Combinatorics

Assignment 3

September 19, 2017

Reading list

- The slides
- Section 6.5

Note:

- If the numbers are smaller than 10000000, please write out the formulas as normal numbers.
- You can hand in your solutions as a single PDF via the assignment module in Blackboard. Note that the document should be typeset with LaTeX, Word or a similar program. It should not be a scan or picture of your handwritten notes.
- Make sure that your name, student number and group number are on top of the first page!
- **Note that your submission should be an individual submission because it can influence your final grade for this course. If we detect that your work is not completely your own work, we will ask the exam committee to investigate whether it is plagiarism or not!**

Exercises to be presented during the exercise hours

Exercise 1
A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, and plain bagels. How many ways are there to choose

a) six bagels?

b) a dozen bagels?

c) two dozen bagels?

d) a dozen bagels with at least one of each kind?

e) a dozen bagels with at least three egg bagels and no more than two salty bagels?

Exercise 2
How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11$$

where x_1, x_2, and x_3 are nonnegative integers? [*Hint:* Introduce an auxiliary variable x_4 such that $x_1 + x_2 + x_3 + x_4 = 11$.]

Exercise 3
How many ways are there to distribute six indistinguishable balls into nine distinguishable bins?

Exercise 4
How many ways are there to distribute twelve indistinguishable balls into six distinguishable bins?

Exercise 5
How many ways are there to distribute twelve distinguishable objects into six distinguishable boxes so that two objects are placed in each box?

Exercise 6
A professor packs her collection of 40 issues of a mathematics journal in four boxes with 10 issues per box. How many ways can she distribute the journals if

a) each box is numbered, so that they are distinguishable?

b) the boxes are identical, so that they cannot be distinguished?

Exercise 7
How many ways are there to travel in xyz space from the origin $(0, 0, 0)$ to the point $(4, 3, 5)$ by taking steps one unit in the positive x direction, one unit in the positive y direction, or one unit in the positive z direction? (Moving in the negative x, y, or z direction is prohibited, so that no backtracking is allowed.)

Exercise 8
Prove Theorem 4 by first setting up a one-to-one correspondence between permutations of n objects with n_i indistinguishable objects of type i, $i = 1, 2, 3, \ldots, k$, and the distributions of n objects in k boxes such that n_i objects are placed in box i, $i = 1, 2, 3, \ldots, k$ and then applying Theorem 3.

Exercise 9
How many ways are there to distribute six distinguishable objects into four indistinguishable boxes so that each of the boxes contains at least one object?

Exercise 10
How many ways are there to distribute five indistinguishable objects into three indistinguishable boxes?

Exercises to be handed in

> You are expected to explain your answers, even if this is not explicitly stated in the exercises themselves.

Exercise 11
How many positive integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 13? [2 pt]

Exercise 12
How many ways are there to deal hands of seven cards to each of five players from a standard deck of 52 cards? [2 pt]

Exercise 13
In how many ways can a dozen books be placed on four distinguishable shelves [4 pt]

a) if the books are indistinguishable copies of the same title?

b) if no two books are the same, and the positions of the books on the shelves matter? [*Hint:* Break this into 12 tasks, placing each book separately. Start with the sequence 1, 2, 3, 4 to represent the shelves. Represent the books by b_i, $i = 1, 2, \ldots, 12$. Place b_1 to the right of one of the terms in 1, 2, 3, 4. Then successively place b_2, b_3, $\ldots$, and b_{12}.]

Exercise 14
How many ways are there to distribute five balls into three boxes if each box must have at least one ball in it if [8 pt]

a) both the balls and boxes are labeled?

b) the balls are labeled, but the boxes are unlabeled?

c) the balls are unlabeled, but the boxes are labeled?

d) both the balls and boxes are unlabeled?

Exercise 15
How many different terms are there in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$ after all terms with identical sets of exponents are added? [2 pt]

Exercise 16
Assume we have one bill of each of the following types: $1, $2, $5, $10, $20, $50, and $100. In how many ways can we divide all these bills over three identical purses if there is at most one empty purse? [3 pt]

Exercise 17
This exercise combines several aspects of the theory of this week.

a) A student has four apples, three bananas, and two pears. If the student eats one piece of fruit each day, and only the type of fruit matters, in how many different ways can these fruits be consumed? Explain your answer. [2 pt]

b) An ice cream parlor has 16 different flavors, 7 different kinds of sauce, and 5 toppings. How many different kinds of medium sundaes are there if a medium sundae contains 3 scoops of ice cream, where each flavor can be used more than once and the order of the scoops does not matter; 2 kinds of sauce, where each sauce can be used only once and the order of the sauces does not matter; and 2 toppings, where each topping can be used only once and the order of the toppings does not matter? Explain your answer. 4 pt

c) Compute the number of surjective functions from the set $\{0, 1, 2, 3, 4, 5\}$ to the set $\{a, b, c, d, e\}$. Explain your answer. 3 pt

Your final grade is the sum of your scores divided by 3.0.