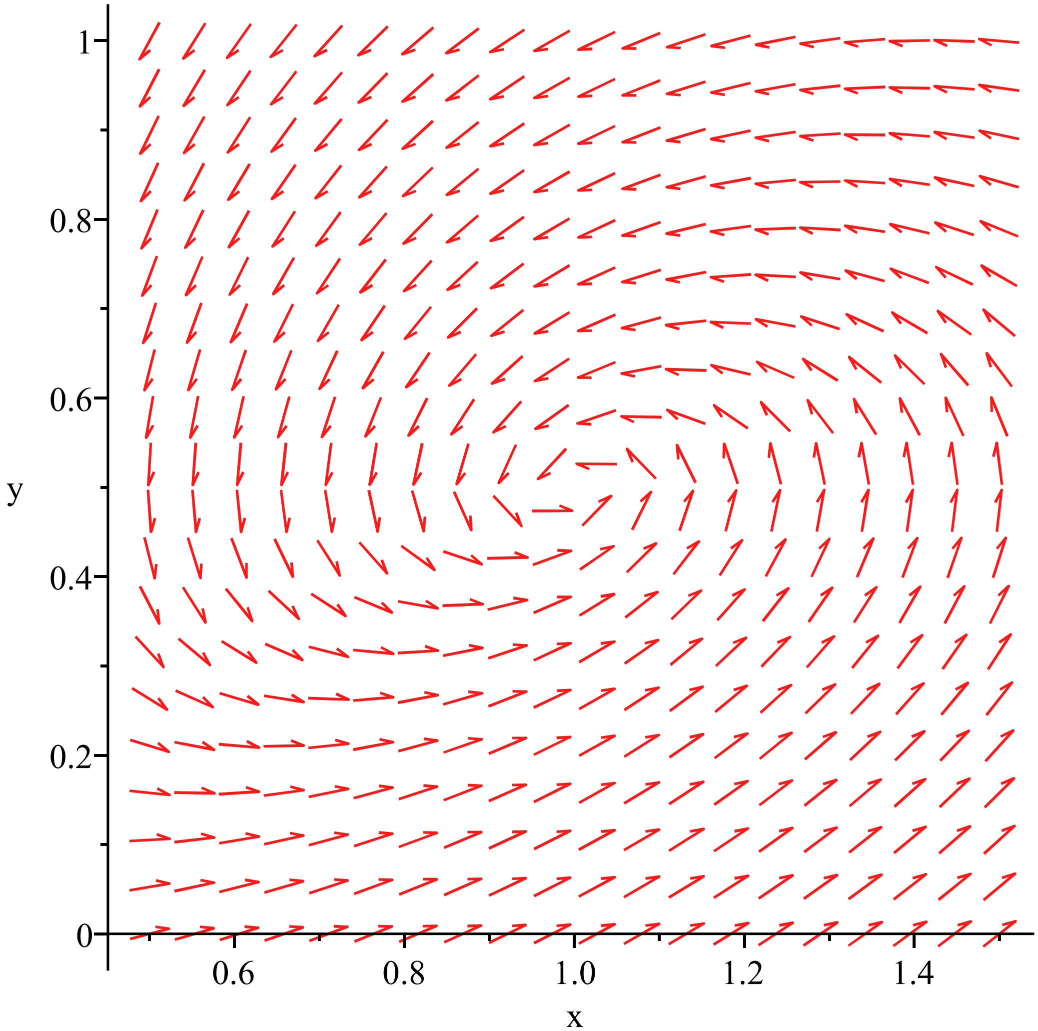
The only equilibrium point here is (0,0). It seems that the shape of the orbits in this small box look like the orbits of a linear system with a saddle.

$$dfieldplot\left(\left[diff(x(t), t) = x(t) - 2\,x(t)\cdot y(t), diff(y(t), t) = \frac{x(t)^2}{2} - y(t)\right], [x(t), y(t)], t = 0..1, x = -\frac{3}{2}..-\frac{1}{2}, y = 0..1\right)$$



The only equilibrium point here is (-1,1/2). It seems that the shape of the orbits in this small box look like the orbits of a linear system with an attractor.

