Exam June 10, 2020. Complete solutions of selected problems

$\boxed{1.}$ Find the linear homogeneous difference equation with constant coefficients of minimal order that has as solution the sequence $(x_k)_{k\geq 0}$ with

(a) $x_k = 7\,\mathrm{Re}(i^k) + 3\,\mathrm{Im}(i^k)$, $k \geq 0$

(b) the first terms of $(x_k)_{k\geq 0}$ are $1, 0, 0, 1, -2, 5$.

<u>Solution</u> (a) It must be a second order difference equation whose characteristic eq. has the roots $\pm i$.
Then, the characteristic eq. is $r^2 + 1 = 0 \Rightarrow$

the LHDE is $x_{k+2} + x_k = 0$.

(b) Note that this sequence is not a geometric progression, thus, it is not a solution of a first order difference eq.
We have $x_0 = 1$, $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $x_4 = -2$, $x_5 = 5$.
A second order LHDE has the form $x_{k+2} + a x_{k+1} + b x_k = 0$ $\forall k \geq 0$

$k=0$ : $x_2 + a x_1 + b x_0 = 0 \Rightarrow b = 0$

$k=1$ : $x_3 + a x_2 + b x_1 = 0 \Rightarrow 1 = 0$ contradiction.

A third order LHDE has the form $x_{k+3} + a x_{k+2} + b x_{k+1} + c x_k = 0$

$$\left.\begin{array}{ll} k=0: & x_3 + a x_2 + b x_1 + c x_0 = 0 \Rightarrow 1 + c = 0 \\ k=1: & x_4 + a x_3 + b x_2 + c x_1 = 0 \Rightarrow -2 + a = 0 \\ k=2: & x_5 + a x_4 + b x_3 + c x_2 = 0 \Rightarrow 5 - 2a + b = 0 \end{array}\right\} \Rightarrow \begin{array}{l} a = 2 \\ b = -1 \\ c = -1 \end{array}$$

<u>Conclusion</u>: $x_{k+3} + 2x_{k+2} - x_{k+1} - x_k = 0$, $k \geq 0$.

$\boxed{2.}$ $\begin{cases} y' = x^2 + y^2 \\ y(0) = 1 \end{cases}$ $\qquad y(x) = \sum_{n=0}^{\infty} a_n x^n$

Find $a_0, a_1, a_2 \in \mathbb{R}$.

<u>Solution</u> $\quad y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$\Rightarrow \quad y(0) = a_0 \quad$ and $\quad y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots$

$\Rightarrow \begin{cases} a_1 + 2a_2 x + 3a_3 x^2 + \dots = x^2 + (a_0 + a_1 x + a_2 x^2 + \dots)^2, \ \forall x \\ a_0 = 1 \end{cases}$

$\Rightarrow \quad a_1 + 2a_2 x + 3a_3 x^2 + \dots = x^2 + a_0^2 + 2a_0 a_1 x + a_1^2 x^2 + 2a_0 a_2 x^2 + \dots \quad \forall x$

$\Rightarrow \quad a_1 = a_0^2 = 1, \quad 2a_2 = 2a_0 a_1 \quad \Rightarrow a_0 = a_1 = a_2 = 1.$

$\boxed{4.}$ Specify the type and stability of the linear system $\dot{x} = -6x$, $\dot{y} = 3y$. Find a global first integral of this system. Find a global first integral using the Check that H is indeed a global first integral using the definition.

<u>Solution</u> The matrix's system is $A = \begin{pmatrix} -6 & 0 \\ 0 & 3 \end{pmatrix}$, which is diagonal. Hence, its eigenvalues are $\lambda_1 = -6$ and $\lambda_2 = 3$.

Since $\lambda_{1,2} \in \mathbb{R}$ with $\lambda_1 < 0 < \lambda_2$, this linear system has the origin an unstable saddle.

Since we have to use the definition of a first integral, we find now the flow.

Let $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ and consider the IVP $\begin{cases} \dot{x} = -6x \\ \dot{y} = 3y \\ x(0) = \eta_1 \\ y(0) = \eta_2. \end{cases}$

$\Rightarrow \quad \varphi(t, \eta_1, \eta_2) = \left(\eta_1 e^{-6t}, \ \eta_2 e^{3t}\right), \ \forall t \in \mathbb{R}.$

Since $\left(\eta_1 e^{-6t}\right) \cdot \left(\eta_2 e^{3t}\right)^2 = \eta_1 \eta_2^2, \quad \forall t \in \mathbb{R}$ we have that $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x, y) = xy^2$ is a first integral which is global since it is well-defined on $\mathbb{R}^2$, and $H \in C^1(\mathbb{R}^2)$.

it is not locally ct.

