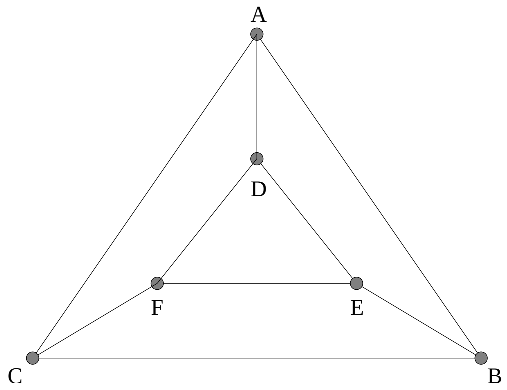
Graph Algorithms

Vertex Coloring

The Input Graph

$G = (V, E)$ a simple and undirected graph:

- $\star$ V: a set of n vertices.
- $\star$ E: a set of m edges.



	A	B	C	D	E	F
A	0	1	1	1	0	0
B	1	0	1	0	1	0
C	1	1	0	0	0	1
D	1	0	0	0	1	1
E	0	1	0	1	0	1
F	0	0	1	1	1	0

Vertex Coloring

Definition I:

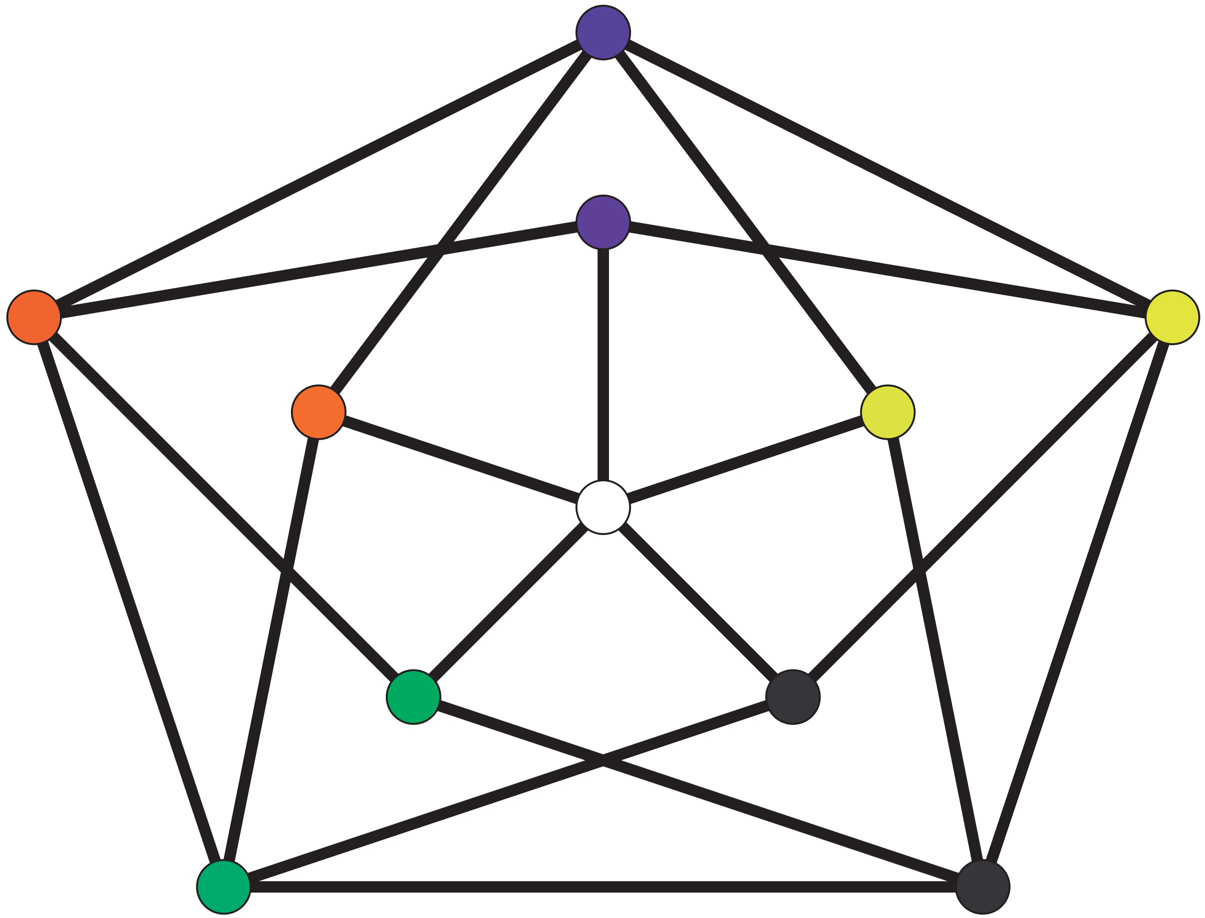
- ⋆ A disjoint collection of independent sets that cover all the vertices in the graph.
- ⋆ A partition $V = I_1 \cup\ I_2 \cup \cdots \cup I_\chi$ such that I_j is an independent set for all $1 \leq j \leq \chi$.

Definition II:

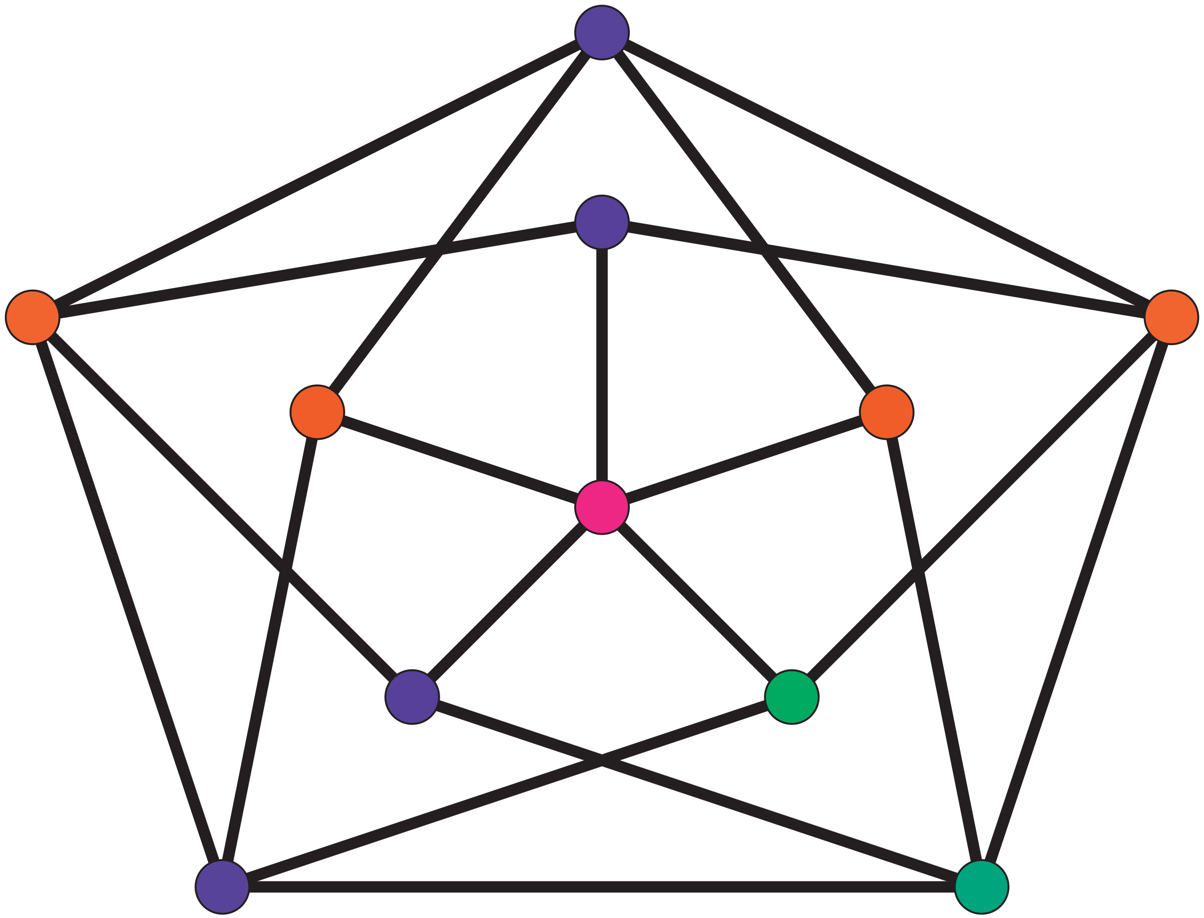
- ⋆ An assignment of colors to the vertices such that two adjacent vertices are assigned different colors.
- ⋆ A function $c : V \to \{1, \ldots, \chi\}$ such that if $(u, v) \in E$ then $c(u) \neq c(v)$.

Observation: Both definitions are equivalent.

Example: Coloring



Example: Coloring with Minimum Number of Colors



The Vertex Coloring Problem

The optimization problem: Find a vertex coloring with minimum number of colors.

Notation: The chromatic number of G, denoted by $\chi(G)$, is the minimum number of colors required to color all the vertices of G.

Hardness: A very hard problem (an NP-Complete problem).

Hardness of Vertex Coloring

- ⋆ It is NP-Hard to color a 3-colorable graph with 3 colors.
- ⋆ It is NP-Hard to construct an algorithms that colors a graph with at most $n^{\varepsilon}\chi(G)$ colors for any constant $0 < \varepsilon < 1$.

Known Algorithms for Vertex Coloring

- ⋆ There exists an optimal algorithm for coloring whose running time is $O\left(mn\left(1+3^{1/3}\right)^{n}\right) \approx mn1.442^{n}$.
- ⋆ There exists a polynomial time algorithm that colors any graph with at most $O(n/\log n)\chi(G)$ colors.
- ⋆ There exists an algorithm that colors a 3-colorable graph with $O(n^{1/3})$ colors.

Properties of Vertex Coloring

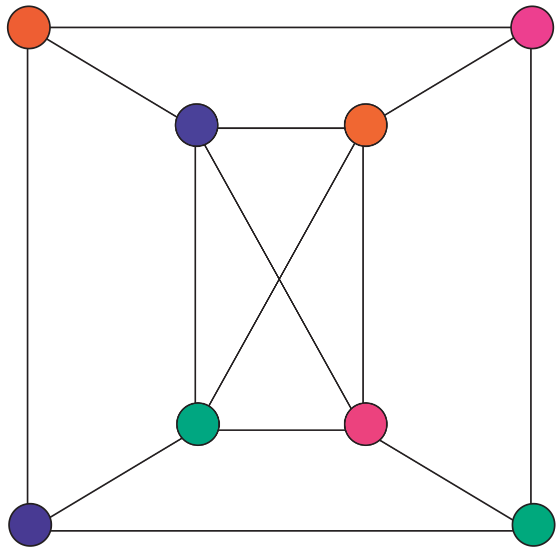
Observation: $K(G) \leq \chi(G)$.

- ⋆ Because in any vertex coloring, each member of a clique must be colored by a different color.

Observation: $\chi(G) \geq \left\lceil \frac{n}{I(G)} \right\rceil$.

