Report for exercise 3 from group H

Tasks addressed:	5
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Last compiled:	2021–05–26
Source code:	https://github.com/Combo1/MLCMS/tree/main/exercise3

The work on tasks was divided in the following way:

Taiba Basit (03734212)	Task 1	33%
	Task 2	33%
	Task 3	33%
	Task 4	33%
	Task 5	33%
Zeenat Farheen (03734213)	Task 1	33%
	Task 2	33%
	Task 3	33%
	Task 4	33%
	Task 5	33%
Fabian Nhan (03687620)	Task 1	33%
	Task 2	33%
	Task 3	33%
	Task 4	33%
	Task 5	33%

Report on task 1/5, Vector Fields, orbits, and visualization

For task 1 we used matplotlib and numpy to visualize/calculate our results. We started off with the given matrix A_{alpha}. To represent the arrows of the meshgrid we used two two-dimensional numpy-ndarray with a shape of (21,21) for the arrows in intervals of 0.1 between -1.0 and 1.0. To get the direction of those arrows we define two variables fx and fy, which we initialize with x/y. To calculate their value we matrix multiply A times the stacked matrix of x and y to get the direction of the arrows. And then bring fx and fy into numpy.ndarray form. To plot the phase portrait + trajectory we use pyplot, via the streamplot method we ploted the trajectory with a starting point, which we read from the images in the exercise sheet. The phase portrait was plotted by the quiver method. We end up with exactly the same plots as in the exercise sheet. The first figure 1 is a focus phase portrait with two positive eigenvalues (one positive in the complex numbers and negative), which tends away from the fixed point → unstable. Figure 2 is the same.

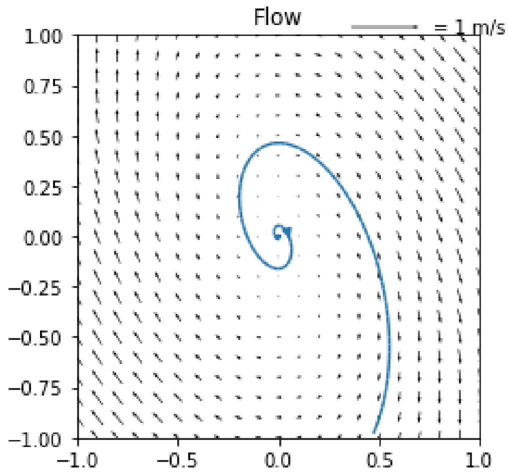

Figure 1: Phase Portrait 1a)

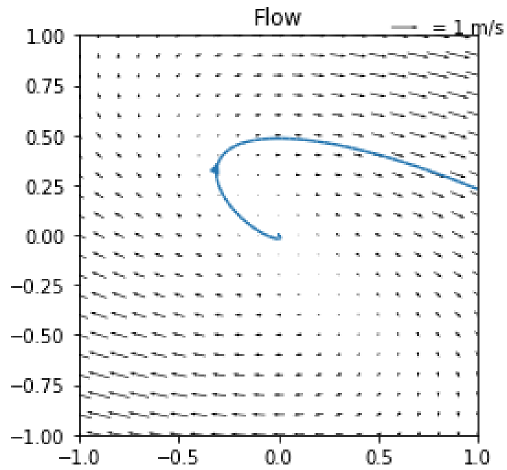

Figure 2: Phase Portrait 1b)

Figure 3 has two positive eigenvalues with no complex part. According to [1] Kuznetsov this is a node phase portrait, which is unstable. The same applies for figure 4

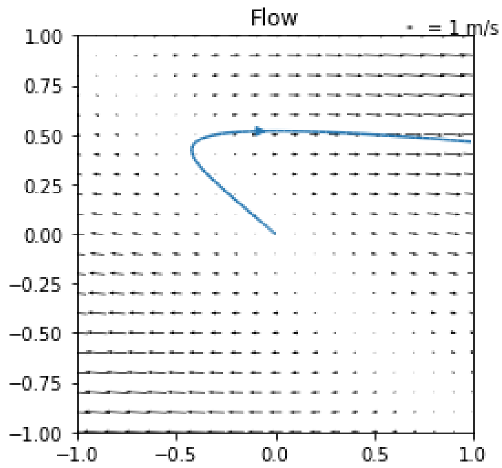

Figure 3: Phase Portrait 1c)

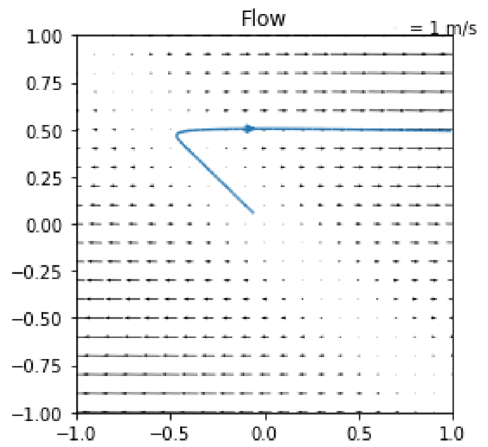

Figure 4: Phase Portrait 1d)

To replicate the phase portraits in the [1] Kuznetsov book we used a matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$. This matrix has the eigenvalues -1 and -2 (both negative, real, no imaginary part) → We get a node phase portrait, which is stable. The resulting phase portrait is shown in Figure 5.

$A = \begin{pmatrix} -1 & -1 \\ 5 & -1 \end{pmatrix}$ This matrix has the eigenvalues -1+2.23606798i and -1-2.23606798i (both negative, real + imaginary) → stable, focus phase portrait.The resulting phase portrait is shown in Figure 6.

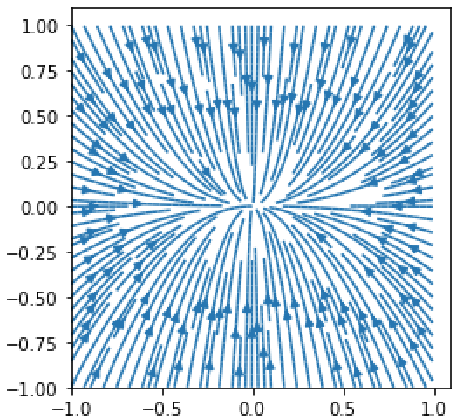

Figure 5: Phase Portrait 1e)

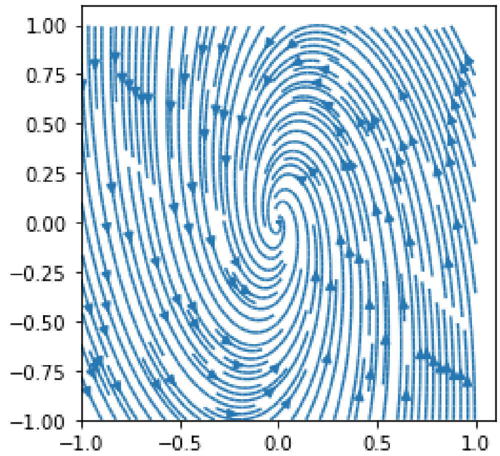

Figure 6: Phase Portrait 1f)

$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ This matrix has the eigenvalues -1 and 1 (one negative real and one positive real, no imaginary part) → unstable, saddle phase portrait. The resulting phase portrait is shown in Figure 7.

$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ This matrix has the eigenvalues 1 and 2 (both positive real, no imaginary part) → unstable, node phase portrait.The resulting phase portrait is shown in Figure 8.

