

Rational expectations

School of Economics, University College Dublin

Spring 2018

Lucas (1976)

Given that the structure of an econometric model consists of optimal decision rules of economic agents, and that optimal decision rules vary systematically with changes in the structure of series relevant to the decision maker, it follows that any change in policy will systematically alter the structure of econometric models.

$$y_t = a\tilde{y}_{t+1} + bx_t \quad (1)$$

$$|a| < 1; b \neq 0$$

\tilde{y}_{t+1} is expectation of y_{t+1} : how to specify?

$$\tilde{y}_{t+1} = y_t \quad (2)$$

$$y_t = ay_t + bx_t \quad (3)$$

$$y_t = \frac{b}{1-a}x_t$$

$$\tilde{y}_{t+1} = \mathbb{E}_t y_{t+1} \tag{4}$$

$$y_t = a\tilde{y}_{t+1} + bx_t \tag{5}$$

$$y_t = a\mathbb{E}_t y_{t+1} + bx_t \tag{6}$$

DSGE models have both backward and forward looking elements:
Backward looking dynamics come from identities

$$K_t = (1 - \gamma)K_{t-1} + I_t \quad (7)$$

Forward looking dynamics from optimising behaviour

- ▶ What agents expect to happen tomorrow is very important for what they decide to do today.

Modeling this requires assumptions about how people formulate expectations

- ▶ DSGE approach relies on the idea that people have **rational expectations**

Rational expectations

Agents formulate expectations in such a way that their subjective probability distribution of economic variables - conditional on the available information - coincides with the objective probability distribution of the same variable - according to a measure of the state of nature- in an equilibrium.

In other words agents

1. Use publicly available information efficiently (i.e. no systematic mistakes)
2. Understand the structure of the economy and base their expectations on this knowledge

Rational expectations (RE) is a **baseline** assumption about people's behaviour.

$$y_t = x_t + a\mathbb{E}_t y_{t+1} \quad (8)$$

Today's value of y is determined by

1. x
2. Tomorrow's expected value of y

Under RE agents formulate expectations that is consistent with

$$\mathbb{E}_t y_{t+1} = \mathbb{E}_t x_{t+1} + a \mathbb{E}_t \mathbb{E}_{t+1} y_{t+2} \quad (9)$$

$$\mathbb{E}_t y_{t+1} = \mathbb{E}_t x_{t+1} + a \mathbb{E}_t y_{t+2}$$

$$\mathbb{E}_t \mathbb{E}_{t+1} y_{t+2} = \mathbb{E}_t y_{t+2} \quad (10)$$

This is the **Law of Iterated Expectations**

- It is not rational for me to expect to have a different expectation next period for y_{t+2} than the one that I have today (based on the information set available at t).

Use repeated substitution to get solution for model

$$y_t = x_t + a\mathbb{E}_t x_{t+1} + a^2\mathbb{E}_t y_{t+2} \quad (11)$$

Include additional periods

$$\begin{aligned} y_t &= x_t + a\mathbb{E}_t x_{t+1} + \dots + a^{N-1}\mathbb{E}_t x_{t+N-1} + a^N\mathbb{E}_t y_{t+N} \\ &= \sum_{k=0}^{N-1} a^k \mathbb{E}_t x_{t+k} + a^N \mathbb{E}_t y_{t+N} \end{aligned} \quad (12)$$

Assuming

$$\lim_{N \rightarrow \infty} a^N \mathbb{E}_t y_{t+N} = 0 \quad (13)$$

We get

$$y_t = \sum_{k=0}^{\infty} a^k \mathbb{E}_t x_{t+k} \quad (14)$$

Asset pricing

1. Asset with price p_t , yielding dividend d_t
2. Close alternative with guaranteed rate of return r

Risk neutral investor will only hold the asset if the yield is the same as the rate of return

$$\frac{d_t + \mathbb{E}_t p_{t+1}}{p_t} = 1 + r \quad (15)$$

Rearranging

$$\frac{d_t + \mathbb{E}_t p_{t+1}}{p_t} = 1 + r \quad (16)$$

Gives

$$p_t = \frac{d_t}{1 + r} + \frac{\mathbb{E}_t p_{t+1}}{1 + r} \quad (17)$$

Repeated substitution solution

$$p_t = \sum_{k=0}^{\infty} \left(\frac{1}{1 + r} \right)^{k+1} \mathbb{E}_t d_{t+k} \quad (18)$$

Dividend-discount model: Asset prices should equal a discounted present-value sum of expected future dividends.