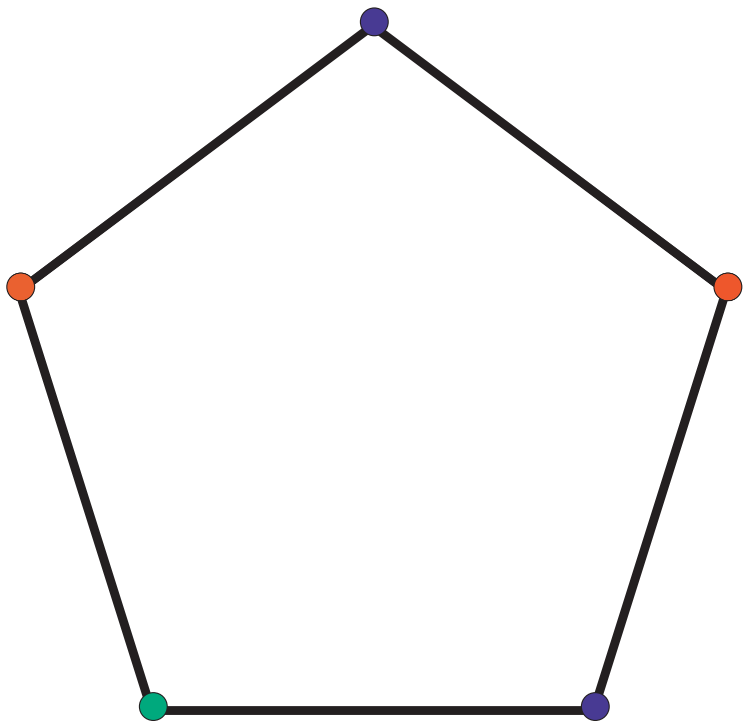
- ⋆ A pigeon hole argument: the size of each color-set is at most $I(G)$.

Example: $\chi(G) = K(G)$



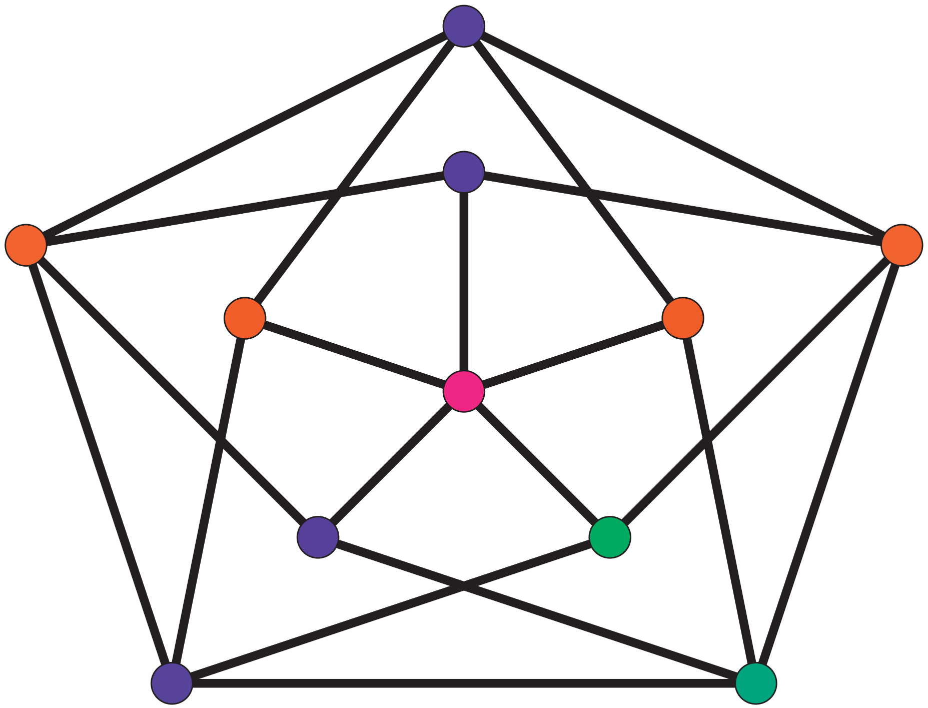
- ⋆ $K(G) = 4$ and $\chi(G) = 4$.
- ⋆ Every member of the only clique of size 4 must be colored with a different color.

Example: $\chi(G) > K(G)$



$K(G) = 2$ and $\chi(G) = 3$

Example: $\chi(G) > K(G)$



$K(G) = 2$ and $\chi(G) = 4$

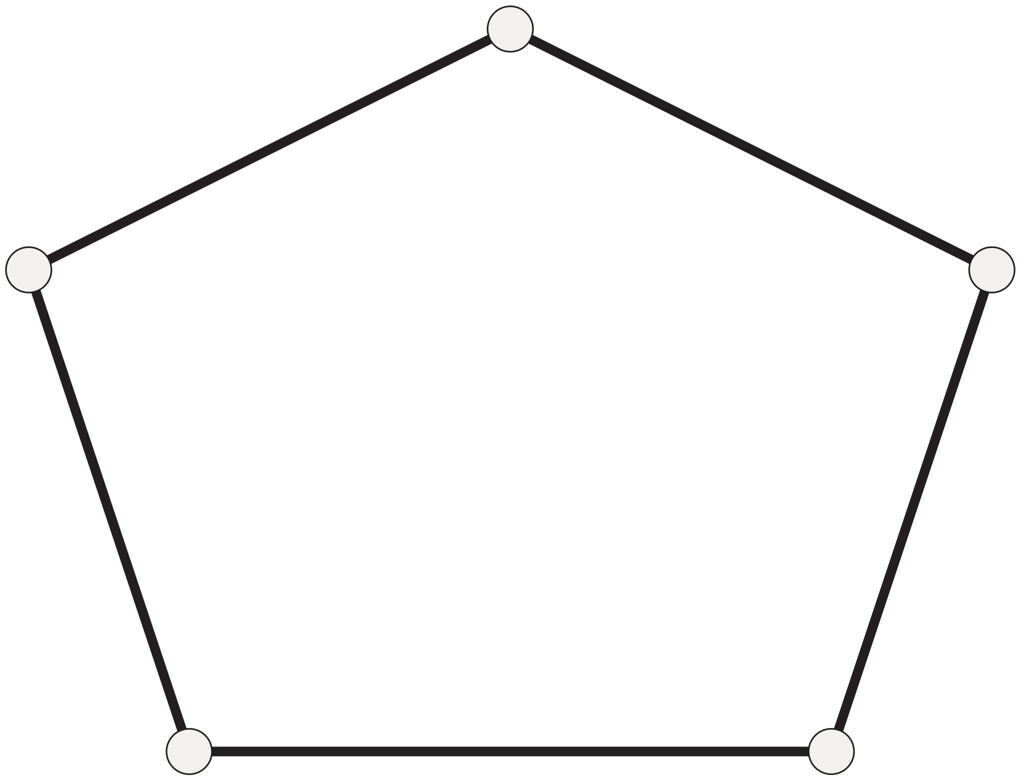
$\chi(G) >> K(G)$

Theorem: For any $k \geq 3$, there exists a triangle-free graph G_k ($K(G_k) = 2$) for which $\chi(G_k) = k$.

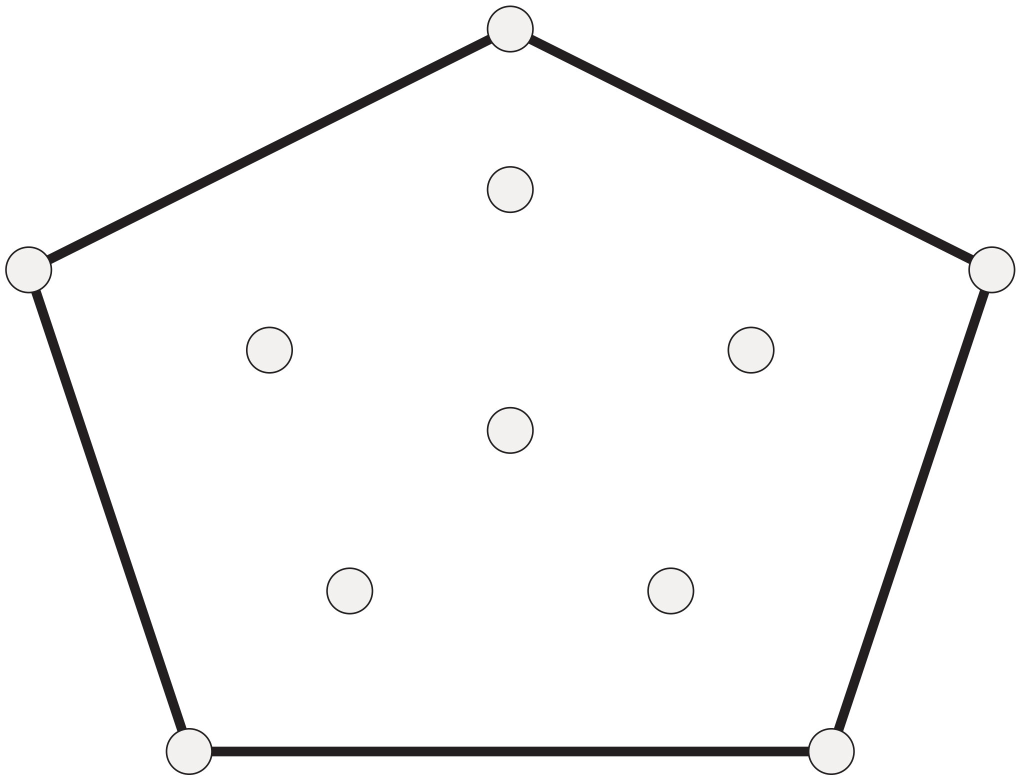
A construction: G_3 and G_4 are the examples above. Construct G_{k+1} from G_k.

- ⋆ Let $V = \{v_1, \ldots, v_n\}$ be the vertices of G_k.
- ⋆ The vertices of G_{k+1} include V, a new vertex w, and a new set of vertices $U = \{u_1, \ldots, u_n\}$ for a total of $2n+1$ vertices.
- ⋆ The edges of G_{k+1} include all the edges of G_k, w is connected to all the vertices in U, and $u_i \in U$ is connected to all the neighbors of v_i in G_k.

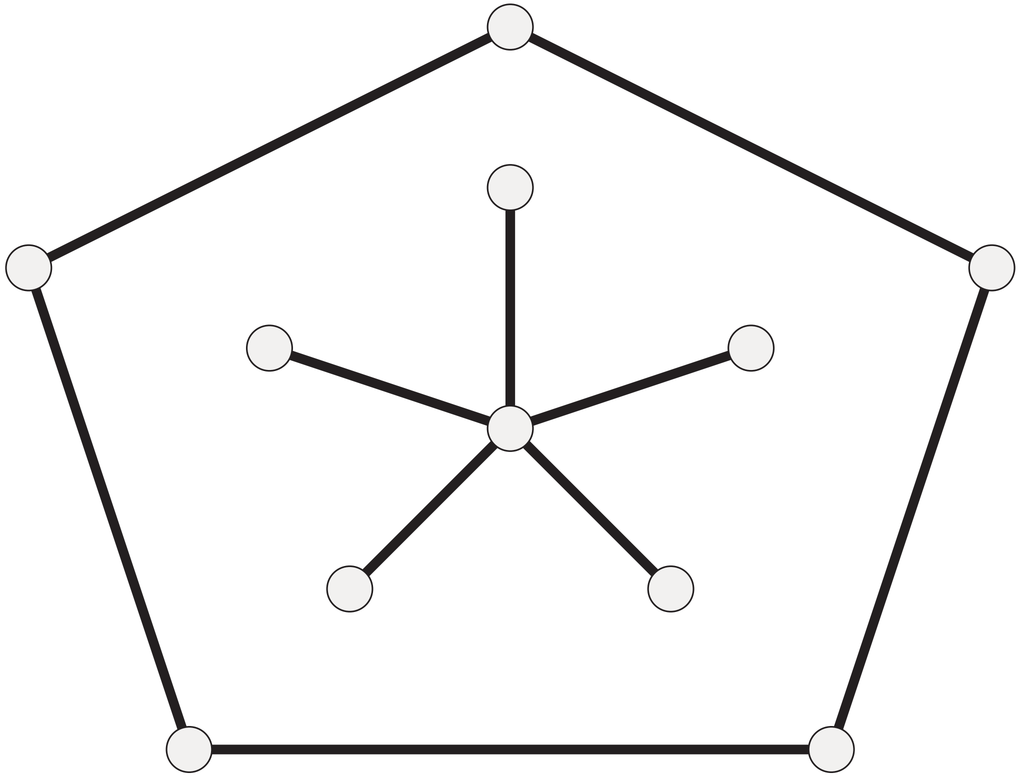
Constructing G_4 from G_3.



