Combinatorics

Assignment 2

September 12, 2017

Reading list

- The slides
- Section 6.3
- Section 6.4

Note:

- You can hand in your solutions as a single PDF via the assignment module in Blackboard. Note that the document should be typeset with LaTeX, Word or a similar program. It should not be a scan or picture of your handwritten notes.
- Make sure that your name, student number and group number are on top of the first page!
- **Note that your submission should be an individual submission because it can influence your final grade for this course. If we detect that your work is not completely your own work, we will ask the exam committee to investigate whether it is plagiarism or not!**

Exercises to be presented during the exercise hours

Exercise 1
How many subsets with an odd number of elements does a set with 10 elements have?

Exercise 2
Thirteen people on a softball team show up for a game.

a) How many ways are there to choose 10 players to take the field?

b) How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?

c) Of the 13 people who show up, three are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

Exercise 3
How many 4-permutations of the positive integers not exceeding 100 contain three consecutive integers k, $k+1$, $k+2$, in the correct order

a) where these consecutive integers can perhaps be separated by other integers in the permutation?

b) where they are in consecutive positions in the permutation?

Exercise 4
Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

Exercise 5
Write the following expressions as a real number whenever well-defined.

a) $\binom{4}{6}$ **b)** $\binom{\sqrt{7}}{3}$ **c)** $\binom{-17}{\sqrt{256}}$

Exercise 6
Find the coefficient of x^4y^7 in $(x+y)^{11}$.

Exercise 7
Prove that if n and k are integers with $1 \leq k \leq n$, then $k\binom{n}{k} = n\binom{n-1}{k-1}$.

a) using a combinatorial proof. [*Hint:* Show that the two sides of the identity count the number of ways to select a subset with k elements from a set with n elements and then an element of this subset.]

b) using an algebraic proof based on the formula for $\binom{n}{r}$ given in Theorem 2 in Section 6.3.

Exercise 8
In this exercise we will count the number of paths in the xy plane between the origin $(0,0)$ and point (m,n), where m and n are nonnegative integers, such that each path is made up of a series of steps, where each step is a move one unit to the right or a move one unit upward. (No moves to the left or downward are allowed.)

a) Show that each path of the type described can be represented by a bit string of length $m+n$.

b) Conclude from part (a) that there are $\binom{m+n}{n}$ paths of the desired type.

Exercise 9
Use Exercise 8 to give an alternative proof of Corollary 2 in Section 6.3, which states that $\binom{n}{k} = \binom{n}{n-k}$ whenever k is an integer with $0 \le k \le n$. [*Hint:* Consider the number of paths of the type described in Exercise 8 from $(0, 0)$ to $(n - k, k)$ and from $(0, 0)$ to $(k, n - k)$.]

Exercises to be handed in

You are expected to explain your answers, even if this is not explicitly stated in the exercises themselves.

Exercise 10
Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have more women than men? 3 pt

Exercise 11
There are six runners in the 100-yard dash. How many ways are there for gold, silver and bronze medals to be awarded if ties are possible? (The runner or runners who finish with the fastest time receive gold medals, the runner or runners who finish with exactly one runner ahead receive silver medals, and the runner or runners who finish with exactly two runners ahead receive bronze medals.) 6 pt

Exercise 12
The row of Pascal's triangle containing the binomial coefficients $\binom{8}{k}$ where $0 \le k \le 8$, is: 3 pt

$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

Use Pascal's identity to produce the row immediately following this row in Pascal's triangle.

Exercise 13
Prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, whenever n, r, and k are nonnegative integers with $r \le n$ and $k \le r$, 6 pt

a) using a combinatorial argument.

b) using an argument based on the formula for the number of r-combinations of a set with n elements.

Exercise 14
For which $n \in \mathbb{N}$ is $\left(1 + \sqrt{2}\right)^n + \left(1 - \sqrt{2}\right)^n$ a natural number? 3 pt

Exercise 15
This exercise combines several aspects of the theory of this week.

a) Write the following expressions as a real number whenever well-defined. 3 pt

(i) $\binom{123456}{123457}$ (ii) $\binom{3\pi}{2\pi}$ (iii) $\binom{-\sqrt{5}}{4}$

b) Give a combinatorial proof of the identity $\boxed{3 \text{ pt}}$

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

[*Hint:* What did you learn about $\mathcal{P}$ in the course Mathematical Structures?]

c) What is the coefficient of x^5y^8 in the polynomial $(3x + 4y)^{13}$? $\boxed{3 \text{ pt}}$

Your final grade is the sum of your scores divided by 3.0.