Backward solution

$$y_t = x_t + a\mathbb{E}_t y_{t+1} \quad (19)$$

$$y_t = x_t + ay_{t+1} + a\epsilon_{t+1}$$

Forecast error ϵ_{t+1} cannot be predicted at time t .

Can move time index back one period

$$y_{t-1} = x_{t-1} + ay_t + a\epsilon_t \quad (20)$$

$$ay_t = y_{t-1} - x_{t-1} - a\epsilon_t$$

$$y_t = a^{-1}y_{t-1} - a^{-1}x_{t-1} - \epsilon_t$$

$$y_t = -\sum_{k=0}^{\infty} a^{-k} \epsilon_{t-k} - \sum_{k=1}^{\infty} a^{-k} x_{t-k} \quad (21)$$

Forward & backward

$$y_t = \sum_{k=0}^{\infty} a^k \mathbb{E}_t x_{t+k}$$

$$y_t = - \sum_{k=0}^{\infty} a^{-k} \epsilon_{t-k} - \sum_{k=1}^{\infty} a^{-k} x_{t-k}$$

Both are correct; which one we use depends on a

$$|a| > 1$$

Weights on forward solution are explosive

- ▶ Forward solution will not converge to finite sum
- ▶ y_t depends more on values of x_t far in the distant future than on today's values

⇒ Use backward solution; will be indeterminate though

$$\mathbb{E}_{t-1}\epsilon_t = 0 \tag{22}$$

Cannot predict y_t even if we know full path of x_t

$$|a| < 1$$

Weights on backward solution are explosive; use forward solution

- ▶ Path of x_t will tell path of y_t

Most cases assume that $|a| < 1$

$$\lim_{N \rightarrow \infty} a^N \mathbb{E}_t y_{t+N} = 0 \quad (23)$$

y_t can't grow too fast (*transversality condition*)

Rational bubbles

If transversality condition is not imposed, model can have any other solution

$$y_t^* = \sum_{k=0}^{\infty} a^k \mathbb{E}_t x_{t+k} \quad (24)$$

Consider other solution

$$y_t = y_t^* + b_t \quad (25)$$

With bubble component b_t model must satisfy

$$y_t^* + b_t = x_t + a \mathbb{E}_t y_{t+1}^* + a \mathbb{E}_t b_{t+1} \quad (26)$$

Model is by construction

$$y_t^* = x_t + a\mathbb{E}_t y_{t+1}^* \quad (27)$$

Bubble component will satisfy

$$b_t = a\mathbb{E}_t b_{t+1} \quad (28)$$

Because $|a| < 1$, $|b|$ is always expected to get larger

$$b_t = \mathbb{E}_t \left[\frac{1}{1+r} b_{t+1} \right] \quad (29)$$

b_t grows in expectations at rate r

There may be restrictions in the real economy that stop b growing forever

- ▶ e.g. constant growth

Following would do the trick

$$b_{t+1} \begin{cases} (aq)^{-1}b_t + e_{t+1} & \text{with } \Pr(q) \\ e_{t+1} & \text{with } \Pr(1-q) \end{cases} \quad (30)$$

With

$$\mathbb{E}_t e_{t+1} = 0 \quad (31)$$

Imposing

$$\lim_{N \rightarrow \infty} a^N \mathbb{E}_t y_{t+N} = 0 \quad (32)$$

rules out bubbles of this or any other form.

