Lecture 1: Bus engine replacement model (Rust, 1987)

Short course "Dynamic programming and structural estimation"

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Origins of Dynamic Programming

Dynamic Programming is a recursive method for solving dynamic optimization problems

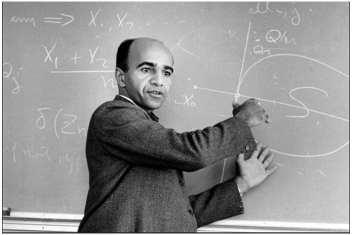
- Origins: Massé 1946 *Les réserves et la régulation de l'avenir* (optimal regulation of hydro reservoirs)
- Sequential analysis and statistical decision theory: Arrow, Blackwell and Girshick 1949 *Econometrica* "Bayes and Minimax Solution of Sequential Decision Problems"
- They were "trying to understand systematically sequential analysis from a decision-theoretic viewpoint. We wrote a paper in which was clearly displayed the recursive nature of the optimization problem." (Arrow, 2002)

Founding fathers

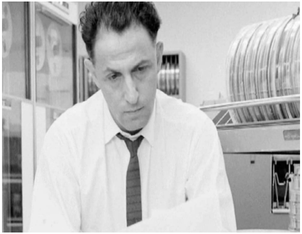
Pierre Massé

Kenneth A. Arrow

David Blackwell



Richard Bellman



Origin of the term "Dynamic Programming"

- From Bellman's autobiography *The Eye of the Hurricane*
- "In the first place, I was interested in planning, in decision making, in thinking. But planning, is not a good word for various reasons."
- "I decided therefore to use the word, 'programming'. I wanted to get across the idea that this was dynamic, this was multistage, this was time-varying."
- "Let's take a word which has an absolutely precise meaning, namely dynamic, in the classical physical sense. It has a very interesting property as an adjective, and that is it's impossible to use the word, dynamic, in the pejorative sense."
- "Thus, I thought dynamic programming was a good name. It was something not even a Congressman could object to. So I used it as an umbrella for my activities."

Bellman's "Principle of Optimality"

- *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.* Bellman, 1957 *Dynamic Programming*
- In game-theoretic language, the Principle of Optimality is equivalent to the concept of a *subgame-perfect equilibrium of a game against "nature."*
- These solutions can be computed via *backward induction on the game tree* where the "game tree" is an extensive-form representation of the game against nature.
- Bellman equation: the embodiment of the recursive way of formulating and solving these problems

Overview of Rust (1987)

The economic question: For how long one should continue to operate and maintain a bus before it is optimal to replace or rebuild the engine?
The model: The optimal replacement decision is the solution to a dynamic optimization problem that formalizes the trade-off between two conflicting objectives:

- *Minimizing maintenance and replacement costs, versus*
- *Minimizing unexpected engine failures*

Empirical question: Did the decision maker (the superintendent of maintenance, Harold Zurcher) behave according to the optimal replacement rule implied by the theory model?
Structural estimation: Using data on *monthly mileage and engine replacements* for a sample of GMC busses, Rust estimates the structural parameters in the engine replacement model using full solution maximum likelihood estimator.

Overview of Rust (1987)

This is a path-breaking paper that introduces a methodology to estimate a single-agent dynamic discrete choice models.

Main contributions

1. Nested Fixed Point Algorithm (NFXP) (full solution MLE)
2. Formulation of assumptions, that makes dynamic discrete choice models tractable.
3. An illustrative application in a simple model of engine replacement.
4. The first researcher to obtain ML estimates of discrete choice dynamic programming models
5. Policy experiments: How does changes in replacement cost affect the distribution of mileage, the demand for engines.

