

Kalman filter

School of Economics, University College Dublin

Spring 2018

Latent variables: Variables that are not directly observed but inferred from other variables

- ▶ Unobserved, but variable can still play important role in theoretical model

Example: Potential output

- ▶ Keynesian model: inflationary pressure determined by deviation of output from potential output
- ▶ Consider GDP increase in last quarter but no sign of inflationary pressure: potential output increased

Q: Can we assume that there has been a change in potential output?

A: Potential output probably stable from quarter to quarter while there is likely random noise fluctuations in inflation

- ▶ Signal vs. noise: Need to have a method that extracts useful signal from data that also contains lot of noise

One way to extract a signal is the **Kalman filter**

- ▶ Recursive method to estimate state of process; minimises MSE

Conditional expectations:

We want an estimate of the value of variable X

- Problem: we don't observe X

Do observe Z which is correlated with X : Assume that X, Z are jointly normally distributed

$$\begin{pmatrix} X \\ Z \end{pmatrix} \sim N \left(\begin{pmatrix} \mu_X \\ \mu_Z \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \sigma_{XZ} \\ \sigma_{XZ} & \sigma_Z^2 \end{pmatrix} \right) \quad (1)$$

We get

$$\mathbb{E}(X|Z) = \mu_X + \frac{\sigma_{XZ}}{\sigma_Z^2}(Z - \mu_Z) \quad (2)$$

Alternatively, define ρ as correlation between X and Z

$$\rho = \frac{\sigma_{XZ}}{\sigma_X \sigma_Z} \quad (3)$$

Inserting in (2) we get

$$\mathbb{E}(X|Z) = \mu_X + \rho \frac{\sigma_X}{\sigma_Z} (Z - \mu_Z) \quad (4)$$

Weight put on information from Z depends on

1. Correlation between X and Z (ρ)
2. The relative standard deviation ($\frac{\sigma_X}{\sigma_Z}$)

If Z has high standard deviation, it is a poor signal.

Multivariate conditional expectations

Can generalise from 2 to n variables

- ▶ Let X be $1 \times n$ vector of variables and Z $1 \times m$

$$\begin{pmatrix} X \\ Z \end{pmatrix} \sim N \left(\begin{pmatrix} \mu_X \\ \mu_Z \end{pmatrix}, \begin{pmatrix} \Sigma_{XX} & \Sigma_{XZ} \\ \Sigma'_{XZ} & \Sigma_{ZZ} \end{pmatrix} \right) \quad (5)$$

Expected value of X conditional on Z is

$$\mathbb{E}(X|Z) = \mu_X + \Sigma_{XZ} \Sigma_{ZZ}^{-1} (Z - \mu_Z) \quad (6)$$

State-space models: Linear time-series models that mix observable and unobservable variables.

$$S_t = FS_{t-1} + u_t \quad (7)$$

State equation - or transition equation - describes how unobservables S_t evolve over time

$$Z_t = HS_t + v_t \quad (8)$$

Measurement equation, relates set of observable variables Z_t to unobservable variables S_t

Errors

Both u_t and v_t can include either

1. Normally distributed errors
2. Zeros, if the described equation is an identity

$$u_t \sim N(0, \Sigma^u) \quad (9)$$

$$v_t \sim N(0, \Sigma^v) \quad (10)$$

Σ might not have full matrix rank

- ▶ Or rank deficient
- ▶ i.e. not enough information in data to estimate equation

Estimation

Observed data described by

$$Z_t = HS_t + v_t \quad (11)$$

Cannot observe S_t but can replace it with unbiased guess based on information available at time t

$$S_{t|t-1} \quad (12)$$

Assume that errors are normally distributed with known covariance matrix

$$S_t - S_{t|t-1} \sim N(0, \Sigma_{t|t-1}^S) \quad (13)$$

Can express observed variables as

$$Z_t = HS_{t|t-1} + v_t + H(S_t - S_{t|t-1}) \quad (14)$$

1. $S_{t|t-1}$ is observable
2. $v_t, S_t - S_{t|t-1}$ are unobservable but normally distributed

Estimate model using ML

$$Z_t = HS_{t|t-1} + v_t + H(S_t - S_{t|t-1}) \quad (15)$$

Variance of error, after conditioning on $t - 1$ state-variable estimate, is given by

$$v_t + H(S_t - S_{t|t-1}) \sim N(0, \Omega_t) \quad (16)$$

$$\Omega_t = \Sigma^v + H \Sigma_{t|t-1}^S H' \quad (17)$$

Parameters of model are given by

$$\theta = (F, H, \Sigma^u, \Sigma^v) \quad (18)$$

Log-likelihood function for Z_t given observables at $t - 1$ is

$$\log f(Z_t | Z_{t-1}, \theta) = -\log 2\pi - \log |\Omega_t| - \frac{1}{2}(Z_t - HS_{t|t-1})'\Omega_t^{-1}(Z_t - HS_{t|t-1}) \quad (19)$$

Combined likelihood is given by

- ▶ Based on initial estimate of first period unobservable state $S_{1|0}$

$$f(Z_1, \dots, Z_T | S_{1|0}, \theta) = f(Z_1 | S_{1|0}, \theta) \prod_{i=2}^{i=T} f(Z_i | Z_{i-1}, \theta) \quad (20)$$

$$\begin{aligned} \log f(Z_1, \dots, Z_T | S_{1|0}, \theta) = & -T \log 2\pi - \sum_{i=2}^T \log |\Omega| \\ & - \frac{1}{2} \sum_{i=1}^T (Z_i - HS_{i|i-1})' \Omega_i^{-1} (Z_i - HS_{i|i-1}) \end{aligned} \quad (21)$$

