Lagrangian

1 Introduction

Let's suppose that we have a vector $\boldsymbol{X}$ of variables:

$$\boldsymbol{X} = [x_i] \Rightarrow \dot{\boldsymbol{X}} = [\dot{x}_i]$$

We have the given energy function defined like

$$E = \frac{1}{2}\dot{\boldsymbol{X}} \cdot M \cdot \dot{\boldsymbol{X}} + \dot{\boldsymbol{X}} \cdot V \cdot \boldsymbol{X} + \frac{1}{2}\boldsymbol{X} \cdot K \cdot \boldsymbol{X} + \boldsymbol{A} \cdot \dot{\boldsymbol{X}} + \boldsymbol{B} \cdot \boldsymbol{X} + C \tag{1}$$

Or using the indical notation

$$\begin{aligned} E = C + \sum_j \left(A_j \dot{x}_j + B_j x_j\right) + \\ \sum_j \sum_k \left(\frac{1}{2} M_{jk} \dot{x}_j \dot{x}_k + V_{jk} \cdot \dot{x}_j x_k + \frac{1}{2} K_{jk} \cdot x_j x_k\right) \end{aligned}$$

The Lagrangian operator, applied in E with the variable $\boldsymbol{X}$ is defined by

$$\boldsymbol{L} = \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\boldsymbol{X}}}\right) - \frac{\partial E}{\partial \boldsymbol{X}} \tag{2}$$

Or using the indicial notation

$$L_i = \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{x}_i}\right) - \frac{\partial E}{\partial x_i} \tag{3}$$

Using the given definition and supposing that M, V, K, A, B and C are only function of $\boldsymbol{X}$, we can write $\boldsymbol{L}$ like

$$\boldsymbol{L} = M_a \cdot \ddot{\boldsymbol{X}} + M_v \cdot \dot{\boldsymbol{X}} + M_p \cdot \boldsymbol{X} + \boldsymbol{M_c} + M_{vv} : \dot{\boldsymbol{X}}\dot{\boldsymbol{X}} + M_{vp} : \dot{\boldsymbol{X}}\boldsymbol{X} + M_{pp} : \boldsymbol{X}\boldsymbol{X} \tag{4}$$

2 Calculating the matrix

2.1 Derivating E to respect of $\dot{x}_i$

$$\left[\frac{\partial E}{\partial \dot{x}_i}\right]_M = \frac{\partial}{\partial \dot{x}_i}\sum_{j,k}\left(\frac{1}{2}M_{jk}\dot{x}_j\dot{x}_k\right) = \sum_{j,k}\left(\frac{1}{2}M_{jk}\delta_{ij}\dot{x}_k + \frac{1}{2}M_{jk}\delta_{ik}\dot{x}_j\right) = \sum_j \frac{1}{2}\left(M_{ij}+M_{ji}\right)\dot{x}_j$$

$$\left[\frac{\partial E}{\partial \dot{x}_i}\right]_V = \frac{\partial}{\partial \dot{x}_i}\sum_{j,k} V_{jk}\dot{x}_j x_k = \sum_{j,k}\left(V_{jk}\cdot\delta_{ij}x_k\right) = \sum_j V_{ij}x_j$$

$$\left[\frac{\partial E}{\partial \dot{x}_i}\right]_K = \frac{\partial}{\partial \dot{x}_i}\sum_{j,k} K_{jk}x_j x_k = 0$$

$$\left[\frac{\partial E}{\partial \dot{x}_i}\right]_A = \frac{\partial}{\partial \dot{x}_i}\sum_j A_j\dot{x}_j = \sum_j A_j\cdot\delta_{ij} = A_i$$

$$\left[\frac{\partial E}{\partial \dot{x}_i}\right]_B = \frac{\partial}{\partial \dot{x}_i}\sum_j B_j x_j = 0$$

$$\left[\frac{\partial E}{\partial \dot{x}_i}\right]_C = \frac{\partial}{\partial \dot{x}_i}C = 0$$

2.2 Derivating previous result to respect of t

And so

$$\begin{aligned}
\left[\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{x}_i}\right)\right]_M &= \frac{d}{dt}\sum_j \frac{1}{2}\left[M_{ij}+M_{ji}\right]\dot{x}_j \\
&= \sum_{j,k}\frac{1}{2}\left[\frac{\partial M_{ij}}{\partial x_k}+\frac{\partial M_{ji}}{\partial x_k}\right]\cdot\dot{x}_j\dot{x}_k + \sum_j\frac{1}{2}\left[M_{ij}+M_{ji}\right]\cdot\ddot{x}_j \\
\left[\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{x}_i}\right)\right]_V &= \frac{d}{dt}\sum_j V_{ij}x_j = \sum_j\left(V_{ij}\dot{x}_j\right) + \sum_{j,k}\frac{\partial V_{ij}}{\partial x_k}\cdot x_j\cdot\dot{x}_k \\
\left[\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{x}_i}\right)\right]_K &= \frac{d}{dt}0 = 0 \\
\left[\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{x}_i}\right)\right]_A &= \frac{d}{dt}A_i = \sum_j \frac{\partial A_i}{\partial x_k}\cdot\dot{x}_k \\
\left[\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{x}_i}\right)\right]_B &= \frac{d}{dt}0 = 0 \\
\left[\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{x}_i}\right)\right]_C &= \frac{d}{dt}0 = 0
\end{aligned}$$