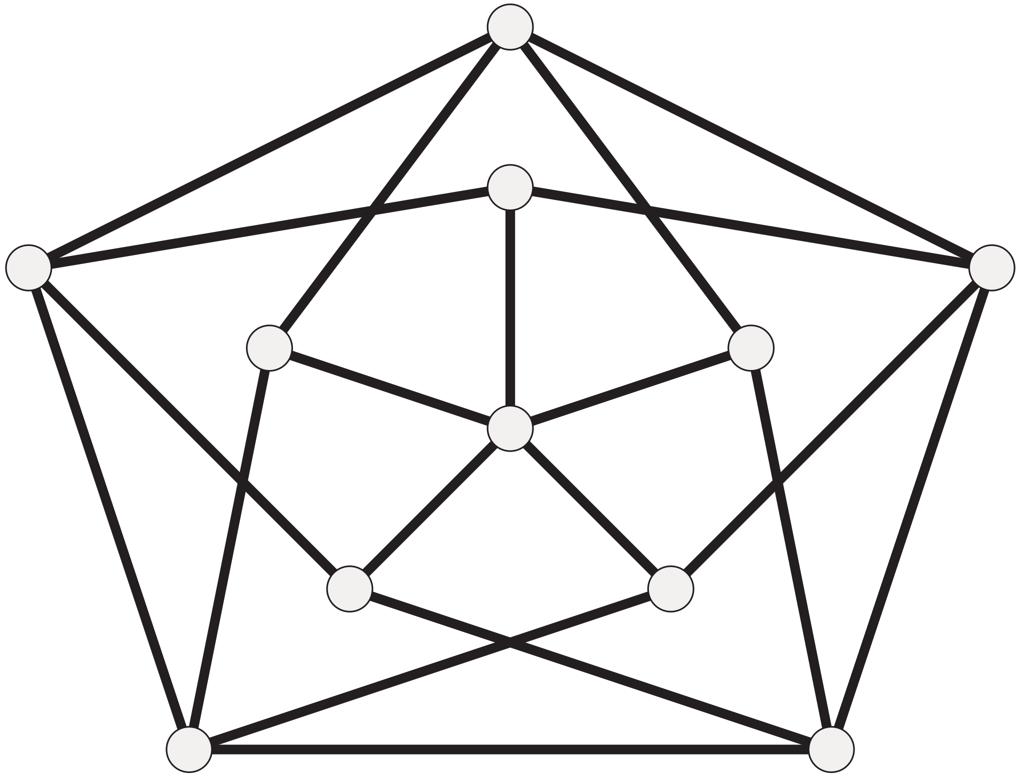
Constructing G_4 from G_3.



Constructing G_4 from G_3.



Constructing G_4 from G_3.



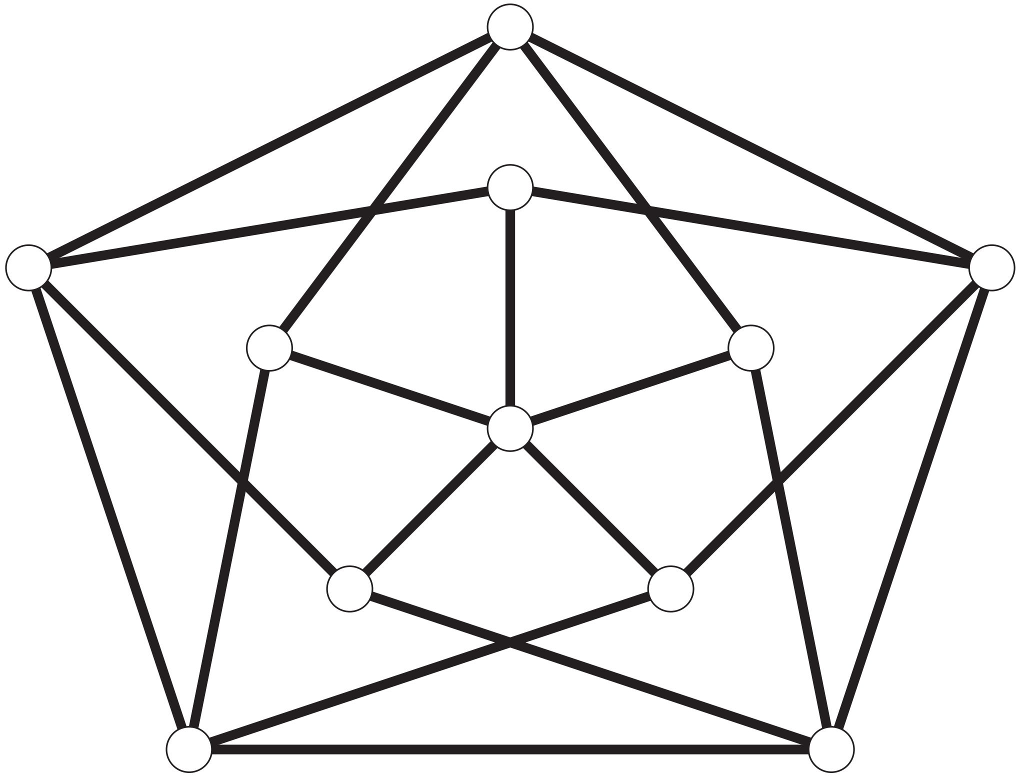
G_{k+1} is a Triangle-Free graph

- $\star$ U is an independent set in G_{k+1} and therefore there is no triangle with at least 2 vertices from U.
- $\star$ w is not adjacent to V and is adjacent to the independent set U. Therefore w cannot be a member in a triangle.
- $\star$ V contains no triangles because G_k is a triangle-free graph.
- $\star$ The remaining case is a triangle with 1 vertex $u_i \in U$ and 2 vertices $v, v' \in V$.
- $\star$ This is impossible since u_i is connected to the neighbors of v_i and therefore the triangle $u_i v v'$ would imply the triangle $v_i v v'$ in the triangle-free graph G_k.

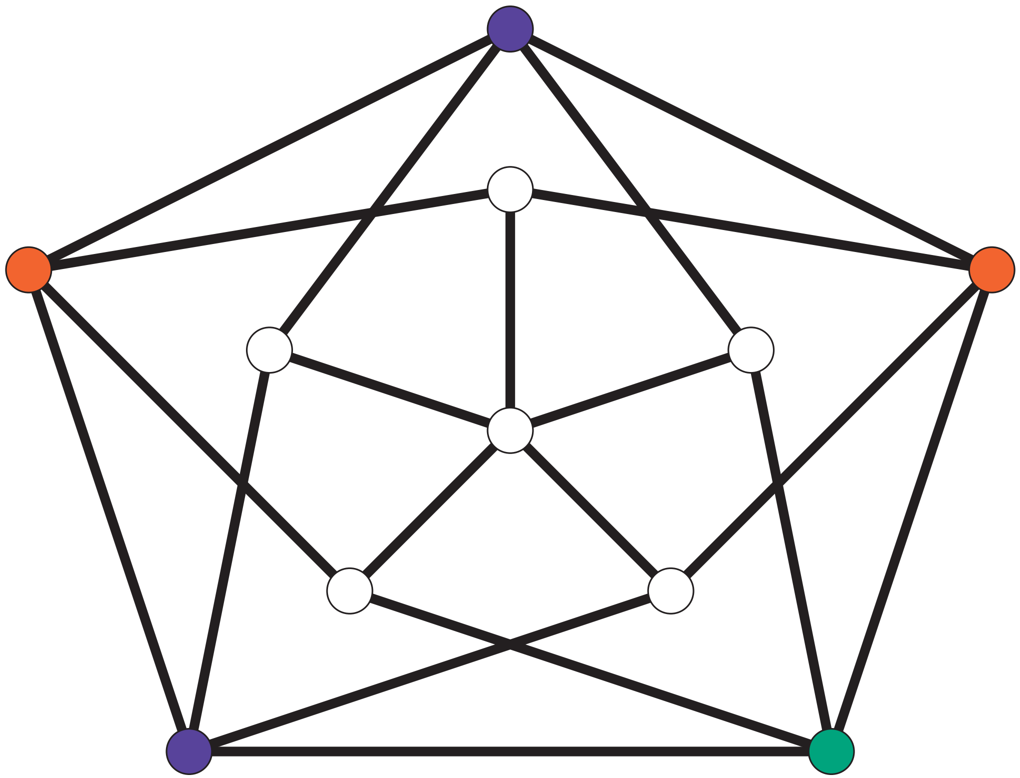
$\chi(G_{k+1}) \le k+1$

- ⋆ Color the vertices in V with k colors as in G_k.
- ⋆ Color u_i with the color of v_i. This is a legal coloring since u_i is connected to the neighbors of v_i.
- ⋆ Color w with a new color.

