C++ Project - Partial Differential Equation Solution

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1 Introduction

This document aims at describing the resolution of the Partial Differential Equation used in the framework of the option pricer developed in C++.

2 Equation solution

The mesh of Figure 1 describes the dependencies of the value of the option at a time t_i for $i \in [0, T]$ ($t_0 = 0$ and $t_T = T$) and for a spot price x_n for $n \in [0, N]$ of the underlying with respect to the price of the option at time t_{i-1} and t_{i+1} and spot x_{n+1}.

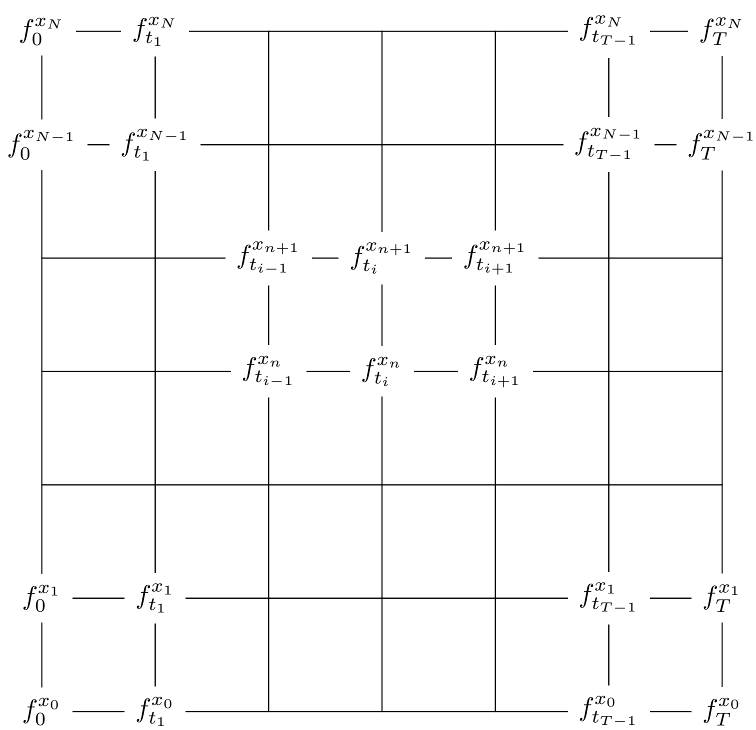

Figure 1: A finite element mesh

The dependencies are expressed by equation (1), (2) and (3).

$\forall i \in [\![2; T-2]\!], \forall n \in [\![0; N-1]\!]$,

$$\begin{aligned}
& f_{t_{i-1}}^{x_n} \quad \left[\theta dt \left(-\frac{\sigma^2}{2dx^2} - \frac{\sigma^2}{4dx} + \frac{r}{2dx}\right)\right] \\
& + f_{t_i}^{x_n} \quad \left[\theta dt \left(\frac{\sigma^2}{dx^2} + r\right) + 1\right] \\
& + f_{t_{i+1}}^{x_n} \quad \left[\theta dt \left(-\frac{\sigma^2}{2dx^2} + \frac{\sigma^2}{4dx} - \frac{r}{2dx}\right)\right] \\
& + f_{t_{i-1}}^{x_{n+1}} \left[(1-\theta) dt \left(-\frac{\sigma^2}{2dx^2} - \frac{\sigma^2}{4dx} + \frac{r}{2dx}\right)\right] \\
& + f_{t_i}^{x_{n+1}} \left[(1-\theta) dt \left(\frac{\sigma^2}{dx^2} + r\right) - 1\right] \\
& + f_{t_{i+1}}^{x_{n+1}} \left[(1-\theta) dt \left(-\frac{\sigma^2}{2dx^2} + \frac{\sigma^2}{4dx} - \frac{r}{2dx}\right)\right] = 0
\end{aligned} \tag{1}$$

For $i = 1, \forall n \in [\![0; N-1]\!]$,

$$\begin{aligned}
& f_0^{x_n} \quad \left[\theta dt \left(-\frac{\sigma^2}{2dx^2} - \frac{\sigma^2}{2dx} + \frac{r}{dx} + r\right) + 1\right] \\
& + f_{t_1}^{x_n} \quad \left[\theta dt \left(\frac{\sigma^2}{dx^2} + \frac{\sigma^2}{2dx} - \frac{r}{dx}\right)\right] \\
& + f_{t_2}^{x_n} \quad \left[\theta dt \left(-\frac{\sigma^2}{2dx^2}\right)\right] \\
& + f_0^{x_{n+1}} \left[(1-\theta) dt \left(-\frac{\sigma^2}{2dx^2} - \frac{\sigma^2}{2dx} + \frac{r}{dx} + r\right) - 1\right] \\
& + f_{t_1}^{x_{n+1}} \left[(1-\theta) dt \left(\frac{\sigma^2}{dx^2} + \frac{\sigma^2}{2dx} + \frac{r}{dx}\right)\right] \\
& + f_{t_2}^{x_{n+1}} \left[(1-\theta) dt \left(-\frac{\sigma^2}{2dx^2}\right)\right] = 0
\end{aligned} \tag{2}$$