2.3 Derivating E to respect of x_i

And the last part

$$\begin{aligned}
\left[\frac{\partial E}{\partial x_i}\right]_M &= \frac{\partial}{\partial x_i}\sum_{j,k}\frac{1}{2}M_{jk}\cdot\dot{x}_j\dot{x}_k = \sum_{j,k}\frac{1}{2}\,\frac{\partial M_{jk}}{\partial x_i}\cdot\dot{x}_j\dot{x}_k \\
\left[\frac{\partial E}{\partial x_i}\right]_V &= \frac{\partial}{\partial x_i}\sum_{j,k}V_{jk}\cdot\dot{x}_j x_k = \sum_j V_{ji}\dot{x}_j + \sum_{jk}\frac{\partial V_{jk}}{\partial x_i}\cdot\dot{x}_j x_k \\
\left[\frac{\partial E}{\partial x_i}\right]_K &= \frac{\partial}{\partial x_i}\sum_{j,k}\frac{1}{2}K_{jk}x_j x_k \\
&= \frac{1}{2}\sum_{j,k}\left(K_{jk}x_j\delta_{ik} + K_{jk}\delta_{ij}x_k + \frac{\partial K_{jk}}{\partial x_i}\right) \\
&= \frac{1}{2}\sum_j (K_{ij}+K_{ji})\cdot x_j + \frac{1}{2}\sum_{jk}\frac{\partial K_{jk}}{\partial x_i}\cdot x_j x_k \\
\left[\frac{\partial E}{\partial x_i}\right]_A &= \frac{\partial}{\partial x_i}\sum_j A_j\dot{x}_j = \sum_j\frac{\partial A_j}{\partial x_i}\dot{x}_j \\
\left[\frac{\partial E}{\partial x_i}\right]_B &= \frac{\partial}{\partial x_i}\sum_j B_j x_j = B_i + \sum_j\frac{\partial B_j}{\partial x_i}x_j \\
\left[\frac{\partial E}{\partial x_i}\right]_C &= \frac{\partial C}{\partial x_i}
\end{aligned}$$

2.4 Result for each matrix

$$\begin{aligned}
\left[\frac{d}{dt}\frac{\partial E}{\partial \dot{x}_i} - \frac{\partial E}{\partial x_i}\right]_M &= \sum_j\frac{1}{2}\left[M_{ij}+M_{ji}\right]\ddot{x}_j + \sum_{jk}\frac{1}{2}\left[\frac{\partial M_{ij}}{\partial x_k} + \frac{\partial M_{ji}}{\partial x_k} - \frac{\partial M_{jk}}{\partial x_i}\right]\dot{x}_j\dot{x}_k \\
\left[\frac{d}{dt}\frac{\partial E}{\partial \dot{x}_i} - \frac{\partial E}{\partial x_i}\right]_V &= \sum_j\left[V_{ij}-V_{ji}\right]\dot{x}_j + \sum_{jk}\frac{\partial V_{jk}}{\partial x_i}\dot{x}_j x_k \\
\left[\frac{d}{dt}\frac{\partial E}{\partial \dot{x}_i} - \frac{\partial E}{\partial x_i}\right]_K &= \sum_j\frac{-1}{2}\left[K_{ij}+K_{ji}\right]x_j + \sum_{jk}\frac{-1}{2}\cdot\frac{\partial K_{jk}}{\partial x_i}x_j x_k \\
\left[\frac{d}{dt}\frac{\partial E}{\partial \dot{x}_i} - \frac{\partial E}{\partial x_i}\right]_A &= \sum_j\left[\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right]\cdot\dot{x}_j \\
\left[\frac{d}{dt}\frac{\partial E}{\partial \dot{x}_i} - \frac{\partial E}{\partial x_i}\right]_B &= -B_i - \sum_j\frac{\partial B_j}{\partial x_i}\cdot x_j \\
\left[\frac{d}{dt}\frac{\partial E}{\partial \dot{x}_i} - \frac{\partial E}{\partial x_i}\right]_C &= -\frac{\partial C}{\partial x_i}
\end{aligned}$$

3 The relation between E and $\boldsymbol{L}$ using matrix

After all this calculation, we have

$$E = \frac{1}{2}\dot{\boldsymbol{X}} \cdot M \cdot \dot{\boldsymbol{X}} + \dot{\boldsymbol{X}} \cdot V \cdot \boldsymbol{X} + \frac{1}{2}\boldsymbol{X} \cdot K \cdot \boldsymbol{X} + A \cdot \dot{\boldsymbol{X}} + B \cdot \boldsymbol{X} + C$$

and we transform it to

$$\boldsymbol{L} = M_a \cdot \ddot{\boldsymbol{X}} + M_v \cdot \dot{\boldsymbol{X}} + M_p \cdot \boldsymbol{X} + \boldsymbol{M_c} + M_{vv} : \dot{\boldsymbol{X}}\dot{\boldsymbol{X}} + M_{vp} : \dot{\boldsymbol{X}}\boldsymbol{X} + M_{pp} : \boldsymbol{X}\boldsymbol{X}$$

where

$$[M_a]_{ij} = \frac{1}{2}\left(M_{ij} + M_{ji}\right) \tag{5}$$

$$[M_v]_{ij} = V_{ij} - V_{ji} + \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \tag{6}$$

$$[M_p]_{ij} = \frac{-1}{2}\left[K_{ij} + K_{ji}\right] - \frac{\partial B_j}{\partial x_i} \tag{7}$$

$$[M_c]_i = -B_i - \frac{\partial C}{\partial x_i} \tag{8}$$

$$[M_{vv}]_{ijk} = \frac{1}{2}\left[\frac{\partial M_{ij}}{\partial x_k} + \frac{\partial M_{ji}}{\partial x_k} - \frac{\partial M_{jk}}{\partial x_i}\right] \tag{9}$$

$$[M_{vp}]_{ijk} = \frac{\partial V_{jk}}{\partial x_i} \tag{10}$$

$$[M_{pp}]_{ijk} = \frac{-1}{2} \cdot \frac{\partial K_{jk}}{\partial x_i} \tag{11}$$