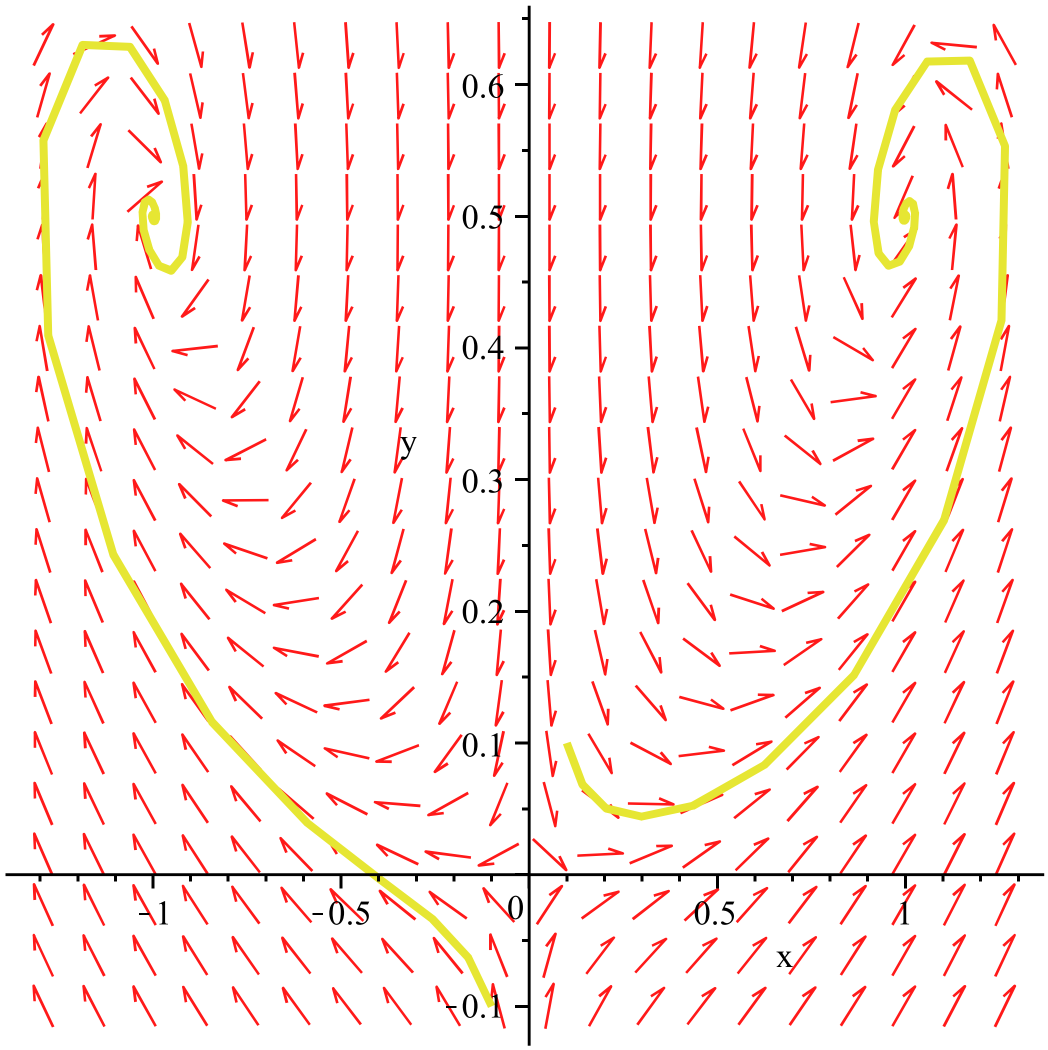
$$dfieldplot\left(\left[diff(x(t),t) = x(t) - 2\,x(t)\cdot y(t),\ diff(y(t),t) = \frac{x(t)^2}{2} - y(t)\right], [x(t), y(t)], t = 0..1, x = \frac{1}{2}..\frac{3}{2}, y = 0..1\right)$$



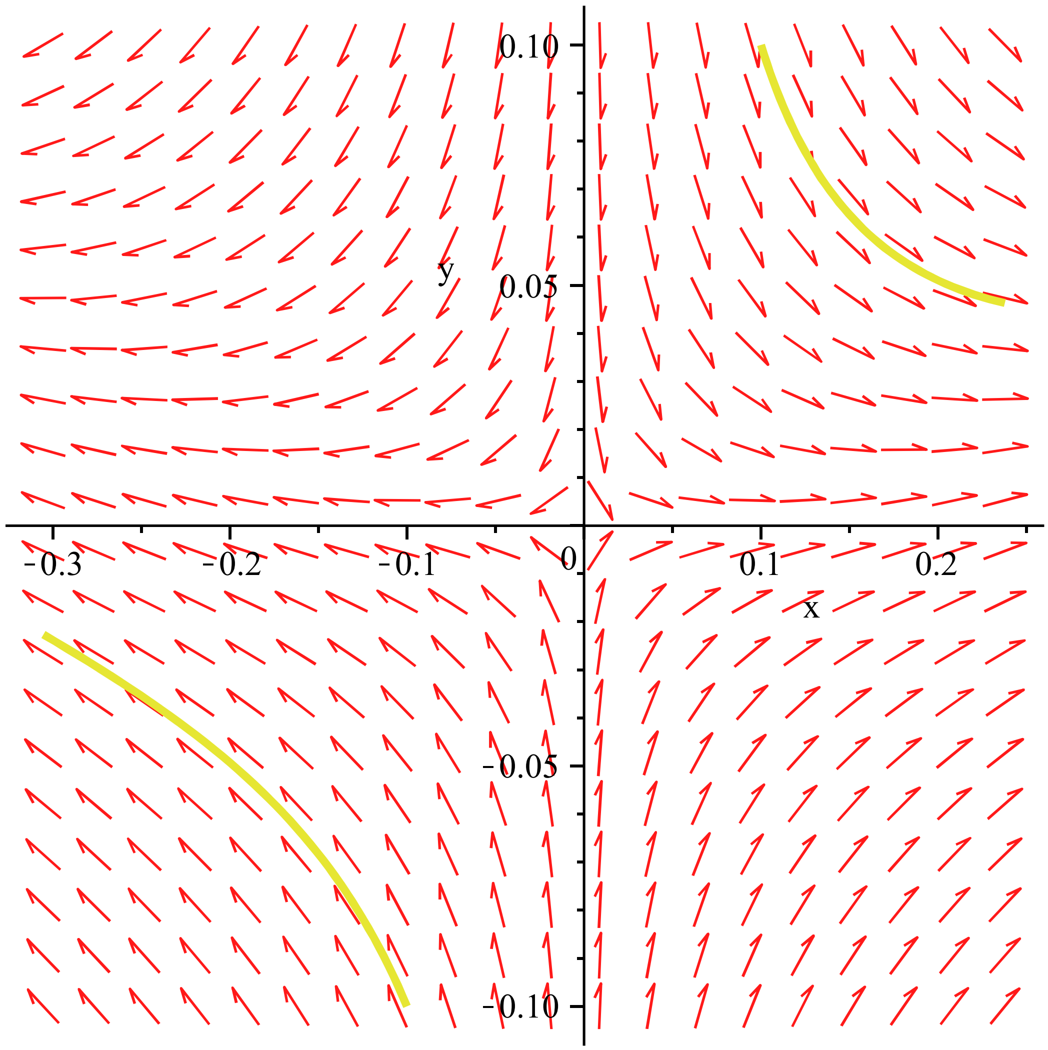
The only equilibrium point here is (1,1/2). It seems that the shape of the orbits in this small box look like the orbits of a linear system with an attractor.

