9.) Let D_1 denote the number on the first die and D_2 the number on the second die.

Therefore their pdf's are $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$

and $X = \min(D_1, D_2)$, $Y = \max(D_1, D_2)$

$\Rightarrow X \leq Y$

Let $x < y$

$\Rightarrow P(X = x, Y = y) = P((D_1 = x, D_2 = y) \cup (D_1 = y, D_2 = x))$

$\overset{m.e.}{=} P(D_1 = x, D_2 = y) + P(D_1 = y, D_2 = x)$

$\overset{D_1, D_2 \text{ are independent}}{=} P(D_1 = x) \cdot P(D_2 = y) + P(D_1 = y) \cdot P(D_2 = x)$

$\overset{\text{pdf of } D_1 \text{ is the same as } D_2}{=} P(D_1 = x) P(D_2 = y) + P(D_2 = y) \cdot P(D_1 = x)$

$= 2 P(D_1 = x) P(D_2 = y)$

$= 2 \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{18}$ because $\forall x \in \overline{1,6}$

That is because either min or max $D_1 \in X | = D_2(x)$

number can appear on either the first or second die $= \frac{1}{6}$

Now for $x = y$ (both dices show the same number)

$P(X = x, Y = x) = P(D_1 = x, D_2 = x) = P(D_1 = x) \cdot P(D_2 = x)$

$= \frac{1}{36}$

$\Rightarrow P(X=x, Y=y) = \begin{cases} \frac{1}{18}, & x \neq y \\ \frac{1}{36}, & x = y \end{cases}$, also if $x > 1$ the probability is 0 therefore it is not inculd in the joint pdf

Check if the sum of all probabilities is 1: $C_6^2 \cdot \frac{1}{18} + \frac{6}{36} = \frac{15}{18} + \frac{6}{36}$

$= \frac{30+6}{36}$

$= 1$ ✓

b) $P(X=1) = P(X=1, Y=1) +$

$P(X=1, Y=2) +$

$- - -$

$P(X=1, Y=6)$

$= \frac{1}{36} + \frac{5}{18} = \frac{11}{36}$

$P(Y=2) = P(X=1, Y=2) + P(X=2, Y=2)$

$= \frac{1}{18} + \frac{1}{36} = \frac{3}{36}$

$P(X=1) \cdot P(Y=2) = \frac{33}{36} + \frac{2}{36} = P(X=1, Y=2)$

$\Rightarrow$ X and Y are not independent

because $P(X=1, Y=2) \neq P(X=1) \cdot P(Y=2)$

c) Denote the following events:

A - smaller number is 2

B - larger one is 5

We need to compute $P(B|A)$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(X=2, Y=5)}{P(X=2)}$$

$$= \frac{\frac{1}{18}}{\frac{1}{36} + \frac{4}{18}} = \frac{\frac{1}{18}}{\frac{9}{36}} = \frac{2}{9}$$