INFINITE GROUP PROBLEM CODE:
FIGURES TEST SUITE

ABSTRACT. The purpose of this file is to verify that changes to the code do not cause regressions in the plotting output. In particular, we verify that the code samples in figure captions of published papers can create the same (or improved) figures. This file should be visually inspected periodically.

1. FIGURES FROM *Light on the Infinite Group Relaxation*

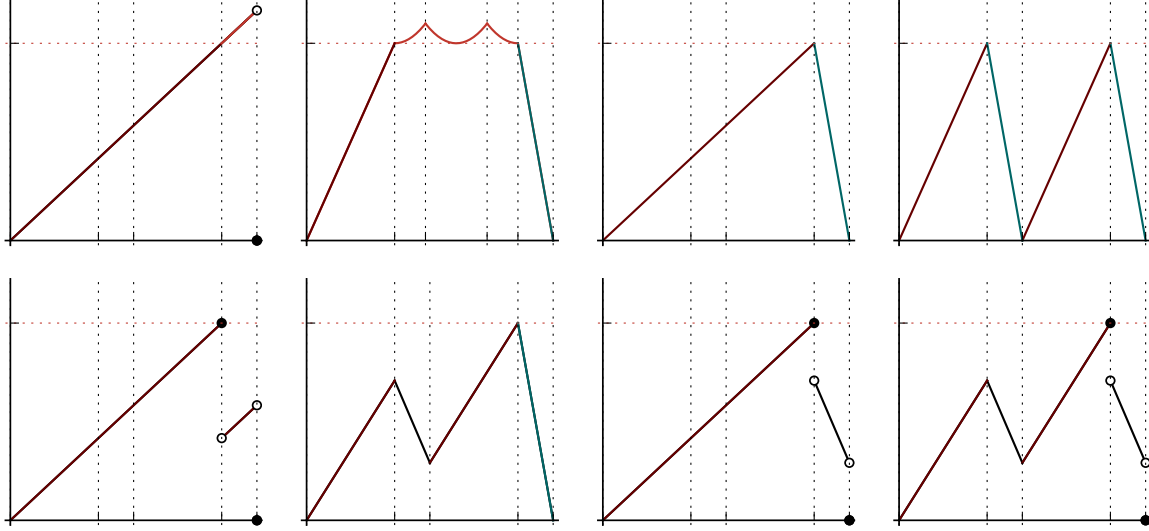FIGURE 1. The hierarchy of valid, minimal, and extreme functions by example... Even without checking the dominance, it is easy to see that some functions cannot be minimal: they have some function values larger than 1 (*international orange*), but minimal valid functions are upper bounded by 1.
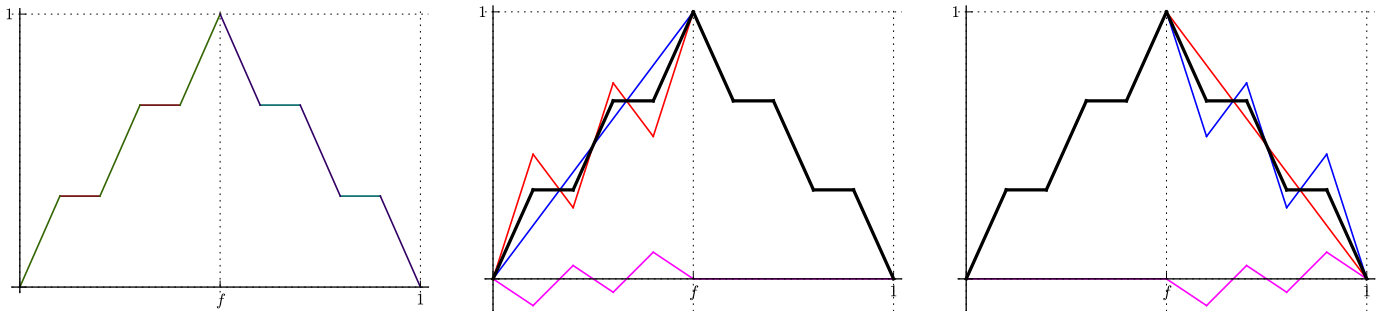

FIGURE 2. This function (`h = not_extreme_1()`) is minimal, but not extreme (and hence also not a facet), as proved by `extremality_test(h, show_plots=True)`. The procedure first shows that for any distinct minimal $\pi^1 = \pi + \bar{\pi}$ (*blue*), $\pi^2 = \pi - \bar{\pi}$ (*red*) such that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$, the functions π^1 and π^2 are continuous piecewise linear with the same breakpoints as π. A finite-dimensional extremality test then finds two linearly independent perturbations $\bar{\pi}$ (*magenta*), as shown.
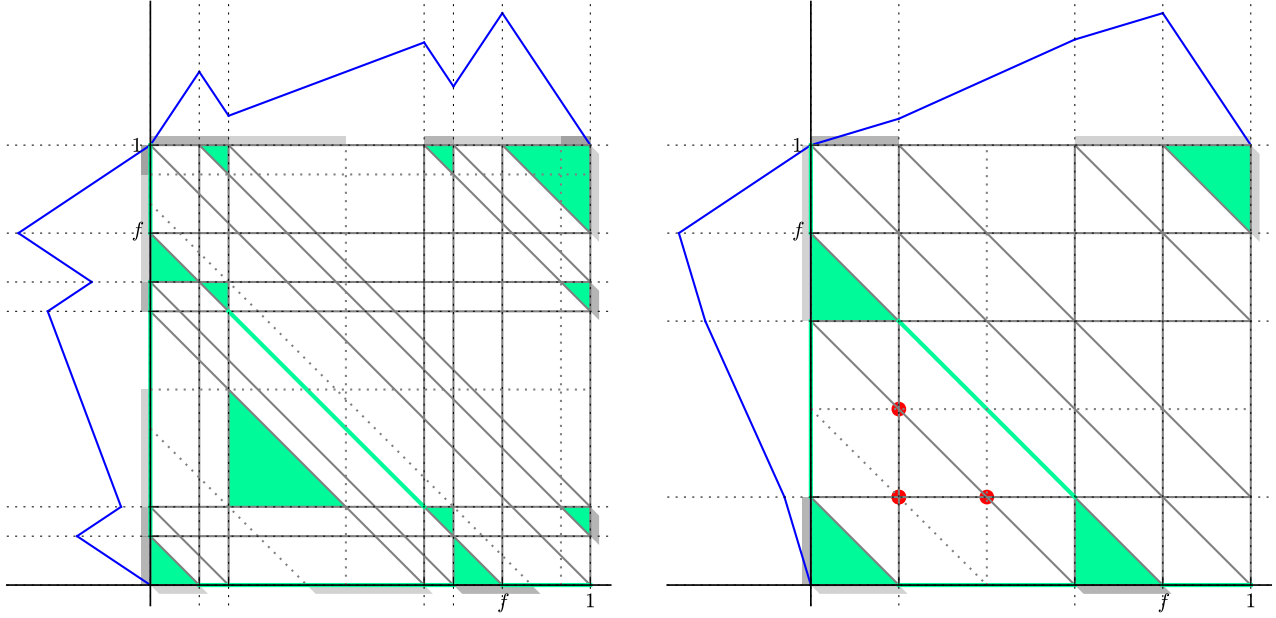
FIGURE 3.   Two diagrams of a function (*blue graphs on the top and the left*) and its polyhedral complex $\Delta\mathcal{P}$ (*gray solid lines*), as plotted by the command `plot_2d_diagram(h)`. *Left*, `h = gj_forward_3_slope()` *(left)*. *Right*, `h = not_minimal_2()`. The set $E(\pi)$ in both cases is the union of the faces shaded in green. The *heavy diagonal green line* $x + y = f$ corresponds to the symmetry condition. Vertices of $\Delta\mathcal{P}$ do not necessarily project (*dotted gray lines*) to breakpoints. Vertices of the complex on which $\Delta\pi < 0$ are shown as *red dots*. At the borders, the projections $p_i(F)$ of two-dimensional additive faces are shown as *gray shadows*: $p_1(F)$ at the top border, $p_2(F)$ at the left border, $p_3(F)$ at the bottom and the right borders.
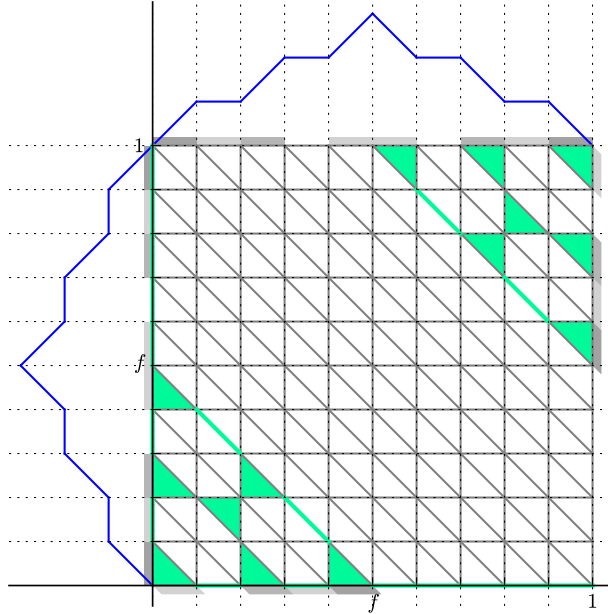

