Combinatorics

Assignment 5

October 3, 2017

Reading list

- The slides
- Section 8.2

> Note:
>
> - If the numbers are smaller than 10000000, please write out the formulas as normal numbers.
> - Give exact answers including fractions and square roots! Note that expressions should be simplified to the canonical form $a + b\sqrt{c}$ where $a, b \in \mathbb{Q}$ and $c \in \mathbb{N}$.
> - You can hand in your solutions as a single PDF via the assignment module in Blackboard. Note that the document should be typeset with LaTeX, Word or a similar program. It should not be a scan or picture of your handwritten notes.
> - Make sure that your name, student number and group number are on top of the first page!
> - **Note that your submission should be an individual submission because it can influence your final grade for this course. If we detect that your work is not completely your own work, we will ask the exam committee to investigate whether it is plagiarism or not!**

Exercises to be presented during the exercise hours

Exercise 1
Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$.

Exercise 2
What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has the roots -1, -1, -1, 2, 2, 5, 5, 7?

Exercise 3
Consider the nonhomogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.

a) Show that $a_n = -2^{n+1}$ is a solution of this recurrence relation.

b) Use Theorem 5 (from the book) to find all solutions of this recurrence relation.

c) Find the solution with $a_0 = 1$.

Exercise 4
What is the general form of the particular solution guaranteed to exist by Theorem 6 (from the book) of the linear nonhomogeneous recurrence relation $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if

a) $F(n) = n^3$?

b) $F(n) = n2^n$?

c) $F(n) = (n^2 - 2)(-2)^n$?

d) $F(n) = 2$?

Exercise 5
a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 3^n$.

b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 5$.

Exercise 6
a) Find the characteristic roots of the linear homogeneous recurrence relation $a_n = a_{n-4}$. [Note: These include complex numbers.]

b) Find the solution of the recurrence relation in part (a) with $a_0 = 1$, $a_1 = 0$, $a_2 = -1$, and $a_3 = 1$.

Exercise 7

Consider the following recurrence relation:

$$\begin{aligned} a_0 &= 2 \\ a_1 &= 4 \\ a_2 &= 8 \\ a_n &= 3a_{n-1} - 3a_{n-2} + a_{n-3} \text{ for } n \geq 3 \end{aligned}$$

a) What is the type of this recurrence relation?

b) Find a closed formula for this recurrence relation.

Exercises to be handed in

You are expected to explain your answers, even if this is not explicitly stated in the exercises themselves.

Exercise 8 [4 pt]
Prove Theorem 2 (from the book).

Exercise 9

a) Determine values of the constants A and B such that $a_n = An + B$ is a solution of recurrence relation $a_n = 2a_{n-1} + n + 5$. [1 pt]

b) Use Theorem 5 (from the book) to find all solutions of this recurrence relation. [1 pt]

c) Find the solution of this recurrence relation with $a_0 = 4$. [1 pt]

Exercise 10
What is the general form of the particular solution guaranteed to exist by Theorem 6 (from the book) of the linear nonhomogeneous recurrence relation $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if [6 pt]

a) $F(n) = (-2)^n$?

b) $F(n) = n^2 4^n$?

c) $F(n) = n^4 2^n$?

Exercise 11

a) Find all solutions of the recurrence relation $a_n = -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$. [3 pt]

b) Find the solution of this recurrence relation with $a_1 = 56$ and $a_2 = 278$. [1 pt]

Exercise 12
Find the solution of the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$ with $a_0 = 1$ and $a_1 = 4$. [*Hint:* How can Theorem 6 be applied if the inhomogeneous part is a linear combination of a polynomial function and an exponential function?] [4 pt]

Exercise 13
This exercise combines several aspects of the theory of the last two weeks.

a) Assume you have an unlimited amount of red, blue and yellow marbles. Let s_n denote the number of different bags that we can fill with n marbles such that the bags contain an even number of blue marbles. Determine s_1 and s_2. Explain your answer. [2 pt]

b) Give a recurrence relation for this s_n where $n \geq 3$. Explain your answer. [3 pt]

c) Find a closed formula for the recurrence relation given by: $\boxed{\text{4 pt}}$

$$\begin{aligned} t_0 &= 1 \\ t_1 &= 2 \\ t_{n+2} &= t_n + n + 1 \text{ for } n \geq 0 \end{aligned}$$

[*Hint:* Note that by applying Theorem 6 you get immediately the proper degree for the particular solution.]

Your final grade is the sum of your scores divided by 3.0.