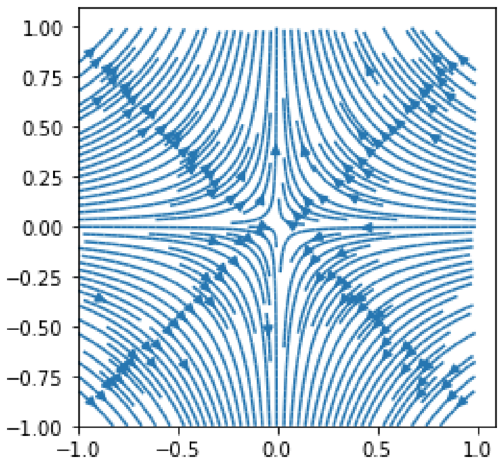

Figure 7: Phase Portrait 1g)

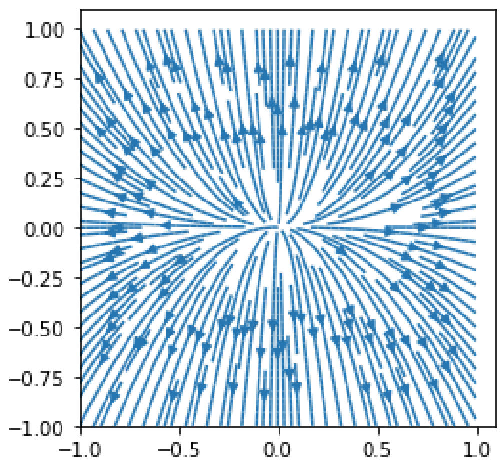

Figure 8: Phase Portrait 1h)

$A = \begin{pmatrix} 1 & 1 \\ -5 & 1 \end{pmatrix}$ This matrix has the eigenvalues 1+2.23606798i and 1-2.23606798i (both positive real, with imaginary part) → unstable, focus phase portrait. The resulting phase portrait is shown in figure 9

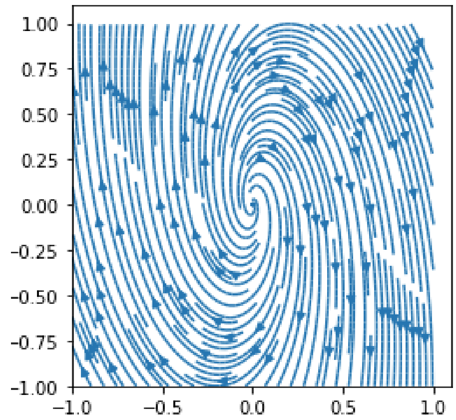

Figure 9: Phase Portrait 1i)

Topological equivalence:

A dynamical system (I, X, ϕ) is topologically equivalent to another dynamical system (I, Y, ψ) if there is a homeomorphism $h : X \rightarrow Y$ mapping orbits of the first system onto orbits of the second system, preserving the direction of time.

In our first four examples from the exercise sheet we can say that a and b/c and d are topologically equivalent since they are the same phase portrait in terms of shape (spirals/straight lines) just with different speeds, their center are all unstable.

From the [1] Kuznetsov Book we can only consider of phase portraits with positive, positive/negativ and negative eigenvalues to have the same topological equivalence, because their direction of arrows are not the same and this would lead to a problem in terms of "preserving the direction of time" of the topological equivalence definition.

For the phase portraits negative, real eigenvalues/negative, real and complex eigenvalues, they are topologically equivalent. Their center is stable with curved + straight lines/spirals as trajectories. Their focus point is a node/focus. And their behavior towards the center can be described as monotonous or close to monotonous/oscillatory. They are not orbitally or smoothly equivalent, but topologically equivalent let us consider the function. h : $\begin{cases} (\mathrm{p}_0 + theta_0)\ \sin(p_0 + theta_0) \\ (\mathrm{p}_0 + theta_0)\ \cos(p_0 + theta_0) \end{cases}$

Which maps from the phase portrait 1e) to 1f). $\rightarrow$ We keep information on the amount, stability and topology of invariant sets. Whereas we lose information about the time-dependent behavior.

For the phase portraits with positive, real eigenvalues/positive, real and complex eigenvalues, they are topologically equivalent, because they are unstable with curved + straight lines/spirals as trajectories. Their focus point is node/focus. Their behavior towards the center is close to monotonous/oscillatory. Their topological equivalence can be seen by the function. h : $\begin{cases} (\mathrm{p}_0 + theta_0)\ \sin(p_0 + theta_0) \\ (\mathrm{p}_0 + theta_0)\ \cos(p_0 + theta_0) \end{cases}$

This function maps from 1h) to 1i). $\rightarrow$ We keep information on the amount, stability and topology of invariant sets. Whereas we lose information about the time-dependent behavior.

Report on task 2/5, Common bifurcations in nonlinear systems

For the dynamical system described by $\dot{x} = a - x^2$, we obtain the bifurcation diagram as shown in Figure 10 for α in the range (-1,1). The resulting bifurcation is called *saddle-node bifurcation*.

Saddle-node bifurcation:

It is a local bifurcation in which two fixed points (or equilibria) of a dynamical system collide and annihilate each other.

For the given dynamical system, steady states exist only for $\alpha > 0$. We obtain 2 steady states, $x = \sqrt{\alpha}$ and $x = -\sqrt{\alpha}$. To find out if the stability of the steady states, we calculate the first derivative of the normal form.

$$\frac{d\dot{x}}{dx} = -2x = f'(\dot{x})$$

Evaluating the first derivative $f'(\dot{x})$ at steady state $x = \sqrt{\alpha}$, we get $f'(\sqrt{\alpha}) = -2\sqrt{\alpha} < 0$. Thus steady state at $x = \sqrt{\alpha}$ is stable. Similarly, evaluating at second steady state $x = -\sqrt{\alpha}$, $f'(-\sqrt{\alpha}) = 2\sqrt{\alpha} > 0$, which means that steady state at $x = -\sqrt{\alpha}$ is unstable. At $\alpha = 0$, we get one steady state state $x = 0$. Evaluating $f'(0) = 0$, which means that this is a saddle point. We have visualized it by an open red circle at (0,0) in figure 10.

At $\alpha = 0$, saddle node bifurcation happens and 2 steady states collide into one steady state i.e. $x = 0$.

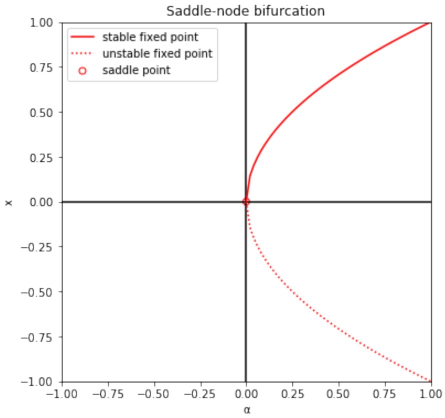

Figure 10: Plot for $\dot{x} = \alpha - x^2$

For the second dynamical system, $\dot{x} = \alpha - 2x^2 - 3$ We get 2 steady states for $\alpha > 3, x = \sqrt{\frac{\alpha-3}{2}}$ and $x = -\sqrt{\frac{\alpha-3}{2}}$. No steady states exist for $\alpha < 3$. To find out the stability of these states we calculate the first derivative of the given equation, to get

$$\frac{d\dot{x}}{dx} = -4x = f'(\dot{x})$$

Evaluating the first derivative $f'(\dot{x})$ at steady state $x = \sqrt{\frac{\alpha-3}{2}}$, we get $f'(\sqrt{\frac{\alpha-3}{2}}) = -4\sqrt{\frac{\alpha-3}{2}} < 0$. Thus steady state at $x = \sqrt{\frac{\alpha-3}{2}}$ is stable. Similarly, evaluating at second steady state $x = -\sqrt{\frac{\alpha-3}{2}}$, $f'(-\sqrt{\frac{\alpha-3}{2}}) = 4\sqrt{\frac{\alpha-3}{2}} > 0$, which means that steady state at $x = -\sqrt{\frac{\alpha-3}{2}}$ is unstable.At $\alpha = 3$, we get one steady state state $x = 0$. Evaluating $f'(0) = 0$, which means that this is a saddle point. We have visualized it by an open red circle at (3,0) in figure 11.

At $\alpha = 3$, saddle node bifurcation happens and 2 steady states collide into one steady state i.e. $x = 0$.