FIGURE 4.   Diagram of a function (*blue graphs on the top and the left*) on the evenly spaced complex $\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ and the corresponding complex $\Delta\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ (*gray solid lines*), as plotted by the command `plot_2d_diagram(h)`, where `h = not_extreme_1()`. Faces of the complex on which $\Delta\pi = 0$, i.e., additivity holds, are *shaded green*. The *heavy diagonal green lines* $x + y = f$ and $x + y = 1 + f$ correspond to the symmetry condition. At the borders, the projections $p_i(F)$ of two-dimensional additive faces are shown as *gray shadows*: $p_1(F)$ at the top border, $p_2(F)$ at the left border, $p_3(F)$ at the bottom and the right borders. Since the breakpoints of $\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ are equally spaced, also $\Delta\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ is very uniform, consisting only of points, lines, and triangles, and the projections are either a breakpoint in $\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$ or an interval in $\mathcal{P}_{\frac{1}{10}\mathbb{Z}}$; compare with Figure 3.
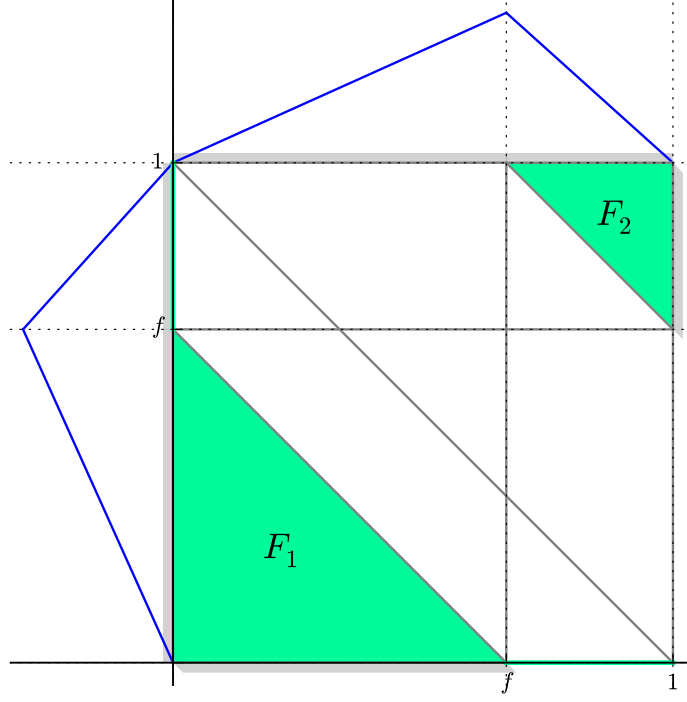
FIGURE 5. A diagram of a function of the type `gmic` (*blue graphs on the top and the left*) and its polyhedral complex $\Delta\mathcal{P}$ (*gray solid lines*), as plotted by the command `plot_2d_diagram(gmic(f=2/3))`. There are three combinatorial types of these diagrams, depending on whether $f < \frac{1}{2}$, $f = \frac{1}{2}$, or $f > \frac{1}{2}$. No matter what f is, the additivity domain $E(\pi)$ is the union of the faces $F_1 = F([0, f], [0, f], [0, f])$ and $F_2 = F([f, 1], [f, 1], [1 + f, 2])$, *shaded in green.* At the borders of each diagram, the projections $p_i(F)$ of two-dimensional additive faces are shown as *gray shadows*: $p_1(F)$ at the top border, $p_2(F)$ at the left border, $p_3(F)$ at the bottom and the right borders.
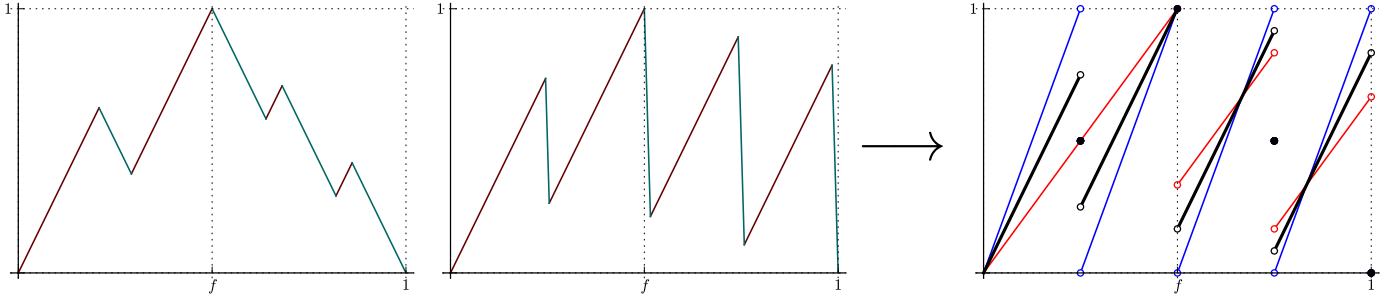

FIGURE 6. A pointwise limit of extreme functions that is not extreme. Consider the sequence of continuous extreme functions of type `gj_2_slope_repeat` set up for any $n \in \mathbb{Z}_+$ by `h = drlm_gj_2_slope_extreme_limit_to_nonextreme(n)`. For example, $n = 3$ *(left)* and $n = 50$ *(center)*. This sequence converges to a non-extreme discontinuous minimal valid function, set up with `h = drlm_gj_2_slope_extreme_limit_to_nonextreme()` *(right)*. The limit function π (*black*) is shown with two minimal functions π^1 (*blue*), π^2 (*red*) such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$.
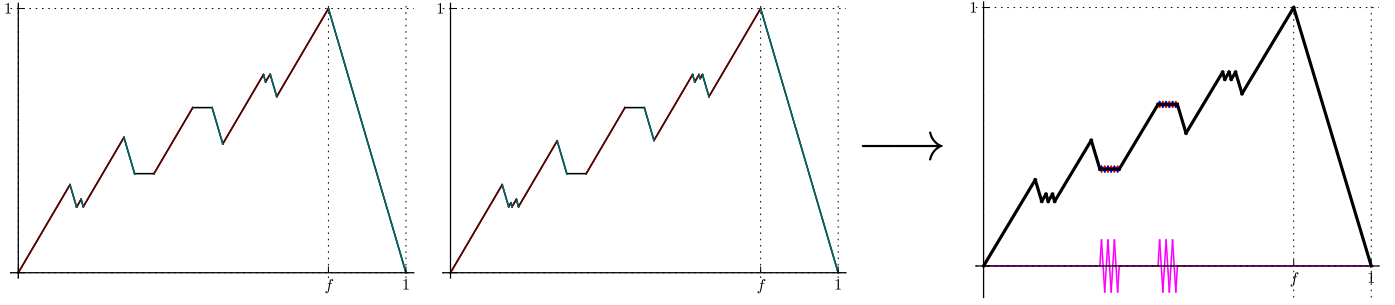
FIGURE 7. A uniform limit of extreme functions that is not extreme. The sequence of extreme functions of type `bhk_irrational`, set up with `h = bhk_irrational_extreme_limit_to_rational_nonextreme(n)` where $n = 1$ *(left)*, $n = 2$ *(center)*, ... converges to a non-extreme function, set up with `h = bhk_irrational_extreme_limit_to_rational_nonextreme()` *(right)*. The limit function π *(black)* is shown with two minimal functions π^1 *(blue)*, π^2 *(red)* such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$ and a scaling of the perturbation function $\bar{\pi} = \pi^1 - \pi$ *(magenta)*.
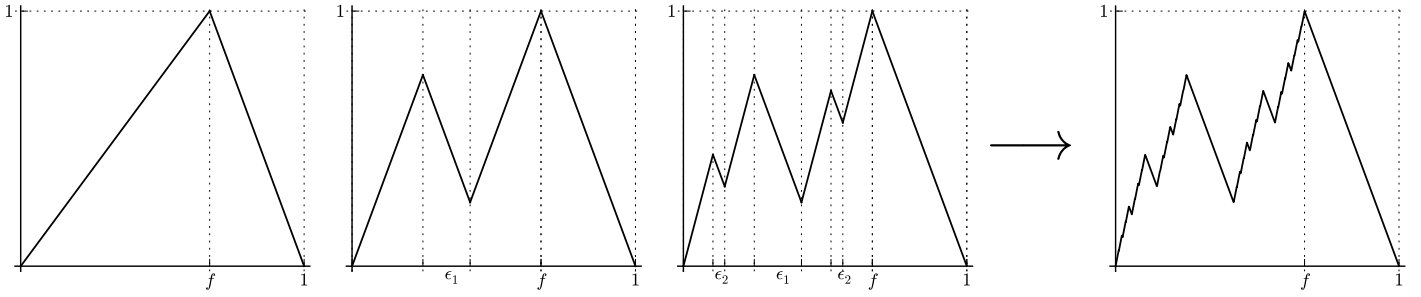
FIGURE 8. First steps ($\psi_0 = $ `gmic()`, ψ_1, ψ_2) in the construction of the continuous non–piecewise linear limit function $\psi = $ `bccz_counterexample()`.
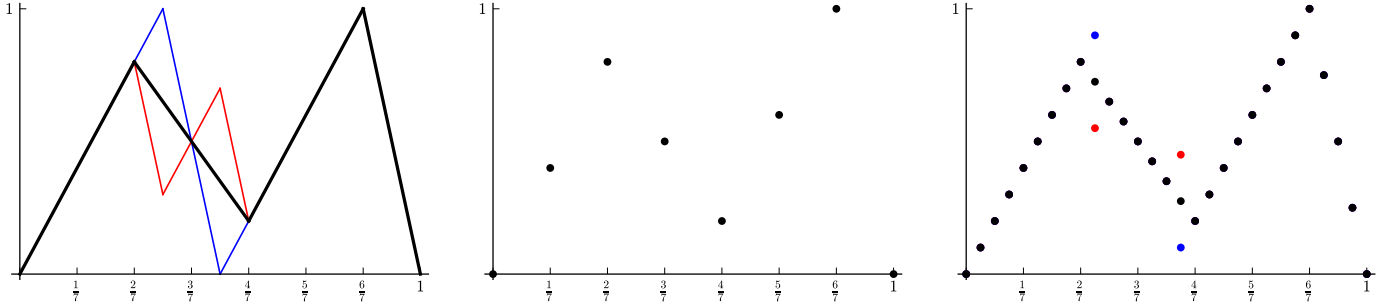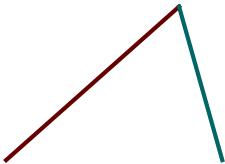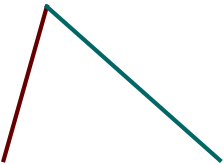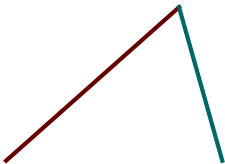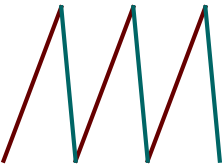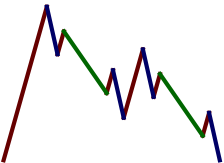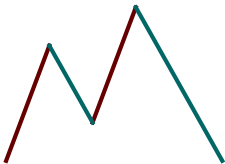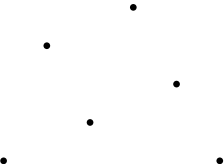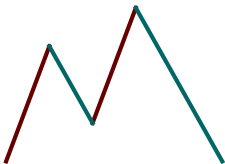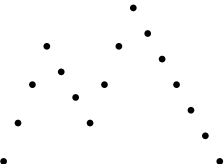

