# Combinatorics

Assignment 7

October 17, 2017

## Reading list

- The slides
- Section 10.1
- Section 10.2
- Section 10.3, except
  - Isomorphism of Graphs

Note:

- Because the list of the exercises is quite long this week, it will not be possible to present all exercises. Therefore the focus will be on the exercises that other students have questions about.
- If it is too difficult to draw graphs directly in LaTeX, please don't waste your time on that, but simply include a clear picture of a graph drawn by hand.
- Note that your homework will be printed in black and white, so do not rely on colors to distinguish between nodes, but use different shapes.
- You can hand in your solutions as a single PDF via the assignment module in Blackboard. Note that the document should be typeset with LaTeX, Word or a similar program. It should not be a scan or picture of your handwritten notes.
- Make sure that your name, student number and group number are on top of the first page!
- **Note that your submission should be an individual submission because it can influence your final grade for this course. If we detect that your work is not completely your own work, we will ask the exam committee to investigate whether it is plagiarism or not!**

# Exercises to be presented during the exercise hours

**Exercise 1**
What kind of graph (from Table 1) can be used to model a highway system between major cities where

**a)** there is an edge between the vertices representing cities if there is an interstate highway between them?

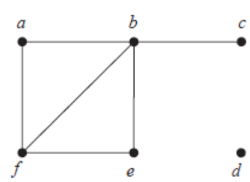
**b)** there is an edge between the vertices representing cities for each interstate highway between them?

**c)** there is an edge between the vertices representing cities for each interstate highway between them, and there is a loop at the vertex representing a city if there is an interstate highway that circles this city?

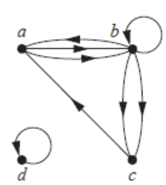
**TABLE 1** **Graph Terminology.**

| *Type* | *Edges* | *Multiple Edges Allowed?* | *Loops Allowed?* |
|---|---|---|---|
| Simple graph | Undirected | No | No |
| Multigraph | Undirected | Yes | No |
| Pseudograph | Undirected | Yes | Yes |
| Simple directed graph | Directed | No | No |
| Directed multigraph | Directed | Yes | Yes |
| Mixed graph | Directed and undirected | Yes | Yes |

**Exercise 2**
Find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.




**Exercise 3**
Determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph.




**Exercise 4**
What does the degree of a vertex represent in the acquaintanceship graph, where vertices represent all the people in the world? What does the neighborhood of a vertex in this graph represent? What do isolated and pendant vertices in this graph represent? In one study it was estimated that the average degree of a vertex in this graph is 1000. What does this mean in terms of the model?

**Exercise 5**
Suppose that there are five young women and six young men on an island. Each woman is willing to marry some of the men on the island and each man is willing to marry any woman who is willing to marry him. Suppose that Anna is willing to marry Jason, Larry, and Matt; Barbara is willing to marry Kevin and Larry; Carol is willing to marry Jason, Nick, and Oscar; Diane is willing to marry Jason, Larry, Nick, and Oscar; and Elizabeth is willing to marry Jason and Matt.

**a)** Model the possible marriages on the island using a bipartite graph.

**b)** Find a matching of the young women and the young men on the island such that each young woman is matched with a young man whom she is willing to marry.

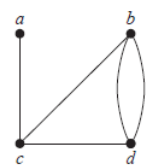
**c)** Is the matching you found in part (b) a complete matching? Is it a maximum matching?

**Exercise 6**
Let $G$ be a graph with $v$ vertices and $e$ edges. Let $M$ be the maximum degree of the vertices of $G$, and let $m$ be the minimum degree of the vertices of $G$. Show that

**a)** $2e/v \geq m$.

**b)** $2e/v \leq M$.

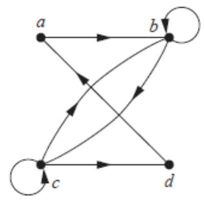
**Exercise 7**
Represent the given graph using an adjacency matrix.




**Exercise 8**
Draw an undirected graph represented by the given adjacency matrix.

$$\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix} \end{array}$$

**Exercise 9**
Find the adjacency matrix of the given directed multigraph with respect to the vertices listed in alphabetic order.




**Exercise 10**
What is the sum of the entries in a row of the adjacency matrix for an undirected graph? And for a directed graph?

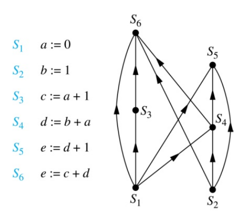
**Exercise 11**
What is the sum of the entries in a row of the incidence matrix for an undirected graph?