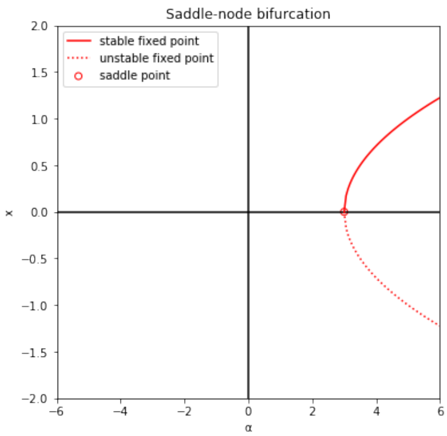

Figure 11: Plot for $\dot{x} = \alpha - 2x^2 - 3$

Topological Equivalence:

As can be seen in fig 12, both dynamical systems have the same parabolic bifurcation diagram. Slight differences between the 2 exists. The second dynamical system does not have any steady states for $\alpha < 3$, while 2 steady states exists for the first dynamical system for $\alpha > 0$. Based on this evidence, we can conclude that the 2 systems are not topologically equivalent at $\alpha = 1$.

However, at $\alpha = -1$, both dynamical systems do not have any visible steady states. Moreover, as both the systems behave in the same quadratic manner , we can say that the 2 systems are indeed topologically equivalent at $\alpha = -1$. The second dynamic system varies in the additional constant term of -3 and a constant multiplier of 2 in the quadratic term. This difference can be mapped by a homeomorphism, thus making the two systems topologically equivalent in region where $\alpha > 3$ and $\alpha < 0$. Our argument is also supported by the fact that both the systems have the same normal form. Both the systems are locally topologically equivalent to the normal form because they both have one saddle node and split into 2 steady states with increasing α. The 2 steady states for the dynamic systems are also very similar in nature, one stable and one unstable. As both the systems are locally indistinguishable and follow the same parabolic nature , we can conclude that they both have the same normal form.

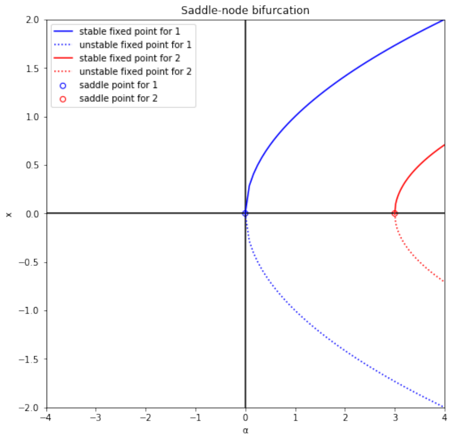

Figure 12: Combined plot for both systems

Report on task 3/5, Bifurcations in higher dimensions

Andronov-Hopf Bifurcation describes the vector field which has the following normal form:

$$\dot{x}_1 = \alpha x_1 - x_2 - x_1(x_2^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + \alpha x_2 - x_2(x_2^2 + x_2^2)$$

Plotting the phase portrait of the above system at three representative values of $\alpha = -1, \alpha = 0$ and $\alpha = -1$, we get the plots as shown in Figure 13

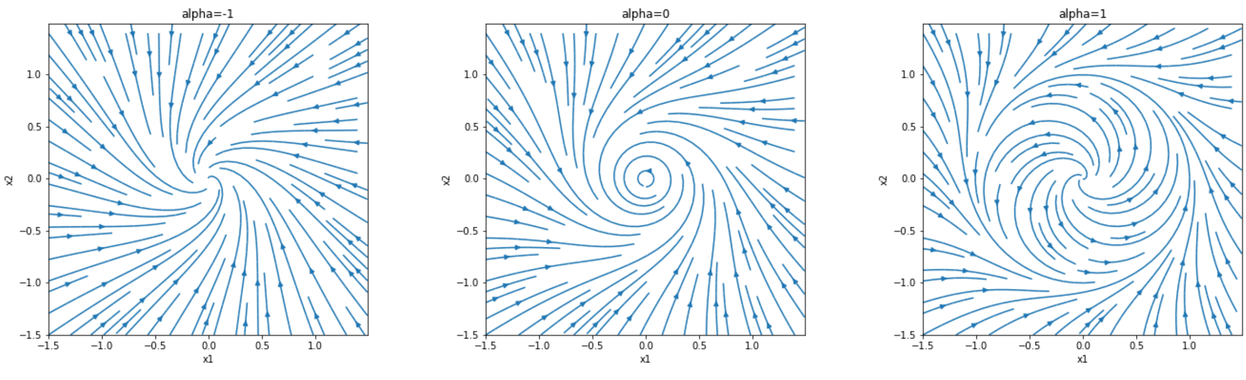

Figure 13: Andronov-Hopf Bifurcation

This system has the equilibrium $x_0 = (0, 0)$ for all parameter values α, and the Jacobi matrix of the system evaluated at x_0 is:

$$A := \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix}$$

The eigenvalues of A are:

$$\lambda_{1,2}(\alpha) = \alpha \pm i$$

For $\alpha < 0$, we see a stable equilibrium point at the origin (0,0) as shown in figure 13 A. At $\alpha = 0$, the system goes through a bifurcation and the stable fixed point at origin turns into a limit cycle depending on the parameter. This is known as Andronov-Hopf bifurcation. This limit cycle grows in diameter and becomes clearer as $\alpha > 0$. This can be clearly seen in figure 13 C which shows a limit cycle for $\alpha = 1$. One can approximate that the radius of the limit cycle might be $\sqrt{\alpha}$. The above system represents the supercritical Andronov-Hopf bifurcation at the transition from $\alpha < 0$ to $\alpha > 0$.

To analyze how the limit cycle form for case $\alpha > 0$, we plot the orbit of the system forward in time with initial point at (2,0) and $\alpha = 1$ in figure 14. To compute the orbit, we used **scipy.integrate.solve ivp** initial value problem solver. Similarly, in figure 15, we plot the orbit of the system forward in time with initial point at (0.5,0) and $\alpha = 1$. In both the figures, we can see how the orbit progresses towards and converges to become the limit cycle of radius 1.

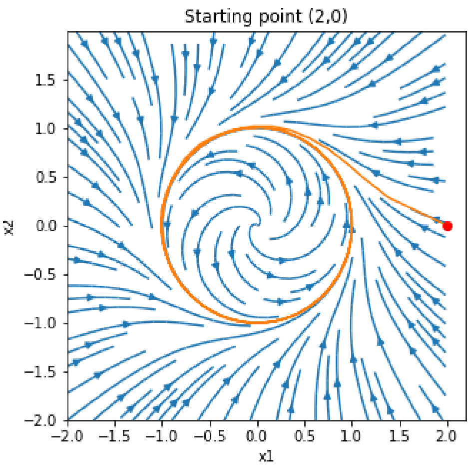

Figure 14: Plot with initial value (2,0)

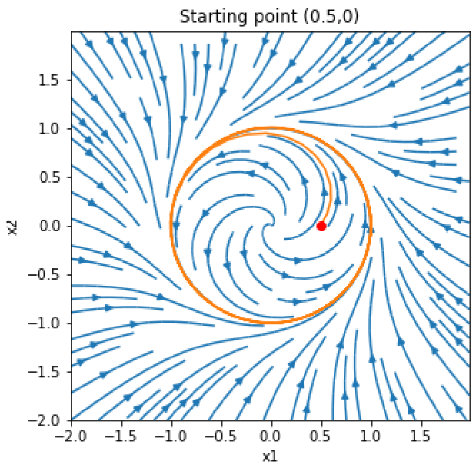

Figure 15: Plot with initial value (0.5,0)

Bifurcation Analysis We now analyze the bifurcation in one state space dimension X = R, but with 2 parameters α belongs to R^2. This is called the cusp bifurcation and has the normal form of:

$$\dot{x} = \alpha_1 + \alpha_2 x - x^3$$

The figure 16 and 17 shows the graph for $\dot{x} = 0$ or $\alpha_1 = x^3 - \alpha_2 x$. It can be inferred from the figure, that when $\alpha_1 < 0$ and $\alpha_2 < 0$, we see only one stable equilibrium point at x = 0. At $\alpha_1 = 0$ and $\alpha_2 = 0$, we see the cusp bifurcation happening and the single fixed point now splitting into 3 steady states. Out of these 3 steady states, it can be found out that 2 are stable and one is unstable. Thus, in this case, when (α_1, α_2) passes through zero from negative to positive values, the asymptotically stable equilibrium x = 0 becomes unstable and two curves of asymptotically stable equilibria are branching off. This is visually represented as a cusp and is therefore called the cusp bifurcation.

