2017 TJIMO Shortlist

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1 Counting/Numbers

1. What is the cube root of 1030301?

2. Challen Eng wants to paint the addresses of all the houses on Jhomas Tefferson Ave. However, because Jhomas Tefferson Ave is a long street, he gets tired after he finishes painting the odd-numbered addresses on one side of the road and does not paint the even-numbered addresses at all. If the house addresses are all distinct integers ranging from 597 to 1706 and no integers in this range are skipped, then how many digits does Challen Eng paint in total?

3. How many of the following statements in the box below are false?

Exactly one of these statements is false.
Exactly two of these statements are false.
Exactly three of these statements are false.
Exactly four of these statements are false.
Exactly five of these statements are false.

4. What integer value is $4 \cdot \sqrt[0.4]{4}^{\sqrt{4}} + 444 - 4 \cdot 4!$ equal to? (Note: $\sqrt[r]{a} = a^{\frac{1}{r}}$ and $n! = 1 \cdot 2 \cdot \ldots \cdot n$.)

5. The diagram to the right is a *net* composed of twenty congruent equilateral triangles. A sheet of paper in its shape can be folded along the interior crease lines shown to create a regular polyhedron with V vertices, E edges, and F faces. Calculate the value of $V+E+F$ for this polyhedron.

6. Ozzy writes the number 1 on his chalkboard. Each subsequent number Ozzy writes on the board is one more than three times the last number he wrote on the board. Therefore, the first three numbers Ozzy writes down are 1, 4, and 13, in that order. What is the 8th number Ozzy writes down? You may find it helpful to know that $3^8 = 6561$.

7. A three-digit representation of an integer between 0 and 999 inclusive is formed by adding zeros to the left of the number until it has three digits. For instance, the three-digit representation of 49 would be 049, and the three-digit representation of 9 would be 009. How many square-number three-digit representations can be reversed (read backwards) to form another square number? (E.g., $900 \to 009$ works, $009 \to 900$ works, and $000 \to 000$ works.)

8. A room has a rectangular 48' $\times$ 33' floor that is tiled with square 1' $\times$ 1' tiles. A line is drawn on the floor from one corner of the room to the opposite corner. How many tiles does the line pass through (in the tile's interior)?

9. To celebrate the completion of the TJHSST renovation in April, 2017, estimate the first six digits after the decimal point when $\left(4+\sqrt{17}\right)^{4\cdot 2017}$ is evaluated in decimal form. Give your answer as a six-digit positive integer, including leading zeros if necessary. Any answer with at least three correct digits will receive full credit.

2 Algebra

1. Mr. E has 40 students in his math class. If $(11x-7)\%$ of Mr. E's students passed his final exam and the other $(2x+16)\%$ of them failed, then how many students in Mr. E's class passed the final exam?

2. Sally, being an active mathlete, has a large collection of frisbees. After Sally gives half of her frisbees to her friend Jennifer, they both have 2018 frisbees. How many frisbees did Jennifer originally have?

3. A one-of-a-kind circus bike has two circular wheels, but the larger wheel's area is 125% larger than that of the smaller wheel. In one trip across the circus ring, the larger wheel makes 50 revolutions. How many revolutions does the smaller wheel make?

4. Evaluate the expression $(x-a)(x-b)(x-c)\cdots(x-z)$, where each of the 26 factors contains a different letter of the alphabet being subtracted.

5. What is the (simplified) fractional value equivalent to the repeating decimal expression $3.703703703\ldots$?

6. Some Haos are working together to build some caos. All Haos work at the same rate. If eight Haos can build eight caos in eight minutes, how many minutes does it take for one Hao to build one cao?

7. We define functions f and g by the equations $f(x)=\frac{x}{x-1}$ and $g(x)=\frac{x-1}{x}$. Suppose $a_1=b_1=2017$. We generate sequences $a_1,a_2,\ldots$ and $b_1,b_2,\ldots$ according to the following rules: $a_{n+1}=f\left(a_n\right), b_{n+1}=g\left(b_n\right)$. What is the largest value of k less than 100 for which $a_k=b_k$?

8. Harry and William are speedracers. In their race, both of them line up at the same start line. However, Harry knows that William can run slightly faster than him, so William allows Harry to have a five second head start when they race (i.e., Harry runs for five seconds before William starts running). If William can run 20% faster than Harry, then how many seconds does Harry run before William passes him?

