8. Let C_1, C_2 be 2 uniformly distributed random variables, between 2pm and 8pm, corresponding to the first customer and to the second customer.

It follows that $f_{C_1}(x) = \frac{1}{6} = f_{C_2}(x)$, and because the customers are independent

$$f_{(C_1,C_2)}(x,y) = f_{C_1}(x) f_{C_2}(y) = \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36} \quad (*)$$

For the first arrival we need to compute the expected value of the minimum of C_1, C_2 and for the last arrival we need to compute the expected value of the maximum of C_1, C_2.

Therefore in what follows, we need:

$$\min(x,y) = \frac{x+y-|x-y|}{2} \quad (1)$$

$$\max(x,y) = \frac{x+y+|x-y|}{2} \quad (2)$$

$$E(h(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) f_{(X,Y)}(x,y)\, dy\, dx$$

a) $E(\min(C_1,C_2)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \min(x,y) f_{(C_1,C_2)}(x,y)\, dy\, dx$

$$\overset{(1)}{\underset{(*)}{=}} \int_2^8\int_2^8 \frac{x+y-|x-y|}{2}\cdot\frac{1}{36}\, dy\, dx$$ (outside $[2,8]\times[2,8]$ the integral is 0 because the customers arrive between 2pm and 8pm)

$$= \frac{1}{36}\int_2^8 \frac{1}{2}\left[\int_2^x [x+y-(x-y)]\,dy + \int_x^8 [x+y-(y-x)]\,dy\right]dx$$

$$= \frac{1}{36} \int_2^8 \frac{1}{2} \left[2 \int_2^x y \, dy + 2 \int_x^8 x \, dy \right] dx$$

$$= \frac{1}{36} \int_2^8 \left(\int_2^x y \, dy + \int_x^8 x \, dy \right) dx$$

$$= \frac{1}{36} \int_2^8 \left(\frac{y^2}{2} \Big|_{y=2}^{y=x} + xy \Big|_{y=x}^{y=8} \right) dx$$

$$= \frac{1}{36} \int_2^8 \left(\frac{x^2}{2} - 2 + 8x - x^2 \right) dx$$

$$= \frac{1}{36} \int_2^8 \left(8x - \frac{x^2}{2} - 2 \right) dx$$

$$= \frac{1}{36} \left(4x^2 - \frac{x^3}{6} - 2x \right) \Big|_2^8$$

$$= \frac{1}{36} \left(4 \cdot 64 - \frac{512}{6} - 16 - 16 + \frac{8}{6} + 4 \right)$$

$$= \frac{1}{36} \left(256 - \frac{256}{3} - 28 + \frac{4}{3} \right)$$

$$= \frac{1}{36} \left(228 - \frac{252}{3} \right)$$

$$= \frac{1}{36} \left(228 - 84 \right) = \frac{1}{36} \cdot 144 = \frac{\cancel{12} \cdot \cancel{12}^{4}}{\cancel{36}_{3}} = 4$$

$\Rightarrow$ The expected time of the first arrival is 4 pm.

b) $E[\max(C_1, C_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \max(x,y) f_{(C_1,C_2)}(x,y)\, dy\, dx$

$$\overset{(2)}{\underset{(*)}{=}} \int_2^8 \int_2^8 \frac{x+y+|x-y|}{2} \cdot \frac{1}{36}\, dy\, dx$$

$$= \frac{1}{36} \int_2^8 \frac{1}{2} \left[\int_2^x (x+y+x-y)\, dy + \int_x^8 (x+y+y-x)\, dy \right] dx$$

$$= \frac{1}{36} \int_2^8 \frac{1}{2} \left(2\int_2^x x\, dy + 2\int_x^8 y\, dy \right) dx$$

$$= \frac{1}{36} \int_2^8 \left(\int_2^x x\, dy + \int_x^8 y\, dy \right) dx$$

$$= \frac{1}{36} \int_2^8 \left(xy \Big|_{y=2}^{y=x} + \frac{y^2}{2} \Big|_{y=x}^{y=8} \right) dx$$

$$= \frac{1}{36} \int_2^8 \left(x^2 - 2x + 32 - \frac{x^2}{2} \right) dx$$

$$= \frac{1}{36} \int_2^8 \left(\frac{x^2}{2} - 2x + 32 \right) dx$$

$$= \frac{1}{36} \left(\frac{x^3}{6} - x^2 + 32x \right) \Big|_2^8$$

$$= \frac{1}{36} \left(\frac{512}{6} - 64 + 256 - \frac{8}{6} + 4 - \cancel{64} \; 64 \right)$$

$$= \frac{1}{36} \left(\frac{256}{3} + 256 - 124 - \frac{4}{3} \right)$$

$$= \frac{1}{36} \left(\frac{252}{3} + 132 \right) = \frac{1}{36} (84 + 132)$$

$$= \frac{216}{36} = 6$$

$\Rightarrow$ The expected time of the last arrival is 6 pm.