For $i = T-1, \forall n \in [\![0; N-1]\!]$,

$$
\begin{aligned}
&f_{t_{T-2}}^{x_n} \left[\theta dt \left(-\frac{\sigma^2}{2dx^2}\right)\right] \\
&+f_{t_{T-1}}^{x_n} \left[\theta dt \left(\frac{\sigma^2}{dx^2} - \frac{\sigma^2}{2dx} + \frac{r}{dx}\right)\right] \\
&+f_{T}^{x_n} \left[\theta dt \left(-\frac{\sigma^2}{2dx^2} + \frac{\sigma^2}{2dx} - \frac{r}{dx} + r\right) + 1\right] \\
&+f_{t_{T-2}}^{x_{n+1}} \left[(1-\theta) dt \left(-\frac{\sigma^2}{2dx^2}\right)\right] \\
&+f_{t_{T-1}}^{x_{n+1}} \left[(1-\theta) dt \left(\frac{\sigma^2}{dx^2} - \frac{\sigma^2}{2dx} + \frac{r}{dx}\right)\right] \\
&+f_{T}^{x_{n+1}} \left[(1-\theta) dt \left(-\frac{\sigma^2}{2dx^2} + \frac{\sigma^2}{2dx} - \frac{r}{dx} + r\right) - 1\right] = 0
\end{aligned}
\tag{3}
$$

From equations (1), (2) and (3), we can derive the matrix formulation of equation (4).

$\forall n \in [\![0; N-1]\!]$,

$$
\left(I_{N+1} + \theta dt A\right) F^n = \left(I_{N+1} + (\theta - 1) dt A\right) F^{n+1} \tag{4}
$$

With, $F^n = \begin{pmatrix} f_0^{x_n} \\ f_{t_1}^{x_n} \\ \vdots \\ f_{t_{T-1}}^{x_n} \\ f_T^{x_n} \end{pmatrix}$, $I_{N+1} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}$ and A is defined in the Appendix.

3 Boundary conditions

The set $\{f_T^{x_n}, \forall n \in [\![0; N]\!]\}$ are the payoffs at maturity for all the $N+1$ underlying spot prices. This set is a parameter of the solver. The payoff function can be given by the user.

In the previous equations, the interest rate r and the volatility of the underlying σ can be given as vectors such that it represents the variation of the parameters during the solving period between time 0 and time T.

4 Appendix

The matrix A of size $(N+1, N+1)$ used in this document is defined as follows:

$$A = \begin{pmatrix}
-\frac{\sigma^2}{2dx^2} - \frac{\sigma^2}{2dx} + \frac{r}{dx} + r & \frac{\sigma^2}{dx^2} + \frac{\sigma^2}{2dx} - \frac{r}{dx} & -\frac{\sigma^2}{2dx^2} & 0 & \cdots & 0 \\
-\frac{\sigma^2}{2dx^2} - \frac{\sigma^2}{4dx} + \frac{r}{2dx} & \frac{\sigma^2}{dx^2} + r & -\frac{\sigma^2}{2dx^2} + \frac{\sigma^2}{4dx} - \frac{r}{2dx} & 0 & & \vdots \\
0 & -\frac{\sigma^2}{2dx^2} - \frac{\sigma^2}{4dx} + \frac{r}{2dx} & \frac{\sigma^2}{dx^2} + r & -\frac{\sigma^2}{2dx^2} + \frac{\sigma^2}{4dx} - \frac{r}{2dx} & & \vdots \\
\vdots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \vdots \\
\vdots & & -\frac{\sigma^2}{2dx^2} - \frac{\sigma^2}{4dx} + \frac{r}{2dx} & \frac{\sigma^2}{dx^2} + r & -\frac{\sigma^2}{2dx^2} + \frac{\sigma^2}{4dx} - \frac{r}{2dx} & 0 \\
\vdots & & 0 & -\frac{\sigma^2}{2dx^2} - \frac{\sigma^2}{4dx} + \frac{r}{2dx} & \frac{\sigma^2}{dx^2} + r & -\frac{\sigma^2}{2dx^2} + \frac{\sigma^2}{4dx} - \frac{r}{2dx} \\
0 & \cdots & 0 & -\frac{\sigma^2}{2dx^2} & \frac{\sigma^2}{dx^2} - \frac{\sigma^2}{2dx} + \frac{r}{dx} & -\frac{\sigma^2}{2dx^2} + \frac{\sigma^2}{2dx} - \frac{r}{dx} + r
\end{pmatrix}$$