1. Find the flow of $\dot{x} = -2(x-5)$. This dynamical system has a global attractor?

$\square$ Let $\eta \in \mathbb{R}$ and consider the IVP $\dot{x} = -2(x-5)$, $x(0) = \eta$.

$\dot{x} = -2(x-5) \Leftrightarrow \dot{x} + 2x = 10$ is a first order linear non-hom. d.e. with c.c.

$\dot{x} + 2x = 0$ has the gen. sol. $x_h = ce^{-2t}$, $c \in \mathbb{R}$

$\dot{x} + 2x = 10$ has a constant partic. sol. $x_p = 5$

$\Rightarrow$ the gen. sol is $x = ce^{-2t} + 5$, $c \in \mathbb{R}$ $\Big\} \Rightarrow$

$x(0) = \eta \Rightarrow c + 5 = \eta \Rightarrow c = \eta - 5$

$\Rightarrow \varphi(t,\eta) = (\eta - 5)e^{-2t} + 5$, $\forall (t,\eta) \in \mathbb{R}^2$.

Note that, for any $\eta \in \mathbb{R}$, $\lim\limits_{t\to\infty} \varphi(t,\eta) = 5$.

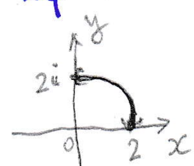
Also, $\varphi(t,5) = 5$ $\forall t \in \mathbb{R}$ $\Big\}$ Then $\Rightarrow$

the equilibrium point $\eta^* = 5$ is a global attractor. $\blacksquare$

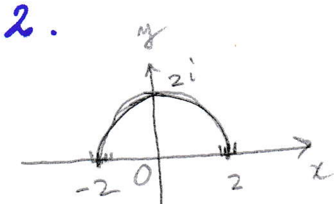
2. Represent in the complex plane the curves $\{2e^{it} : t \in [0, \frac{\pi}{2}]\}$, $\{2e^{it} : t \in [0,\pi]\}$, $\{2e^{it} : t \in [0,2\pi)\}$

$\square$ $z = 2e^{it} = 2(\cos t + i \sin t) \Rightarrow |z| = 2$ and the $\arg(z) = t$ $\Rightarrow$

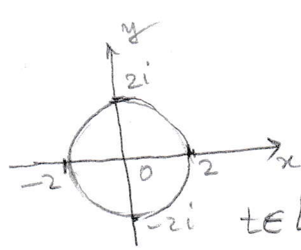
each curve is a part of the circle centered in the origin of radius 2.



$t \in [0, \frac{\pi}{2}]$



$t \in [0, \pi]$



$t \in [0, 2\pi)$

3. $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$ (a) Find the principal matrix solution of $X' = AX$.

(b) Compute e^{At}. (c) Find $a, b \in \mathbb{R}$ s.t. $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = x^2 + ay^2 + bxy$ is a global first integral of $X' = AX$.

□ First we find the general sol. of the system. $X' = AX \iff$ where $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{cases} x' = 2x - 5y \\ y' = x - 2y \end{cases}$$

we apply the reduction method.

$x = y' + 2y$; $y'' = x' - 2y' = 2x - 5y - 2y' = 2(y'+2y) - 5y - 2y'$

$\Rightarrow y'' + y = 0$ this is a second order lin. hom. d.e. with c.c.

$r^2 + 1 = 0$ $r_{1,2} = \pm i \longmapsto \cos t, \sin t$

$\Rightarrow y = c_1 \cos t + c_2 \sin t$

$x = y' + 2y = -c_1 \sin t + c_2 \cos t + 2c_1 \cos t + 2c_2 \sin t =$
$= c_1(2\cos t - \sin t) + c_2(\cos t + 2\sin t)$

So, the general sol. is $\begin{cases} x = c_1(2\cos t - \sin t) + c_2(\cos t + 2\sin t) \\ y = c_1 \cos t + c_2 \sin t \end{cases}$, $c_1, c_2 \in \mathbb{R}$.

In order to find the principal matrix sol., we need the solution of each of the following 2 IVP's.

$X' = AX$, $x(0) = 1$, $y(0) = 0$. (1)

and

$X' = AX$, $x(0) = 0$, $y(0) = 1$. (2)

we have $x(0) = 2c_1 + c_2$, $y(0) = c_1$

$x(0) = 1$ and $y(0) = 0 \Rightarrow c_1 = 0$, $c_2 = 1$

$x(0) = 0$ and $y(0) = 1 \Rightarrow c_1 = 1$, $c_2 = -2$

Thus, the sol. of the IVP (1) is $X_1(t) = \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$

and the sol of the IVP (2) is $X_2(t) = \begin{pmatrix} -5\sin t \\ \cos t - 2\sin t \end{pmatrix}$

Then, the principal matrix solution is

$$E(t) = \begin{pmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix}.$$

(d) We know that e^{At} is the unique principal matrix sol. of $X' = AX$. Then $e^{At} = E(t)$.

(e) $\begin{cases} x' = (2x-5y) \\ y' = x - 2y \end{cases}$. We have that $H \in C^1(\mathbb{R}^2)$ for any $a, b \in \mathbb{R}$.

