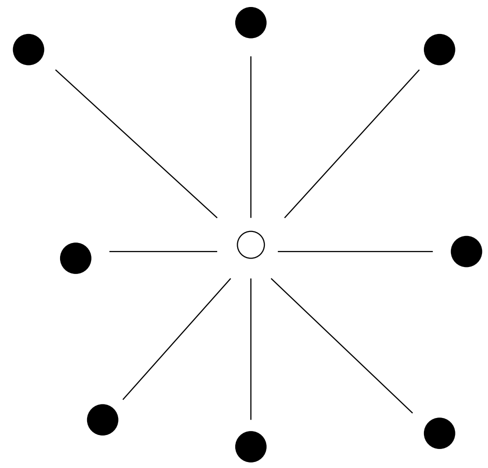
Vertex Coloring

Consider a graph $G = (V, E)$

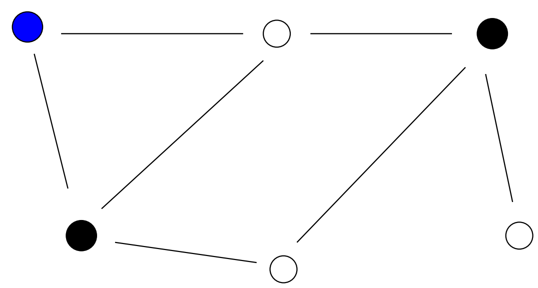


Edge coloring: no two edges that share an endpoint get the same color

Vertex coloring: no two vertices that are adjacent get the same color

Use the minimum amount of colors

This is the chromatic number



Number between 1 and $|V|$ (why?)

Lower bound

It is hard to approximate the chromatic number with approximation ratio of at most

$$n^{1-\varepsilon}$$

for every fixed $\varepsilon > 0$, unless NP=ZPP

ZPP = Zero-error Probabilistic Polynomial time

Problems for which there exists a probabilistic Turing machine that

- always gives the correct answer,
- has unbounded running time,
- runs in polynomial-time on average

Additive approximations

- Instead of

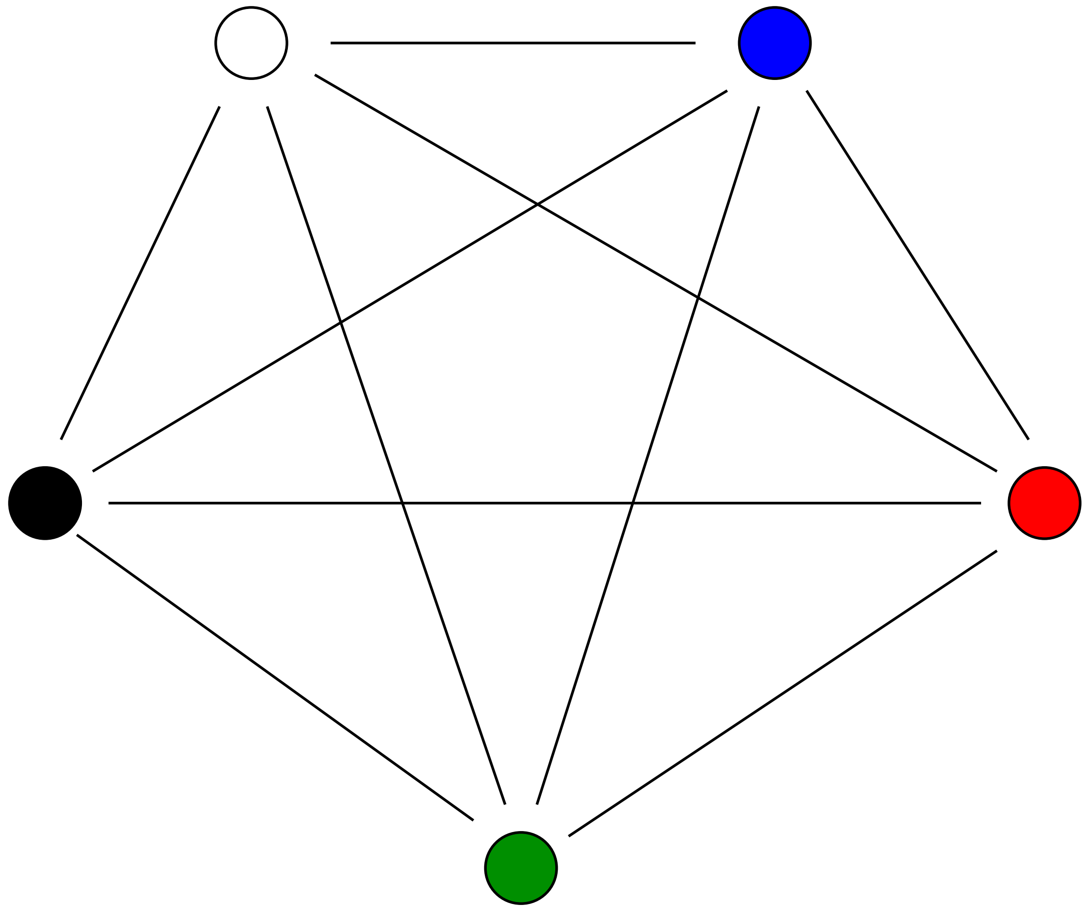
$$A(\sigma) \leq R \cdot \mathrm{OPT}(\sigma)$$