d)

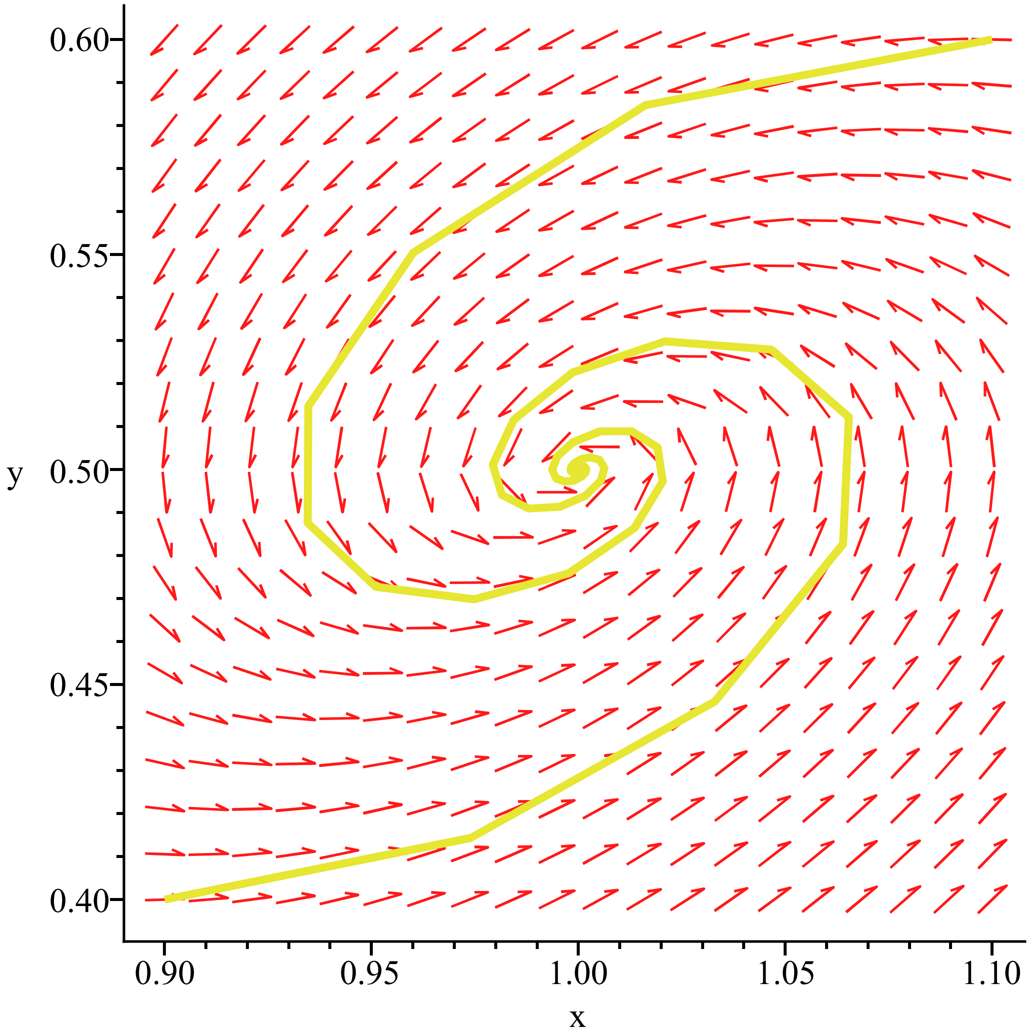
$$DEplot\left(\left[diff(x(t),t) = x(t) - 2\,x(t)\cdot y(t),\ diff(y(t),t) = \frac{x(t)^2}{2} - y(t)\right], [x(t), y(t)], t = 0..20,\right.$$
$$\left.[[x(0) = 0.1, y(0) = 0.1], [x(0) = -0.1, y(0) = -0.1]]\right)$$