$\boxed{5}$ $\boxed{(a)}$ Represent in the complex plane the curve
$$\left\{ \frac{1+i}{\sqrt{2}}\, e^{-3t-it} \;:\; t \geq 0 \right\}.$$

<u>Solution.</u> Denote $\quad z(t) = \frac{1+i}{\sqrt{2}}\, e^{-3t-it}, \quad \forall\, t \geq 0.$

we have $\quad z(0) = \frac{1+i}{\sqrt{2}}, \quad e^{-3t-it} = e^{-3t}(\cos t - i \sin t)$

$\Rightarrow \quad z(0) = \cos\frac{\pi}{4} + i \sin\frac{\pi}{4} = e^{i\frac{\pi}{4}}$

$\Rightarrow \quad z(t) = e^{i\frac{\pi}{4}} \cdot e^{-3t-it} = e^{-3t} \cdot e^{i\left(\frac{\pi}{4} - t\right)} =$

$$= e^{-3t}\left[\cos\left(\frac{\pi}{4} - t\right) + i \sin\left(\frac{\pi}{4} - t\right)\right] \quad \forall\, t \geq 0.$$

$\Rightarrow \quad |z(t)| = e^{-3t}, \quad \forall\, t \geq 0 \quad$ and $\quad \arg(z(t)) = \frac{\pi}{4} - t, \ \forall t \geq 0.$

$\Rightarrow$ as $t$ increases from $0$ to $\infty$,

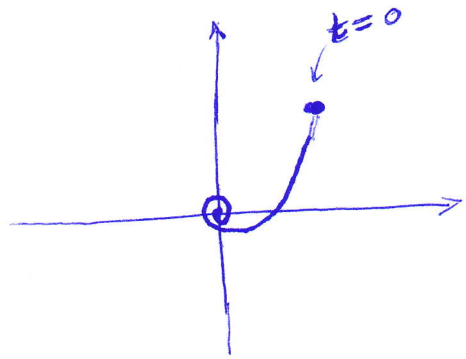
$|z(t)|$ decreases exponentially to $0$

and $\arg(z(t))$ is strictly decreasing

$\Rightarrow$ when moving on the curve as $t$ increases from $0$ to $\infty$,

~~the~~ we approach the origin and rotate around

the origin in the ~~clo~~ clockwise direction.




<u>Remark.</u> $\quad z(t) = \frac{e^{-3t}}{\sqrt{2}}(1+i)(\cos t - i \sin t) =$

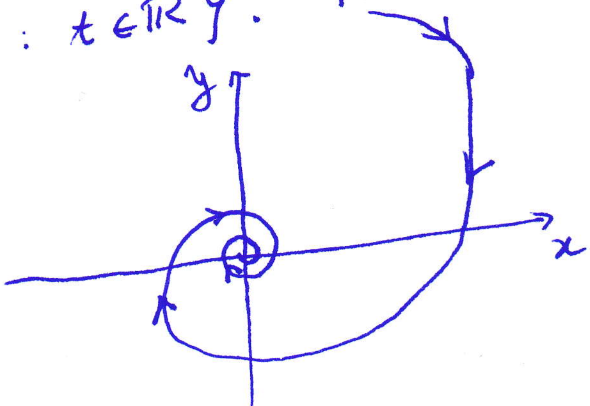
$= \frac{1}{\sqrt{2}} e^{-3t}(\cos t - i \sin t + i \cos t + \sin t) = \frac{1}{\sqrt{2}} e^{-3t}\left[\cos t + \sin t + i(\cos t - \sin t)\right]$

$\boxed{5(b)}$ Find the solution of the IVP $\begin{cases} \dot{x} = -3x + y, \ \dot{y} = -x - 3y \\ x(0) = y(0) = -1. \end{cases}$

Represent in the phase plane the corresponding orbit.

Solution. I will not solve the first part. You have enough written examples (similar exercises). The ~~sol~~ following sol. is obtained:
$$\begin{cases} x = e^{-3t}\cos t + e^{-3t}\sin t \\ y = e^{-3t}\cos t - e^{-3t}\sin t \end{cases}, \quad t \in \mathbb{R}. \qquad (*)$$

By definition, the corresponding orbit is the curve of parametric equations (*). Using the notation at (a), we have that this orbit is, in fact, the curve $\{ \sqrt{2}\, z(t) : t \in \mathbb{R} \}$. Thus it looks like




$\boxed{5(c)}$ Specify the type and stability of the linear system $\begin{cases} \dot{x} = -3x + y \\ \dot{y} = -x - 3y \end{cases}$. If this system has a global first integral, find it.

Solution. $A = \begin{pmatrix} -3 & 1 \\ -1 & -3 \end{pmatrix}$ $\quad \begin{vmatrix} -3-\lambda & 1 \\ -1 & -3-\lambda \end{vmatrix} = 0 \iff (3+\lambda)^2 + 1 = 0$

$\lambda_{1,2} = -3 \pm i \Rightarrow$ focus, global attractor.

We know from the lecture that a global attractor does not have a global first integral.

$\boxed{5(d)}$ Let $\eta = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. Find $e^{2A}\eta$, where $A$ is the matrix of the system from (c).

<u>Solution.</u> We know from the lecture that $e^{tA}\eta$ is the unique solution of the IVP $\dot{X} = AX$, $X(0) = \eta$.

Thus, $e^{tA}\eta$ is the solution found at b) in our case.

$$\Rightarrow \qquad e^{2A}\eta = e^{\begin{pmatrix} -6 & 2 \\ -2 & -6 \end{pmatrix}} \cdot \begin{pmatrix} -1 \\ -1 \end{pmatrix} =$$

$$= \begin{pmatrix} e^{-6}\cos 2 + e^{-6}\sin 2 \\ e^{-6}\cos 2 - e^{-6}\sin 2 \end{pmatrix}.$$