  we require

$$A(\sigma) \leq \mathrm{OPT}(\sigma) + c$$

- Denote the maximum degree of a node in G by $\Delta(G)$
- We can always color a graph with $\Delta(G) + 1$ colors
- This is sometimes required
- Some graphs require far less colors

A graph that requires $\Delta(G) + 1$ colors

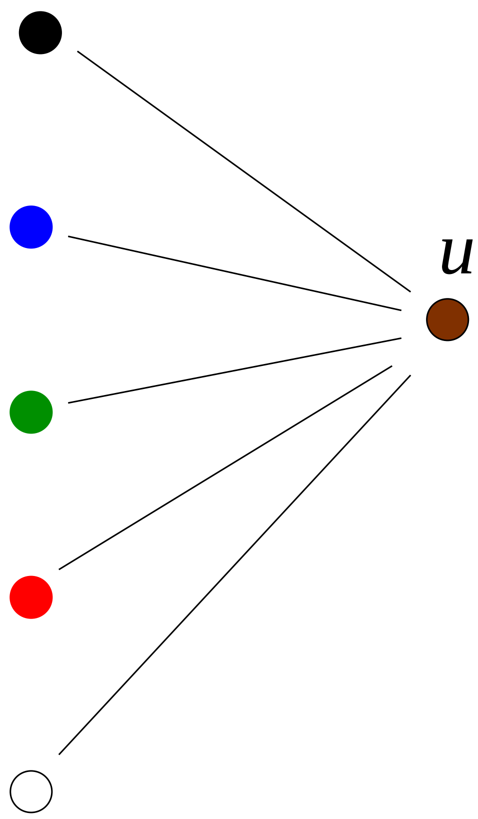


$\Delta(G) = 4$

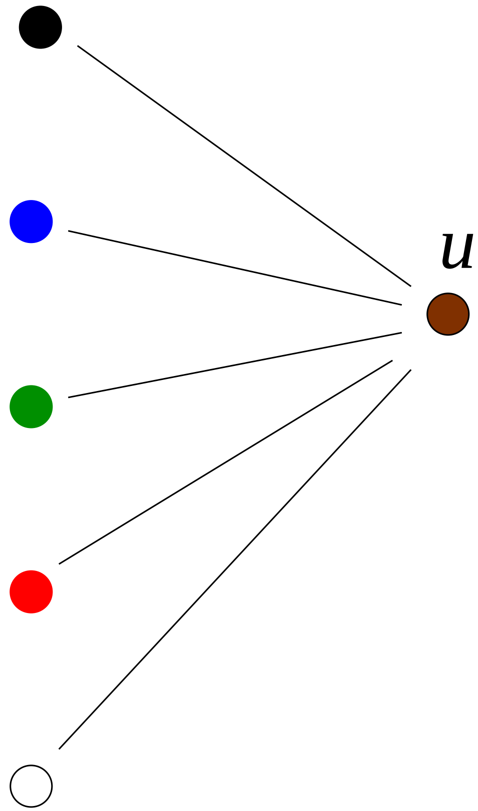
Greedy Algorithm 1

Colors are indicated by numbers $1,2,\ldots$

- Consider the nodes in some order
- At the start, each node is uncolored (has color 0)
- Give each node the smallest color that is not used to color any neighbor