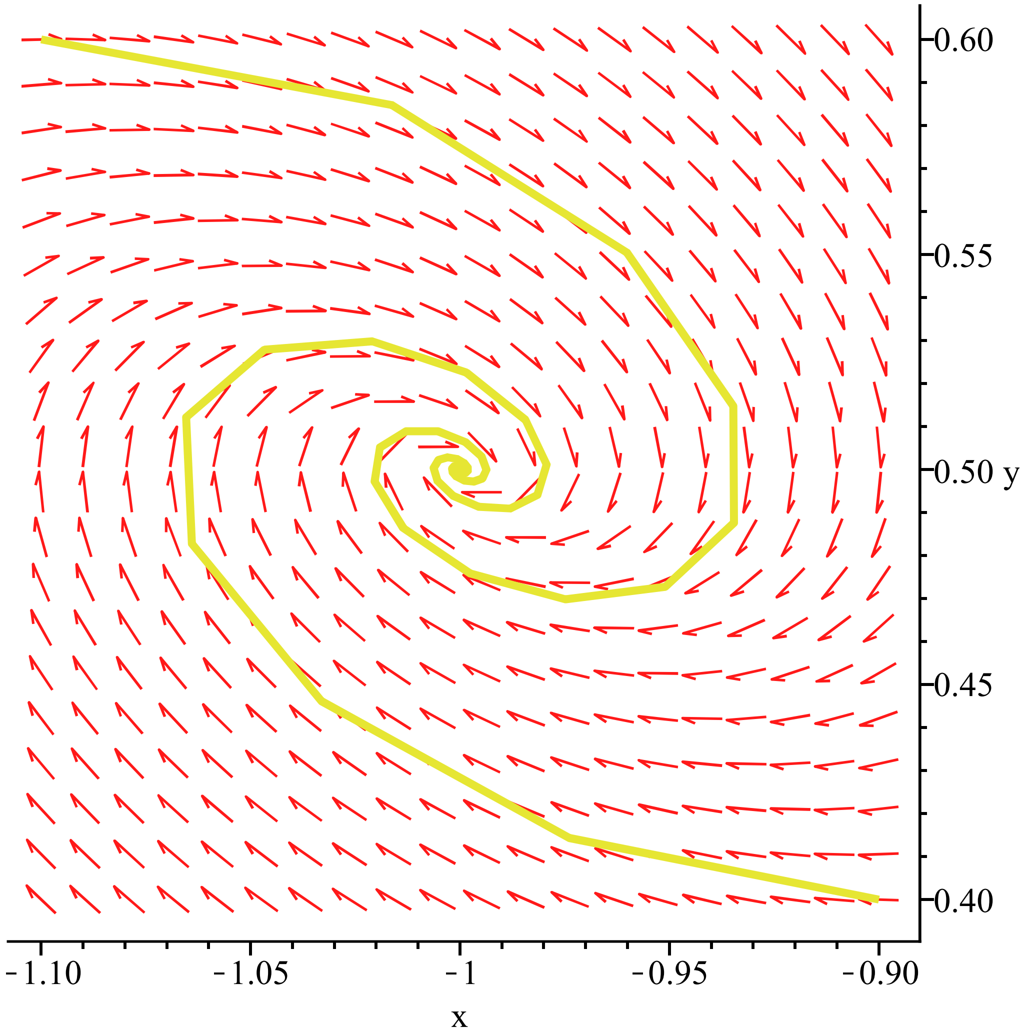
$$DEplot\left(\left[diff\left(x(t),t\right)=x(t)-2\,x(t)\cdot y(t),\ diff\left(y(t),t\right)=\frac{x(t)^2}{2}-y(t)\right],\ [x(t),y(t)],\ t=0..1,\ [[x(0)=0.1,\ y(0)=0.1],\ [x(0)=-0.1,\ y(0)=-0.1]]\right)$$



$$DEplot\left(\left[diff\,(x(t),t)=x(t)-2\,x(t)\cdot y(t),\ diff\,(y(t),t)=\frac{x(t)^2}{2}-y(t)\right],\ [x(t),y(t)],\ t=0\,..20,\right.$$
$$\left.[\,[x(0)=1.1,\ y(0)=0.6],\ [x(0)=0.9,\ y(0)=0.4]\,]\right)$$



$$DEplot\left(\left[diff(x(t),t)=x(t)-2\,x(t)\cdot y(t),\, diff(y(t),t)=\frac{x(t)^2}{2}-y(t)\right],\,[x(t),y(t)],\,t=0..20,\right.$$
$$\left.[[x(0)=-1.1,\,y(0)=0.6],\,[x(0)=-0.9,\,y(0)=0.4]]\right)$$



II.4

a)

$solve(\{x - x \cdot y = 0,\ -0.3\,y + 0.3\,x \cdot y = 0\})$

$$\{x = 0.,\ y = 0.\},\ \{x = 1.,\ y = 1.\} \tag{9}$$

The equilibrium points of the nonlinear sysyem are (0,0), (1,1)