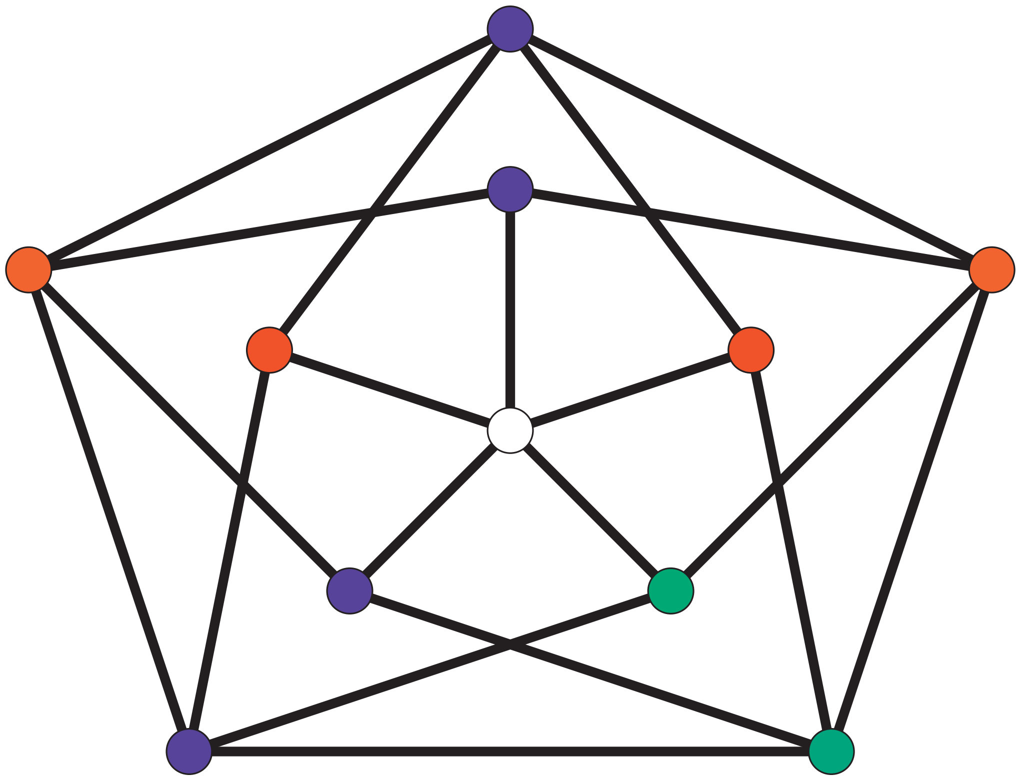
Coloring G_4



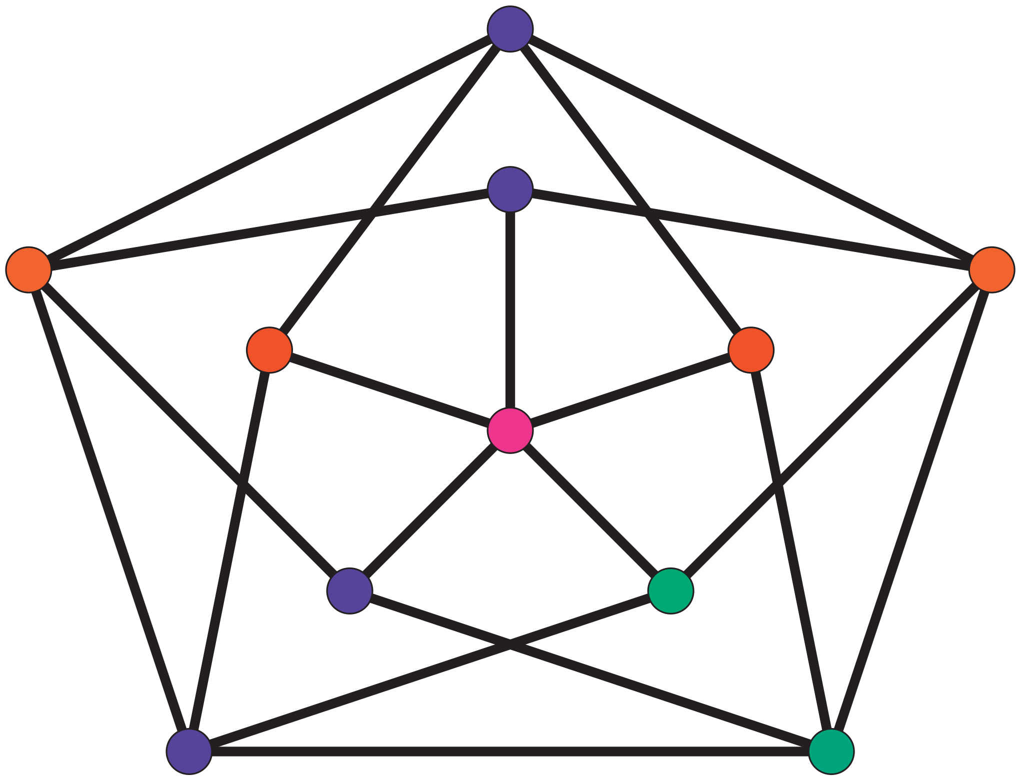
Coloring G_4



Coloring G_4



Coloring G_4



$\chi(G_{k+1}) > k$

- ⋆ Assume that G_{k+1} is colored with the colors $1, \ldots, k$.
- ⋆ Let the color of w be k.
- ⋆ Since w is adjacent to all the vertices in U it follows that the vertices in U are colored with the colors $1, \ldots, k-1$.
- ⋆ Color each v_i that is colored by k with the color of u_i.
- ⋆ This produces a legal coloring of the G_k subgraph of the G_{k+1} graph because u_i is adjacent to all the neighbors of v_i and the set of all the k-colored v_i is an independent set.
- ⋆ A contradiction since $\chi(G_k) = k$.

Perfect Graphs

- In a perfect graph $\chi(G) = K(G)$ for any "induced" subgraph of G.
- Coloring is not Hard for perfect graphs.
- The complement of a perfect graph is a perfect graph.
- Interval graphs are perfect graphs.

The Trivial Cases

Observation: A graph with $n \geq 1$ vertices needs at least 1 color and at most n colors.

- ⋆ $1 \leq \chi(G) \leq n$.

Null Graphs: No edges $\Rightarrow$ 1 color is enough.

- ⋆ $\chi(N_n) = 1$.

Complete Graphs: All edges $\Rightarrow$ n colors are required.

- ⋆ $\chi(K_n) = n$.

The Easy Case

Theorem: The following three statements are equivalent for a simple undirected graph G:

1. G is a bipartite graph.

2. There are no odd length cycles in G.

3. G can be colored with 2 colors.

Proof: $1 \Rightarrow 2$

- $\star$ The vertices of G can be partitioned into 2 sets A and B such that each edge connects a vertex from A with a vertex from B.
- $\star$ The vertices of any cycle alternate between A and B.
- $\star$ Therefore, any cycle must have an even length.

Proof: $2 \Rightarrow 3$

- ⋆ Run BFS on G starting with an arbitrary vertex.
- ⋆ Color odd-levels vertices 1 and even-level vertices 2.
- ⋆ Tree edges connect vertices with different colors.
- ⋆ In a BFS there are no forward and backward edges and a cross edge connects level ℓ with level ℓ' only if $|\ell - \ell'| \leq 1$.
- ⋆ If $\ell = \ell' + -1$ then the cross edge connects vertices with different colors.
- ⋆ If $\ell = \ell'$ then the cross edge closes an odd-length cycle contradicting the assumption.
- ⋆ Thus, all the edges connect vertices with different colors.

Proof: $3 \Rightarrow 1$

- $\star$ Let A be all the vertices with color 1 and let B be all the vertices with color 2.

- $\star$ By the definition of coloring, any edge connects a vertex from A with a vertex from B.

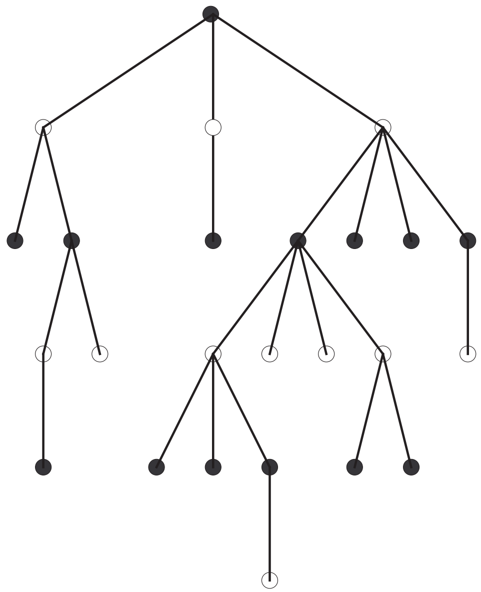
- $\star$ Therefore, the graph is bipartite.

Coloring 2-colorable graphs

- ⋆ Apply the BFS algorithm from the $2 \Rightarrow 3$ proof.
- ⋆ $O(n+m)$-time complexity using adjacency lists.
- ⋆ Can be used to recognize bipartite graphs: If there exists an edge connecting vertices with the same color then the graph is not bipartite.

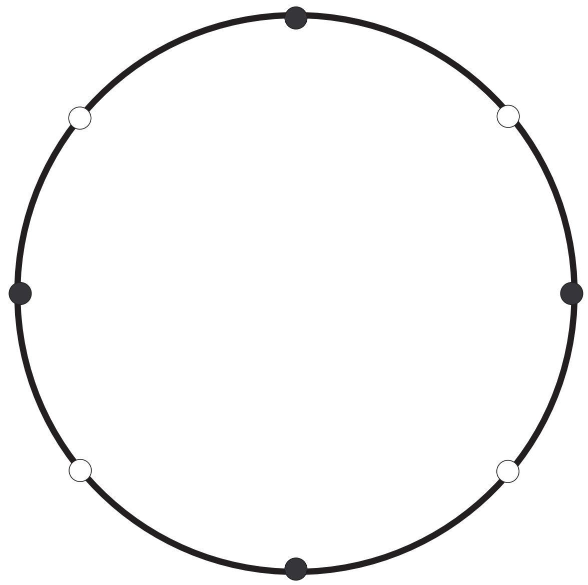
Trees

⋆ A tree is a bipartite graph and therefore can be colored with 2 colors.



Even Length Cycles

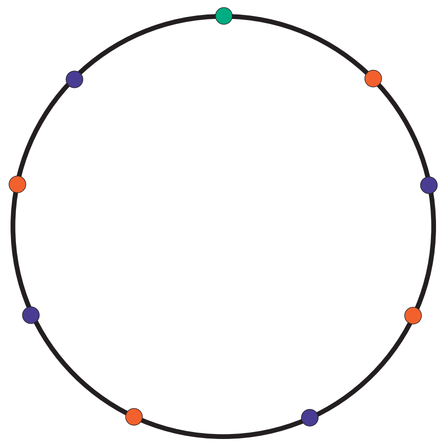
⋆ A cycle graph with an even number of vertices is a bipartite graph and therefore can be colored with 2 colors.



Odd Length Cycles

⋆ A cycle graph with an odd number of vertices is not a bipartite graph $\Rightarrow$ it cannot be colored with 2 colors.

⋆ 3-Coloring: Color one vertex 3. The rest of the vertices induce a bipartite graph and therefore can be colored with colors 1 and 2.



Greedy Vertex Coloring

Theorem: Let Δ be the maximum degree in G. Then G can be colored with $\Delta + 1$ colors.

Proof:

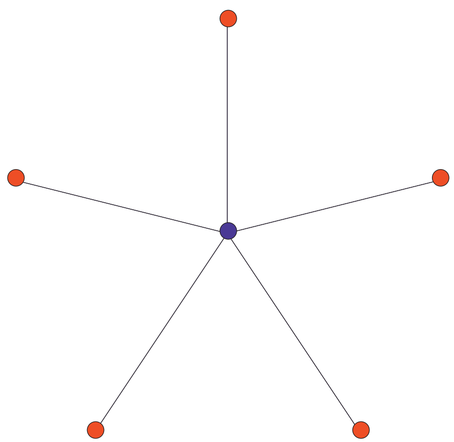
- ⋆ Color the vertices in a sequence.
- ⋆ A vertex is colored with a free color, one that is not the color of one of its neighbors.
- ⋆ Since the maximum degree is Δ, there is always a free color among $1, 2, \ldots, \Delta + 1$ (a pigeon hole argument).

A Proof by Induction

- ⋆ Assume G has n vertices.
- ⋆ The theorem is true for a graph with 1 vertex since $\Delta = 0$.
- ⋆ Let $n \geq 2$ and assume that the theorem is correct for any graph with $n - 1$ vertices.
- ⋆ Omit an arbitrary vertex and all of its edges from the graph.
- ⋆ By the induction hypothesis the remaining graph can be colored with at most $\Delta + 1$ colors.
- ⋆ Since the degree of the omitted vertex is at most Δ it follows that one of the colors $1, \ldots, \Delta + 1$ will be available to color the omitted vertex (a pigeon hole argument).

$\Delta >> \chi(G)$

- ⋆ The star graph is a bipartite graph and therefore can be colored with 2 colors.
- ⋆ $\Delta = n-1$ in a star graph. The above theorem guarantees a performance that is very far from the optimal performance.



First-Fit Implementation

⋆ Consider the vertices in any sequence.

⋆ Color a vertex with the smallest available color.

Greedy Coloring (G)

for $i = 1$ to n

$c = 1$

while $(\exists_j \{(i, j) \in E\})$ AND $(c(j) = c)$

$c = c + 1$

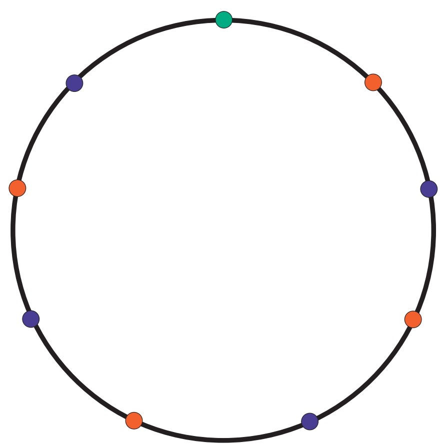
$c(i) = c$

Complexity: Possible in $O(m + n)$ time.

Sometimes Greedy is optimal

Complete graphs: $\Delta = n - 1$ and $n = \Delta + 1$ colors are required.

Odd-length cycles: $\Delta = 2$ and 3 colors are required.