## Exercises to be handed in

> You are expected to explain your answers, even if this is not explicitly stated in the exercises themselves.

**Exercise 12**
Let $G$ be an undirected graph with a loop at every vertex. Show that the relation $R$ on the set of vertices of $G$ such that $u\,R\,v$ if and only if there is an edge associated to $\{u, v\}$ is a symmetric, reflexive relation on $G$. 3 pt

**Exercise 13**
Which statements must be executed before $S_6$ is executed in the program defined by this precedence graph? 2 pt




**Exercise 14**
Do only b, e and f.
Draw these graphs. 3 pt

**a)** $K_7$ **c)** $K_{4,4}$ **e)** $W_7$

**b)** $K_{1,8}$ **d)** $C_7$ **f)** $Q_4$

**Exercise 15**

For which values of n are these graphs bipartite? $\boxed{\text{4 pt}}$

**a)** $K_n$

**b)** $C_n$

**c)** $W_n$

**d)** $Q_n$

**Exercise 16**

For the graph $G$ in Exercise 2 (of this assignment) find $\boxed{\text{2 pt}}$

**a)** the subgraph induced by the vertices $a$, $b$, $c$, and $f$.

**b)** the new graph $G_1$ obtained from $G$ by contracting the edge connecting $b$ and $f$.

**Exercise 17**

Do only a, e and f.

The degree sequence of a graph is the sequence of the degrees of the vertices of the graph in nonincreasing order. A sequence $d_1, d_2, \ldots, d_n$ is called graphic if it is the degree sequence of a simple graph. $\boxed{\text{3 pt}}$

Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

**a)** 5, 4, 3, 2, 1, 0

**b)** 6, 5, 4, 3, 2, 1

**c)** 2, 2, 2, 2, 2, 2

**d)** 3, 3, 3, 2, 2, 2

**e)** 3, 3, 2, 2, 2, 2

**f)** 1, 1, 1, 1, 1, 1

**g)** 5, 3, 3, 3, 3, 3

**h)** 5, 5, 4, 3, 2, 1

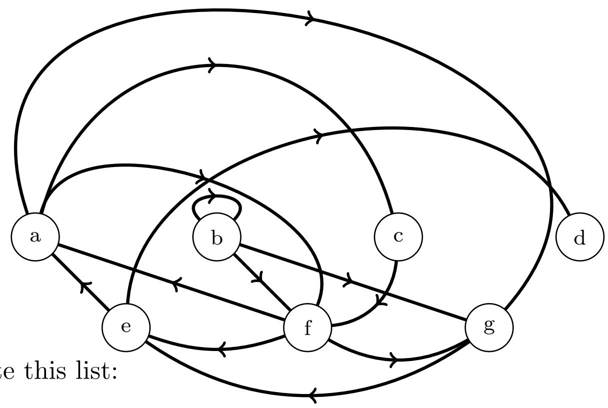
**Exercise 18**

Show that if a bipartite graph $G = (V, E)$ is $n$-regular for some positive integer $n$ (see the preamble to Exercise 10.2.39) in the book and $(V_1, V_2)$ is a bipartition of $V$, then $|\,V_1\,| = |\,V_2\,|$. That is, show that the two sets in a bipartition of the vertex set of an $n$-regular graph must contain the same number of vertices. $\boxed{\text{2 pt}}$

**Exercise 19**

This exercise combines several aspects of the theory of this week.

**a)** Consider the following graph: 3 pt




Complete this list:

- This is a planar graph: ... (yes/no)
- The number of edges is ...
- The in-degree of $b$ is ...
- The out-degree of $b$ is ...

**b)** Are there any wheel graphs that are complete? Explain your answer. 2 pt

**c)** Let $G$ be an undirected graph with $v$ vertices and $e$ edges. Let $M$ be the maximum degree of the vertices of $G$. Is it true that $\frac{2e}{v} \leq M$. If it is, prove it! If it is not, provide an explicit counterexample. 3 pt

**d)** Given the following adjacency matrix, describe the characteristics of the corresponding graph $G$. 3 pt

$$
A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array}
\begin{array}{c}
\begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\
\begin{bmatrix}
0 & 2 & 3 & 0 & 4 & 1 & 2 \\
2 & 0 & 1 & 1 & 1 & 1 & 0 \\
3 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 \\
4 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

- Edges: directed / undirected, because...
- Multiple edges: yes / no, because...
- Loops: yes / no, because...

Your final grade is the sum of your scores divided by 3.0.