General Behavioral Framework

The decision problem

- The decision maker chooses a sequence of actions to maximize expected discounted utility over a finite horizon

$$V_\theta\left(s_t\right) = \sup_{\Pi} E\left[\sum_{j=0}^{\infty} \beta^j U\left(s_{t+j}, d_{t+j}; \theta_1\right) | s_t, d_t\right]$$

where

- $\Pi = \left(f_t, f_{t+1}, .., \right), d_t = f_t\left(s_t, \theta\right) \in C\left(x_t\right) \subset \{1, 2, .., J\}$
- $\beta \in (0, 1)$ is the discount factor
- $U\left(s_t, d_t; \theta_1\right)$ is a choice and state specific utitty function
- E summarizes expectations of future states given s_t and d_t

Rust's Assumptions

Assumption (CI)

State variables, $s_t = (x_t, \varepsilon_t)$ obeys a (conditional independent) controlled Markov process with probability density

$$p(x_{t+1}, \varepsilon_{t+1} | x_t, \varepsilon_t, d, \theta_2, \theta_3) = q(\varepsilon_{t+1} | x_{t+1}, \theta_3) p(x_{t+1} | x_t, i, \theta_2)$$

Assumption (AS (additive separability))

$$U(s_t, d) = u(x_t, d; \theta_1) + \varepsilon_t(d)$$

Assumption (XV)

The unobserved state variables, ε_t are assumed to be multivariate iid. extreme value distributed

Object of interest: vector $\theta = (\beta, \theta_1, \theta_2, \theta_3)$

The Dynamic Programming Problem

- Under AS, the agent solves the following DP problem

$$V_\theta(x_t, \varepsilon_t) = \max_{d \in C(x_t)} [u(x_t, d, \theta_1) + \varepsilon_t(d) + \beta EV_\theta(x_{t+1}, \varepsilon_{t+1} | x_t, \varepsilon_t, d)]$$

- Under (CI) and (XV) we can integrate out the unobserved state variables, such that the unknown function, EV_θ, no longer depends on ε_t.

$$EV_\theta\left(x, d\right) = \Gamma_\theta(EV_\theta)\left(x, d\right)$$

$$= \int_y \ln \left[\sum_{d' \in D(y)} \exp\left[u(y, d'; \theta_1) + \beta EV_\theta\left(y, d'\right)\right] \right] p\left(dy | x, d, \theta_2\right)$$

Zurcher's Bus Engine Replacement Problem

- Choice set: Binary choice set, $C(x_t) = \{0, 1\}$. Each bus comes in for repair once a month and Zurcher chooses between ordinary maintenance ($d_t = 0$) and overhaul/engine replacement ($d_t = 1$).
- State variables: Harold Zurcher observes:
  - x_t: mileage at time t since last engine overhaul
  - $\varepsilon_t = [\varepsilon_t(d_t = 0), \varepsilon_t(d_t = 1)]$: other state variable
- Utility function:

$$u(x_t, d, \theta_1) + \varepsilon_t(d_t) = \begin{cases} -RC - c(0, \theta_1) + \varepsilon_t(1) & \text{if } d_t = 1 \\ -c(x_t, \theta_1) + \varepsilon_t(0) & \text{if } d_t = 0 \end{cases} \tag{1}$$

- State variables process x_t (mileage since last replacement)

$$p(x_{t+1}|x_t, d_t, \theta_2) = \begin{cases} g(x_{t+1} - 0, \theta_2) & \text{if } d_t = 1 \\ g(x_{t+1} - x_t, \theta_2) & \text{if } d_t = 0 \end{cases} \tag{2}$$

- If engine is replaced, state of bus regenerates to $x_t = 0$.

Likelihood Function

Likelihood

- Under assumption (CI) the likelihood function ℓ^f has the particular simple form

$$\ell^f\left(x_1, \dots x_T, d_1, \dots d_r | x_0, d_0, \theta\right) = \prod_{t=1}^{T} P\left(d_t | x_t, \theta\right) p\left(x_t | x_{t-1}, d_{t-1}, \theta_2\right)$$

where $P\left(d_t | x_t, \theta\right)$ is the choice probability given the observable state variable, x_t.