FIGURE 9. This function (`h = drlm_not_extreme_1()`) is minimal, but not extreme (and hence also not a facet), as proved by `extremality_test(h, show_plots=True)` by demonstrating a perturbation. The red and blue perturbations des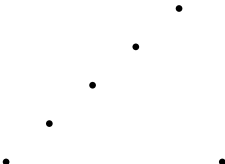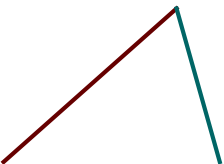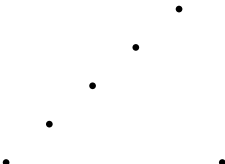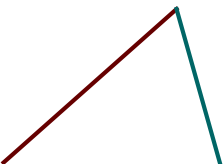cribe the minimal functions π^1, π^2 that verify that π is not extreme. These minimal functions necessarily have more breakpoints than π. This is because $\pi|_{\frac{1}{q}\mathbb{Z}}$ with $q = 7$, as depicted in the middle figure, is extreme for the finite group problem $R_f(\frac{1}{q}\mathbb{Z}, \mathbb{Z})$. However, $\pi|_{\frac{1}{2q}\mathbb{Z}}$ is not extreme for $R_f(\frac{1}{2q}\mathbb{Z}, \mathbb{Z})$. The discrete perturbations, depicted on the right, are interpolated to obtain the continuous functions π^1, π^2.

TABLE 1. An updated compendium of known extreme functions for the infinite group problem V. Procedures.

Procedure[a]	Graphs		Notes
	From	To	
automorphism			From Johnson
multiplicative_homomorphism			
projected_sequential_merge			Operation $\Diamond_n^1$ from Dey–Richard
restrict_to_finite_group			Restrictions to finite group problems $R_f(\frac{1}{q}\mathbb{Z}, \mathbb{Z})$ preserve extremality if f and all breakpoints lie in $\frac{1}{q}\mathbb{Z}$.
restrict_to_finite_group (oversampling=3)			If oversampling by a factor $m \geq 3$, the restriction is extreme for $R_f(\frac{1}{mq}\mathbb{Z}, \mathbb{Z})$ if and only if the original function is extreme.
interpolate_to_infinite_group			Interpolation from finite group problems $R_f(\frac{1}{q}\mathbb{Z}, \mathbb{Z})$ preserves minimality, but in general not extremality.
two_slope_fill_in			Described by Gomory–Johnson, Johnson. For $k = 1$, if minimal, equal to interpolate_to_infinite_group (above).

[a]A procedure name shown in typewriter font is the name of the corresponding function in the accompanying Sage program.
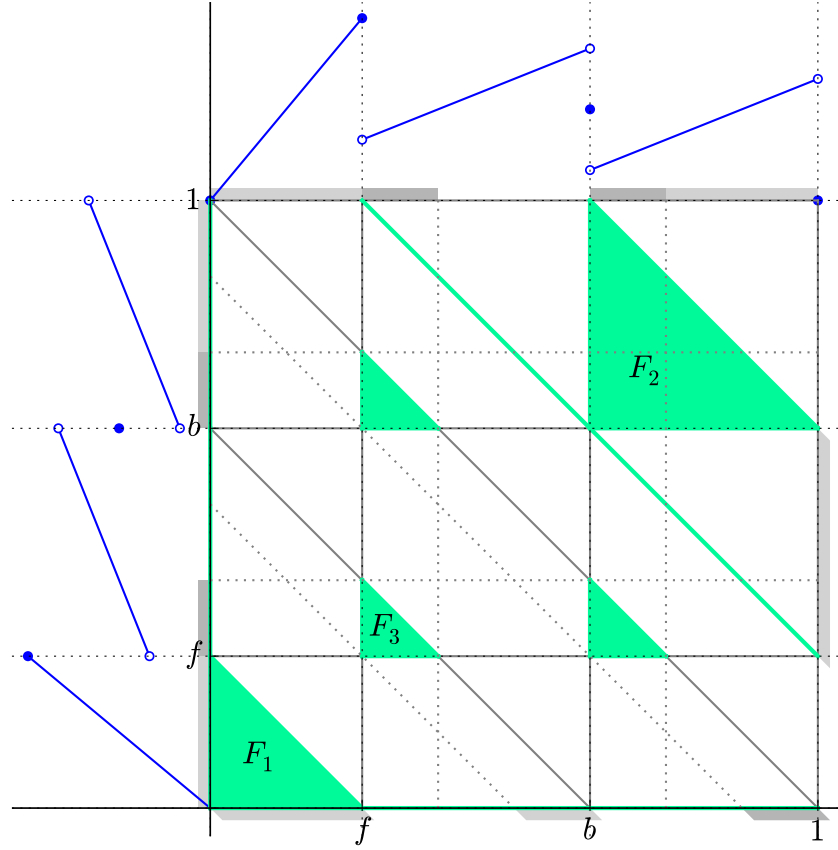
FIGURE 1. Diagram of the function `rlm_dpl1_extreme_3a` (*blue graphs on the top and the left*) and its polyhedral complex $\Delta\mathcal{P}$ (*gray solid lines*). The set $E(\pi)$ is the union of the faces shaded in green. The *heavy diagonal green line* $x + y = 1 + f$ corresponds to the symmetry condition (the line $x + y = f$ appears as an edge of F_1). Vertices of $\Delta\mathcal{P}$ do not necessarily project (*dotted gray lines*) to breakpoints. At the borders, the projections $p_i(F)$ of two-dimensional additive faces are shown as *gray shadows*: $p_1(F)$ at the top border, $p_2(F)$ at the left border, $p_3(F)$ at the bottom and the right borders.
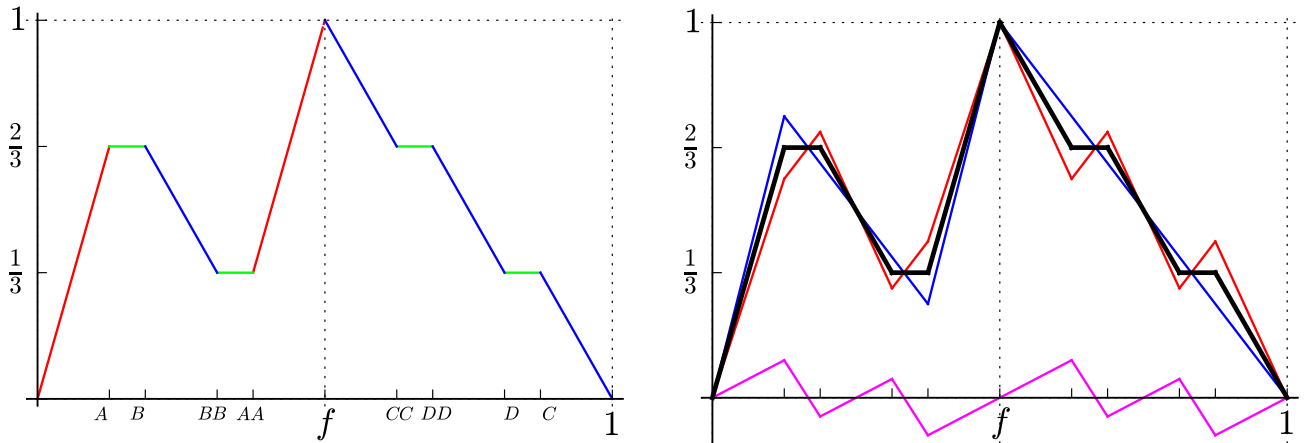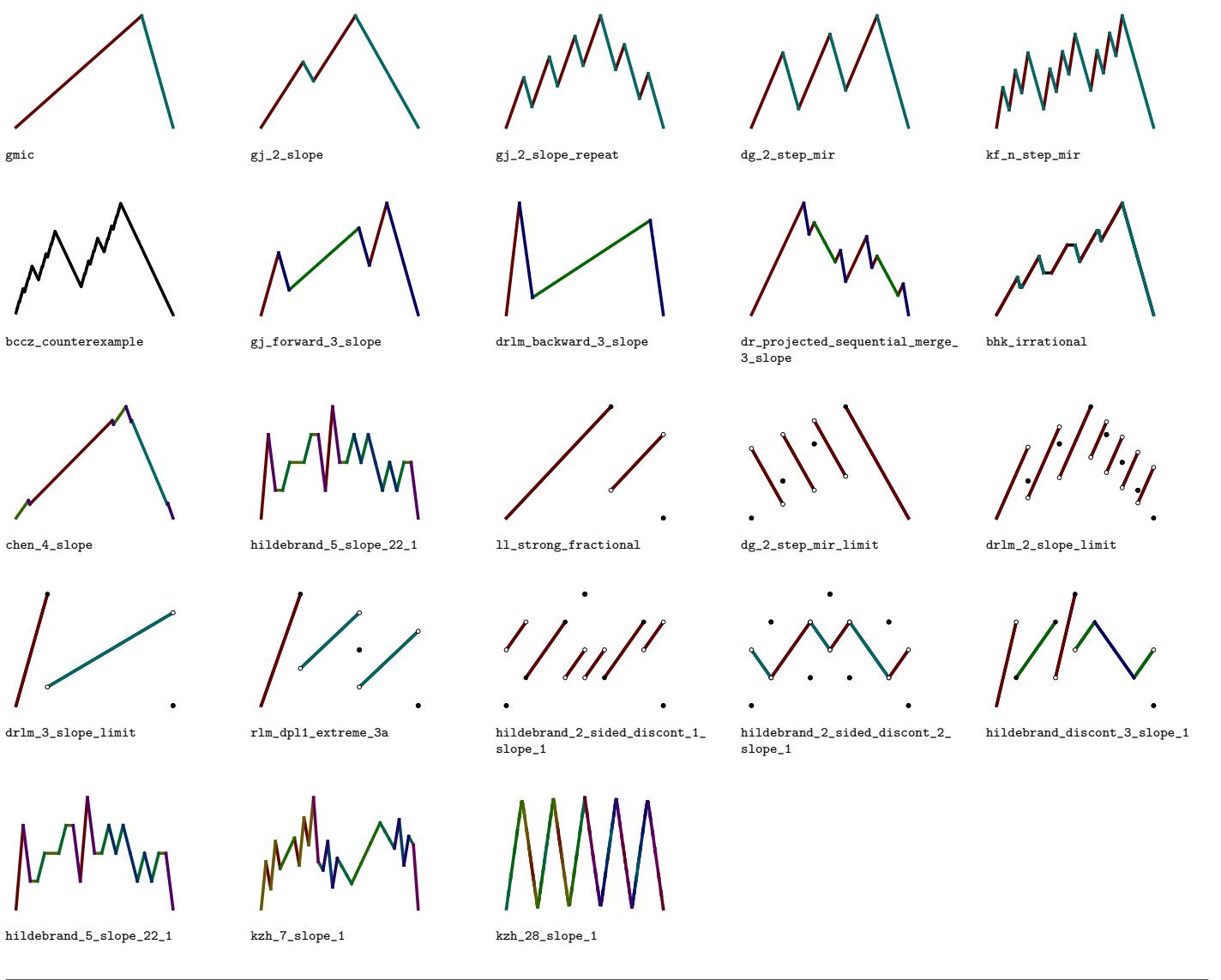