Analysis



- Running time: $O(|V| + |E|)$ (how?)
- Needs at most $\Delta(G) + 1$ colors:
  - Consider a node u
  - It has at most $\Delta(G)$ neighbors
  - Among the colors $1, \ldots, \Delta(G) + 1$, there must be an unused color

Analysis

What is the difference with $\mathrm{OPT}(G)$?

We only consider graphs with at least one edge.

Then $\mathrm{OPT}(G) \geq 2$.

But then $\mathrm{Greedy}(G) - \mathrm{OPT}(G) \leq \Delta(G) + 1 - 2 = \Delta(G) - 1$.

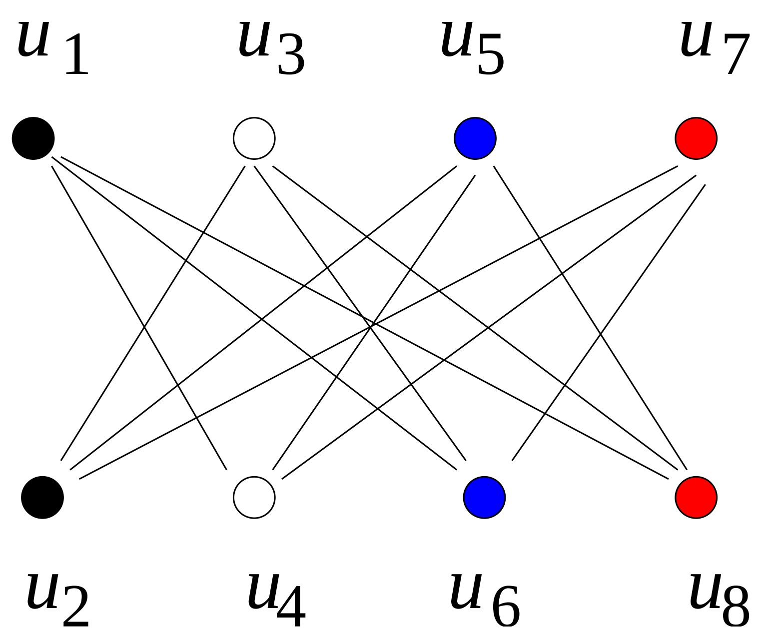
This bound is tight!

There are graphs G such that $\mathrm{Greedy}(G) - \mathrm{OPT}(G) = \Delta(G) - 1$.

Lower bound

We use a nearly complete bipartite graph

Greedy considers the nodes in order from left to right, OPT = 2.



This example can be generalized

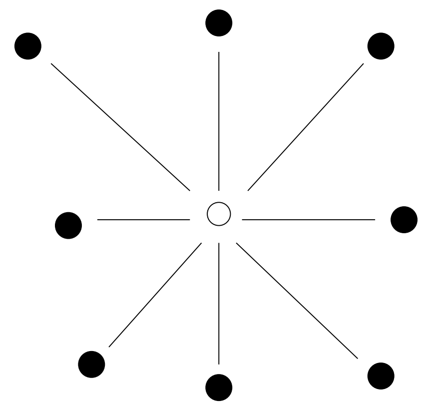
Greedy needs $\Delta(G)+1$ colors

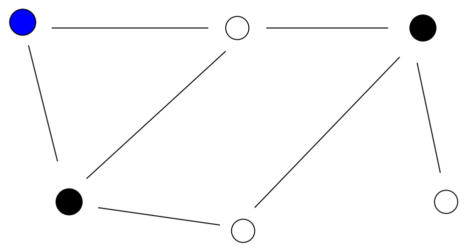
Analysis

- The chromatic number $\Delta(G)$ can be $\Theta(n)$
- For such graphs, Greedy performs very poorly
- However, nothing much better is possible
  (unless NP = ZPP)
- We show an algorithm that uses $O(n/\log n)$ colors
- On planar graphs, we can do much better