9. If all values of x that satisfy $bx-2x^2=18$ are real numbers, then find the largest possible value of b.

10. Within the 12-hour period from 12:00 noon to 12:00 midnight, how many times will an analog clock make a right angle with its hour and minute hands?

11. Suppose x and y satisfy $x = \frac{9}{y} - 1$ and $y = \frac{22}{x} + 1$. What is the value of $x + y$?

12. Beaker A starts out full with 150 mL of distilled water, and beaker B starts out empty. I pour half of the water from beaker A into beaker B. Then, I pour half of the water from beaker B into beaker A. Dedicatedly, I repeat this process 2018 times, ending by pouring half of the water from B into A. What whole number of milliliters is the best estimate for the volume of water in beaker A?

13. Alice and Bob are running back and forth on a 20-meter strip to practice for the 20-meter Fitnessgram™ PACER test. The strip is marked with 21 cones, labeled from 0 to 20 and spaced out in 1-meter intervals. Alice starts at cone 0, and Bob starts at cone 20. Instead of stopping when they reach the endpoints, Alice and Bob immediately turn around and run in the opposite direction, maintaining constant speeds. After both of them start at the same time, Alice and Bob first meet each other at cone 12. What is the number of the cone at which they will meet next?

14. The seventh-degree polynomial P has a leading coefficient of 1. In other words, takes the form $P(x) = 1 \cdot x^7 + a_6x^6 + a_5x^5 + \cdots + a_1x + a_0$. All of the polynomial's coefficients are integers, and P has six distinct roots, located at $x = 1, 2, 3, 4, 5, 6$. Determine the smallest possible value of $P(0)$. (Note: a is *smaller* than b if $a < b$.)

15. What positive value of x satisfies

$$\sqrt[3]{x + 3\sqrt[3]{x + 3\sqrt[3]{x + \cdots}}} = 3?$$

16. Roads A, B, C, and D have intersections at P, Q, R, and S, as shown in the diagram. All cars traveling on these roads travel straight through all intersections, not making left or right turns. Within a given timespan, an observer records 26 cars passing through P, 17 cars passing through Q, 13 cars passing through R, and 18 cars passing through S. How many cars in total are seen during this observation period?

3 Number Theory

1. What is the least common multiple of the first seven positive integers?

2. What is the smallest perfect square greater than 1 that cannot be written as the sum of two prime numbers?

3. Let a, b, c, and d be four nonzero digits ranging from 1 to 9. Find the largest four-digit number $\underline{abcd}$ such that the three-digit numbers $\underline{abc}$ and $\underline{bcd}$ are both perfect squares.

4. Let x be the least common multiple of $1!, 2!, 3!, \ldots, 97!$, and $98!$. Let y be the least common multiple of $1!, 2!, 3!, \ldots, 99!$, and $100!$. Compute the value of $\frac{y}{x}$.

5. There are no positive two-digit integers $\underline{ab}$ that are exactly equal to $a^2 + b^2$, but there are two values of $\underline{ab}$ that are one larger than the value of $a^2 + b^2$. One of these values is 75, which equals $1 + (7^2 + 5^2)$. What is the other value of $\underline{ab}$ satisfying the same condition?

6. For any three-digit integer $\underline{abc}$ from 100 to 999 inclusive, let its reverse be the number $\underline{cba}$. For example, 100's reverse would be 1, 940's reverse would be 49, and 123's reverse would be 321. How many positive three-digit integers satisfy the condition that the integer itself minus its reverse is some multiple of 9? (Note: the multiple of 9 can be positive, zero, or negative.)

7. When evaluated as a base-10 integer, how many consecutive zeros does $3^6 \cdot 4^5 \cdot 5^4 \cdot 6^3$ end with? (For clarification, 200000 ends with five consecutive zeros, but 200020 ends with only one.)

8. Find the sum of all positive factors of 144, including 1 and 144.

9. How many positive base-ten integers N satisfy the property that $5 \cdot N$ has two digits, but $6 \cdot N$ has three digits?

10. The year 2016 satisfies the property that its value can be written as the difference between a perfect square number and the sum of the digits of that square number. Specifically, $2025 = 45^2$, and $2025 - (2 + 0 + 2 + 5) = 2016$. What is the next year that satisfies this property?