$Jm := Jacobian([x - x \cdot y,\ -0.3 \cdot y + 0.3 \cdot x \cdot y],\ [x, y])$

$$\begin{bmatrix} 1 - y & -x \\ 0.3\,y & -0.3 + 0.3\,x \end{bmatrix} \tag{10}$$

$A1 := subs([x = 1, y = 1], Jm)$

$$\begin{bmatrix} 0 & -1 \\ 0.3 & 0. \end{bmatrix} \tag{11}$$

$eigenvalues(A1)$

$$0. + 0.547722557505166\,I,\ 0. - 0.547722557505166\,I \quad \textbf{(12)}$$

The real part of both eigenvalues is 0 thus the equilibrium pont (1,1) is non-hyperbolic.

b)

$expr := y - \ln(y) + 0.3\,(x - \ln(x))$

$$y - \ln(y) + 0.3\,x - 0.3\,\ln(x) \quad \textbf{(13)}$$

$H := unapply(expr, [x, y])$

$$(x, y) \rightarrow y - \ln(y) + 0.3\,x - 0.3\,\ln(x) \quad \textbf{(14)}$$

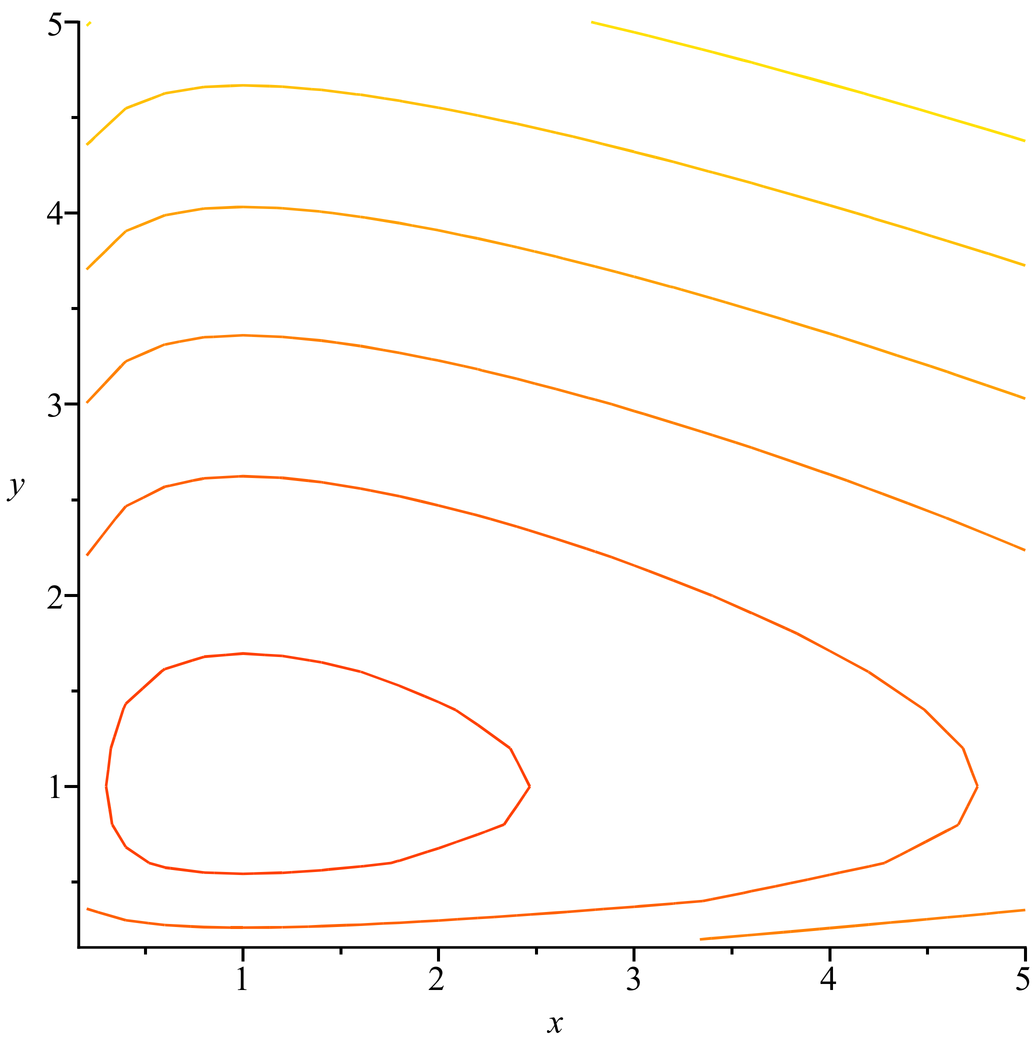
$expand(diff(H(x, y), x) \cdot (x - x \cdot y) + diff(H(x, y), y) \cdot (-0.3 \cdot y + 0.3\,x \cdot y))$

$$0. \quad \textbf{(15)}$$

Because the partial derivative with respect to x of H times f1 plus the partial derivative with respect to y of H times f2 is 0 it means that H is a first integral

c)