Greedy algorithm 2

- For any color, the vertices with this color form an independent set
- Recall that we can find a maximal independent set in polynomial time
- We look for a large independent set U in a greedy fashion
- U gets one color, is removed from the graph, and we repeat
- Continue until the graph is empty

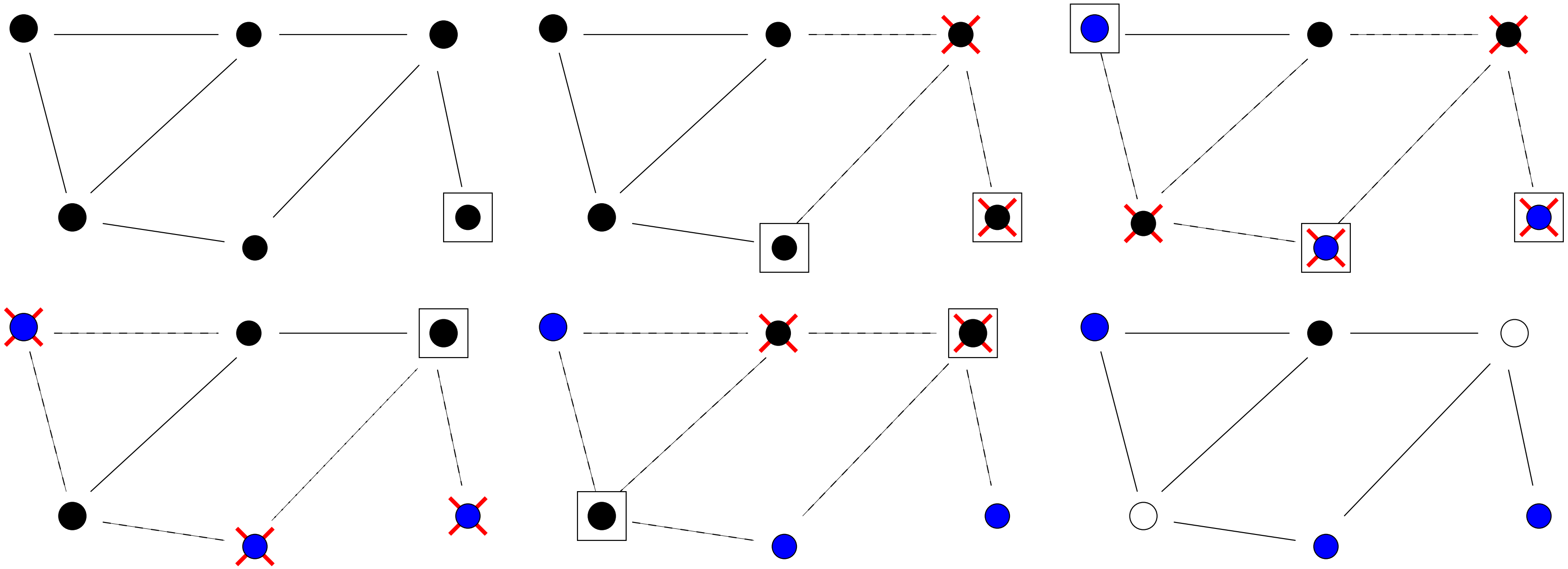




Subroutine: finding a large independent set (GreedyIS)

- Take some node u with minimum degree
- Remove u and all its neighbors from the graph, put u in U
- Repeat until graph is empty
- Return U

Finding a large independent set (GreedyIS)



How well does this work?