FIGURE 2. The function `chen_3_slope_not_extreme` is minimal, but not extreme, as proved by `extremality_test(h, show_plots=True)`. The procedure first shows that for any distinct minimal $\pi^1 = \pi + \bar{\pi}$ (*blue*), $\pi^2 = \pi - \bar{\pi}$ (*red*) such that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$, the functions π^1 and π^2 are continuous piecewise linear with the same breakpoints as π. A finite-dimensional extremality test then finds a perturbation $\bar{\pi}$ (*magenta*), as shown.
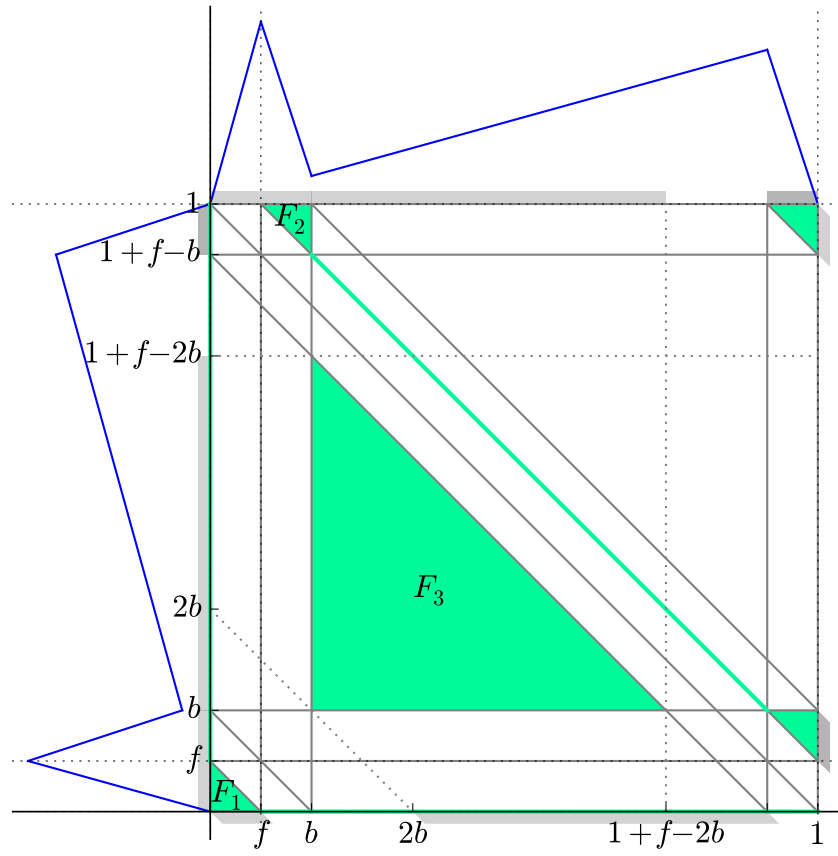
FIGURE 3. The `drlm_backward_3_slope` function
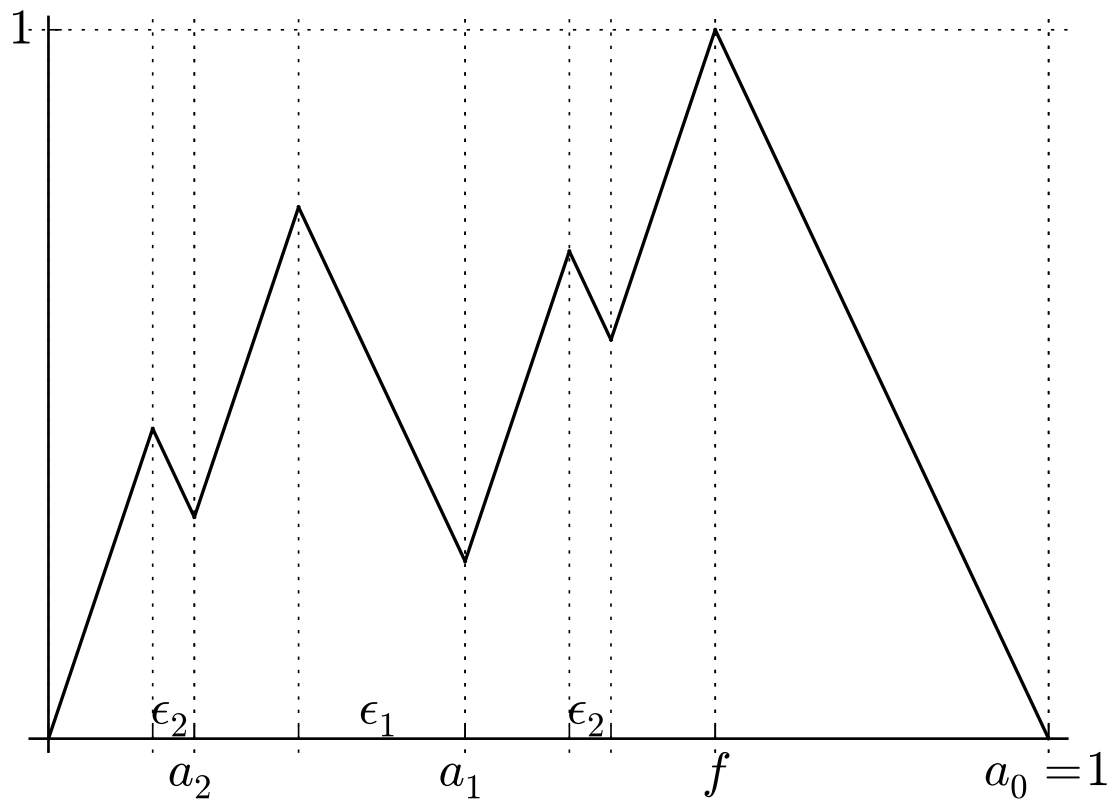
FIGURE 4. The `kf_n_step_mir` function