So, H is a global first integral if and only if

$$\frac{\partial H}{\partial x}(x,y)\cdot(2x-5y) + \frac{\partial H}{\partial y}(x,y)\cdot(x-2y) = 0 \quad \forall (x,y) \in \mathbb{R}^2.$$

we replace $H(x,y) = x^2 + ay^2 + bxy$ and obtain

$$(2x+by)(2x-5y) + (2ay+bx)(x-2y) = 0 \quad \forall (x,y)\in\mathbb{R}^2$$

$$\Leftrightarrow \quad 4x^2 + \cancel{2bxy} - 10xy - 5by^2 + 2axy + bx^2 - 4ay^2 - \cancel{2bxy} = 0 \quad \forall (x,y)\in\mathbb{R}^2$$

$$\Leftrightarrow \quad (4+b)x^2 + 2(a-5)xy - (4a+5b)y^2 = 0 \quad \forall (x,y)\in\mathbb{R}^2$$

$$\Leftrightarrow \quad 4+b=0,\ a-5=0,\ 4a+5b=0 \quad \Leftrightarrow \quad a=5,\ b=-4,\ 4\cdot 5 - 5\cdot 4 = 0$$

$$\Leftrightarrow \quad a = 5 \text{ and } b = -4.$$

We found that $H(x,y) = x^2 + 5y^2 - 4xy$ is a global f.i.

4. How many solutions has the following problem?

(a) $x''' + t^2 x = 0, \quad x(0) = x'(0) = 1.$

(b) $x''' + t^2 x = 0, \quad x(0) = x'(0) = x''(0) = 1.$

□ We have that (b) is an IVP for a linear d.e. of third order with the coeff. $a(t) = t^2$ that satisfies $a \in C(\mathbb{R})$.
Then it has a unique solution in $C^3(\mathbb{R})$.

(a) is not an IVP. If we take $\eta \in \mathbb{R}$ (arbitrary, fixed) and we add the condition $x''(0) = \eta$ we get an IVP, which has a unique sol. $\varphi(\cdot, \eta)$. Thus, (a) has ~~at least~~ as many solutions as points in $\mathbb{R}$. ■

5. We consider the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{50} x(100 - x)$.

(a) Find its fixed points and study their stability.

(b) Estimate the basin of attraction of the attractor fixed point.

(c) If $(x_k)_{k \geq 0}$ represent the number of fish in some lake at month k and $x_{k+1} = \frac{1}{50} x_k (100 - x_k)$, $x_0 = \eta$ try to predict the fate of the fish in the case $\eta = 80$ and also in the case $\eta = 10$.

□ (a) $f(x) = x \iff \frac{1}{50} x(100 - x) = x \iff 100x - x^2 = 50x$

$\iff x^2 - 50x = 0 \iff x(x - 50) = 0$

the fixed points are $\eta_1^* = 0$ and $\eta_2^* = 50$.

$f(x) = 2x - \frac{1}{50} x^2 \Rightarrow f'(x) = 2 - \frac{2x}{50} \Rightarrow$

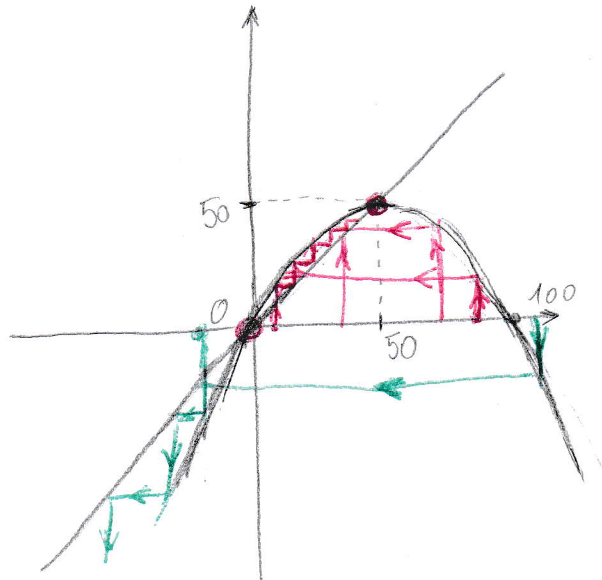
$|f'(0)| = 2 > 1 \Rightarrow \eta_1^* = 0$ is unstable

$|f'(50)| = 0 < 1 \Rightarrow \eta_2^* = 50$ is an attractor.

(b) We intend to represent the graph of f.

Since f is a quadratic polynomial function, its graph is a parabola.

x	$-\infty$		0		50		100		$+\infty$
f	$-\infty$	$\nearrow$	0	$\nearrow$	50	$\searrow$	0	$\searrow$	$-\infty$
f'		$+ + +$			0	$- - - -$			



$\eta_1^* = 0$ is a fixed point

$\eta = 100 \quad x_1 = f(100) = 0 \quad x_2 = f(0) = 0 \ldots$

It seems that the basin of attraction of $\eta_2^* = 50$ is $A_{50} = (0, 100)$.

(c) From (b) we have that for any $\eta \in (0,100)$ the sequence $(x_k)_{k \geq 0}$ converges to 50. Also from the cobweb diagram we can say the following:

- for $\eta = 10$ the number of fish will increase. In few months ~~so~~ there will be around 50, but the lake can not support more
- for $\eta = 80$, in the next month there will be $f(80) = \frac{80 \cdot 20}{50} = 32$. Then their number will increase, but no more than 50. ■