We will prove a bound that depends on k, the optimal number of colors required to color the vertices

Note that k is not part of the input of GreedyIS

Lemma 1. *If G can be vertex colored with k colors, there exists a vertex u with degree at most* $\lfloor(1-\frac{1}{k})|V|\rfloor$

Recall: We do not know k, we only use that k is the optimal number of colors and that $k \geq 2$

Proof. Consider a k-coloring

This partitions the vertices of the graph into k independent sets

Take the largest set: it has at least $\lceil\frac{1}{k}\cdot|V|\rceil$ vertices

Any vertex u in this set can only have edges to vertices in other sets

Therefore u has degree at most $|V|-\lceil\frac{1}{k}|V|\rceil \leq \lfloor(1-\frac{1}{k})|V|\rfloor$ $\square$

Lemma 1. *If G can be vertex colored with k colors, there exists a vertex u with degree at most* $\lfloor(1-\frac{1}{k})|V|\rfloor$

Lemma 2. *If G can be vertex colored with k colors, the size of the independent set found by GreedyIS is at least* $\lceil\log_k(|V|/3)\rceil$.

Proof. In each step t, we remove the vertex u_t with minimum degree and all its neighbors

Denote the number of vertices remaining in step t by n_t

By Lemma 1, u_t has degree at most $\lfloor(1-\frac{1}{k})n_t\rfloor$

At least $n_t - \lfloor(1-\frac{1}{k})n_t\rfloor - 1 \geq \frac{n_t}{k} - 1$ vertices remain

So $n_{t+1} \geq \frac{n_t}{k} - 1$.

We find

$$
\begin{aligned}
n_{t+1} &\geq \frac{n_t}{k} - 1 \\
&\geq \frac{n_{t-1}/k - 1}{k} - 1 = \frac{n_{t-1}}{k^2} - \frac{1}{k} - 1 \\
&\geq \dots \\
n_t &\geq \frac{n}{k^t} - \frac{1}{k^{t-1}} - \frac{1}{k^{t-2}} - \dots - 1 \\
&\geq \frac{n}{k^t} - 2
\end{aligned}
$$

using that $k \geq 2$.

Lemma 1. *If G can be vertex colored with k colors, there exists a vertex u with degree at most* $\lfloor (1 - \frac{1}{k})|V| \rfloor$

Lemma 2. *If G can be vertex colored with k colors, the size of the independent set found by GreedyIS is at least* $\lfloor \log_k(|V|/3) \rfloor$.

Proof. In each step t, we remove the vertex u_t with minimum degree and all its neighbors

Denote the number of vertices remaining in step t by n_t

We have seen that $n_t \geq \frac{n}{k^t} - 2$

We have $\frac{n}{k^t} - 2 \geq 1$ as long as $t \leq \log_k(n/3)$

So GreedyIS certainly takes $\lfloor \log_k(n/3) \rfloor$ steps. In every step $1, \ldots, \lfloor \log_k(n/3) \rfloor$, one node is added to the independent set $\square$

Greedy algorithm 2 (repeat)

- We look for a large independent set U using GreedyIS
- U gets one color, is removed from the graph along with adjacent edges, and we repeat
- Continue until the graph is empty

We are now ready to analyze this algorithm.

Let n_t be the number of remaining vertices after step t of Greedy 2

By Lemma 2, in step t at least $\log_k(n_t/3)$ vertices are colored and removed (we ignore $\lfloor\cdot\rfloor$)

Greedy 2 stops when $n_t = 0$, i.e. when $n_t < 1$. When is this?

Suppose we have $n_t \geq \frac{n}{\log_k(n/16)}$. Then by Lemma 2, the amount of vertices colored in each step is at least

$$\begin{aligned}
\log_k(n_t/3) &\geq \log_k\left(\frac{n}{3\log_k n}\right) \\
&\geq \log_k\left(\sqrt{\frac{n}{16}}\right) \qquad \frac{n}{\log_k n} \geq \frac{n}{\log_2 n} \geq \frac{3}{4}\sqrt{n} \\
&= \frac{1}{2}\log_k\left(\frac{n}{16}\right) =: x.
\end{aligned}$$

So in this case it would take at most n/x steps to color **all** vertices

Theorem 3. *The approximation ratio of Greedy 2 is $O(n/\log n)$*

Proof. We have seen that after at most $\frac{n}{\frac{1}{2}\log_k(n/16)}$ steps (maybe less!), at most $\frac{n}{\log_k(n/16)}$ uncolored vertices remain