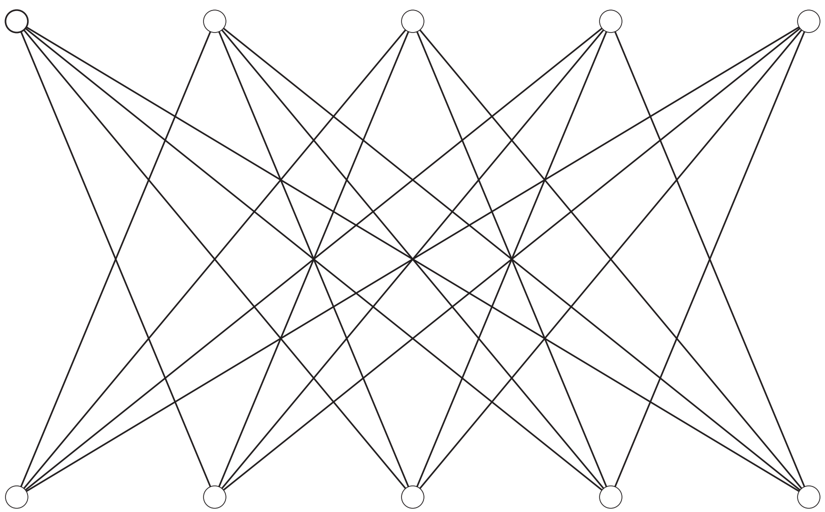


The order of the vertices is crucial

⋆ A bipartite graph G.

– $2k$ vertices $v_1, v_2, \ldots, v_k$ and $u_1, u_2, \ldots, u_k$.

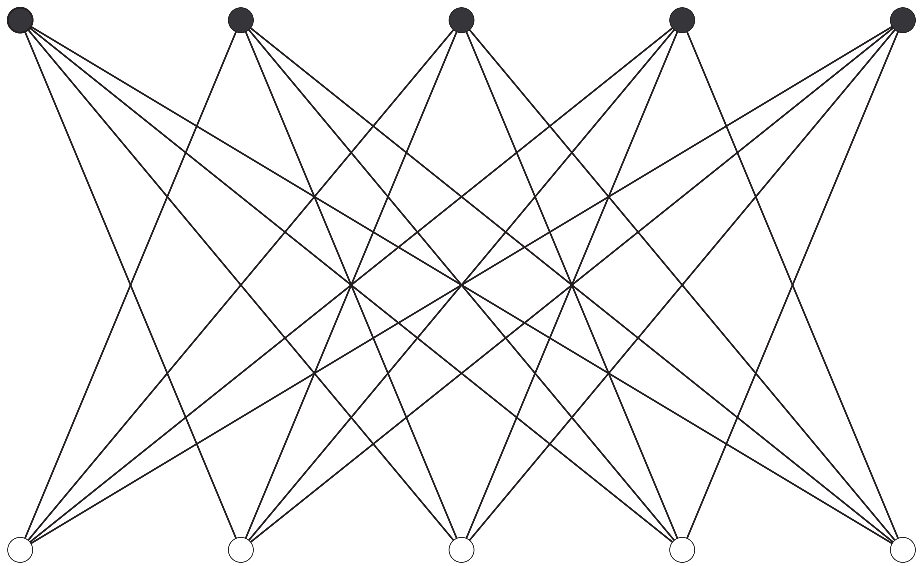
– All (v_i, u_j) edges for $1 \leq i \neq j \leq k$.



A Good Order

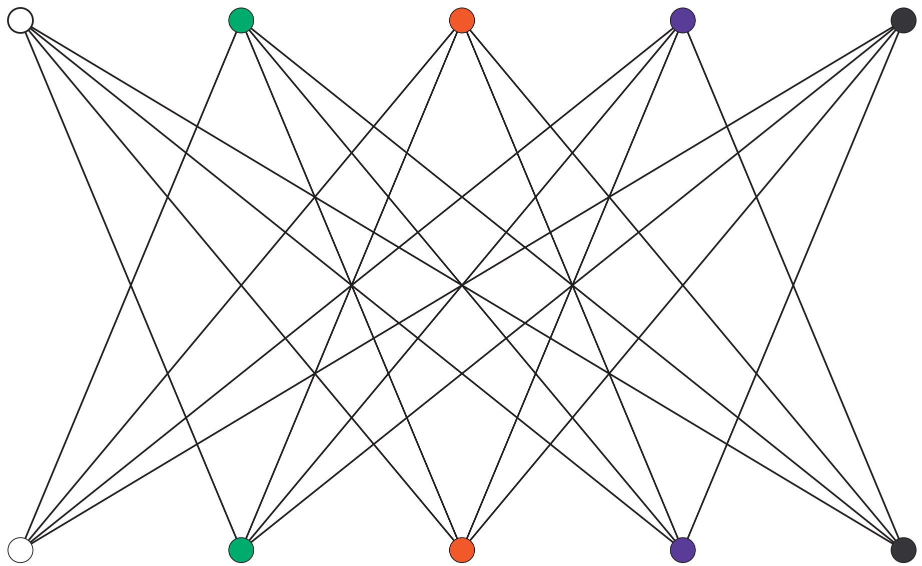
⋆ Suppose the order is $v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_k$.

– The algorithm colors G with 2 colors.



A Bad Order

- Suppose the order is $v_1, u_1, , v_2, u_2, \ldots, v_k, u_k$.
  - The algorithm colors G with k colors.



Greedy with a Decreasing Order of Degrees

Notation: Let the vertices be $v_1, v_2, \ldots, v_n$ and let their degrees be $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n$.

Theorem: $\chi(G) \leq \max_{1 \leq i \leq n} \min \{d_i + 1, i\}$.

Proof:

- ⋆ The input order for greedy is $v_1, v_2, \ldots, v_n$.
- ⋆ When coloring v_i at most $i - 1$ colors are used by its neighbors since greedy has colored only $i - 1$ vertices.
- ⋆ When coloring v_i at most d_i colors are used by its neighbors because the degree of v_i is d_i.

Back Degrees

Notation:

- ⋆ Let the vertices be $v_1, v_2, \ldots, v_n$ and let their degrees be $d_1, d_2, \ldots, d_n \leq \Delta$.
- ⋆ Let $d_i' \leq d_i$ be the number of neighbors of v_i among $v_1, \ldots, v_{i-1}$ (in particular: $d_i' \leq i-1$).

Theorem: $\chi(G) \leq \max_{1\leq i\leq n}\{d_i' + 1\}$.

Proof:

- ⋆ The input order for greedy is $v_1, v_2, \ldots, v_n$.
- ⋆ When coloring v_i at most d_i' colors are used by its neighbors.

A Marginal Improvement to the Greedy Algorithm

Theorem: A connected non-clique G can be colored with Δ colors where $\Delta \geq 3$ is the maximum degree in G.

Cliques: K_n requires $n = \Delta + 1$ colors.

Cycles: C_n, for an odd n, requires $3 = \Delta + 1$ colors.

Proof

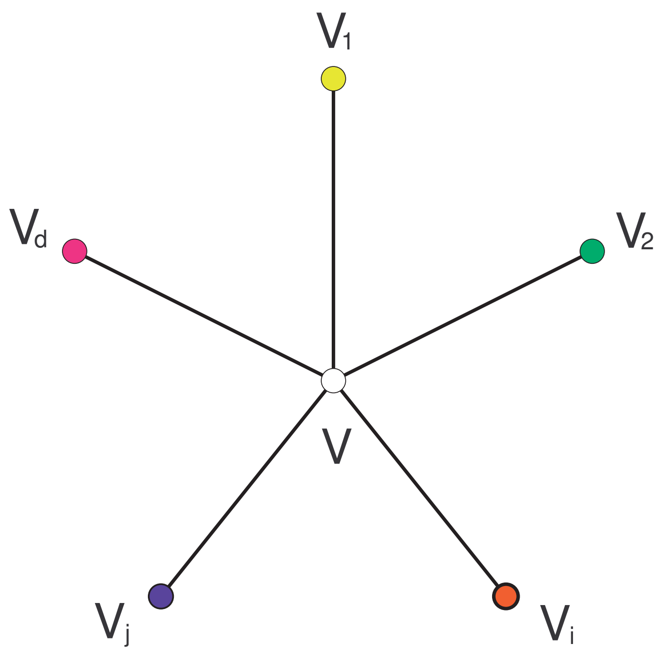
- By induction implying an algorithm.
- Let v be an arbitrary vertex with degree $d(v)$.
- Let $G' = G \setminus \{v\}$:
  - If G' is not a clique or a cycle, then color it recursively with Δ colors.
  - If G' is a clique, then it is a K_Δ graph that can be colored with Δ colors. G' cannot be a $K_{\Delta+1}$ graph since then the neighbors of v would have degree $\Delta + 1$.
  - If G' is a cycle, then it can be colored with $3 \leq \Delta$ colors.

Proof Continue

- ⋆ If $d(v) \leq \Delta - 1$, then color v with a free color (pigeon hole argument).
- ⋆ If v has 2 neighbors colored with the same color, then color v with a free color (pigeon hole argument).
- ⋆ From now on assume that $d(v) = \Delta$ and that each neighbor of v is colored with a different color.

Proof Continue

⋆ Let the neighbors of v be v_1,v_2,. . . ,v_Δ and let their colors be c_1,c_2,. . . ,c_Δ respectively.



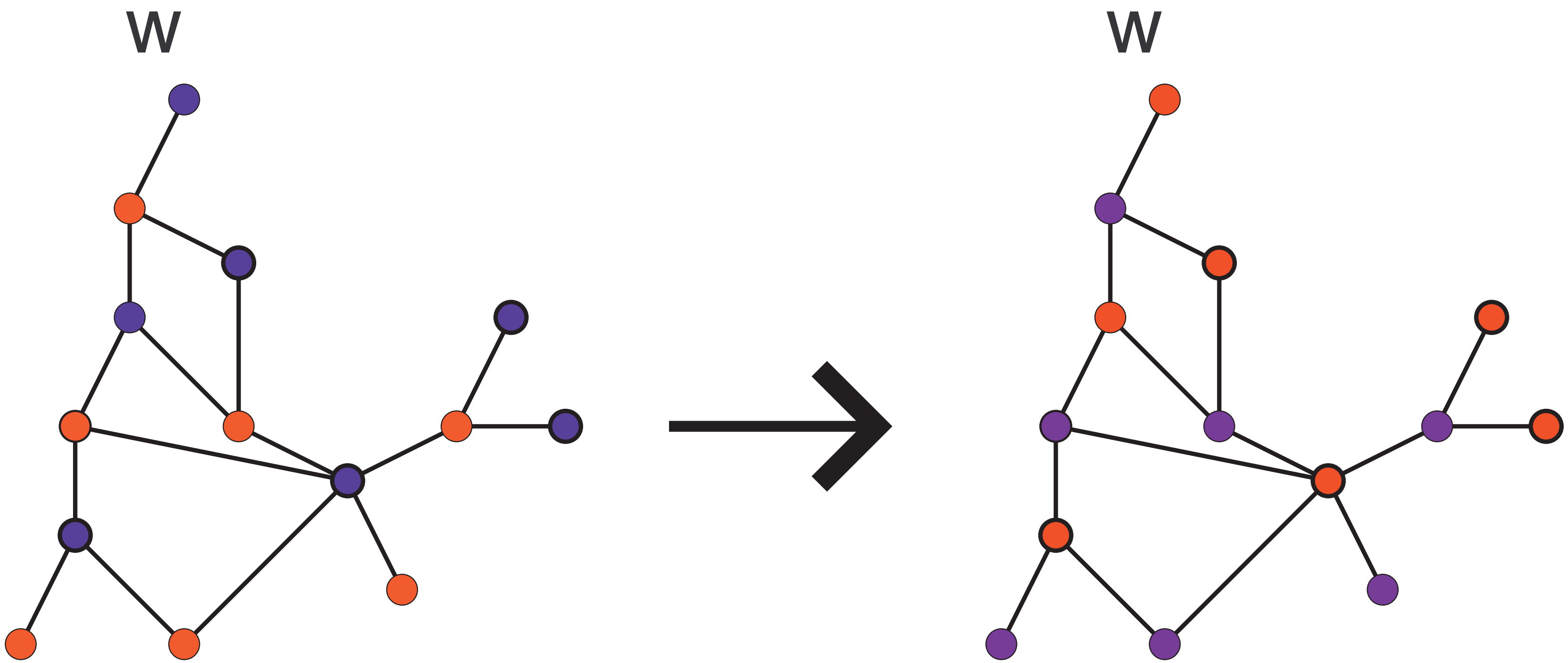
Definitions and an Observation

- ⋆ For colors x and y, let $G(x, y)$ be the subgraph of G containing only the vertices whose colors are x or y.