Structural to reduced form

$$y_t = \sum_{k=0}^{\infty} a^k \mathbb{E}_t x_{t+k} \quad (33)$$

Requires assumption on how x_t evolves over time

- ▶ Otherwise cannot be used for predictions on y_t dynamics

Assumption that exogenous variables x_t are generated by backward-looking time-series models

- ▶ Therefore strong link between DSGE and VAR models

Consider x_t with DGP

$$\begin{aligned}x_t &= \rho x_{t-1} + \epsilon_t \\ |\rho| &< 1\end{aligned}\tag{34}$$

$$\mathbb{E}_t x_{t+k} = \rho^k x_t\tag{35}$$

Model solution given by

$$y_t = \left[\sum_{k=0}^{\infty} (a\rho)^k \right] x_t\tag{36}$$

Assuming $|a\rho| < 1$: infinite sum converges to

$$\sum_{k=0}^{\infty} (a\rho)^k = \frac{1}{1 - a\rho} \quad (37)$$

Gives reduced-form solution

$$y_t = \frac{1}{1 - a\rho} x_t \quad (38)$$

Can be combined with equation describing x_t and simulated on computer

Reduced-form model has VAR-like representation

$$y_t = \frac{1}{1 - a\rho}(\rho x_{t-1} + \epsilon_t) \quad (39)$$

$$= \rho y_{t-1} + \frac{1}{1 - a\rho} \epsilon_t \quad (40)$$

x_t, y_t series have purely backward-looking representations

- ▶ Theoretical models tend to predict that data can be described by VAR

DSGE recipe

1. Obtain structural equations involving expectations of future driving variables

$$\mathbb{E}_t x_{t+k}$$

2. Make assumptions about time-series process for the driving variables

$$x_t$$

3. Solve for reduced-form solution that can be simulated on a computer

$$y_t = \frac{1}{1 - a\rho} x_t$$

Permanent income hypothesis

Let consumption depend on present discounted value of after-tax income

$$c_t = \gamma \sum_{k=0}^{\infty} \beta^k \mathbb{E}_t y_{t+k} \quad (41)$$

Income generating process is

$$y_t = (1 + g)y_{t-1} + \epsilon_t \quad (42)$$

$$\mathbb{E}_t y_{t+k} = (1 + g)^k y_t \quad (43)$$

Reduced-form representation

$$c_t = \gamma \left[\sum_{k=0}^{\infty} (\beta(1 + g))^k \right] y_t \quad (44)$$

Assuming $\beta(1 + g)^k < 1$ this becomes

$$c_t = \frac{\gamma}{1 - \beta(1 + g)} y_t \quad (45)$$

Lucas critique

Two important implications

1. Structural equation is always true for the model

$$c_t = \gamma \sum_{k=0}^{\infty} \beta^k \mathbb{E}_t y_{t+k}$$

2. Reduced-form depends on process y_t taking particular form

$$c_t = \frac{\gamma}{1 - \beta(1 + g)} y_t$$

Reduced-form parameters change when $\mathbb{E}_t y_{t+k}$ changes

- ▶ Reduced-form models using historical data useless for policy analysis

Temporary tax cut

Suppose following parameter values

$$g = 0.02; \beta = 0.95$$

If consumers expect g , tax cut will increase consumption by

$$\frac{\gamma}{1 - \beta(1 + g)} = \frac{\gamma}{1 - 0.95(1 + 0.02)} \approx 32\gamma$$

Under rational expectations increase will be

$$\gamma$$

Off by a factor of 32.

VAR vs. DSGE

VARs fit data well but cannot be used for policy analysis

- ▶ VARs do not allow reduced-form correlations to change over time

DSGE can as it includes policy equations based on rational expectations

- i Relate interest rates to inflation and unemployment
- ii Dependence of fiscal variables on other macro variables
- iii Exchange rate regime

DSGE can explain pattern as result of structural changes in policy rules

Jump variables

$$y_t = \sum_{k=0}^{\infty} a^k \mathbb{E}_t x_{t+k}$$

- ▶ Depend on what happens today and what's expected to happen tomorrow
- ▶ Will jump if expectations about future change
- ▶ Past does not restrict their movement

Useful to characterise stock prices

For real economy use **second-order stochastic difference equations**

$$y_t = ay_{t-1} + b\mathbb{E}_t y_{t+1} + x_t \quad (46)$$

How to solve these? Suppose there is value for λ such that

$$v_t = y_t - \lambda y_{t-1} \quad (47)$$

follows SDE of form

$$v_t = \alpha \mathbb{E}_t v_{t+1} + \beta x_t \quad (48)$$

Can solve for v_t and back out the values for y_t

Given

$$y_t = v_t + \lambda y_{t-1} \quad (49)$$

Rewrite as

$$\begin{aligned} v_t + \lambda y_{t-1} &= ay_{t-1} + b(\mathbb{E}_t v_{t+1} + \lambda y_t) + x_t \\ &= ay_{t-1} + b\mathbb{E}_t v_{t+1} + b\lambda(v_t + \lambda y_{t-1}) + x_t \end{aligned} \quad (50)$$