11. The sequence $\{a, b, c\}$ is *geometric* if $\frac{c}{b} = \frac{b}{a}$. Find the smallest value of $a + b + c$ such that $\{a, b, c\}$ is a geometric sequence of positive integers greater than 1 with no prime factors common to all three terms. (For example, the sequence $\{2, 4, 8\}$ is disallowed because each term is divisible by 2.)

12. A *perfect* number is a positive integer greater than 1 that equals the sum of its proper divisors (all factors besides itself). The smallest perfect number is 6, whose proper divisors are 1, 2, and 3 $(1 + 2 + 3 = 6)$. Find the second smallest perfect number.

13. What is the smallest positive three-digit prime number that becomes a perfect square when its digits are reversed?

14. Determine the smallest perfect square that is 2700 greater than a positive perfect cube.

15. Harold and his fifteen classmates stand in a circle. Harold starts off with a ball and passes it one person to the left of him. That person then passes the ball two people to the left of them. The next person who receives the ball passes it three people to the left, and so on, the increment of people increasing by one with every pass. How many times is the ball passed until Harold receives the ball from someone? (Note: throwing the ball to oneself and Harold's first pass both count as "passes.")

16. Kyle the man of gates enjoys opening and closing gates. Whenever he stops by an open gate, he closes it and moves on. Whenever he stops by a closed gate, he opens it and moves on. Kyle arrives at a garden with G closed gates in a row, numbered in order from 1 to G. On his first traversal (round), he stops at each gate. On his second traversal, he stops at only the gates with multiples of 2. On any nth traversal, he will stop at only the gates with multiples of n. If Kyle performs G traversals, then 69 gates will remain closed. What is the value of G?

4 Geometry

1. A 5-inch by 7-inch photo of the TJ Varsity Math Team lies inside a rectangular frame, as shown in the diagram. No portion of the frame is covered by the photo, and the margin of the frame is 1 inch on all sides. What is the area of the frame?

2. Six congruent 6×6 squares are stacked to form a symmetric pyramid. Suppose point P is the bottom-left corner of the leftmost square on the lowest tier, and let point Q be the upper corner shared by both squares on the middle tier. Compute the distance between P and Q.

3. The concentric squares $ABCD$ and $EFGH$ satisfy $\overline{AB} \parallel \overline{EF}$, $AB = \sqrt{23}$, and $EF = \sqrt{43}$. Let M_1, M_2, M_3, and M_4 be the midpoints of $\overline{AB}$, $\overline{EF}$, $\overline{HE}$, and $\overline{DA}$, respectively. Determine the area of hexagon $AM_1M_2EM_3M_4$.

4. A specific quadrilateral has an inscribed circle that is tangent to all four of its sides. If three of the sides of the quadrilateral have lengths 17, 13, and 18, in no specific order, then what is the largest possible length of the fourth side?

5. A modified basketball court has a three-point "line" (arc) in the shape of a complete semicircle, as shown in the diagram. The area of the key, the shaded rectangular region, is 228 ft^2, and the area of the small semicircular region directly above the key is 18π ft^2. What is the area of the large semicircle enclosed by the three-point line, in square feet? Express your answer as a common fraction in terms of π.

6. The *diameter* of a regular octahedron is the longest distance between any two of its vertices. Compute the volume of a regular octahedron with diameter 12.

7. In quadrilateral $PQRS$, $m\angle PQS = m\angle PRS = 54^\circ$ and $m\angle RPS = 45^\circ$. What is the measure of $\angle PQR$, in degrees?

8. A rectangle is inscribed in a circle with area 9π. Four squares are drawn from the four sides of the rectangle. What is the total area of all four squares?

9. The figure at the right is made by attaching two 90° arcs each of length 5π to a 180° arc of length 10π at their endpoints so that the two smaller arcs do not lie on the same circle as the larger arc. What is the area of the region bounded by these arcs?

10. In the diagram to the right, $ABCD$ is a trapezoid with right angles at A and B, and DEF is an isosceles right triangle with its right angle at E. As the diagram suggests, A, D, and E are collinear, and C, D, and F are collinear, as well. The lengths AD and BC are positive integers, and the areas of $ABCD$ and DEF are equal. If $AB = 2$ and $EF = 8$, then determine the distance between A and E.

11. A rectangle with an area of 34 is inscribed in a circle with an area of 34π. Calculate the perimeter of the rectangle.