- ⋆ For a vertex w whose color is x, let $G_w(x, y)$ be the connected component of $G(x, y)$ that contains w.

- ⋆ Interchanging the colors x and y in the connected component $G_w(x, y)$ of $G(x, y)$ results with another legal coloring in which the color of w is y.

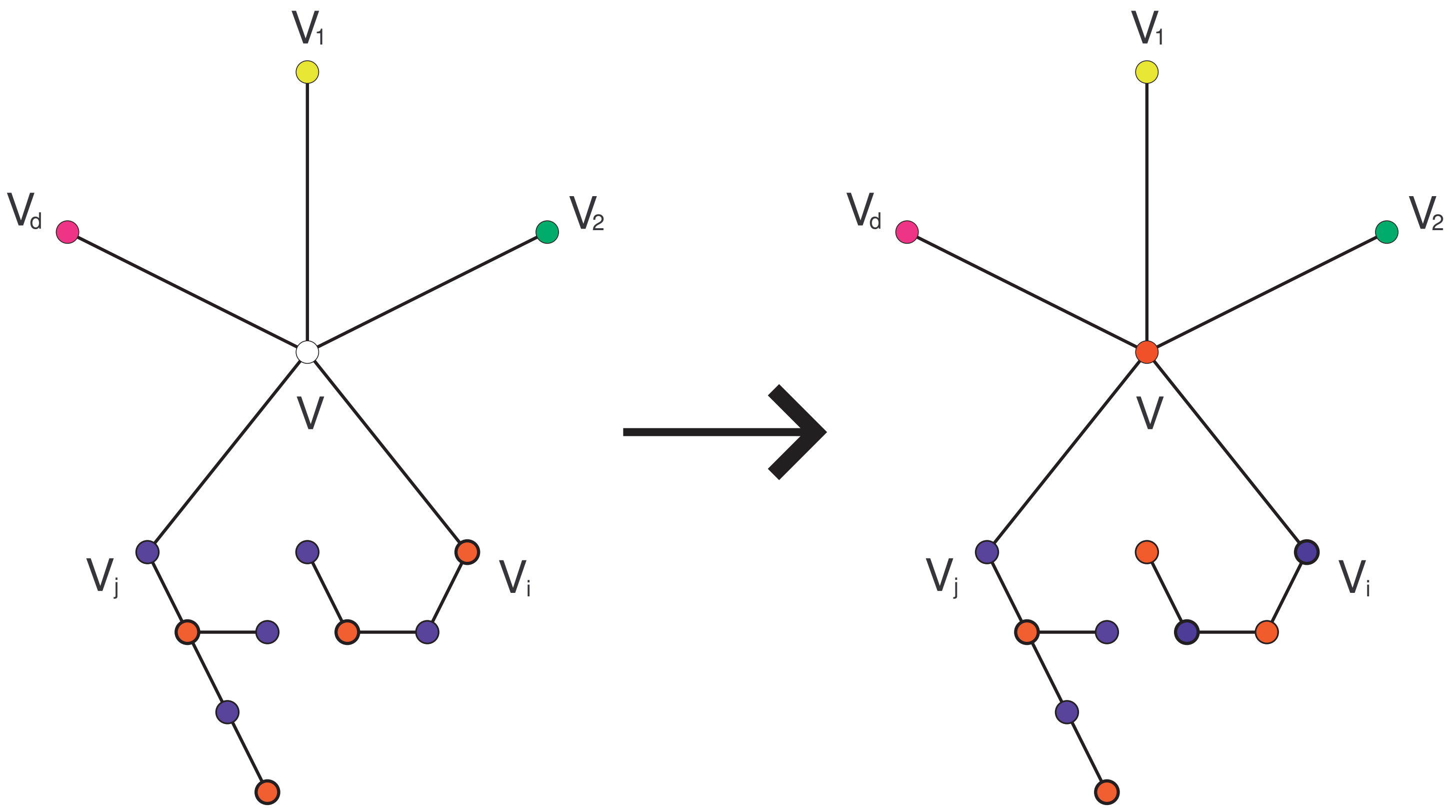
The Observation



Proof Continue

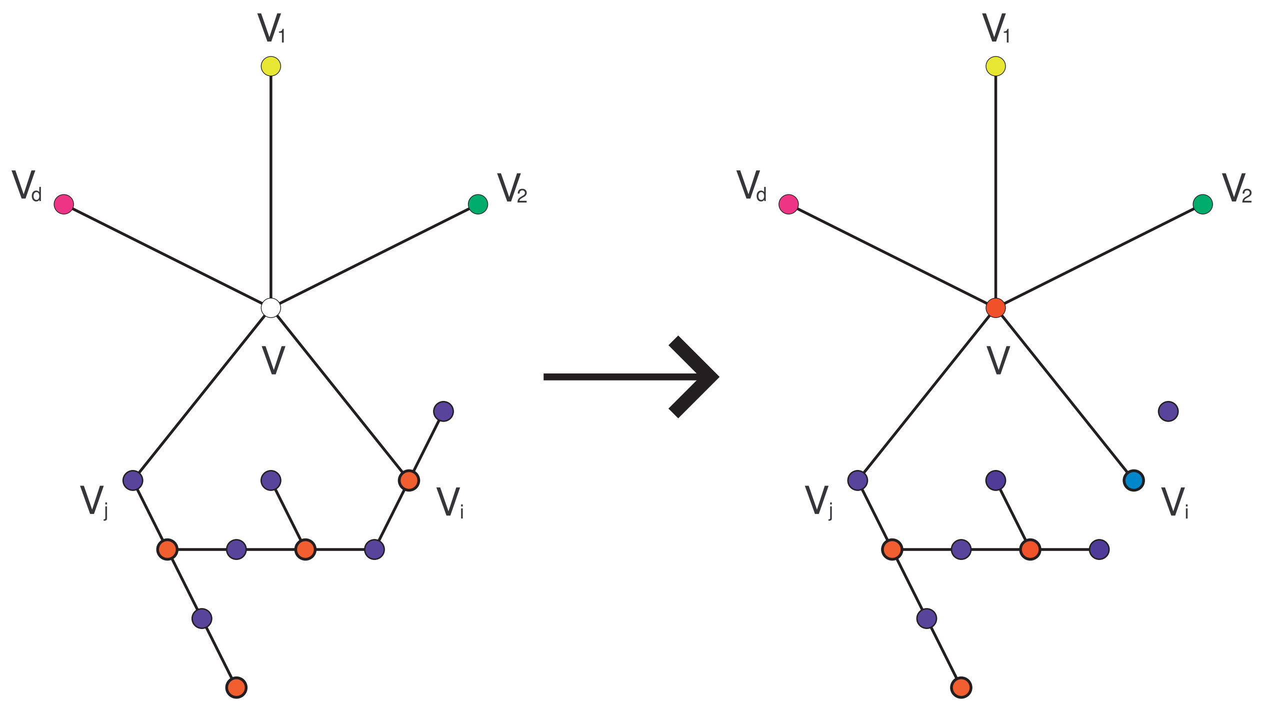
- ⋆ Let v_i and v_j be any 2 neighbors of v.
- ⋆ If $G_{v_i}(c_i, c_j)$ does not contain v_j, then interchange the colors c_i and c_j in $G_{v_i}(c_i, c_j)$.
- ⋆ The color of both v_i and v_j is now c_j and no neighbor of v is colored with c_i.
- ⋆ Color v with c_i.

Proof Continue



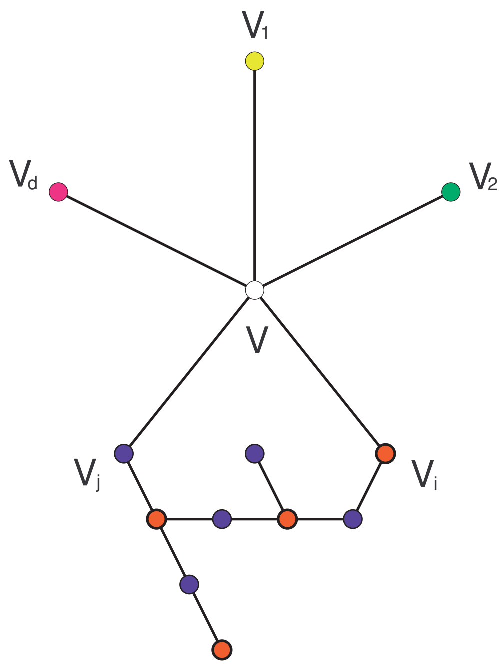
Proof Continue

⋆ If v_i has 2 neighbors colored with c_j, then color v_i with a different color and color v with c_i.



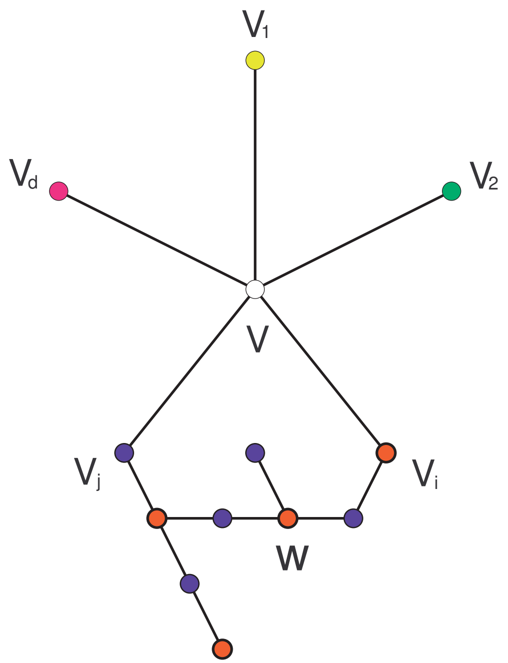
Proof Continue

⋆ From now on assume that v_i and v_j belong to the same connected component in $G(c_i, c_j)$ and that v_i has only 1 neighbor colored with c_j.



Proof Continue

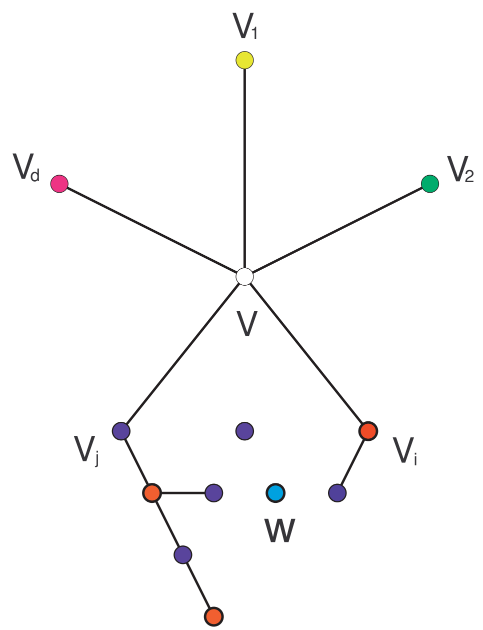
⋆ If $G_{v_i}(c_i, c_j)$ is not a path, then let $w \in G_{v_i}(c_i, c_j)$ be the closest to v_i whose color is c_i (or c_j) and who has more than 2 neighbors whose colors are c_j (or c_i).



