TD Linear Programming - Geometric Approach

Exercise 6.1

1.

Let x_1 and x_2 be two points of H.

Let consider $x_t = t\,x_1 + (1-t)\,x_2$ for some $t \in [0,1]$.

Let show that $x_t \in H$.

We have:

$$a^T x_t = t a^T x_1 + (1-t) a^T x_2 \le tb + (1-t)b = b$$

So $x_t \in H$

Se we have:

$\forall (x_1, x_2) \in H^2, \forall t \in [0,1], t x_1 + (1-t) x_2 \in H$

so by defenition, H is convex.

2.

Let $M \in \mathbb{R}_+$.

Let define $x(\alpha) := \frac{-a}{\|a\|}\alpha$

To have $x(\alpha) \in H$ we need

$a^T x(\alpha) \le b \implies -\|a\|\alpha \le b \implies \alpha \ge \frac{-b}{\|a\|}$

Let choose $\alpha = \max(-b/\|a\|, M+1)$

We have $\|x(\alpha)\| = \alpha \ge M + 1 > M$

So for any $M \in \mathbb{R}_+$, there is an $x \in H$ such that $\|x\| \ge M$, so by defenition H is unbounded.

3.

Let $(x_1, x_2) \in (A \cap B)^2$.

$x_1, x_2 \in A \cap B$ so in particular, $x_1, x_2 \in A$.

A is convex so $[x_1, x_2] \subset A$

We also have $[x_1, x_2] \subset B$

So $[x_1, x_2] \subset A \cap B$

So $\forall (x_1, x_2) \in (A \cap B)^2, [x_1, x_2] \subset A \cap B$.

So $A \cap B$ is convex.

Exercise 6.2

1. cf feuille
2.

$$\max_{x \in \mathbb{R}^2} b^T x$$
$$Ax \geq c$$
$$x \geq 0$$

where $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

and $c = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

3.

$$\max_{x_s \in \mathbb{R}^4} b_s^T x_s$$
$$A_s x_s = c$$
$$x_s \geq 0$$

where

$b_s = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$