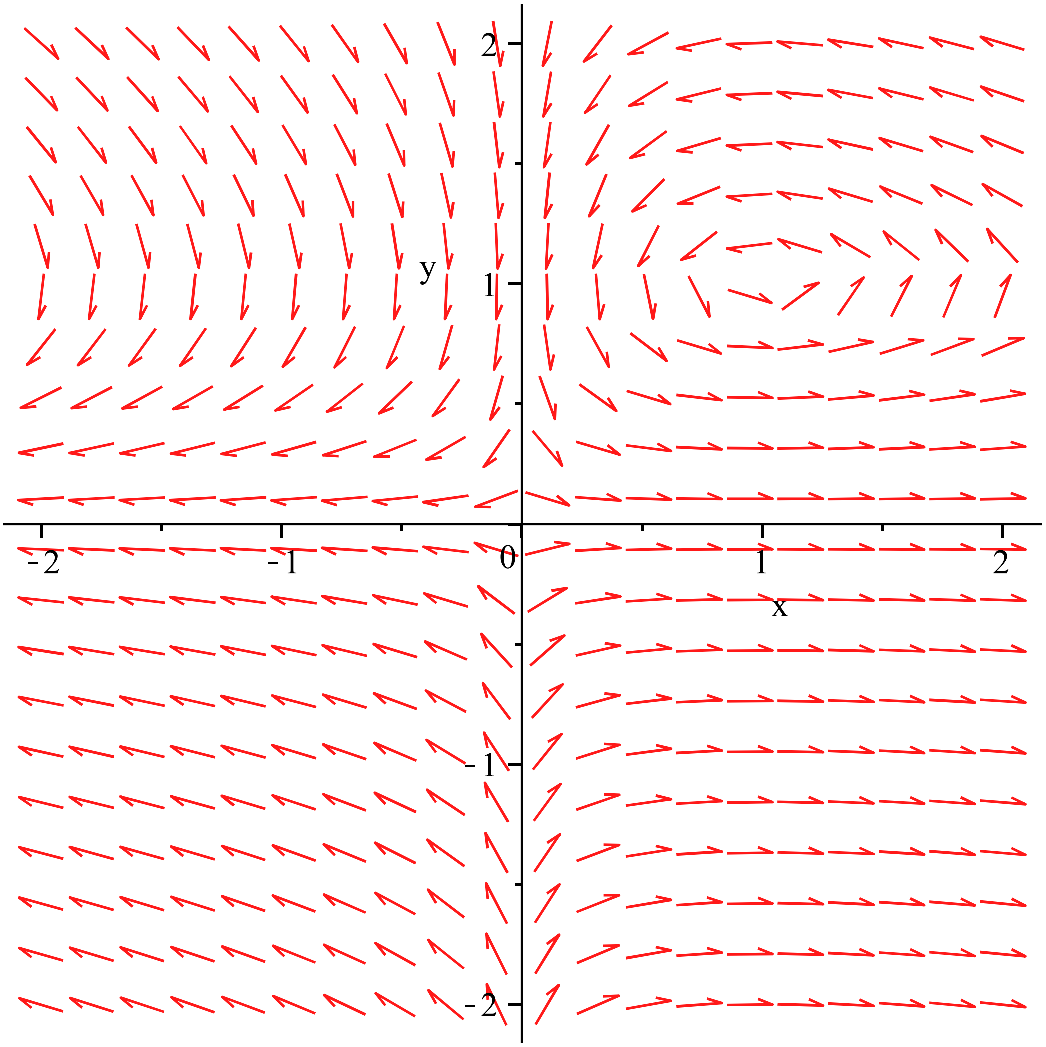
$contourplot(H(x, y), x = 0\,..5, y = 0\,..5)$



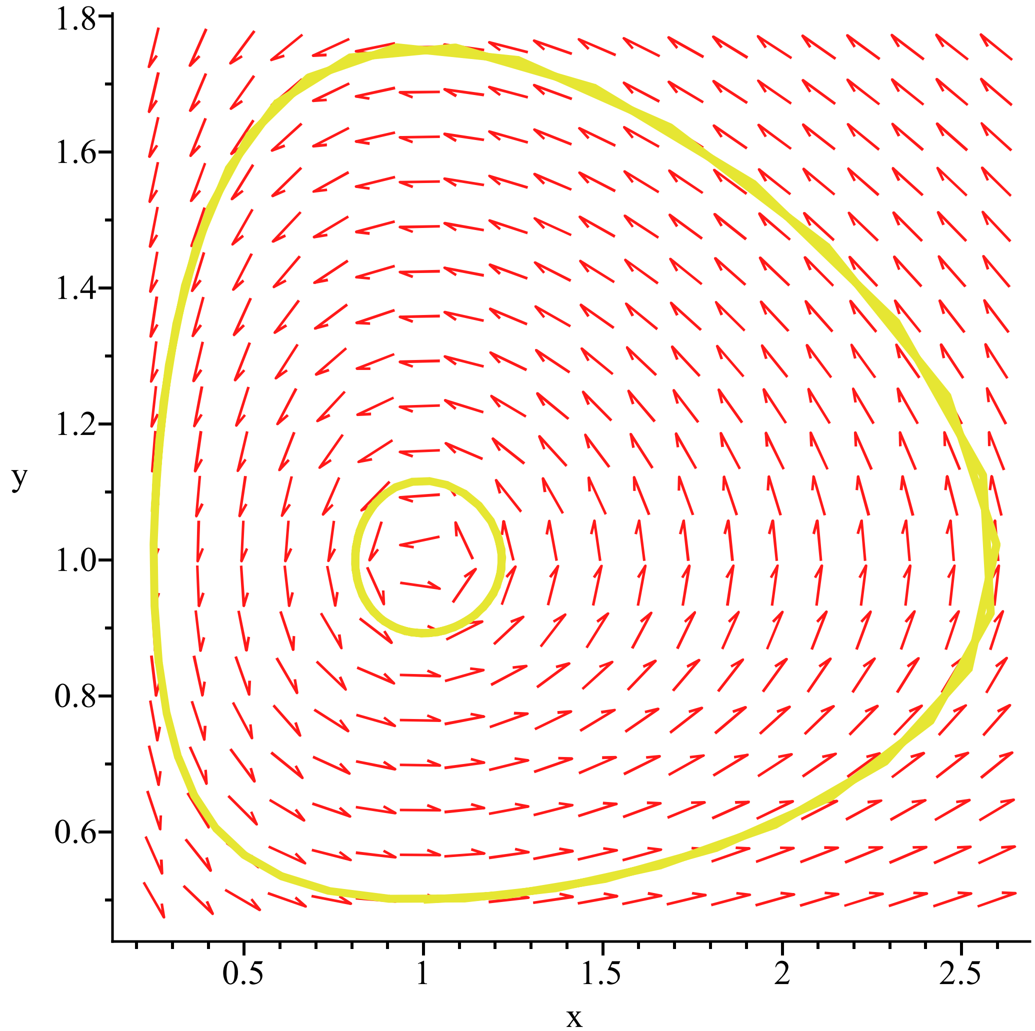
These are the levels curves of H.