Proof Continue

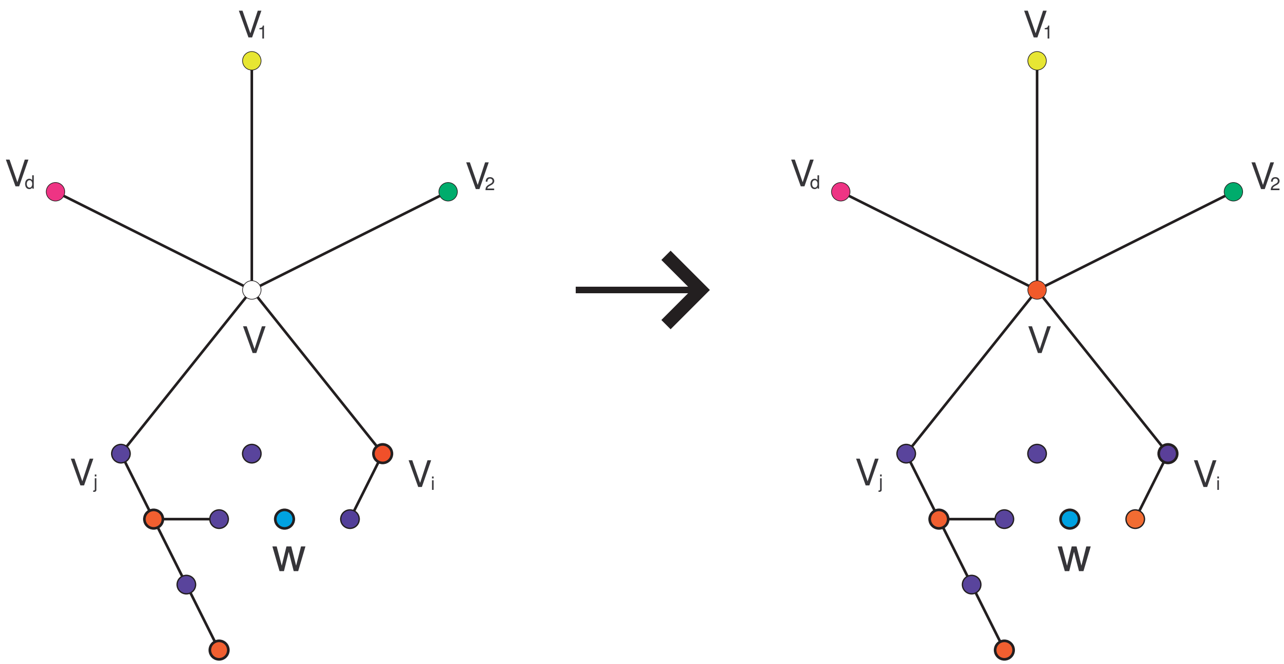
⋆ Color w with a different color.

⋆ v_i and v_j are not anymore in the same connected component of $G(c_i, c_j)$.



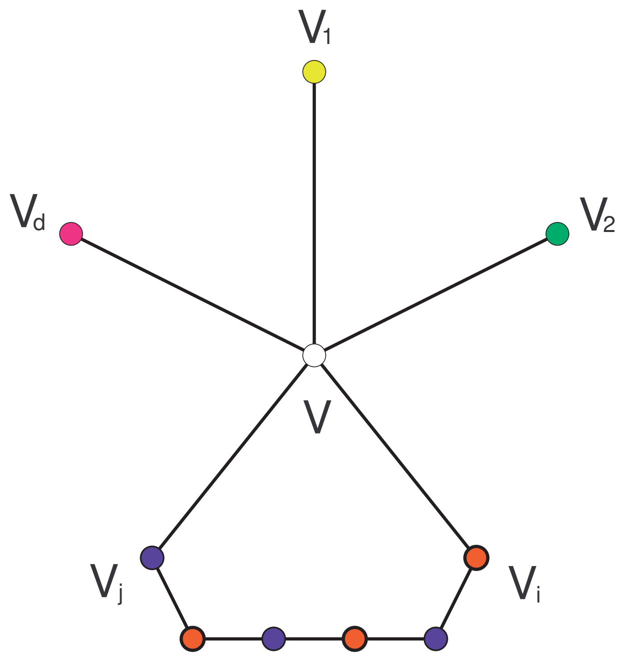
Proof Continue

- ⋆ Interchange the colors c_i and c_j in $G_{v_i}(c_i, c_j)$.
- ⋆ The color of both v_j and v_i is now c_j and no neighbor of v is colored with c_i: Color v with c_i.



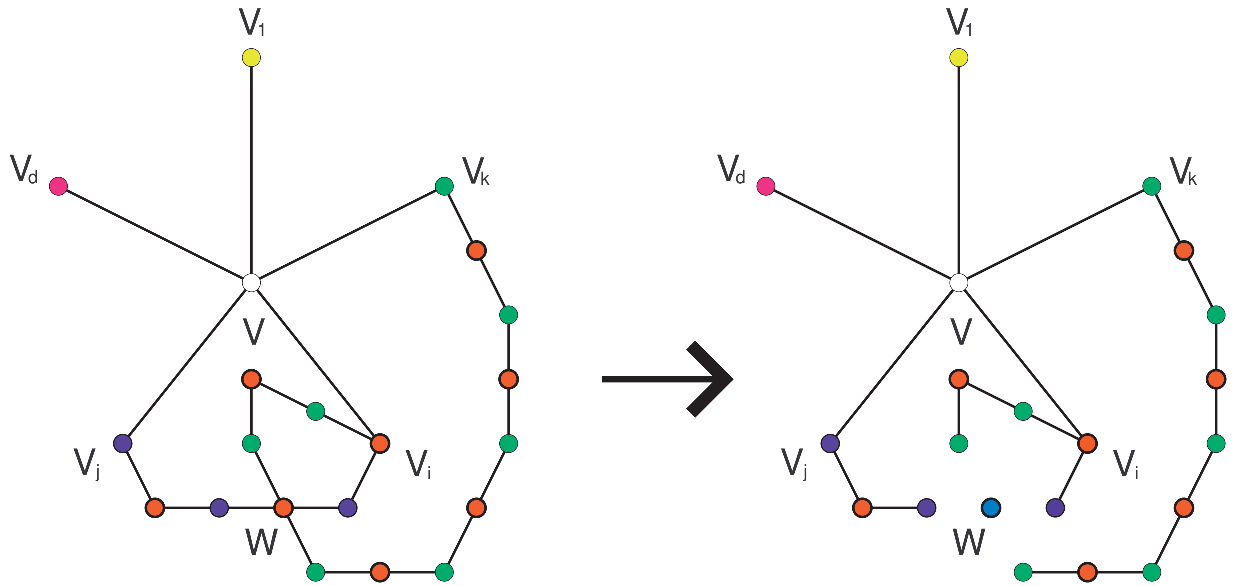
Proof Continue

⋆ From now on assume that for any 2 neighbors v_i and v_j of v, the subgraph $G_{v_i}(c_i, c_j)$ is a path (could be an edge) starting with v_i and ending with v_j.



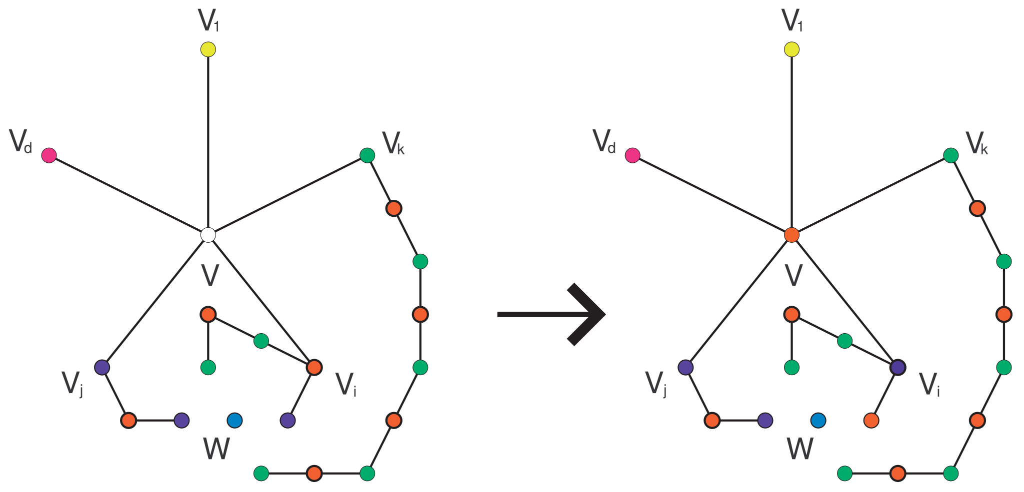
Proof Continue

- ⋆ If for some v_k the path $G_{v_i}(c_i, c_k)$ intersects the path $G_{v_i}(c_i, c_j)$ in a vertex $w \neq v_i$ whose color is c_i, then w has 2 neighbors colored c_k and 2 neighbors colored c_j.
- ⋆ Color w with a different color than c_i, c_j, c_k.



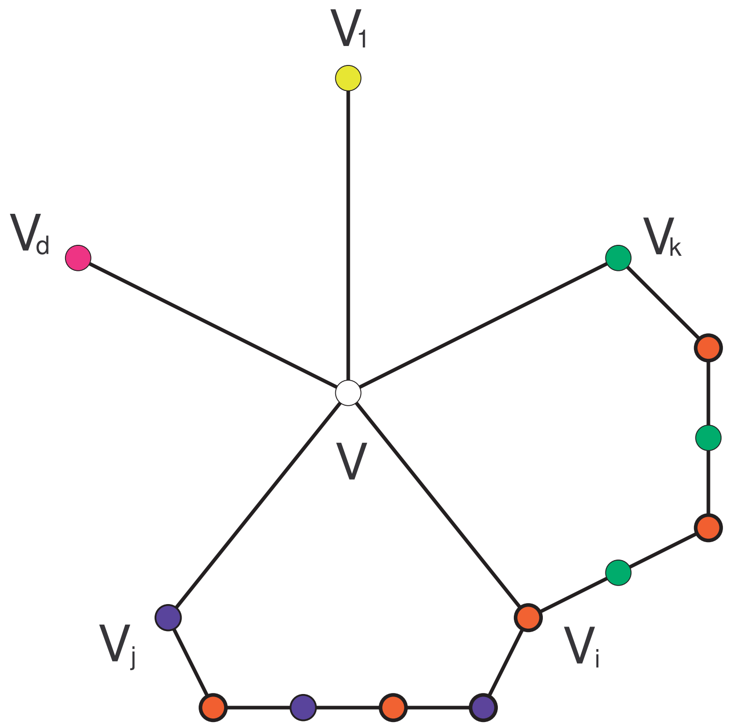
Proof Continue

- ⋆ v_i and v_j are not anymore in the same connected component of $G(c_i, c_j)$.
- ⋆ Interchange the colors c_i and c_j in $G_{v_i}(c_i, c_j)$.
- ⋆ The color of both v_j and v_i is now c_j and no neighbor of v is colored with c_i: Color v with c_i.



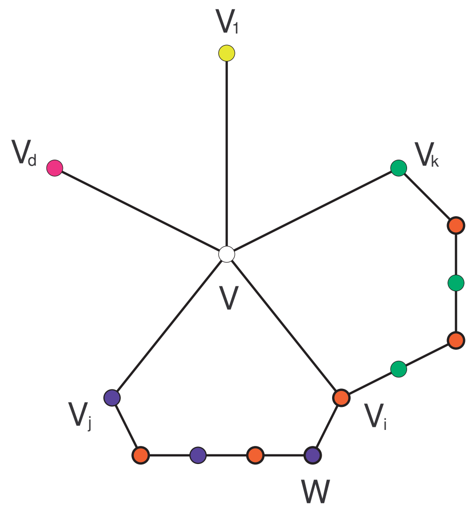
Proof Continue

⋆ From now on assume that the path $G_{v_i}(c_i, c_k)$ intersects the path $G_{v_i}(c_i, c_j)$ only at v_i.