In the worst case, **all** these vertices receive different colors

In total, Greedy 2 thus uses at most
$\frac{n}{\frac{1}{2}\log_k(n/16)} + \frac{n}{\log_k(n/16)} = \frac{3n}{\log_k(n/16)}$ colors

G can be colored with k colors. The approximation ratio is

$$\frac{3n/\log_k(n/16)}{k} = \frac{3n}{\log(n/16)} \cdot \frac{\log k}{k} = O\left(\frac{n}{\log n}\right).$$

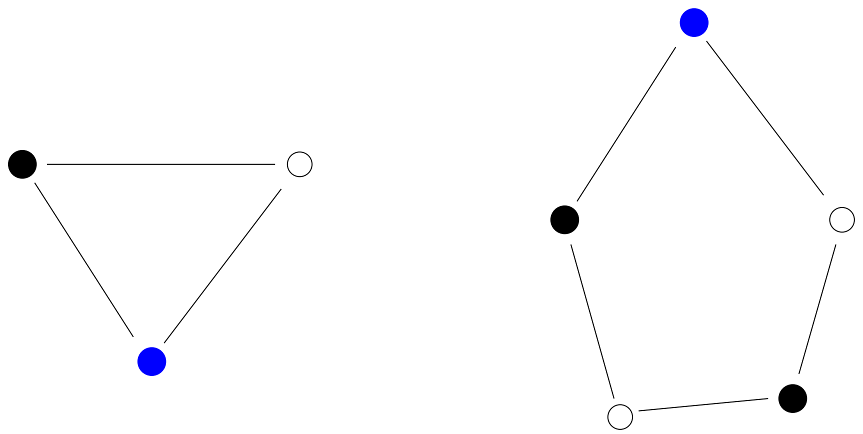
$\square$

Planar graphs

- We can decide in polynomial time whether a planar graph can be vertex colored with only two colors, and also do the coloring in polynomial time if such a coloring exists
- It is NP-complete to determine whether a planar graph can be vertex colored with three colors
- The Four Color Theorem: each planar graph can be vertex colored with only four colors
- We can do this in time $O(|V|^2)$
- We show a simple algorithm that uses at most 6 colors (what is its approximation ratio?)

Two colors

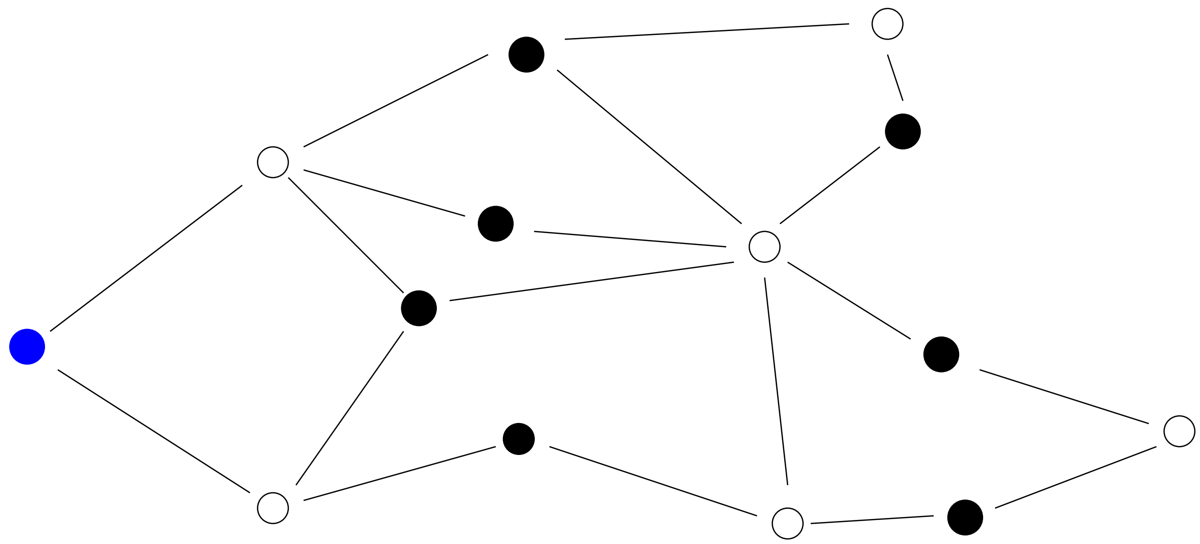
- When are two colors sufficient?
- The graph is not allowed to have a cycle of odd length
- We show that this is a sufficient condition