Rearrange to

$$(1 - b\lambda)v_t = b\mathbb{E}_t v_{t+1} + x_t + (b\lambda^2 - \lambda + a)y_{t-1} \quad (51)$$

By definition λ is such that the v_t it defined followed a first-order SDE: λ satisfies

$$b\lambda^2 - \lambda + a = 0 \tag{52}$$

This is a quadratic equation so there are two values for λ that satisfy it.

For either value can characterise v_t by

$$v_t = \frac{b}{1 - b\lambda} \mathbb{E}_t v_{t+1} + \frac{1}{1 - b\lambda} x_t \quad (53)$$

$$= \frac{1}{1 - b\lambda} \sum_{k=0}^{\infty} \left(\frac{b}{1 - b\lambda} \right)^k \mathbb{E}_t x_{t+k} \quad (54)$$

y_t obeys

$$y_t = \lambda y_{t-1} + \frac{1}{1 - b\lambda} \sum_{k=0}^{\infty} \left(\frac{b}{1 - b\lambda} \right)^k \mathbb{E}_t x_{t+k} \quad (55)$$

Usually have one potential value for which $|\lambda| < 1$: gives unique stable solution

Systems of RE equations:

1. Generalise solution for single equation to vector

$$Z_t = \begin{pmatrix} z_{1t} \\ z_{2t} \\ \vdots \\ z_{nt} \end{pmatrix} \quad (56)$$

2. Specify as macroeconomic model

$$Z_t = B\mathbb{E}_t Z_{t+1} + X_t \quad (57)$$

B is an $n \times n$ matrix

3. Get solution using repeated substitution

$$Z_t = \sum_{k=0}^{\infty} B^k \mathbb{E}_t X_{t+k} \quad (58)$$

NB- Will give stable non-explosive solution under certain conditions

Eigenvalues

λ_i is an eigenvalue of the matrix B if there exists a vector e_i such that

$$Be_i = \lambda_i e_i \quad (59)$$

Many $n \times n$ matrices have n eigenvalues

- Denote P which has columns n eigenvectors corresponding to these eigenvalues

$$BP = P\Omega \quad (60)$$

$$\Omega = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & . & 0 \\ 0 & 0 & 0 & \lambda_n \end{pmatrix} \quad (61)$$

Ω is a diagonal matrix of eigenvalues.

Stability condition

$$BP = P\Omega \quad (62)$$

Implies

$$B = P\Omega P^{-1} \quad (63)$$

Provides information on relation between eigenvalues and higher powers of B

$$B^n = P\Omega^n P^{-1} = P \begin{pmatrix} \lambda_1^n & 0 & 0 & 0 \\ 0 & \lambda_2^n & 0 & 0 \\ 0 & 0 & . & 0 \\ 0 & 0 & 0 & \lambda_n^n \end{pmatrix} P^{-1} \quad (64)$$

Higher powers of B depend on eigenvalues taken to power n

B^n will tend towards zero as $n \rightarrow \infty$

- If all of the eigenvalues are in the unit circle $|\lambda| < 1$

Therefore we can get unique stable forward-looking solution for

$$Z_t = B\mathbb{E}_t Z_{t+1} + X_t \quad (65)$$

if the eigenvalues of B are all inside the unit circle.