d) We observe that around (1,1) is a closed curve therefore the orbits around (1,1) are periodic. It seems that the sysyem cycles in time, in a realistic way because when the population of the prey is higher than the population of predators, the predators don't starve and their population increases, which in turn determines the population of prey to decrease. In this case, when the population of prey is smaller, the population of predators decreases because of starvation, which allows the population of prey to increase, returning to the initial state and the cycle begins again. Therefore we can see four crucial moments: when the population of predators is at it's maximum/minimum and when the population of the prey is at it's maximum/minimum.

$dfieldplot([diff(x(t),t)=x(t)-x(t)\cdot y(t), diff(y(t),t)=-0.3\cdot y(t)+0.3\cdot x(t)\cdot y(t)], [x(t),y(t)], t=0..1, x=-2..2, y=-2..2)$



$DEplot([diff(x(t),t)=x(t)-x(t)\cdot y(t), diff(y(t),t)=-0.3\cdot y(t)+0.3\cdot x(t)\cdot y(t)], [x(t),y(t)], t=0..20, [[x(0)=1, y(0)=0.5], [x(0)=0.9, y(0)=1.1]])$


1.8
1.6
1.4
1.2
1.0
0.8
0.6
y
0.5
1
1.5
2
2.5
x

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I $\begin{cases} \dot{x} = x - 2xy \\ \dot{y} = \frac{x^2}{2} - y \end{cases}$

a) $\begin{cases} x - 2xy = 0 \\ \frac{x^2}{2} - y = 0 \Rightarrow \frac{x^2}{2} = y \end{cases} \Rightarrow x - 2x \cdot \frac{x^2}{2} = 0 \Rightarrow x - x^3 = 0$

$\Leftrightarrow x(1-x^2) = 0$

$\Rightarrow x_1 = 0 \Rightarrow y_1 = 0$

$x_2 = 1 \Rightarrow y_2 = \frac{1}{2}$

$x_3 = -1 \Rightarrow y_3 = \frac{1}{2}$

$\Rightarrow$ The three equilibrium points are $(0,0)$, $(1, \frac{1}{2})$ $(-1, \frac{1}{2})$

b) $J(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} 1-2y & -2x \\ x & -1 \end{pmatrix}$ – the Jacobian matrix

Matrix of the linearised system around $(0,0)$ is $J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The eigenvalues are the solutions of the equation: $\det(J(0,0) - \lambda I_2) = 0 \Leftrightarrow$

$\Leftrightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-1-\lambda) = 0 \Rightarrow \lambda_1 = 1$

$\lambda_2 = -1$

The eigenvalues are real, different from 0, therefore the equilibrium $(0,0)$ is hyperbolic $\xrightarrow[\text{method}]{\text{Linearization}}$ $\dot{X} = J(0,0)X$ has a saddle, so the equilibrium point $(0,0)$ of $\dot{X} = f(x)$ is unstable

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Matrix of the linearised system around $(1, \frac{1}{2})$ is $J(1, \frac{1}{2}) = \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix}$

$\det(J(1, \frac{1}{2}) - \lambda I_2) = 0 \Leftrightarrow \begin{vmatrix} \lambda & -2 \\ 1 & -1-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda(1+\lambda) + 2 = 0$

$\Rightarrow \lambda^2 + \lambda + 2 = 0 \Rightarrow \Delta = 1 - 8 = -7 \Rightarrow \lambda_1 = \frac{-1 + i\sqrt{7}}{2}, \lambda_2 = \frac{-1 - i\sqrt{7}}{2}$

Both eigenvalues have the real part different from 0 therefore the equilibrium $(1, \frac{1}{2})$ is hyperbolic

Linearisation $\Rightarrow$ method Because the eigenvalues are complex conjugates with the real part negative, the linearised system ~~around~~ has an attracting focus, therefore the nonlinear system has a attractor and is stable in $(1, \frac{1}{2})$

