29th TJIMO

Alexandria, Virginia

Round: **Power**

Unlike the other rounds, just getting the answer right is not enough on the Power Round. Make sure you explain your answer and use words to describe how you arrived at your answer. In the words of middle school math teachers across the nation – no work, no credit!

Feel free to use results from previous problems on this round to prove a later problem (that is, you can use Problem 2 to prove Problem 3, but not vice versa). You do not need to have solved the earlier problem to cite its result. You may also use results from the morning's Practice Power Round.

This Power Round is divided into three sections. The first is a short review from the morning's Practice Power round, and is recommended to be completed first. The remaining two sections are on 3×3 magic squares and more on magic rectangles, and can be completed independently of each other. They are equally weighted in terms points. It might be wise to split up your team after finishing section 1, with one half working on section 2 and the other half working on section 3.

1 Review

Definition. An $n \times n$ *magic square* is an $n \times n$ grid with positive integers in each cell such that the sum of the integers in any row, the sum of the integers in any column, and the sum of the integers in either of the two long diagonals are all equal. This sum is called the *magic sum*.
Definition. A *standard* $n \times n$ magic square is one that uses the integers from 1 through n^2, inclusive, each once.
Problem 1 (1 point)**.** Determine all standard 1×1 magic squares.

Problem 2 (5=3+1+1 points)**.** In this problem, you will consider the magic sum for an arbitrary standard magic square.

(a) Determine the magic sum for an $n \times n$ magic square, where n is any positive integer. (Hint: You may find the formula $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$ useful.)

(b) What does your formula give for the magic sum for each of $n = 1, 2, 3, 4$? Are these consistent with what you found this morning? (If not, go back and check your work for your formula.)

(c) Is your expression always guaranteed to be an integer? Why or why not? (Hint: Consider when n is odd or even.)

2 3×3 Magic Squares

Problem 3 (3 points)**.** Fill in the missing entries below to make a magic square, or prove that it is impossible to do so.

Time limit: 45 minutes.

20		
	17	
10	28	

Problem 4 (15=1+3+2+3+3+3 points)**.** In this problem, you will construct a standard 3×3 magic square.

(a) What is the magic sum?

(b) List all the ways you can add three distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ to get the magic sum. (Order doesn't matter here, so $4 + 5 + 6$ is the same as $5 + 6 + 4$.)

(c) How many rows, columns, and diagonals are there? How many ways did you find in part (b)? These two numbers should be the same. What does that mean?

(d) Which number has to go in the center? Why?

(e) Which numbers have to go in the corners? Why?

(f) Draw a complete, standard 3×3 magic square.

Problem 5 (12=4+4+4 points)**.** For each of the following, find a 3×3 magic square whose entries are the given numbers, or prove that it is impossible to do so.

(a) 2, 3, 4, 5, 6, 7, 8, 9, 10

(b) 2, 4, 6, 8, 10, 12, 14, 16, 18

(c) 1, 3, 4, 5, 6, 7, 8, 9, 10

3 Magic Rectangles

Definition. An $m \times n$ *magic rectangle* is a grid with m rows and n columns and positive integers in each cell such that the sum of the integers in each row is the same and the sum of the integers in each column is the same, but these two sums do not necessarily have to be the same. The first is called the *row sum* and the second is called the *column sum*.

Definition. A *standard* $m \times n$ magic rectangle is one that uses the integers from 1 through $m \times n$, inclusive, each once.

Problem 6 (3 points)**.** Fill in the missing entries below to make a standard magic rectangle, or prove that it is impossible to do so.

6		8		10
13		1	11	
	14	15	4	

Problem 7 (8=2+2+4 points)**.** In this problem, you will construct a standard 4×2 magic rectangle.

(a) Using the sum of the entries, determine the row sum and the column sum.

(b) List all the ways you can add two of the possible numbers to obtain the row sum.

(c) Draw a complete, standard 4×2 magic rectangle.

Problem 8 (6=3+2+1 points)**.** In this problem, you will investigate the row and column sums for arbitrary magic rectangles.

(a) Determine the row sum and the column sum for a standard $m \times n$ magic rectangle. (Hint: You may find the formula given in the hint for problem 2 useful.)

(b) What do your formulas give for a 4×2 magic rectangle? What about a 4×3 magic rectangle? Are these correct?

(c) Is your expression always guaranteed to be an integer? Why or why not? What are the implications of this?

Problem 9 (8=4+4 points)**.** For each of the following, find a standard magic rectangle with the given dimensions or prove that it is impossible to do so.

(a) 8×2

(b) 8×3

Problem 10 (5 points)**.** Show that it is impossible to construct a standard $m \times n$ rectangle if $m + n$ is odd.

Time limit: 45 minutes.