12. Point A lies at $(3, 0)$, point B lies at $(0, 3)$, and point C lies at (x, y) in the Cartesian coordinate plane. If the area of triangle ABC equals 9, then determine all possible values of $x + y$.

13. $TJHS$ is a convex quadrilateral satisfying $TJ = JH = 5$, $HS = 1$, and $ST = 7$. If $m\angle TJH = 90°$, then determine the area of $TJHS$.

14. Two unit spheres are inside a $5 \times 6 \times 14$ rectangular box. If the spheres are spread out from each other as far as possible inside the box (i.e., at opposite corners), what is the length of the shortest line segment connecting the surfaces of both spheres? (Note: a unit sphere has a radius of 1.)

15. The product of the three altitude (height) lengths of a certain triangle is 24. If this triangle has an area of 3, then determine the product of the three side lengths of this triangle.

16. $\triangle ABC$ has an obtuse angle at A. Let D be the point on $\overline{BC}$ such that $\overline{BC}$ and $\overline{AD}$ are perpendicular. $AC = 4$, $BC = 13$, and the area of is 24. What is the length of $\overline{AD}$? Express your answer as a common fraction.

17. Two circles are placed in the same plane. Their centers lie three feet apart, and the circles intersect at two points, producing three enclosed regions. The two regions that are only contained in one circle but not the other (labeled as A and B in the diagram) have areas that differ by 9π ft^2. When the circles are moved so that their centers are now seven feet apart, what is the new positive difference in the areas of regions A and B?

18. A bicycle tire can be viewed as the region between two concentric circles (circles that are centered at the same point). For the tire shown in the diagram, the longest straight line segment that can be drawn across the face of the tire has a length of 20 inches. What is the area of the shaded region representing the tire, in square inches? Express your answer in terms of π.

19. A square is inscribed in a semicircle with diameter of length 10. All four vertices of the square lie on the border of the semicircle; two vertices lie on the flat edge, and the other two vertices lie on the arc. Compute the area of the square.

20. In square $ABCD$, let M be the midpoint of $\overline{AB}$, and let $\overline{DM}$ and $\overline{CA}$ intersect at E. If the area of $ABCD$ is 1, then determine the area of triangle BME.

21. Triangle PQR has sides of length 2016, 2017, and 2018, in no particular order. Point X, not in the plane containing the triangle, satisfies $m\angle PXQ = m\angle QXR = m\angle RXP = 90°$. Which of the following choices best describes the set of all possible locations of X? Express your answer as the letter of your choice.

    (A) Nothing; no such point exists. (B) One point (C) Two points

    (D) A line (E) Three lines (F) A circle (G) A sphere

22. A 2-foot by 3-foot sheet of poster paper is folded in a way such that two diagonally opposite corners meet on top of each other, as shown in the diagram. Let point A be either endpoint of the line segment formed by the crease, and let point B be the location of the coinciding corners. How many **inches** long is $\overline{AB}$?

23. The two ends of a 1-meter wire are connected, and the resulting loop is placed on a flat surface, so that it may assume any non-overlapping, closed, two-dimensional shape. What is the largest possible area, in square meters, of the region contained inside of the loop?

24. Mathville consists of four neighborhoods, positioned at $A(0, 3)$, $B(5, 1)$, $C(3, 6)$, and $D(1, 10)$. A hospital is conveniently placed inside quadrilateral $ABCD$ at point E such that the sum of its distances from each neighborhood ($EA + EB + EC + ED$) is as small as possible. If the coordinates of point E are (x, y), then compute the value of $y - x$.

25. A circle intersects each side of a 7×7 square twice, splitting each side into segments of lengths 3, 1, and 3, as shown in the diagram. What is the area of the circle, in terms of π?

26. A spool consists of two circular discs of different sizes and a cylindrical axle that connects the centers of the two discs, as shown in the diagram. The diameter of the smaller disc is 10.5 cm, the diameter of the larger disk is 14 cm, and the length of the axle is 6 cm. If the spool is turned to rest on its side and rolled as a wheel on a flat surface, the spool will trace a circular path. How many full (whole-number) rotations will the wheel make to go around the circle once to end up back at its starting position?

27. A cube has eight vertices, some of which are colored red and the rest of which are colored blue. No two vertices with the same color share an edge on the cube. According to this rule, there will be four red points defining the vertices of a regular tetrahedron (triangular pyramid). If the cube has sides of length 3, then determine the volume of the regular tetrahedron.