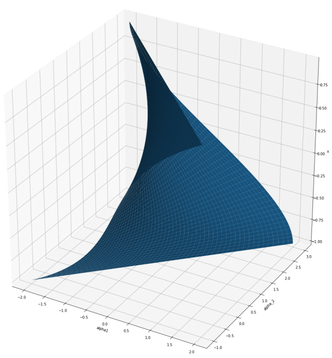

Figure 16: Cusp Bifurcation

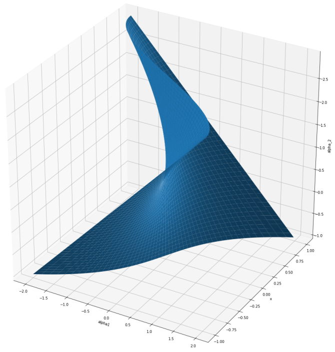

Figure 17: Cusp Bifurcation from a different perspective

Report on task 4/5, Chaotic dynamics

Dynamical systems can behave in very irregular ways, and changes in their parameters can lead to very drastic changes in their behavior. Consider the discrete map:

$$x_{n+1} = rx_n(1 - x_n), n \in N$$

with the parameter $r \in (0, 4]$. For this task we have considered the value of x in the range 0 and 1. Now performing the bifurcation analysis for different values of r:

- **Vary r from 0 to 2:**
  We also plotted the discrete map for various value of r in this range as well as different initial values of x as shown in figure 18. We see that the bifurcation plot remains 0 for r less than 1 as well as the plots in figure 18 also does not show much variation.
  Steady states:
  A point $x_0 \in X$ is called an equilibrium (Fixed point, steady state) if $\phi(t, x_0) = x_0$ for all $t \in I$.
  To evaluate the steady states:

$$\begin{aligned} rx(1-x) &= x \\ rx(1-x) - x &= 0 \\ x(r - rx - 1) &= 0 \end{aligned}$$

From the above we get $x_1^* = 0$ or $x_2^* = \frac{r-1}{r}$. To analyze the stability of these fixed points, we take the derivative of the logistic map function with respect to x as

$$f_r'(x) = r - 2xr$$

Case 1 : $r \in (0, 1)$
In this range the second fixed point x_2^* does not exist for r $<$ 1. Evaluating $f_r'(x)$ at the first fixed point x_1^*, we get

$$|f_r'(0)| = r < 1$$

Thus x_1^* is the only stable state in this interval and acts an attractor i.e. irrespective of the initial value, the orbit will be attracted to x_1^* as n $\rightarrow$ infinity.

Case 2 : 1<=r<=2
In this interval, both x_1^* and x_2^* exist. However, x_1^* becomes unstable and x_2^* is is stabilized,

$$|f_r'(r-1/r)| = |2-r| < 1$$

At r=1 transcritical bifurcation happens as the stability of x_1^* and x_2^* changes at this point.

Limit Cycles: No periodic behaviour is observed in this interval of r. Thus there are no limit cycles.

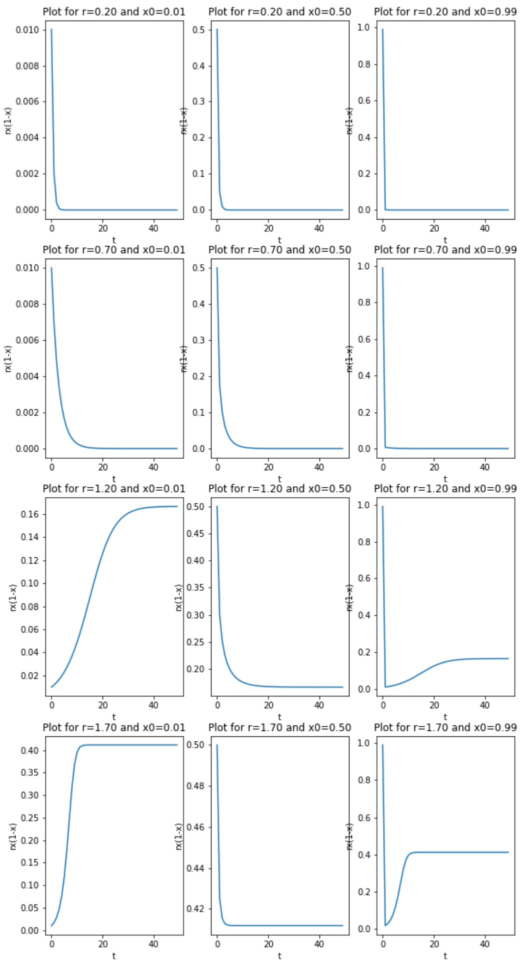

Figure 18: Plots obtained varying values of r from 0 to 2 and x_0

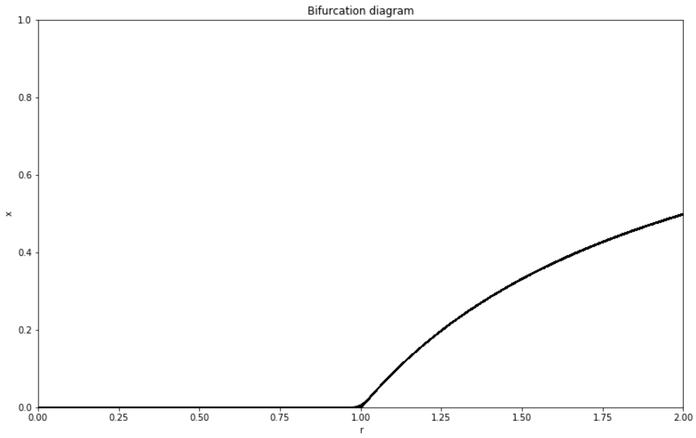

Figure 19: Bifurcation Diagram for r from 0 to 2

- **Vary** $r \in (2, 4]$
  Steady states:
  Interesting behaviour can be seen in this interval of r.

  Case 1 : 2 < r < 3
  As r grows from 2 , we see one stable state $\frac{r-1}{r}$ irrespective of the initial value of x.
  Case 2 : r > 3
  When r>3, pitch-fork bifurcation occurs and two steady state appears thus giving a limit cycle of period 2. The previous steady state i.e x_2^* now becomes unstable as $|f_r'(r - 1/r)| = |2 - r| > 1$ for r > 3.

  When r is between 3.44949 and 3.54409, the value of x oscillates between 4 values. With r increasing beyond 3.54409, from almost all initial conditions x will approach oscillations among 8 values, then 16, 32, etc. The lengths of the parameter intervals that yield oscillations of a given length decrease rapidly, the ratio between the lengths of two successive bifurcation intervals approaches the Feigenbaum constant 4.66920. This behaviour is called period-doubling cascade. At r = 3.56995 is the onset of chaos. From almost all initial conditions, we no longer see oscillations of finite period. However there are short periods of oscillations seen after this as well which is called intermittency. Slight variations in the initial value of x yield dramatically different results over time. The steady states are visualized as black lines in figure 22.

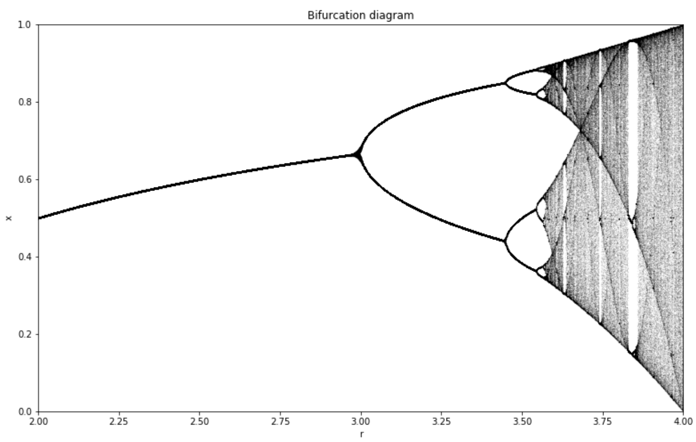

Figure 20: Bifurcation Diagram for r from 2 to 4

Limit cycles:
The limit circles exist at: 3, 3.449, 3.544,... until 3.569 after which "chaos" starts and no finite oscillations are observed. The red marks in figure 22 shows the points where limit cycle appears when r is varied from 0 to 4.