Matrix of the linearised system around $(-1; \frac{1}{2})$ is $J(-1, \frac{1}{2}) = \begin{pmatrix} 0 & 2 \\ -1 & -1 \end{pmatrix} \Rightarrow \det(J(-1, \frac{1}{2}) - \lambda I_2) = \begin{vmatrix} -\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix}$

$\Rightarrow \lambda(1+\lambda) + 2 = 0 \Rightarrow \lambda^2 + \lambda + 1 = 0 \Rightarrow$ the same eigenvalues as before therefore the nonlinear system is stable in $(-1; \frac{1}{2})$ and has an attractor.

c) $x = 0 \Rightarrow 0 = 0$
$\dot{y} = -y$

$\Rightarrow$

y	$-\infty$	0	∞
$-y$	$+$	0	$-$

$\Rightarrow$

$y = 0 \Rightarrow \dot{x} = x$
$0 = \frac{x^2}{2} \Leftrightarrow x = 0$

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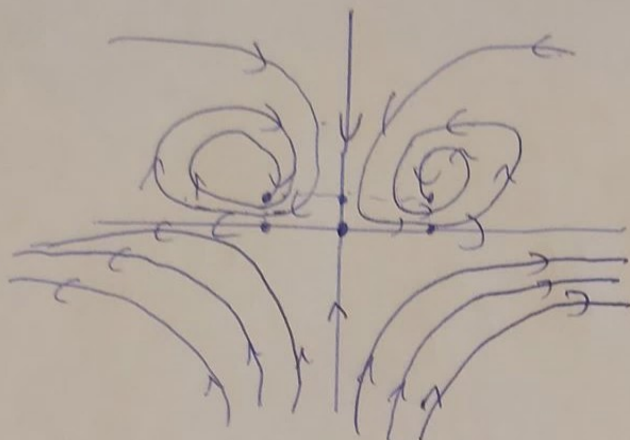

II $\begin{cases} \dot{x} = x - xy \\ \dot{y} = -0{,}3y + 0{,}3xy \end{cases}$

a) $\begin{cases} x - xy = 0 \\ -0{,}3y + 0{,}3xy = 0 \Rightarrow y = xy \end{cases} \Rightarrow x = y$

$\Rightarrow x = x^2 \Rightarrow x_1 = 0,\ y_1 = 0$; $x_2 = 1,\ y_2 = 1$ $\Rightarrow$ the equilibrium points are $(0,0)$ and $(1,1)$

$$J(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} 1-y & -x \\ 0{,}3y & -0{,}3+0{,}3x \end{pmatrix}$$

$$J(1,1) = \begin{pmatrix} 0 & -1 \\ 0{,}3 & 0 \end{pmatrix} \Rightarrow \det(J(1,1) - \lambda I_2) = 0 \Leftrightarrow \begin{vmatrix} -\lambda & -1 \\ 0{,}3 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 + 0{,}3 = 0 \Leftrightarrow \lambda_1 = i\sqrt{0{,}3}$ $\lambda_2 = -i\sqrt{0{,}3}$

The real parts of the eigenvalues are 0 therefor $(1,1)$ is non-hyperbolic

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b) $\frac{dy}{dx} = \frac{-0,3y+0,3xy}{x-xy}$

$\Leftrightarrow \frac{dy}{dx} = \frac{y(-0,3+0,3x)}{x(1-y)} \Leftrightarrow \frac{dy}{dx} = \frac{y}{1-y} \cdot \frac{-0,3+0,3x}{x}$

$\Leftrightarrow \frac{1-y}{y}dy = \frac{0,3x-0,3}{x}dx \Leftrightarrow \int \frac{1-y}{y}dy = 0,3\int \frac{x-1}{x}dx$

$\Leftrightarrow \int \frac{1}{y} - 1\, dy = 0,3\int 1-\frac{1}{x}dx \Leftrightarrow \ln y - y + C_1 = 0,3(x-\ln x)+C_2$

$\Rightarrow$ ~~the~~ first integral $H(x,y)$, $H:(0,\infty)\times(0,\infty)\to\mathbb{R}$

is $H(x,y) = y - \ln y + 0,3x - 0,3\ln x$

Now we check if it is a first integral

$\frac{\partial H}{\partial x}(x,y)\cdot f_1(x,y) + \frac{\partial H}{\partial y} f_2(x,y) = 0$

$\Leftrightarrow (0,3-0,3\frac{1}{x})(x-xy) + (1-\frac{1}{y})(-0,3y+0,3xy) = 0 \mid :0,3$

$\Leftrightarrow (1-\frac{1}{x})(x-xy) + (1-\frac{1}{y})(xy-y) = 0$

$\Leftrightarrow \cancel{x} - \cancel{1} - \cancel{xy} + \cancel{y} + \cancel{xy} - \cancel{y} - \cancel{x} + \cancel{1} = 0$ „A"

$\Rightarrow H(x,y)$ is a first integral

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c) $x = 0 \Rightarrow 0 = 0$

$\dot{y} = -0{,}3y$

y	$-\infty$	0	∞
$\dot{y}$	$+$	0	$-$

$y = 0 \Rightarrow \dot{x} = x$

$0 = 0$

x	$-\infty$	0	∞
$\dot{x}$	$-$	0	$+$