Combinatorics

Assignment 4

September 26, 2017

Reading list

- The slides
- Section 8.1, except
  - Algorithms and Recurrence Relations
- Section 8.2, except
  - Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Note:

- If the numbers are smaller than 10000000, please write out the formulas as normal numbers.
- Give exact answers including fractions and square roots! Note that expressions should be simplified to the canonical form $a + b\sqrt{c}$ where $a, b \in \mathbb{Q}$ and $c \in \mathbb{N}$.
- You can hand in your solutions as a single PDF via the assignment module in Blackboard. Note that the document should be typeset with LaTeX, Word or a similar program. It should not be a scan or picture of your handwritten notes.
- Make sure that your name, student number and group number are on top of the first page!
- **Note that your submission should be an individual submission because it can influence your final grade for this course. If we detect that your work is not completely your own work, we will ask the exam committee to investigate whether it is plagiarism or not!**

Exercises to be presented during the exercise hours

Exercise 1
A vending machine dispensing books of stamps accepts only \$1 coins, \$1 bills, and \$5 bills.

a) Find a recurrence relation for a_n, the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matters.

b) What are the initial conditions?

c) How many ways are there to deposit \$10 for a book of stamps?

Exercise 2

a) Find a recurrence relation for s_n, the number of strictly increasing sequences of positive integers that have 1 as their first term and n as their last term, where n is a positive integer. That is, sequences $a_1, a_2, \ldots, a_k$, where $a_1 = 1$, $a_k = n$, and $a_j < a_{j+1}$ for $j = 1, 2, \ldots, k-1$.

b) What are the initial conditions?

c) How many sequences of the type described in (a) are there when n is an integer with $n \geq 2$?

Exercise 3

a) Find a recurrence relation for a_n, the number of bit strings of length n that contain three consecutive 0s.

b) What are the initial conditions?

c) How many bit strings of length seven contain three consecutive 0s?

Exercise 4

a) Find a recurrence relation for a_n, the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.

b) What are the initial conditions?

c) In how many ways can this person climb a flight of eight stairs?

Exercise 5

a) Find a recurrence relation for a_n, the number of ternary strings of length n that do not contain consecutive symbols that are the same.

b) What are the initial conditions?

c) How many ternary strings of length six do not contain consecutive symbols that are the same?

Exercise 6

a) Find a recurrence relation for a_n, the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes. [*Hint:* Consider separately the coverings where the position in the top right corner of the checkerboard is covered by a domino positioned horizontally and where it is covered by a domino positioned vertically.]

b) What are the initial conditions for the recurrence relation in part (a)?

c) How many ways are there to completely cover a 2×17 checkerboard with 1×2 dominoes?

Exercise 7

Solve these recurrence relations together with the initial conditions given.

a) $a_n = 2a_{n-1}$ for $n \geq 1$, $a_0 = 3$

b) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

c) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$

d) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$

Exercise 8

A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years.

a) Find a recurrence relation for $\{L_n\}$, where L_n is the number of lobsters caught in year n, under the assumption for this model.

b) Find L_n if 100,000 lobsters were caught in year 1 and 300,000 were caught in year 2.

Exercises to be handed in

You are expected to explain your answers, even if this is not explicitly stated in the exercises themselves.

Exercise 9 5 pt

A bus driver pays all tolls, using only nickels and dimes, by throwing one coin at a time into the mechanical toll collector.

a) Find a recurrence relation for a_n, the number of different ways the bus driver can pay a toll of n cents (where the order in which the coins are used matters).

b) In how many different ways can the driver pay a toll of 45 cents?

Exercise 10 4 pt

a) Find a recurrence relation for a_n, the number of ways to lay out a walkway with slate tiles if the tiles are red, green, or gray, so that no two red tiles are adjacent and tiles of the same color are considered indistinguishable.

b) What are the initial conditions for the recurrence relation in part (a)?

c) How many ways are there to lay out a path of seven tiles as described in part (a)?

Exercise 11 6 pt

In the Tower of Hanoi puzzle, suppose our goal is to transfer all n disks from peg 1 to peg 3, but we cannot move a disk directly between pegs 1 and 3. Each move of a disk must be a move involving peg 2. As usual, we cannot place a disk on top of a smaller disk.

a) Find a recurrence relation for a_n, the number of moves required to solve the puzzle for n disks with this added restriction.

b) Solve this recurrence relation to find a closed formula for the number of moves required to solve the puzzle for n disks. Prove your formula using the principle of mathematical induction.

c) How many different arrangements are there of the n disks on three pegs so that no disk is on top of a smaller disk?

d) Show that every allowable arrangement of the n disks occurs in the solution of this variation of the puzzle.

Exercise 12

This exercise deals with a variation of the Josephus problem described by Graham, Knuth, and Patashnik in [GKP94]. This problem is based on an account by the historian Flavius Josephus, who was part of a band of 41 Jewish rebels trapped in a cave by the Romans during the Jewish-Roman war of the first century. The rebels preferred suicide to capture; they decided to form a circle and to repeatedly count off around the circle, killing every third rebel left alive. However, Josephus and another rebel did not want to be killed this way; they determined the positions where they should stand to be the last two rebels remaining alive. The variation we consider begins with n people, numbered 1 to n, standing around a circle. In each stage, every second person still left alive is eliminated until only one survives. We denote the number of the survivor by $J(n)$.

a) Determine the value of $J(n)$ for each integer n with $1 \le n \le 16$ by completing the following table: 2 pt

n	$J(n)$	order of death
1		
2		
$\vdots$		

b) Use the values you have listed in your table to conjecture a formula for $J(n)$. [*Hint:* Write $n = 2^m + k$, where m is a nonnegative integer and k is a nonnegative integer less than 2^m.] 2 pt

Exercise 13

Solve these recurrence relations together with the initial conditions given. 6 pt

a) $a_n = a_{n-1}$ for $n \geq 1$, $a_0 = 2$

b) $a_n + 4a_{n-2} = -4a_{n-1}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$

c) $a_n = \frac{a_{n-2}}{4}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

Exercise 14

A *lucky number* is a natural number that does not contain the pattern 13 in its decimal representation. How many lucky numbers exist that are smaller than or equal to 10^{23}? 5 pt

Your final grade is the sum of your scores divided by 3.0.

References

[GKP94] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics.* Addison-Wesley, Reading, MA, 2d ed. edition, 1994.