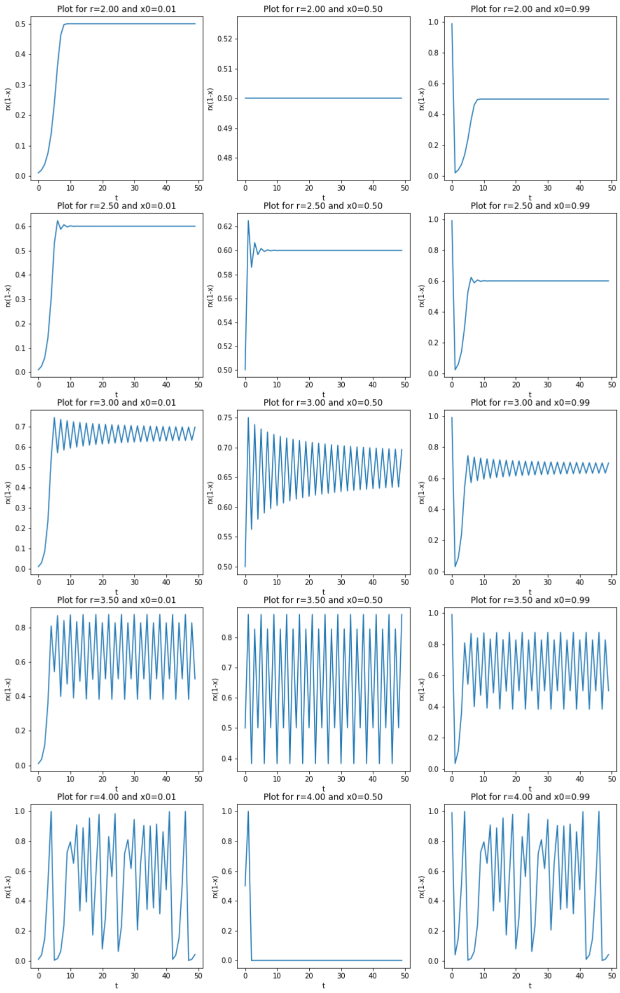

Figure 21: Plots obtained by varying values of r from 2 to 4 and x_0

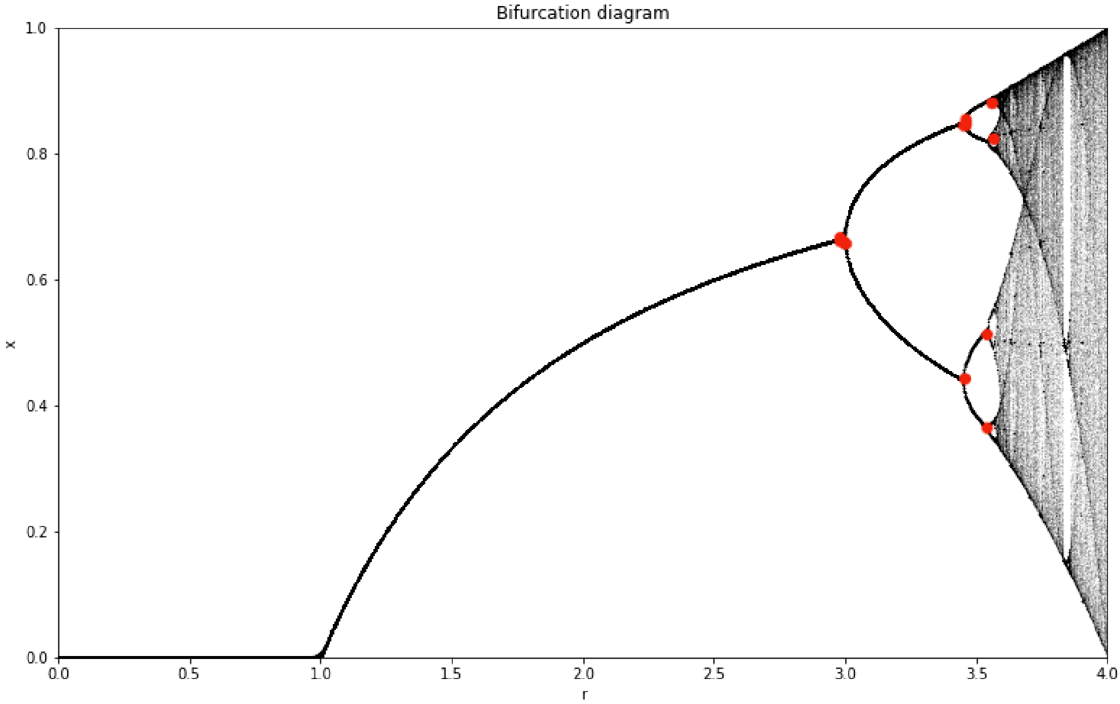

Figure 22: Bifurcation Diagram

Figure below shows the onset how the trajectory converges for the same starting point of x = 0.5 and varying values of parameter r.

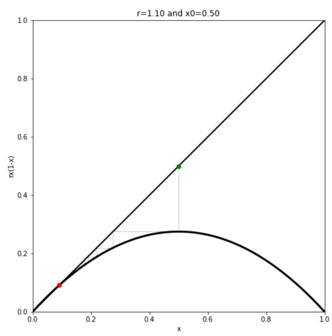

Figure 23: Trajectory reaches the a fixed end point for r = 1.10 and $x_0 = 0.5$

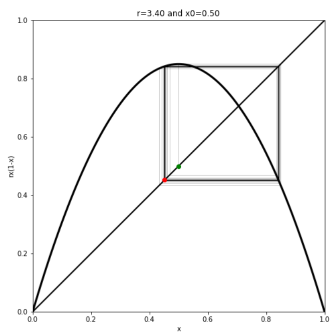

Figure 24: Trajectory oscillates between two fixed points for r = 3.40 and $x_0 = 0.5$

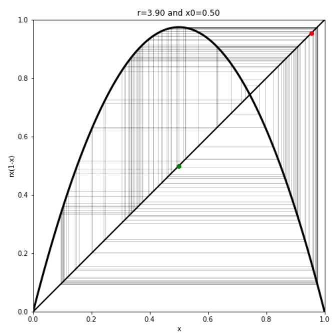

Figure 25: Chaotic behaviour for r = 3.90 and $x_0 = 0.5$

For the next part of this question, we implemented the Lorenz Attractor. For the first part we visualized the trajectory of the Lorenz system starting at $x_0 = (10, 10, 10)$, until the end time of $T_{end} = 1000$. We set the values of $\sigma = 10$, $\beta = 8/3$ and $\rho = 28$. The trajectory is shown in figure 26

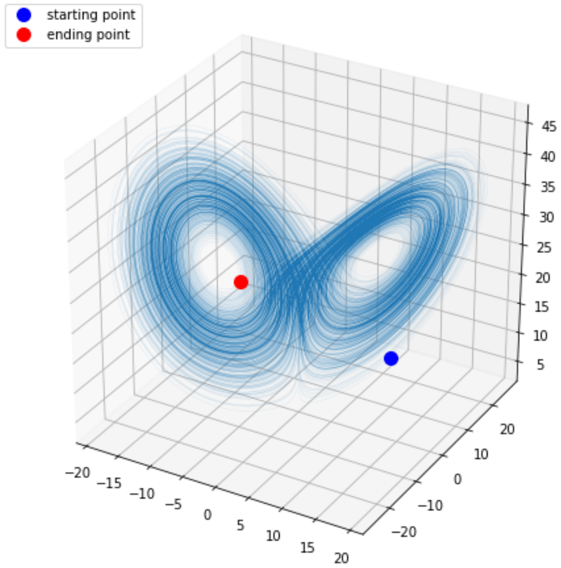

Figure 26: Trajectory of Lorenz System for the initial value (10,10,10) and $\rho = 28$

Next we visualized the trajectory of the Lorenz system starting at $x_0 = (10 + 10^{-8}, 10, 10)$, until the end time of $T_{end} = 1000$. We set the values of $\sigma = 10$, $\beta = 8/3$ and $\rho = 28$. The trajectory is shown in figure 26