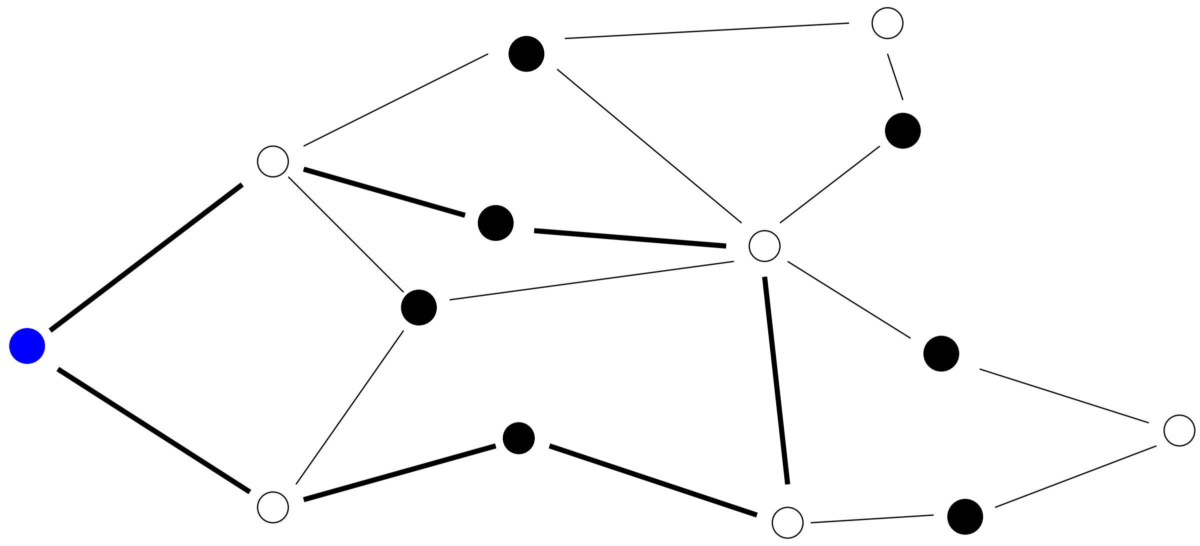
Lemma 4. *If G has no cycle of odd length, it is 2-colorable.*

Proof. Assume G is not 2-colorable. We may assume G is connected.

Take a vertex v. Color vertices at even distances from v white, others black.



Since this is not a valid coloring, we find a circuit of odd length (using an edge that has vertices with the same color at both ends)



If this is a cycle, we have a contradiction. Else, it must contain a smaller circuit of odd length. Use induction. □

Algorithm for planar graphs

- Check whether two colors are sufficient. If so, color the graph with two colors (as in the previous proof!)
- Else, find an uncolored vertex u with degree at most 5
- Remove u and all its adjacent edges and color the remaining graph recursively
- Finally, put u and its adjacent edges back and color u with a color that none of its neighbors has

Question: does such a vertex u exist?

Note: removing a node from a planar graph keeps it planar, so if we can find a node u once, we can do it repeatedly

Properties of planar graphs

- Euler: $n - m + f = 2$ (n is number of vertices, m is number of edges, f is number of faces)

- $m \leq 3n - 6$

  Proof: $3f \leq 2m$ since each face has at least three edges and each edge is counted double

  Thus $3f = 6 - 3n + 3m \leq 2m$ and therefore $m \leq 3n - 6$

- There is a node with degree at most 5

  Proof: if not, then $2m \geq 6n$ (each node has at least 6 outgoing edges, all edges are counted double) and $m \geq 3n$

Algorithm which uses three colors

Find separator of size $\sqrt{m}$

Try all colorings of the separator

Use recursion on both halves of the graph

$T(m) = 2^{O(\sqrt{m})} \cdot T(m/2)$

So $T(m) = 2^{O(\sqrt{m})}$