28. Four ants are positioned on the vertices of square $A_1A_2A_3A_4$. For $i = 1, 2, 3$, the ant who started at vertex A_i will always be traveling directly toward the ant who started at vertex A_{i+1} at a constant rate of 4 meters per minute (similarly, the ant starting at A_4 directly follows the ant starting at A_1, traveling at the same fixed rate). This results in four spiraling paths converging in the center. If the side length of square $A_1A_2A_3A_4$ is 28 meters, then estimate the time, in seconds, it takes for all four ants to meet up in the center. Any estimation no more than 20 (seconds) away from the correct answer will receive full credit.

5 Combinatorics/Probability

1. Four students with different heights are standing in line. Their teacher wants each pair of consecutive students to have the taller student behind the shorter student; however, the teacher will allow at most one pair of neighboring students to be incorrectly arranged. How many different lineups is the teacher okay with?

2. The Varsity Math Team captain committee consists of one captain and two co-captains. If six candidates run in the committee election, then how many different possible committees can be formed? Assume that the two co-captain positions are the same.

3. In the city of Mathalopolis, there is a $\frac{1}{3}$ chance that it rains on any given day. What is the probability that it rains in Mathalopolis at some point during two consecutive days?

4. The TJ Varsity Math Team wants to schedule weekly team practices. The team randomly selects two different days of the 7-day week to hold practice. What is the probability that these practice days include one weekday (i.e., Monday, Tuesday, Wednesday, Thursday, or Friday) and one weekend day (i.e., Saturday or Sunday)?

5. In single-player tic-tac-toe, a player gets to place the letter X on exactly three spaces of a 3×3 tic-tac-toe grid. Recall that a player wins tic-tac-toe if any three of their markers on the board are collinear. If a computer plays single-player tic-tac-toe by moving randomly, then what is the probability that the computer will *not* win the game, despite having an easy winning strategy?

6. How many positive three-digit integers (with no leading zeros such as the number 049) have either three even digits or three odd digits?

7. The world-famous Gumby's Pizza sells whole pizzas with 10 slices each. How many ways can the 10 slices be divvied up among a group of three mathletes such that each member receives at least one slice of pizza? Assume that all 10 slices are indistinguishable but the three mathletes are distinguishable.

8. Each vertex of an octagon is randomly colored green or yellow. By the natural structure of an octagon, each vertex is directly connected to two different vertices by two edges. On average, how many vertices will be directly connected to two yellow vertices?

9. Lawrence is very rich. He enters his local dollar store with one million nickels (5 cents each), one million dimes (10 cents each), and one million quarters (25 cents each). He wants to buy a duck for one dollar (100 cents). How many unique combinations of nickels, dimes, and quarters can Lawrence use to pay the exact price for the duck? (Note: in each set of one million coins, the coins are unordered and indistinguishable.)

10. Tom and Jerry have a set of 96 white cards and want to color some of them purple. Before they color any cards, Tom randomly selects 10 cards to color, and Jerry randomly selects 48 cards to color from the same set of 96 cards, without knowing any of Tom's choices. Both of them then color purple all of the cards they selected. How many cards, on average, will end up purple?

11. George wants to play the Glop Guessing Game. A Glop has a uniformly random weight between 0 and 2 pounds, and George must guess the Glop's weight. If his guess is no more than 0.5 pounds away from the Glop's actual weight, then George wins. However, George is not very smart, for he always guesses a random number between 0 and 2. According to his guessing method, what is the probability that George wins the game?

12. Freddy sells special dice in the shape of a square pyramid, shown to the right. The five vertices of the die are numbered 1, 2, 3, 4, and 5, in no specific order. Two dice are said to be *equivalent* if one can be rotated to appear just like the other. What is maximum number of dice Freddy can sell such that no two of his dice are equivalent?

13. I have three closed boxes, one of which conceals an apple. Your task is to correctly guess which box contains the apple without opening any of them. After your first guess, I will open a box you have not guessed that does not contain the apple. You may then change your guess, but you do not have to. Assuming you utilize the most optimal strategy, what is the chance that your final guess is correct?

14. One percent of all mathletes are legendary. Some mathletes are oracles, whose job is to predict whether a given mathlete is legendary or not. Oracles make correct predictions

95% of the time. Kural the mathlete asks an oracle if he is legendary. If the oracle tells Kural that he is legendary, then what is the probability that he is actually legendary?