3. Electronic compendium

TABLE 1. An overview of the Electronic Compendium of extreme functions, available at `https://github.com/mkoeppe/infinite-group-relaxation-code`

gmic

gj_2_slope

gj_2_slope_repeat

dg_2_step_mir

kf_n_step_mir

bccz_counterexample

gj_forward_3_slope

drlm_backward_3_slope

dr_projected_sequential_merge_3_slope

bhk_irrational

chen_4_slope

hildebrand_5_slope_22_1

ll_strong_fractional

dg_2_step_mir_limit

drlm_2_slope_limit

drlm_3_slope_limit

rlm_dpl1_extreme_3a

hildebrand_2_sided_discont_1_slope_1

hildebrand_2_sided_discont_2_slope_1

hildebrand_discont_3_slope_1

hildebrand_5_slope_22_1
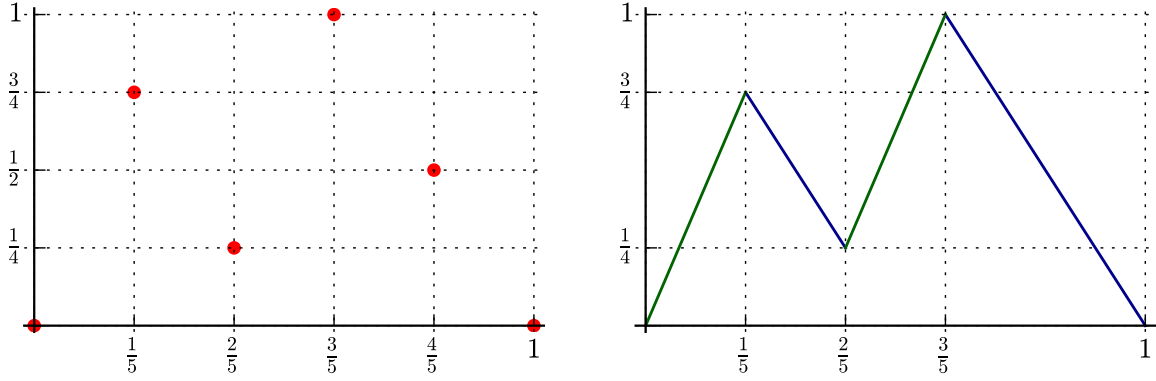
kzh_7_slope_1

kzh_28_slope_1

FIGURE 1. The 2-slope extreme function `gj_2_slope`, discovered by Gomory and Johnson. *Left*, `gj_2_slope` for the finite group problem with $q = 5$ and $f = \frac{3}{5}$, obtained by `restrict_to_finite_group(gj_2_slope())`. It is a discrete function whose interpolation is the right subfigure. *Right*, `gj_2_slope` for the infinite group problem with $f = \frac{3}{5}$. It is a continuous piecewise linear function with two slopes, although it has four pieces. Its restriction to $\frac{1}{5}\mathbb{Z}$ is the left subfigure.
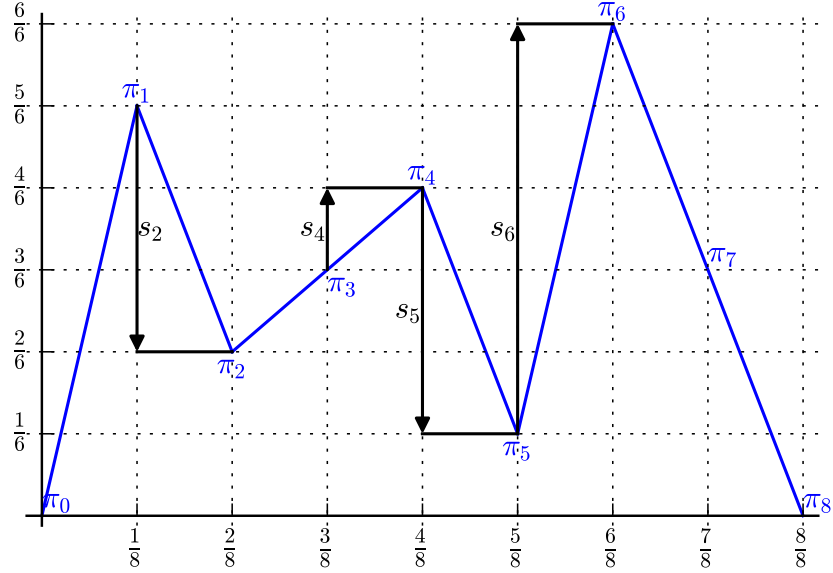

FIGURE 2. The $q \times v$ grid discretization of the space of continuous piecewise linear functions with rational data. Here $q = 8$ and $v = 6$.
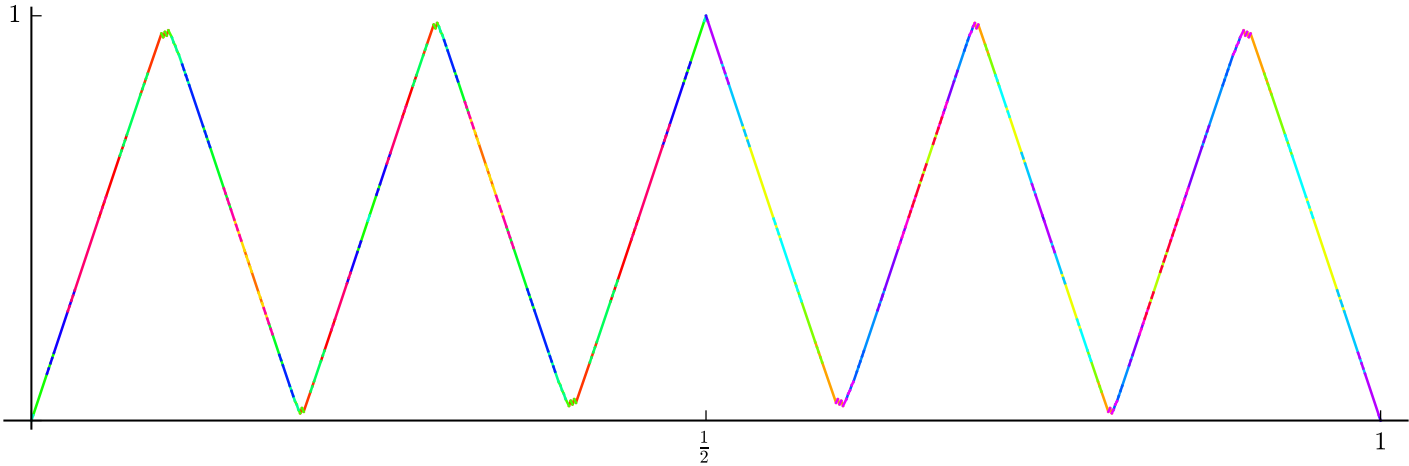
FIGURE 3. A 28-slope extreme function `kzh_28_slope_1` found by our search code. Each color in the plotting corresponds to a different slope value.
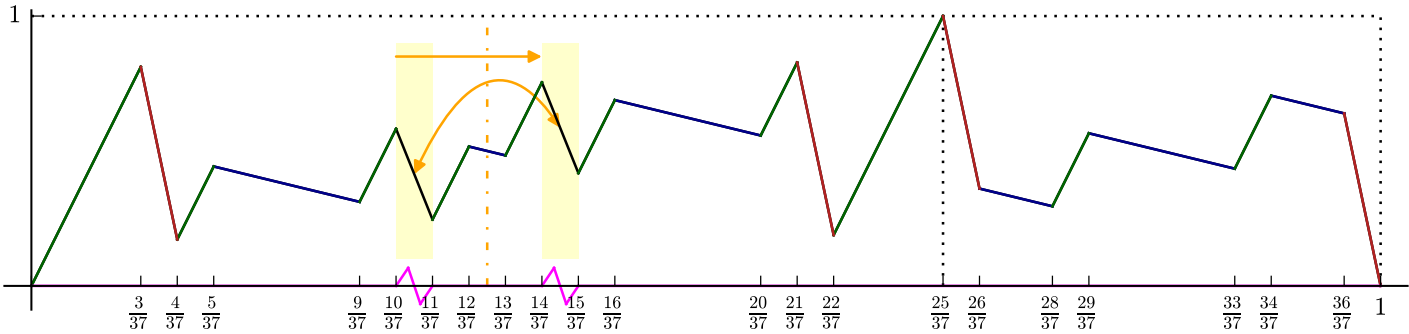