ML parameter estimates will be set of matrices

$\theta = (F, H, \Sigma^v, \Sigma^u)$ that provides largest value for this function

MLE will estimate model's parameters; only need an unbiased guess based on information available at $t - 1$ ($S_{t|t-1}$): Use Kalman filter

- ▶ Iterative method
- ▶ Provides estimates of state variables for t , uses observable data for $t + 1$ to update estimates

Estimating state variables

Formulate estimate of state variable at time t given information at $t - 1$

$$S_t = FS_{t-1} + u_t \Rightarrow S_{t|t-1} = FS_{t-1|t-1} \quad (22)$$

At $t - 1$, expected value for the observables at t are

$$Z_{t|t-1} = HS_{t|t-1} = HFS_{t-1|t-1} \quad (23)$$

At t we observe Z_t ; need to update estimate of state variable given information

$$Z_t - HFS_{t-1|t-1} \quad (24)$$

Model assumptions imply

$$\begin{pmatrix} S_t \\ Z_t \end{pmatrix} \sim N \left(\begin{pmatrix} FS_{t-1|t-1} \\ HFS_{t-1|t-1} \end{pmatrix}, \begin{pmatrix} \Sigma_{t|t-1}^S & (H\Sigma_{t|t-1}^S)' \\ H\Sigma_{t|t-1}^S & \Sigma^V + H\Sigma_{t|t-1}^S H' \end{pmatrix} \right) \quad (25)$$

Use conditional expectations to state that minimum variance unbiased estimate of $S_t|Z_t$ is

$$\mathbb{E}(S_t|Z_t) = S_{t|t} = FS_{t-1|t-1} + K_t(Z_t - HFS_{t-1|t-1}) \quad (26)$$

K_t is the **Kalman gain** matrix

$$K_t = \left(H \Sigma_{t|t-1}^S \right)' \left(\Sigma^V + H \Sigma_{t|t-1}^S H' \right)^{-1} \quad (27)$$

Covariance matrices required to compute K_t are updated by

$$\Sigma_{t|t-1}^S = F \Sigma_{t-1|t-1}^S F' + \Sigma^U \quad (28)$$

$$\Sigma_{t|t}^S = (I - K_t H) \Sigma_{t|t-1}^S \quad (29)$$

Initialising the Kalman filter we still need

1. Initial estimate $S_{1|0}$
2. Covariance matrix

In many macroeconomic models S can be assumed to have zero mean.

For the covariance matrix recall

$$\Sigma_{t|t-1}^S = F \Sigma_{t-1|t-1}^S F' + \Sigma^u \quad (30)$$

Matrix will generally converge: for unconditional covariance matrix use Σ value solves

$$\Sigma = F \Sigma F' + \Sigma^u \quad (31)$$

Kalman smoother: Kalman filter is a one-sided filter

- ▶ \hat{S}_t based on data available at t

Good for real-time, but we often have access to full historical dataset

- ▶ Time-varying models can be estimated using Kalman smoother
- ▶ Two-sided filter using all available data to compute \hat{S}_t

Hodrick-Prescott filter

$$\sum_{t=1}^N [(y_t - y_t^*)^2 + \lambda(\Delta y_t^* - \Delta y_{t-1}^*)] \quad (32)$$

Consider state-space model

$$y_t = y_t^* + C_t \quad (33)$$

$$\Delta y_t^* = \Delta y_{t-1}^* + \epsilon_t^g \quad (34)$$

$$C_t = \epsilon_t^c \quad (35)$$

Here

$$\text{Var}(\epsilon_t^g) = \sigma_g^2; \text{Var}(\epsilon_t^c) = \sigma_c^2 \quad (36)$$

HP filter = Kalman filter when

$$\lambda = \frac{\sigma_c^2}{\sigma_g^2} \quad (37)$$

It is assumed that σ_c^2 is 5 percentage points and σ_g^2 one-eight percentage point: $\lambda = 1600$

Laubach & Williams (2001): Estimate model with two unobservable time-varying series

1. Potential output
2. Natural rate of interest

$$\tilde{y}_t = y_t - y_t^*$$

$$\tilde{y}_t = A_y(L)\tilde{y}_{t-1} + A_r(L)(r_{t-1} - r_{t-1}^*) + \epsilon_{1t}$$

$$\pi_t = B_\pi(L)\pi_{t-1} + B_y(L)\tilde{y}_{t-1} + B_x(L)x_t + \epsilon_{2t}$$

$$r_t^* = c g_t + z_t$$

$$z_t = D_z(L)z_{t-1} + \epsilon_{3t}$$

$$y_t^* = y_{t-1}^* + g_{t-1} + \epsilon_{4t}$$

$$g_t = g_{t-1} + \epsilon_{5t}$$

Figure 1: Smoothed Estimates of Unobserved States

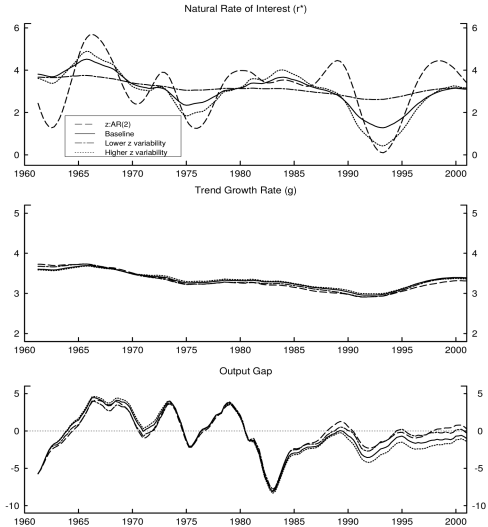


Figure 4: Real-time Estimates of the Natural Rate of Interest