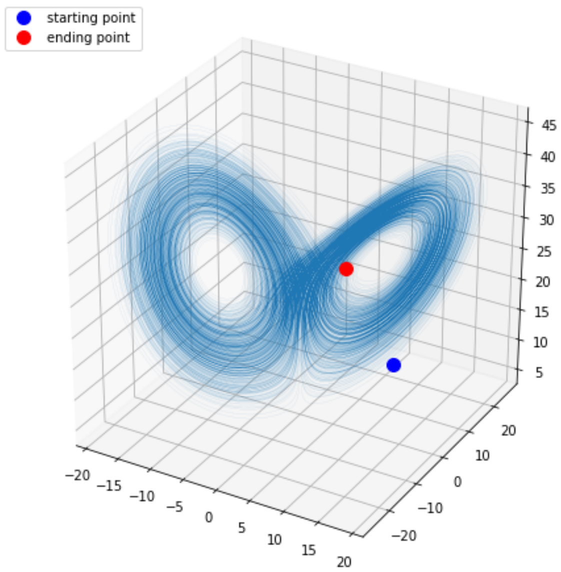

Figure 27: Trajectory of Lorenz System for the initial value $(10 + 10^{-8}, 10, 10)$ and $\rho = 28$

The chaotic nature can be clearly seen from both the images. Although, the difference in the starting point was negligible, the trajectory changed a lot. This is also clear from the ending point of both the trajectory.The chaotic nature leads to the difference larger than 1 between the points of the trajectory at 22.61 simulation time. This is very early as the simulation was run for $T_{end} = 1000$. Thus, we can clearly say that the chaotic nature of the system grows exponentially with the change in initial condition. To analyze the behaviour of the system in relation to the parameter ρ, we visualized both the trajectory for same conditions and only changed the value of ρ from 28 to 0.5. Figure 28 and figure 29 shows both the trajectories.

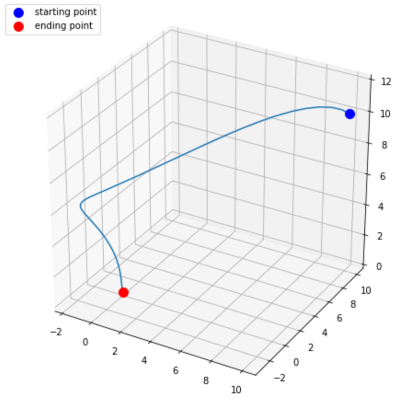

Figure 28: Trajectory of Lorenz System for the initial value $(10 + 10^{-8}, 10, 10)$ and $\rho = 0.5$

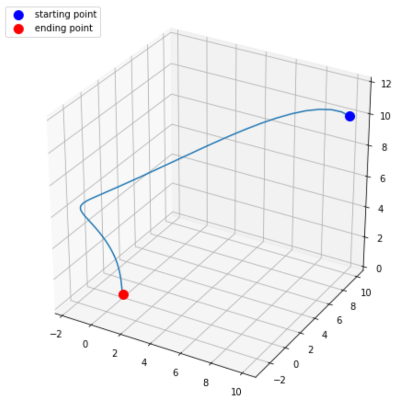

Figure 29: Trajectory of Lorenz System for the initial value $(10 + 10^{-8}, 10, 10)$ and $\rho = 0.5$

For this case the difference between the points in the trajectory does not exceed 1. Also, as can be seen from the plot there is little difference. So, we can say that the sensitivity of the system with changes in the initial condition decreases as we decrease the value of ρ.

Bifurcation:

To analyze the bifurcation of Lorenz system of equations, we plot the orbit at different values of ρ. In all these cases, the initial point is fixed at (10,10,10)

Case : $\rho < 1$

If ρ <1 then there is only one stable equilibrium point, which is at the origin as shown in Figure 30 and 31 . This means that independent of the starting positions, all points move towards the origin.

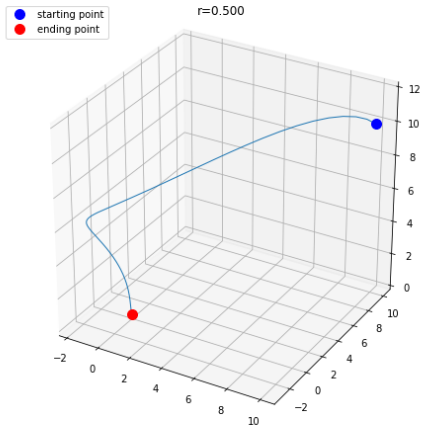

Figure 30: Trajectory of Lorenz System for the initial value $(10, 10, 10)$ and $\rho = 0.5$

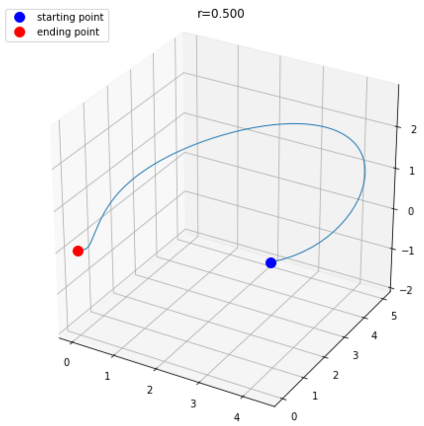

Figure 31: Trajectory of Lorenz System for the initial value $(2.0, 5.0, -2.0)$ and $\rho = 0.5$

Case : At $\rho > 1$

A pitchfork bifurcation occurs at ρ =1, and for ρ >1 two additional critical points appear. This is shown in fig 32 and 33 where we see two different end points for 2 different starting points.

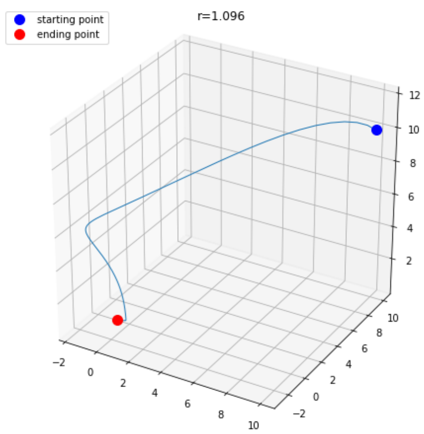

Figure 32: Trajectory of Lorenz System for the initial value $(10, 10, 10)$ and $\rho = 1.096$

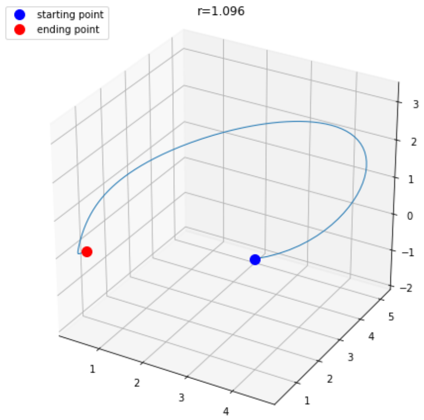

Figure 33: Trajectory of Lorenz System for the initial value $(2.0, 5.0, -2.0)$ and $\rho = 1.096$

Case : $\rho \in (7, 9)$
In this interval we see that the end point of trajectory for same initial point shifts from one fixed point to another.

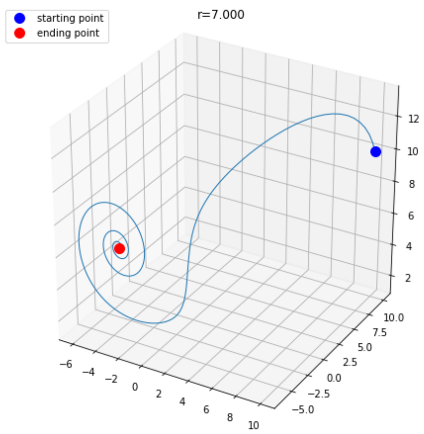

Figure 34: Trajectory of Lorenz System for the initial value $(10, 10, 10)$ and $\rho = 7$

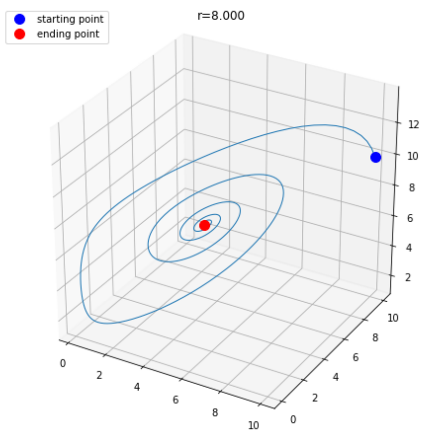

Figure 35: Trajectory of Lorenz System for the initial value $(10, 10, 10)$ and $\rho = 9$

Case : $\rho \in (19, 24)$

At $\rho = 19$, we observe only one fixed point towards which the orbit spirals. As $\rho > 19$, the trajectory changes and now seems to be spiralling around 2 fixed points. This change is not topologically equivalent to the previous state and suggests that a bifurcation has occurred after $\rho = 19$.