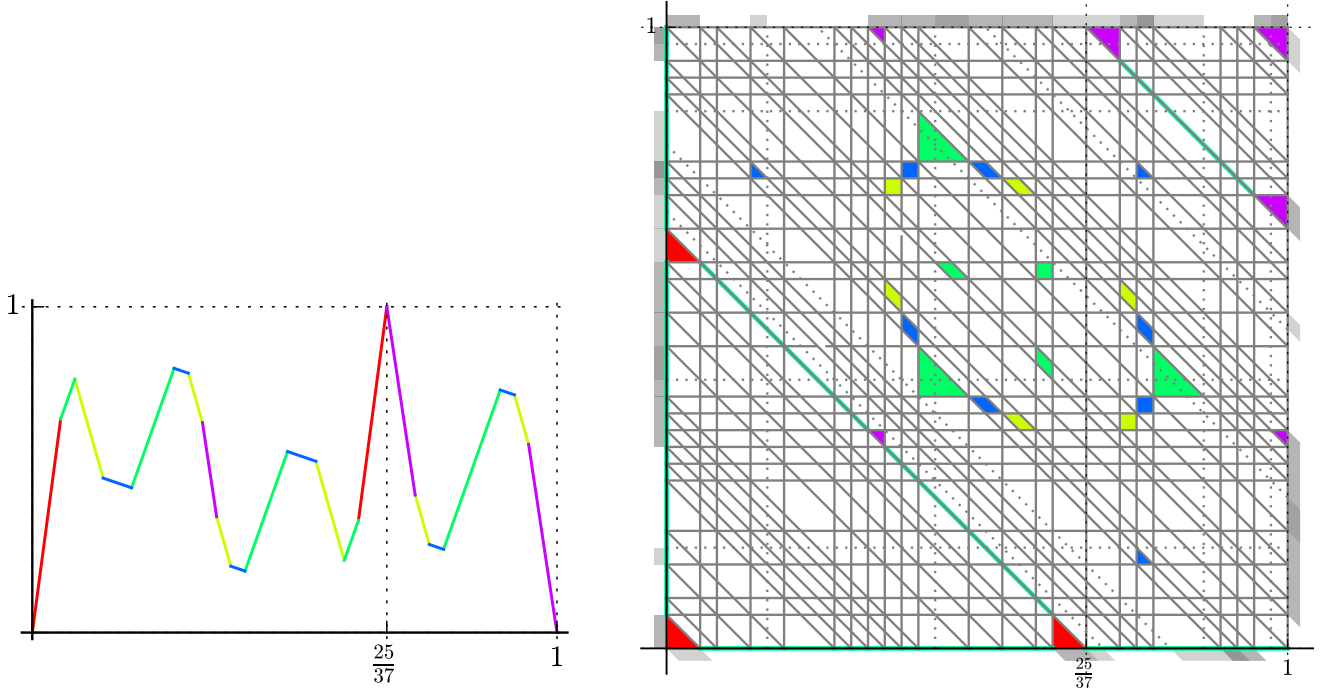FIGURE 4. The example `kzh_2q_example_1`, showing that an oversampling factor of $m = 3$ is best possible.

FIGURE 9. The 5-slope extreme function `kzh_5_slope_fulldim_1` found by our search code (*left*). Its two-dimensional polyhedral complex $\Delta\mathcal{P}$ (*right*), as plotted by the command `plot_2d_diagram(h,colorful=True)`, does not have any lower-dimensional maximal additive faces except for the symmetry reflection or $x = 0$ or $y = 0$.
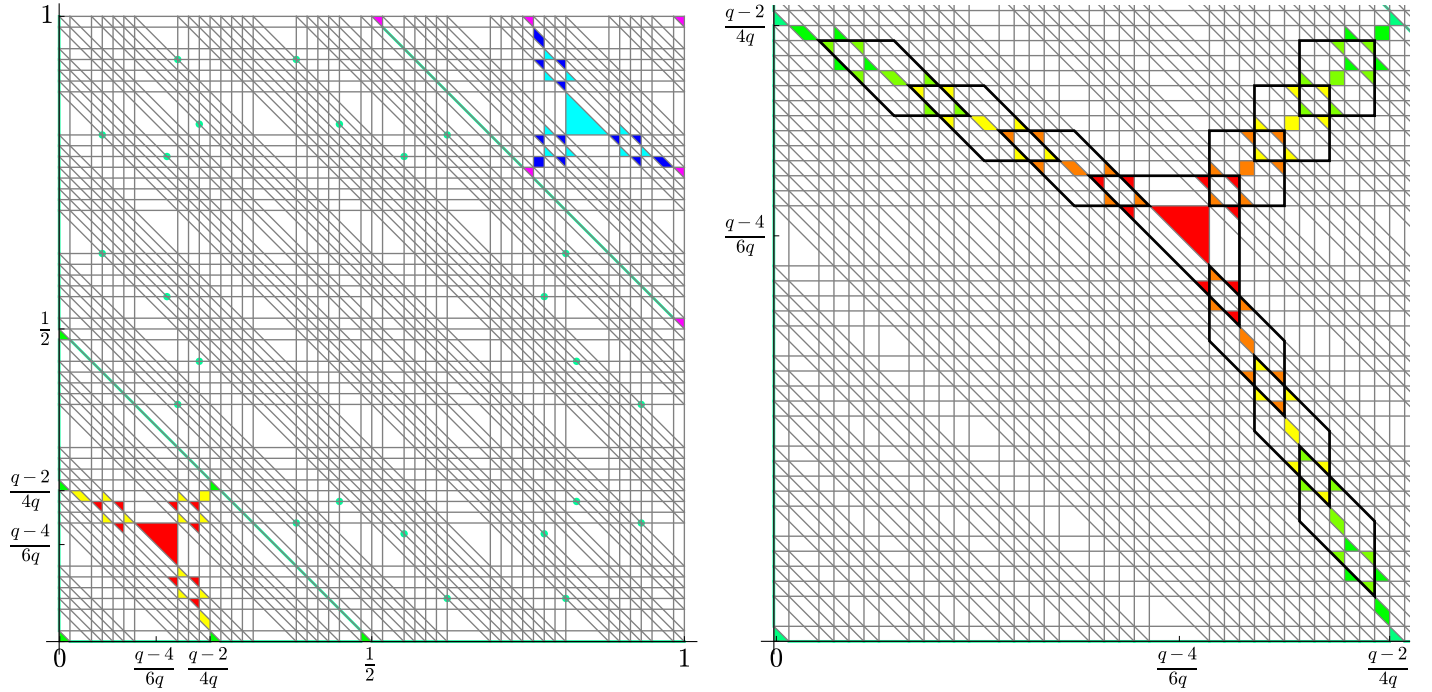

FIGURE 11. Special patterns on the two-dimensional polyhedral complex $\Delta\mathcal{P}_{\frac{1}{q}\mathbb{Z}}$. *Left,* the $\Delta\mathcal{P}_{\frac{1}{q}\mathbb{Z}}$ of the 6-slope extreme function `kzh_6_slope_1` with $q = 58$. We observe that the additive triangles are located in the lower left and upper right corners. The function has the same slopes on the intervals that are projections of the same color additive triangles. The 6-pointed star patterns appear several times. *Right,* the lower-left corner of $\Delta\mathcal{P}_{\frac{1}{q}\mathbb{Z}}$ of the 10-slope extreme function `kzh_10_slope_1` with $q = 166$, where we see that the 6-pointed stars are actually the result of additivity patterns within certain intersecting quadrilaterals (*black*), which connect like links of three chains.
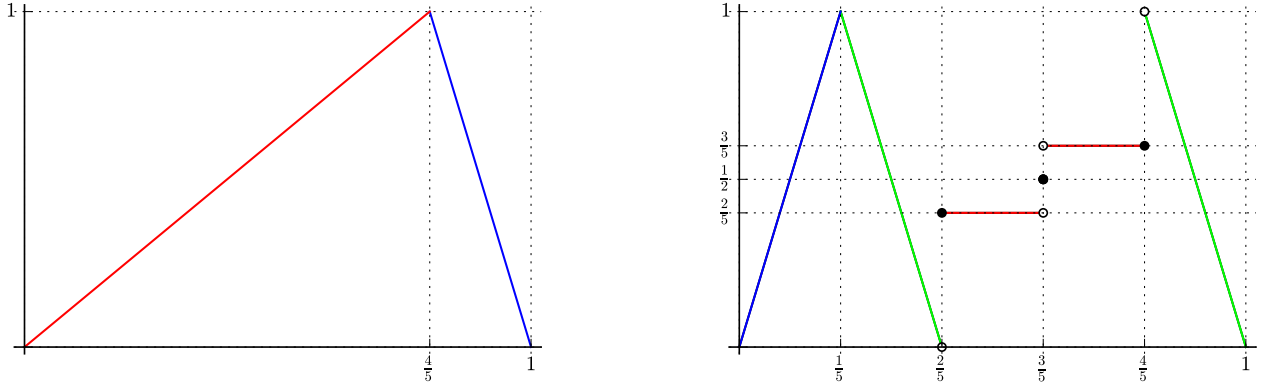
FIGURE 1. Two piecewise linear functions, as plotted by the command `plot_with_colored_slopes(h)`. *Left,* continuous extreme function `h = gmic()`. *Right,* random discontinuous function `h = equiv5_random_discont_1()`, generated by `random_piecewise_function(xgrid=5, ygrid=5, continuous_proba=1/3, symmetry=True)`.