Calculating eigenvalues

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad (66)$$

Suppose two eigenvalues for A : λ_1, λ_2

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \quad (67)$$

There are eigenvectors when multiplied by $A - \lambda I$ equal a vector of zeroes: determinant of matrix

$$A - \lambda I = \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{pmatrix} \quad (68)$$

will equal zero: solving quadratic formula will give the two eigenvalues of A

$$(a_{11} - \lambda_1 a_{12})(a_{22} - \lambda_2) - a_{12} a_{21} = 0 \quad (69)$$

Binder-Pesaran method

$$Z_t = AZ_{t-1} + B\mathbb{E}_t Z_{t+1} + HX_t \quad (70)$$

One-lag/one-lead form is only for illustration

- Equation summarises all possible linear RE models

Binder & Pesaran (1996) solved this model in manner exactly analogous to second-order SDE, for

$$W_t = Z_t - CZ_{t-1} \quad (71)$$

find matrix C such that (70) obeys a first-order matrix

$$W_t = F\mathbb{E}_t W_{t+1} + GX_t \quad (72)$$

i.e. turn second-order into simpler first-order

To determine C can use fact that

$$Z_t = W_t + CZ_{t-1} \quad (73)$$

Can be rewritten as

$$\begin{aligned} W_t + CZ_{t-1} &= AZ_{t-1} + B(\mathbb{E}_t W_{t+1} + CZ_t) + HX_t \\ &= AZ_{t-1} + B(\mathbb{E}_t W_{t+1} + C(W_t + CZ_{t-1})) + HX_t \end{aligned} \quad (74)$$

Rearranges to

$$(I - BC)W_t = B\mathbb{E}_t W_{t+1} + (BC^2 - C + A)Z_{t-1} + HX_t \quad (75)$$

C is matrix such that W_t follow a first-order forward-looking matrix equation

- ▶ No extra Z_{t-1} terms

Follows that

$$BC^2 - C + A = 0 \tag{76}$$

This equation can be solved to give C : which is non-trivial

Can use

$$C = BC^2 + A \quad (77)$$

and solve reiteratively:

1. Provide initial guess $C_0 = I$
2. Iterate on $C_n = BC_{n-1}^2 + A$ until all entries in C_n converge

Once we know C , we have

$$W_t = F\mathbb{E}_t W_{t+1} + GX_t \quad (78)$$

where

$$F = (I - BC)^{-1}B \quad (79)$$

$$G = (I - BC)^{-1}H \quad (80)$$

Has a stable forward-looking solution

- ▶ Assuming all eigenvalues of F are inside the unit circle

$$W_t = \sum_{k=0}^{\infty} F^k \mathbb{E}_t (GX_{t+k}) \quad (81)$$

Reduced form representation

Sub. (81) in original equation

$$Z_t = CZ_{t-1} + \sum_{k=0}^{\infty} F^k \mathbb{E}_t(GX_{t+k}) \quad (82)$$

Suppose variables X_t can be represented using VAR

$$X_t = DX_{t-1} + \epsilon_t \quad (83)$$

D has eigenvalues inside unit circle, implies

$$\mathbb{E}_t X_{t+k} = D^k X_t \quad (84)$$

Model solution is given by

$$Z_t = CZ_{t-1} + \left[\sum_{k=0}^{\infty} F^k GD^k \right] X_t \quad (85)$$

Infinite sum in (85) will converge to matrix P : model has reduced-form representation

- ▶ Can be simulate along with the VAR process driving X_t

$$Z_t = CZ_{t-1} + PX_t \quad (86)$$

Recipe for simulating DSGE model:

1. Specify matrices A, B, H

$$Z_t = AZ_{t-1} + B\mathbb{E}_t Z_{t+1} + HX_t$$

2. Solve for C, F, G

$$W_t = Z_t - CZ_{t-1}$$

$$W_t = F\mathbb{E}_t W_{t+1} + GX_t$$

3. Specify a VAR process for the driving variables

$$X_t = DX_{t-1} + \epsilon_t$$

4. Obtain the reduced-form representations

$$Z_t = CZ_{t-1} + PX_t$$

Finally, there are often multiple values for different variables at t :
Can estimate

$$KZ_t = AZ_{t-1} + B\mathbb{E}_t Z_{t+1} + HX_t \quad (87)$$

Computer will multiply both sides by K^{-1} and solve using
Binder-Pesaran model.

$$Z_t = K^{-1}AZ_{t-1} + K^{-1}B\mathbb{E}_t Z_{t+1} + K^{-1}HX_t \quad (88)$$

Need to figure out what model implies for K, A, B, H ; computer
will provide following representation

$$Z_t = CZ_{t-1} + PX_t \quad (89)$$

$$X_t = DX_{t-1} + \epsilon_t \quad (90)$$

These can be used for further calculations