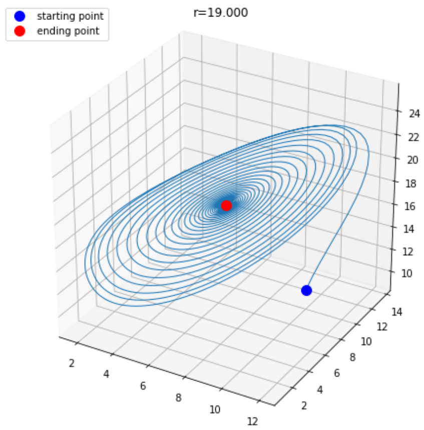

Figure 36: Trajectory of Lorenz System for the initial value $(10, 10, 10)$ and $\rho = 19$

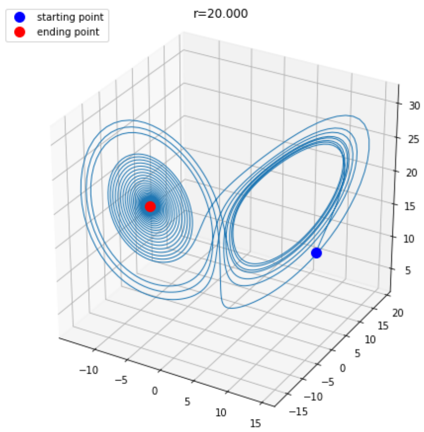

Figure 37: Trajectory of Lorenz System for the initial value $(10, 10, 10)$ and $\rho = 20$

Case : $\rho >=$ **24** We observe that the end point never meets the fixed points suggesting a change in the stability of the fixed points. This means a bifurcation has happened near this value of ρ in the system under consideration.

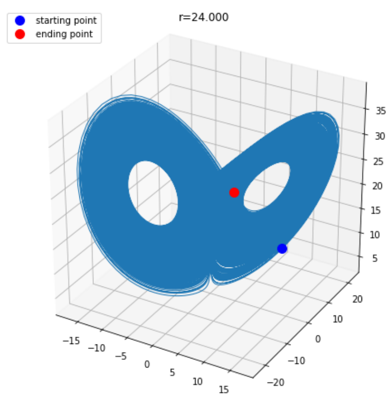

Figure 38: Trajectory of Lorenz System for the initial value $(10, 10, 10)$ and $\rho = 24$

Report on task 5/5, Bifurcations in crowd dynamics

1. For the first subtask we moved the methods of mu, R0, h and model inside of the SIR_Model.py file with their added documentation, which we got from reading the exercise sheet and the annotated papers.
2. For our purposes the imports were all working only the SIR model equations needed to be fixed, which we copied from the exercise sheet.

```
#Task 5.2
dSdt = A - d * S - (beta * S * I) / (S + I + R)
dIdt = - (d + nu) * I - m * I + (beta * S * I) / (S + I + R)
dRdt = m * I - d * R
```

Figure 39: SIR Model Equations

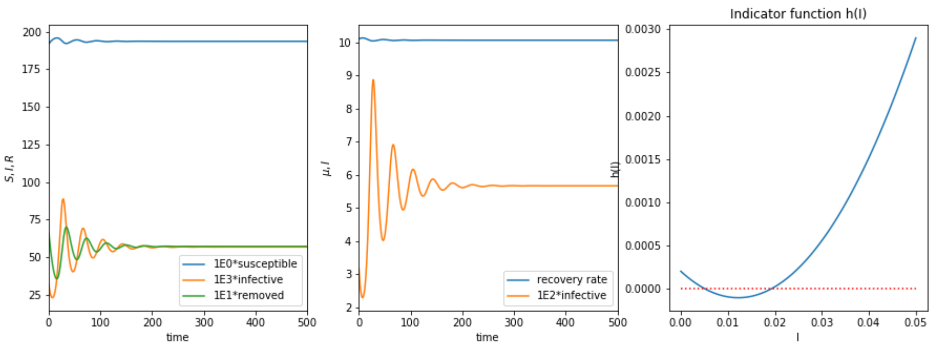

Figure 40: SIR Model Plots

3. We initialized the parameter b with the values in the range 0.01 to 0.03 with a step size of 0.001 and started off with the first starting point $(S_0,\ I_0,\ R_0) = (195.3,\ 0.052,\ 4.4)$. Then we iterated through all those and called the plotting methods. You can see for the 3D as well as in the 2D plots, where we removed the R axis by omitting the projection="3D" attribute in the fig.add_subplot function and removing the value from ax.plot, that towards 0.022 the plot seems to oscillate around the fixed point but never reaches it. At values smaller than 0.022 the trajectory quickly finds the center the smaller b is. For values b bigger than 0.022 the trajectory breaks out and leaves outside of our inspection scope.

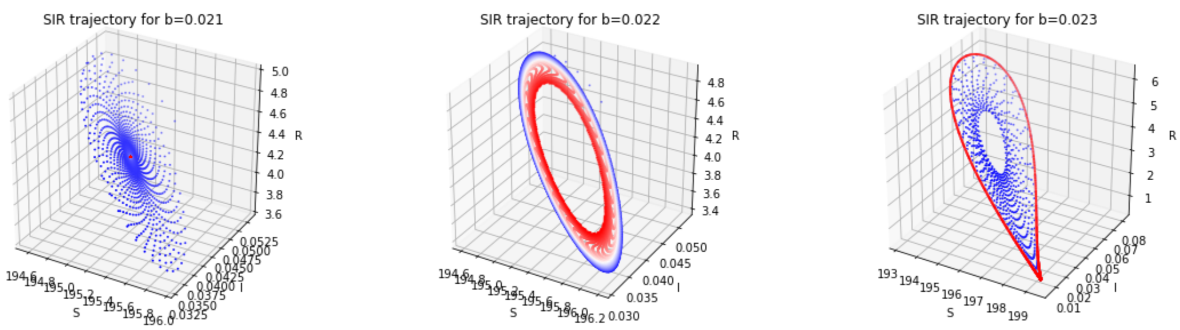

Figure 41: SIR trajectory for (195.3, 0.052, 4.4)

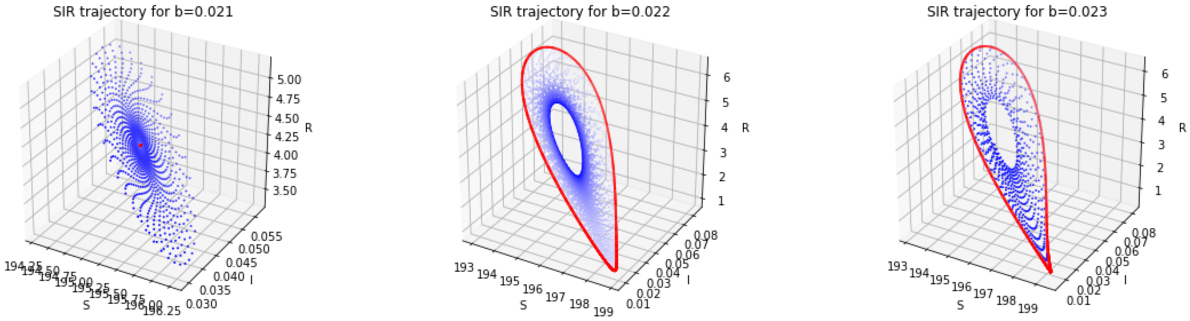

Figure 42: SIR trajectory for (195.7 0.03, 3.92)