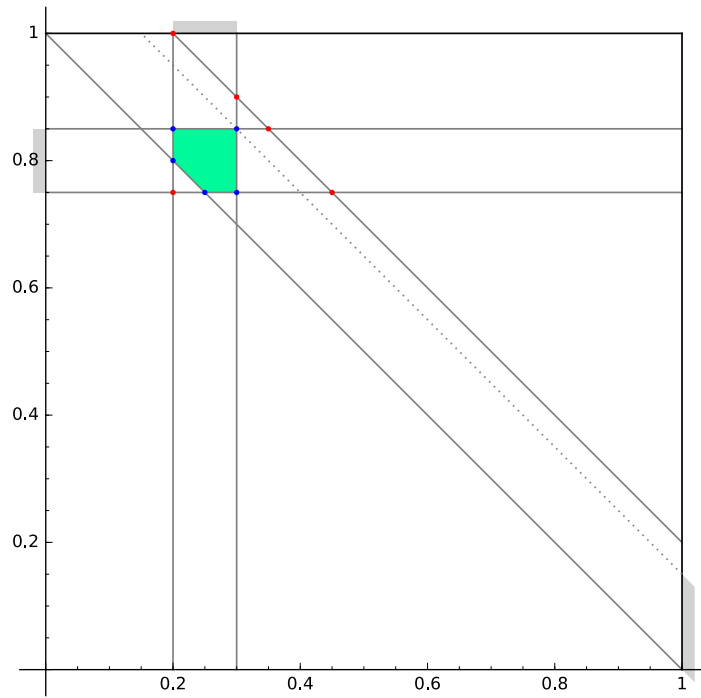FIGURE 2. An example of a face $F = F(I, J, K)$ of the 2-dimensional polyhedral complex $\Delta\mathcal{P}$, set up by `F = Face([[0.2, 0.3], [0.75, 0.85], [1, 1.2]])`. It has vertices (*blue*) $(0.2, 0.85)$, $(0.3, 0.75)$, $(0.3, 0.85)$, $(0.2, 0.8)$, $(0.25, 0.75)$, whereas the other basic solutions (*red*) $(0.2, 0.75)$, $(0.2, 1)$, $(0.3, 0.9)$, $(0.35, 0.85)$, $(0.45, 0.75)$ are filtered out because they are infeasible. The face F has projections (*gray shadows*) $I' = p_1(F) = [0.2, 0.3]$ (*top border*), $J' = p_2(F) = [0.75, 0.85]$ (*left border*), and $K' = p_3(F) = [1, 1.15]$ (*right border*). Note that $K' \subsetneq K$.
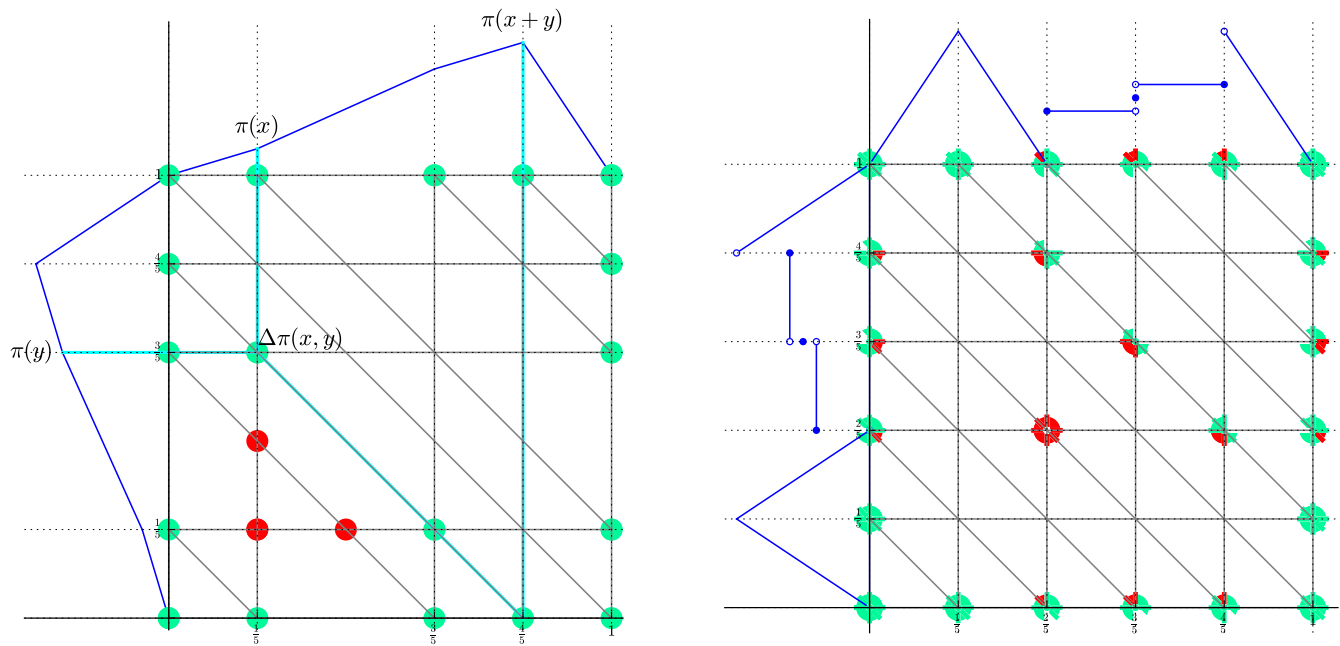
FIGURE 3. Two diagrams of functions and their polyhedral complexes $\Delta\mathcal{P}$ with colored cones at $\mathrm{vert}(\Delta\mathcal{P})$, as plotted by the command `plot_2d_diagram_with_cones(h)`. *Left*, continuous function `h = not_minimal_2()`. *Right*, random discontinuous function `h = equiv5_random_discont_1()`.
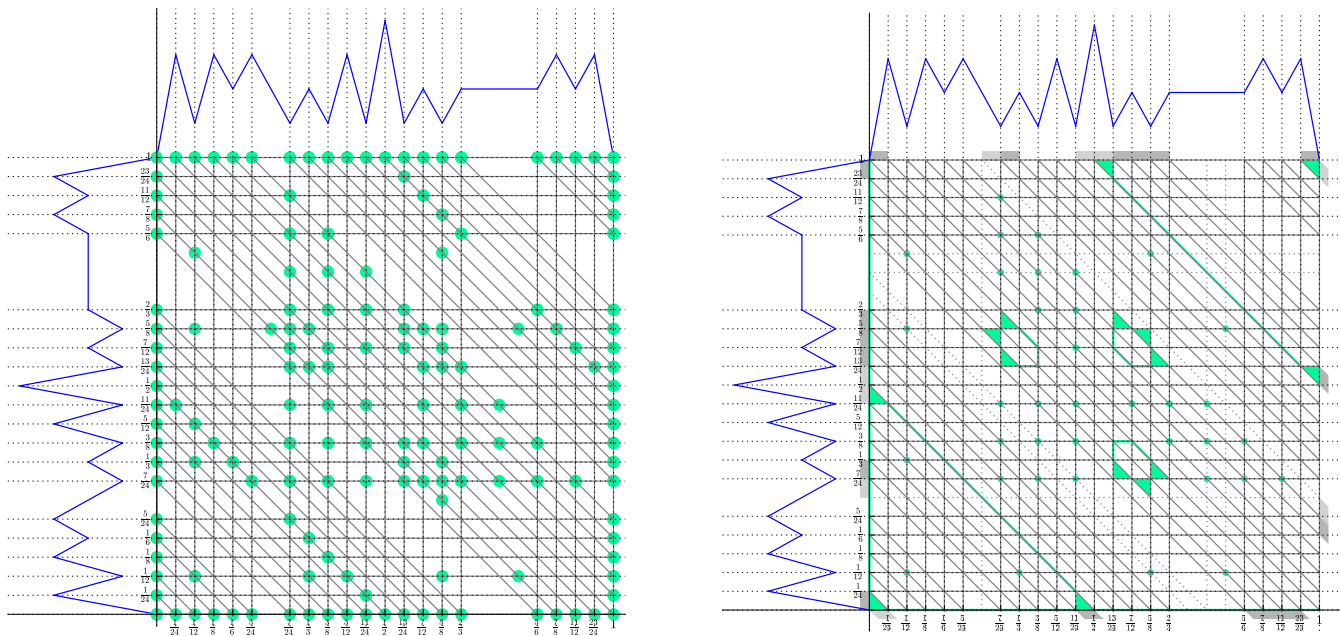

