Combinatorics

Assignment 6

October 10, 2017

Reading list

- The slides
- Section 8.5
- Section 8.6

Note:

- Unless it is explicitly stated that you can use any method, you should use the principle of inclusion and exclusion. Either the version for sets or the version for properties.
- You can hand in your solutions as a single PDF via the assignment module in Blackboard. Note that the document should be typeset with LaTeX, Word or a similar program. It should not be a scan or picture of your handwritten notes.
- Make sure that your name, student number and group number are on top of the first page!
- **Note that your submission should be an individual submission because it can influence your final grade for this course. If we detect that your work is not completely your own work, we will ask the exam committee to investigate whether it is plagiarism or not!**

Exercises to be presented during the exercise hours

Exercise 1
How many elements are in $A_1 \cup A_2$ if there are 12 elements in A_1, 18 elements in A_2, and

a) $A_1 \cap A_2 = \emptyset$?

b) $|\, A_1 \cap A_2 \,| = 1$?

c) $|\, A_1 \cap A_2 \,| = 6$?

d) $A_1 \subseteq A_2$?

Exercise 2
Use the principle of inclusion and exclusion for sets to find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in each set and if

a) the sets are pairwise disjoint.

b) there are 50 common elements in each pair of sets and no elements in all three sets.

c) there are 50 common elements in each pair of sets and 25 elements in all three sets.

d) the sets are equal.

Exercise 3
Find the number of positive integers not exceeding 100 that are either odd or the square of an integer. Use the principle of inclusion and exclusion for two sets.

Exercise 4
Write out the explicit formula given by the principle of inclusion and exclusion for the number of elements in the union of six sets when it is known that no three of these sets have a common intersection.

Exercise 5
Find the probability that when a fair coin is flipped five times tails comes up exactly three times, the first and last flips come up tails, or the second and fourth flips come up heads. Use the principle of inclusion and exclusion for three sets.

Exercise 6
How many solutions does the equation $x_1 + x_2 + x_3 = 13$ have where x_1, x_2, and x_3 are nonnegative integers less than 6? Use the principle of inclusion and exclusion for properties.

Exercise 7
A machine that inserts letters into envelopes goes haywire and inserts letters randomly into envelopes. What is the probability that in a group of 100 letters

a) no letter is put into the correct envelope?

b) exactly one letter is put into the correct envelope?

c) exactly 98 letters are put into the correct envelopes?

d) all letters are put into the correct envelopes?

Exercises to be handed in

You are expected to explain your answers, even if this is not explicitly stated in the exercises themselves.

Exercise 8

In a survey of 270 college students, it is found that 64 like Brussels sprouts, 94 like broccoli, 58 like cauliflower, 26 like both Brussels sprouts and broccoli, 28 like both Brussels sprouts and cauliflower, 22 like both broccoli and cauliflower, and 14 like all three vegetables. How many of the 270 students do not like any of these vegetables? Use the principle of inclusion and exclusion for three sets. 2 pt

Exercise 9

How many elements are in the union of four sets if each of the sets has 100 elements, each pair of the sets shares 50 elements, each three of the sets share 25 elements, and there are 5 elements in all four sets? 3 pt

Exercise 10

In the lecture notes two types of the principle of inclusion and exclusion are explained: the principle for sets and the principle for properties. However, it is also possible to define this principle for *probabilities*. Find a formula for the probability of the union of four events in a finite sample space if no four of them can occur at the same time. In other words find a formula for $p(E_1 \cup E_2 \cup E_3 \cup E_4)$ knowing that the four events cannot occur at the same time, where E_i denotes that event i occurs, and $E_i \cup E_j$ denotes that at least one of the events i or j occurs, and $E_i \cap E_j$ denotes that both events i and j occur. Explain how you derived your formula. 2 pt

Exercise 11

An integer is called squarefree if it is not divisible by the square of a positive integer greater than 1. Find the number of squarefree positive integers less than 100. 4 pt

Exercise 12

In how many ways can eight distinct balls be distributed into three distinct urns if each urn must contain at least one ball? 2 pt

Exercise 13

a) Use a combinatorial argument to show that the sequence $\{D_n\}$, where D_n denotes the number of derangements of n objects, satisfies the recurrence relation 3 pt

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

b) Use the previous result to show that 2 pt

$$D_n = nD_{n-1} + (-1)^n$$

for $n \geq 1$.

Exercise 14

How many derangements of $\{1, 2, 3, 4, 5, 6\}$ end with the integers 1, 2, and 3, in some order? 2 pt

Exercise 15

This exercise combines several aspects of the theory of this week.

a) A machine that inserts letters into envelopes goes haywire and inserts letters randomly into envelopes. What is the probability that in a group of six letters 4 pt

(i) exactly two letters end up in the proper envelope?

(ii) exactly five letters are put into the correct envelopes?

b) In the rest of this exercise we will investigate the equation

$$x + y + z = 14 \tag{1}$$

where $x, y, z \in \mathbb{N}$.

Which distribution problem can we use to find that equation (1) has exactly $\binom{16}{2}$ solutions? 1 pt

c) Find the number of solutions of equation (1) that also adhere to the additional requirements $x < 4$, $y < 5$ and $z < 8$, by using the principle of inclusion and exclusion for three properties. 5 pt

Your final grade is the sum of your scores divided by 3.0.