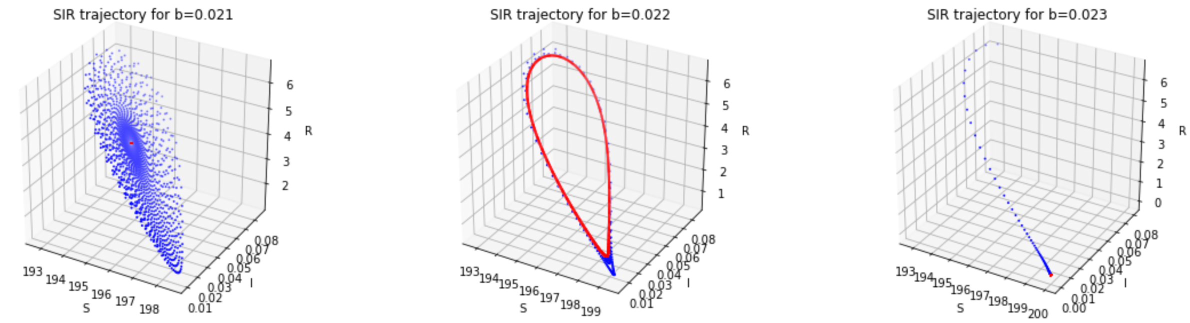

Figure 43: SIR trajectory for (193, 0.08, 6.21)

For our other two starting points on at the value of b = 0.022 an oval oscilation around the same point as in the first initialization can be found, for the other values of b no pattern was found only the single data point at the beginning was plotted.

4. We seem to have a Andronov-Hopf bifurcation here, since we seem to have an attracting point in the middle before b is 0.022 and when b hits 0.022 a limit cycle appears. Our normal form is: dz/dt = z((lambda + i) + b $|z|^2$)

5. According to [3] P. van den Driessche and James Watmough. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission: The basic reproduction number, denoted R0 is 'the expected number of secondary cases produced, in a completely susceptiblepopulation, by a typical infective individual'. If R0 $<$ 1, then on average an infected individual produces less than one new infected individual over the course of its infectious period, and the infection cannot grow. Conversely, if R0 $>$ 1, then each infected individual produces, on average, more than one new infection, and the disease can invade the population. For the case of a single infected compartment, R0 is simply the product of the infection rate and the mean duration of the infection. However, for more complicated models with several infected compartments this simple heuristic definition of R0 is insufficient. A more general basic reproduction number can be defined as the number of new infections produced by a typical infective individual in a population at a DFE.

   $\rightarrow$

   The variables beta, d, nu, mu1 are needed to compute the reproduction rate. If the parameter beta increases the reproduction number increases and if beta decreases the reproduction number decreases. When plotting the number of infective persons we can see a oscillating behavior for e.g. beta = 12.5 or 13.5, but a rapidly decreasing number without oscillations for beta = 11.5.

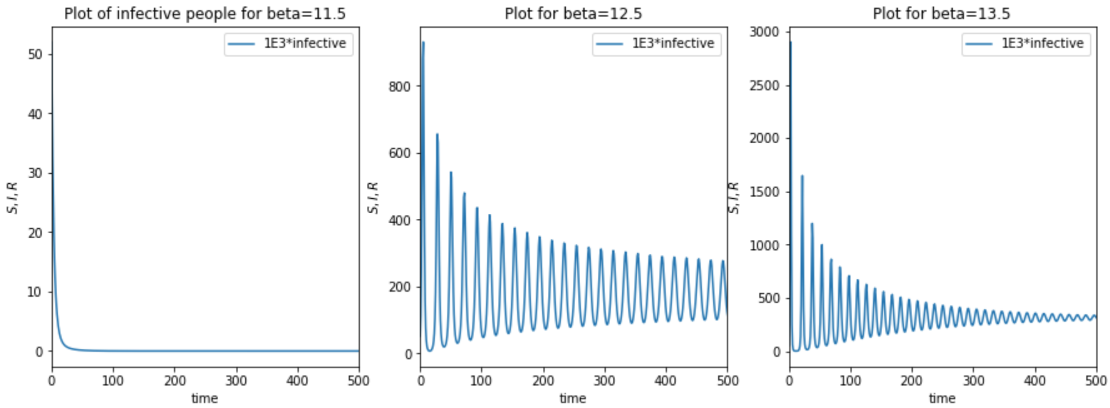

Figure 44: Number of infective people

6. According to [2], an equilibria can be found by setting the right hand side to system (2.2) equal to zero. One of those equilibrium points is the disease free equilibrium (DFE) which is E_0(A/d,0,0). To calculate E_0 we use the formula for the basic reproduction number. In the case of $R_0 < 1$, dependent on beta and v we have multiple different possible outcomes:

   - beta < v → no endemic equilibrium
   - beta = v → no endemic equilibrium
   - beta > v →
     - if beta <= d + v + mu0 → no endemic equilibrium
     - if beta > d + v + mu0 → two endemic equilibria if and only if $delta_0 > 0$, B < 0, and these two equilibria coalesce into E* if and only if $delta_0 = 0$, B < 0 otherwise there is no endemic equilibrium.

   (The endemic equilibrium state is the state where the disease cannot be totally eradicated but remains in the population.)

   For $R_0 < 1$ is an attracting node I could not find any definition for this term, but according to its name, I think for a reproduction number less than 1 the bifurcation converges towards the disease free equilibrium. If the values of (S, I, R) are close to E_0, then we are getting closer to a point at which no disease is present in the population. For values of (S,I,R) close to E_0 there is no change in amount of susceptible, infective or recovered people for b = 0.022.

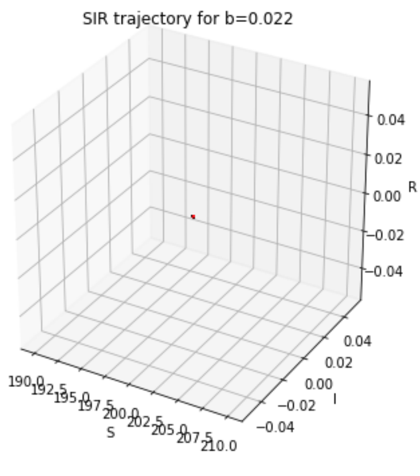

Figure 45: SIR close to E_0

7. According to paper [2], our system should be able to show backward bifurcation, saddle-node bifurcation, Hopf bifurcation and cusp type of Bogdanov–Takens bifurcation of codimension 3.

   We can see a saddle-node bifurcation at C^{-}_{delta}: b = (beta (mu_1 - mu_0) + d_0 (d_1 - beta) - sqrt(betad_1(mu_1 - mu_0)(d_1 - beta))) / ((beta - nu) d_1^2). b = 0.00372

   The backwards bifurcation occurs when mu1 = beta - d - nu and 0 < b < A * (mu1 - mu0) / beta (beta - nu). mu1 = 10.4, b = 0.0562.

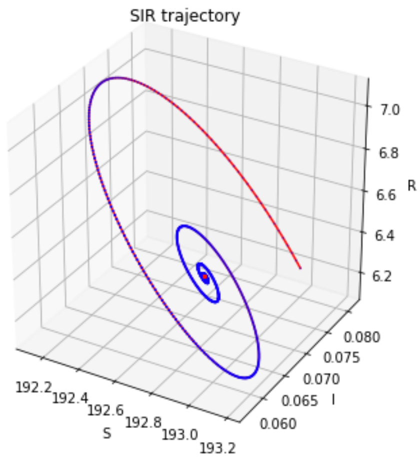

Figure 46: b = 0.00372

References

[1] Yuri A. Kuznetsov. *Elements of Applied Bifurcation Theory*. Springer New York, 2004.

[2] Chunhua Shan and Huaiping Zhu. Bifurcations and complex dynamics of an SIR model with the impact of the number of hospital beds *Journal of Differential Equations*, 257(5):1662–1688, September 2014.

[3] P. van den Driessche and James Watmough. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Mathematical Biosciences*, 180(1-2):29–48, November 2002.