FIGURE 4. Diagrams of $\Delta\mathcal{P}$ of a continuous function `h = example7slopecoarse2()`, with (*left*) additive vertices as plotted by the command `plot_2d_diagram_with_cones(h)`; (*right*) maximal additive faces as plotted by the command `plot_2d_diagram(h)`.
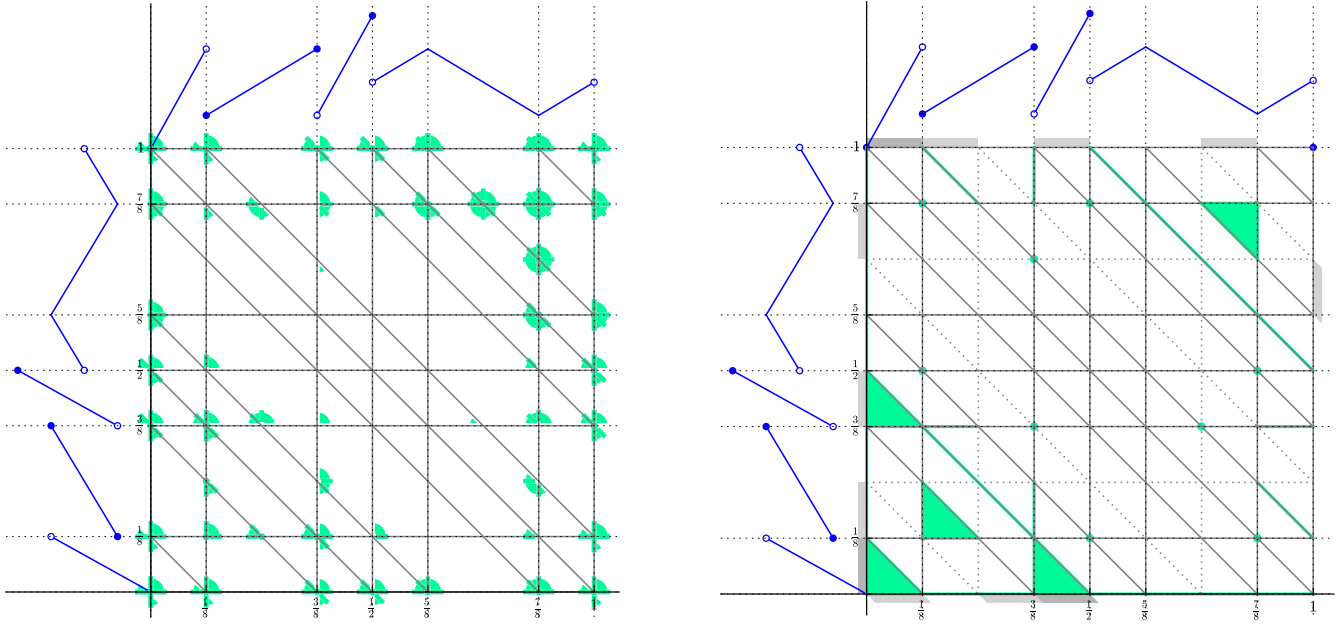
FIGURE 5. Diagrams of $\Delta\mathcal{P}$ of a discontinuous function h = hildebrand_discont_3_slope_1(), with (*left*) additive limiting cones as plotted by the command plot_2d_diagram_with_cones(h); (*right*) additive faces as plotted by the command plot_2d_diagram(h).
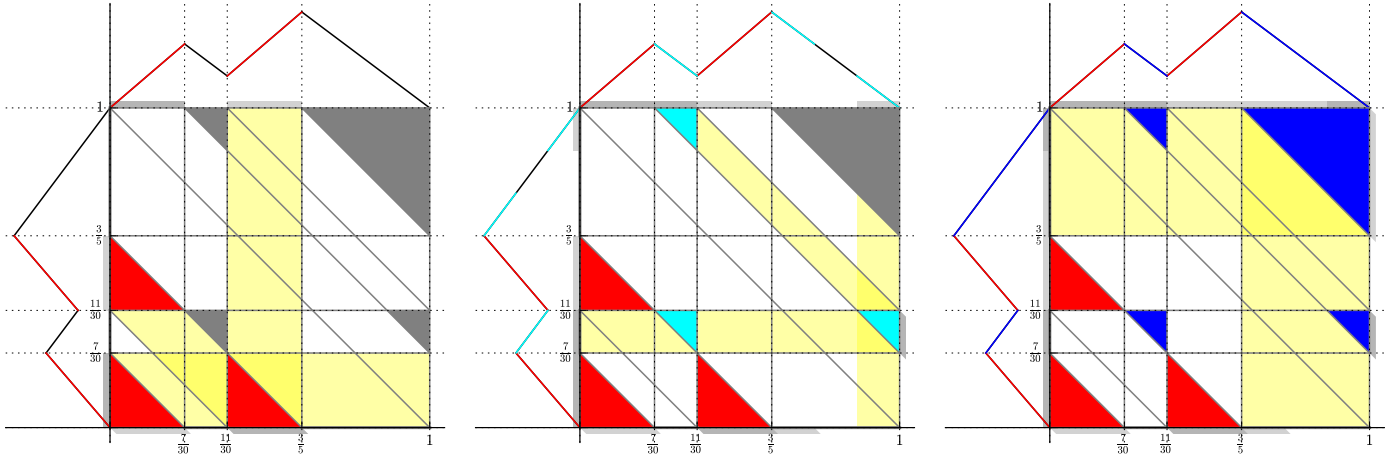

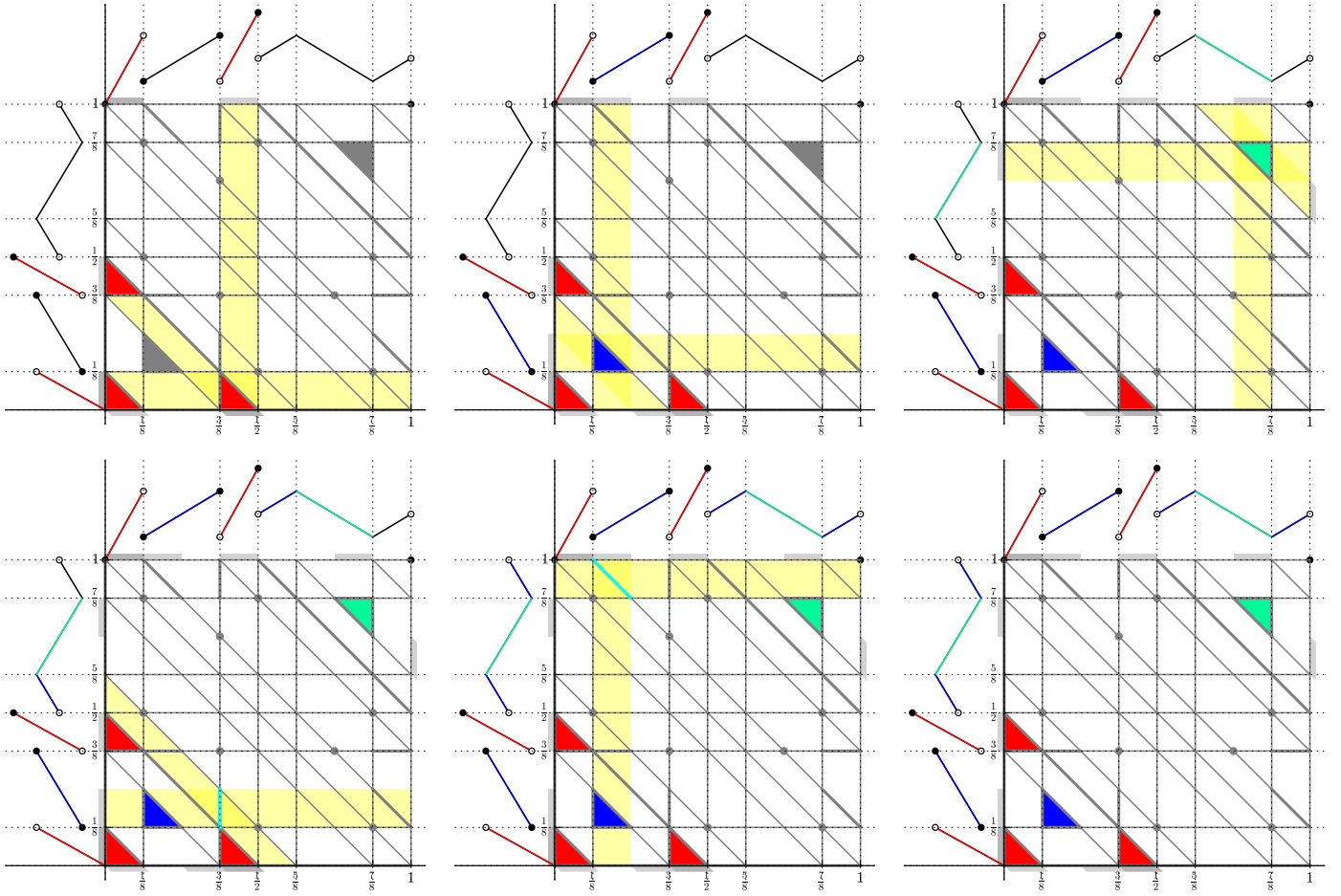FIGURE 6. Compute the (directly) covered intervals for $\pi =$ gj_2_slope(3/5,1/3).

FIGURE 7. Compute the (directly and indirectly) covered intervals for $\pi = $ `hildebrand_discont_3_slope_1()`

TABLE 3. A sample Sage session on the extremality test
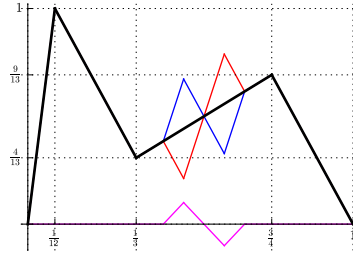

TABLE 4. A sample Sage session on discrete functions for the finite group problem.