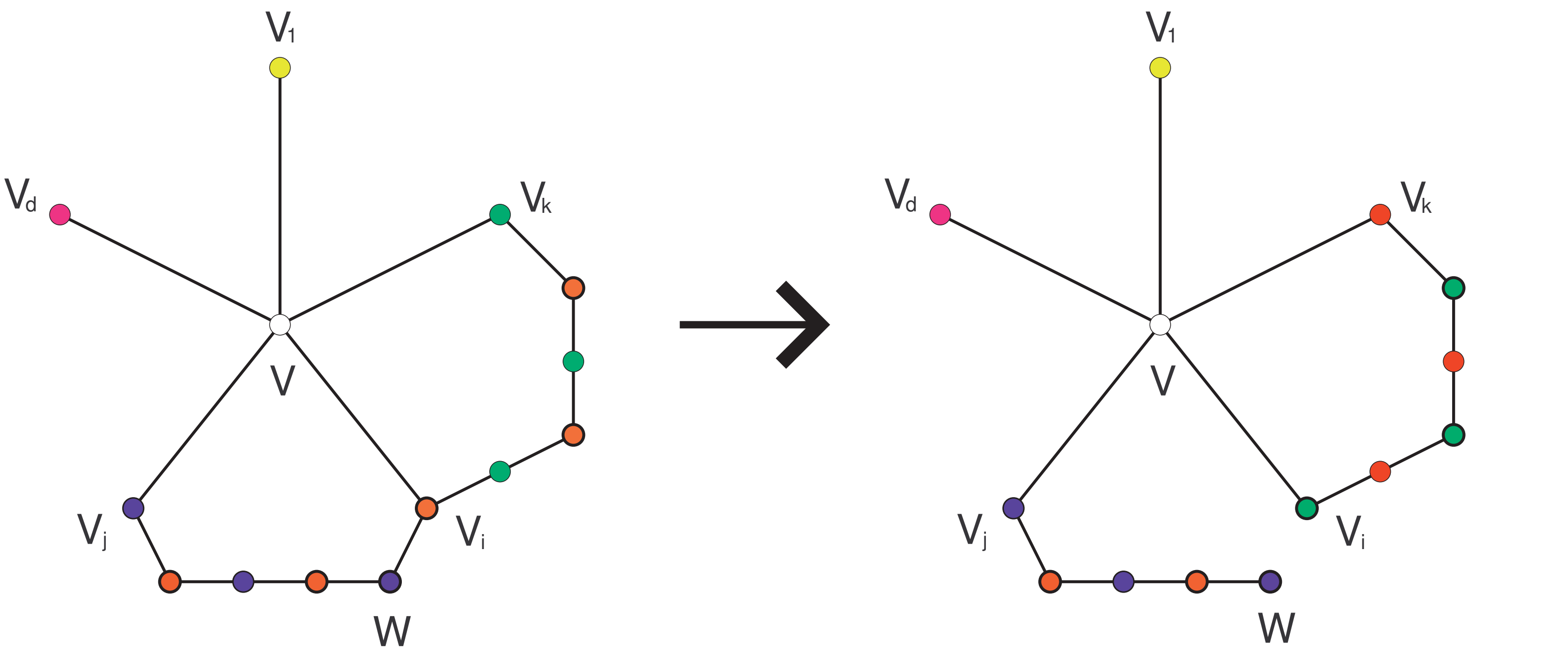
Proof Continue

- ⋆ By assumption the graph is not a clique. Therefore, there exist 2 neighbors of v, v_i and v_j, that are not adjacent. Let w be the c_j neighbor of v_i.
- ⋆ By assumption $\Delta \geq 3$. Therefore, there exists another neighbor of v, v_k that is different than v_i and v_j.



Proof Continue

- ⋆ Interchange the colors c_i and c_k in $G_{v_i}(c_i, c_k)$.
- ⋆ The color of v_i is c_k and the color of v_k is c_i.



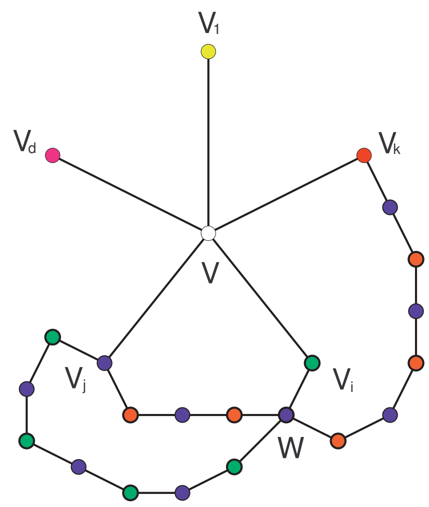
Proof Continue

⋆ Repeat the arguments as before and assume that

– $G_{v_j}(c_j, c_i)$ is a path from v_j to v_k.

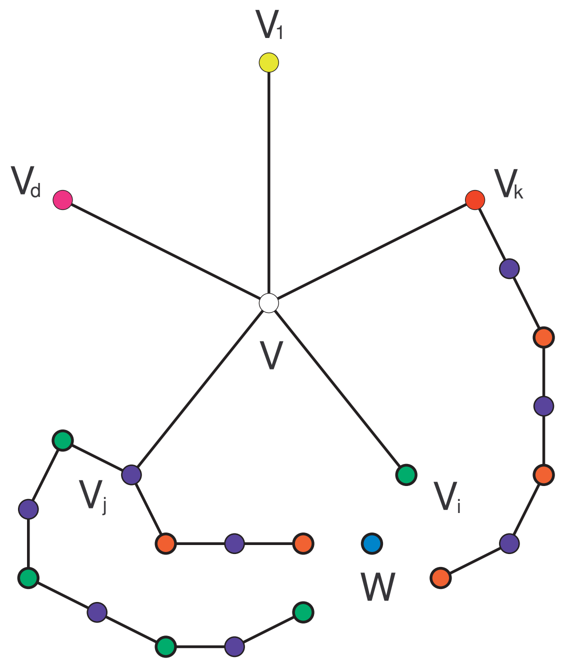
– $G_{v_j}(c_j, c_k)$ ia a path from v_j to v_i.

⋆ These paths must intersect with w because w is the only c_j neighbor of v_i.



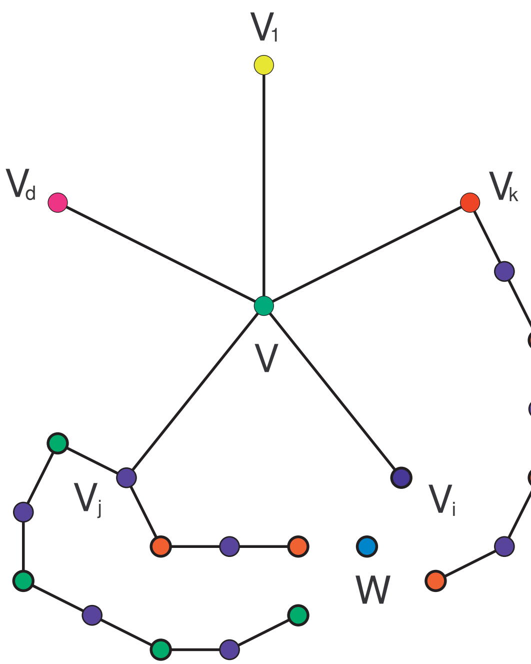
Proof Continue

- ⋆ Color w with a different color than c_i, c_j, c_k.
- ⋆ v_i has no c_j neighbor.



Proof End

- ⋆ Color v_i with c_j.
- ⋆ Color v with c_k.



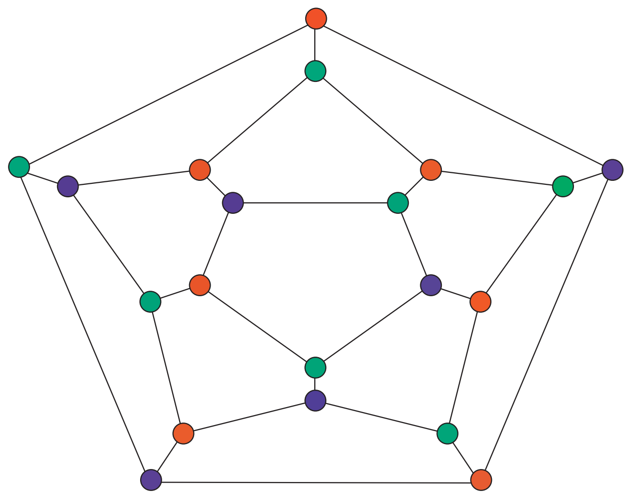
Complexity

- Possible in $O(nm)$.
- Each correction can be done in $O(m)$.

Cubic Graphs

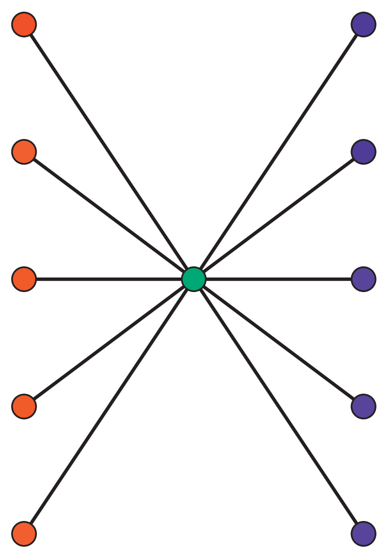
Definition: A cubic graph is a regular graph in which the degree of every vertex is 3.

Corollary: The chromatic number of a non-bipartite cubic graph that is not K_4 is 3.



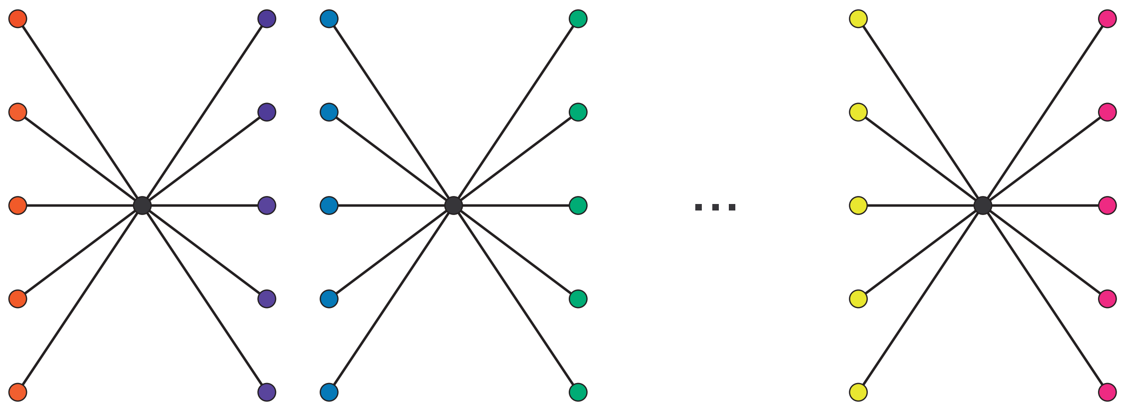
Coloring 3-Colorable Graphs with $O(\sqrt{n})$ colors

Observation: In 3-colorable graphs, the subgraph containing only the neighbors of a particular vertex is a 2-colorable graph (a bipartite graph).



Algorithm

- ⋆ Let G be a 3-colorable graph.
- ⋆ Allocate 3 colors to a vertex and all of its neighbors if the degree of this vertex is larger than $\sqrt{n}$.
- ⋆ There are at most $\sqrt{n}$ such vertices and therefore so far at most $3\sqrt{n}$ colors were used.



Algorithm

- ⋆ Now, all the degrees in the graph are less than $\sqrt{n}$.
- ⋆ The greedy algorithm needs at most $\sqrt{n}$ colors to color the rest of the graph.
- ⋆ All together, the algorithm uses $O(\sqrt{n})$ colors.
  - – If all omitted vertices are colored with the same color, then at most $2\sqrt{n}+1$ colors are used before applying the greedy algorithm.
  - – Therefore, the algorithm uses about $3\sqrt{n}$ colors.