How to compute the choice probability, $P\left(d_t | x_t, \theta\right)$

- Need to solve dynamic program

How to estimate the transition probability, $p\left(x_t | x_{t-1}, d_{t-1}, \theta_2\right)$

- Can be estimated non-parametrically or by NLS

Conditional Choice Probabilities

- Under the extreme value assumption choice probabilities are multinomial logistic

$$P\left(d|x,\theta\right)=\frac{\exp\left\{u\left(x,d,\theta_1\right)+\beta EV_\theta\left(x,d\right)\right\}}{\sum_{j\in C(y)}\exp\left\{u\left(x,j,\theta_1\right)+\beta EV_\theta\left(x,j\right)\right\}}$$

- The expected value function is given by the unique fixed point to the contraction mapping Γ_θ, defined by

$$\begin{aligned} EV_\theta\left(x,d\right) &= \Gamma_\theta(EV_\theta)\left(x,d\right) \\ &= \int_y \ln\left[\sum_{d'\in D(y)}\exp\left[u(y,d';\theta_1)+\beta EV_\theta\left(y,d'\right)\right]\right] \\ &\quad p\left(dy|x,d,\theta_2\right) \end{aligned}$$

- Structural Estimation: Rust's *Nested Fixed Point Algorithm* (NFXP)

Structural Estimation: The Nested Fixed Point Algorithm

Since the contraction mapping Γ always has a unique fixed point, the constraint $EV = \Gamma_\theta(EV)$ implies that the fixed point EV_θ is an *implicit function* of θ.

Hence, NFXP solves the *unconstrained* optimization problem

$$\max_\theta L(\theta, EV_\theta)$$

Outer loop (Hill-climbing algorithm):

- Likelihood function $L(\theta, EV_\theta)$ is maximized w.r.t. θ
- Quasi-Newton algorithm: Usually BHHH, BFGS or a combination.
- Each evaluation of $L(\theta, EV_\theta)$ requires solution of EV_θ

Inner loop (fixed point algorithm):
The implicit function EV_θ defined by $EV_\theta = \Gamma(EV_\theta)$ is solved by:

- Successive Approximations (SA)
- Newton-Kantorovich (NK) Iterations

Solving Dynamic Programs

- DPs sometimes admit *closed form, analytical solutions*
- But it only takes slight twiddles to a problem formulation to destroy them.
- Fortunately, there is a large body of work on *numerical dynamic programming* that shows how to numerically approximate the solutions to dynamic programs. See Rust "Numerical Dynamic Programming in Economics" (1994) *Handbook of Computational Economics*
- In finite horizon problems we do *backward induction* using *numerical quadrature* (to approximate expectation operators), *function approximation* (to approximate value functions), and *numerical optimization* (to approximate the optimal decision rule).

The Bellman Operator is a Contraction Mapping

- In *stationary infinite horizon problems* we need to solve the *Bellman equation* by solving for V as a *fixed point of the Bellman operator* $V = \Gamma(V)$ where

$$\Gamma(V)(s) = \max_{a \in A(s)} \left[u(s,a) + \beta \int_{s'} V(s')p(ds'|s,a) \right]$$

- In *bounded* stationary, infinite horizon problems, the Bellman operator can be shown to be a *contraction mapping* under fairly general conditions

$$\|\Gamma(V) - \Gamma(W)\| \leq \beta \|V - W\|$$

where $\|V\| = \sup_{s \in S} |V(s)|$.

Successive approximations = value function iteration

- When Γ is a contraction mapping the standard method of *successive approximations* can be used to find V, the fixed point of Γ starting from any initial guess V_0

$$V_{t+1} = \Gamma(V_t) \quad t = 0, 1, 2 \ldots$$

- Thus, the method of successive approximations is *globally convergent* from any initial guess V_0.
- Often $V_0 = 0$ and successive approximations is equivalent to *approximating the solution to an infinite horizon problem by solving a finite horizon problem with a sufficiently large horizon T*

Policy Iteration = Newton's Method

- Successive approximations can be slow when β is close to 1, and in problems with short time intervals ∇t we have $\beta = \exp\{-r\nabla t\}$ where r is the discount rate expressed as an annual interest rate. So $\beta \to 1$ as $\nabla t \to 0$.
- A much more effective method, which usually converges in a *very small number of iterations,* is the method of *policy iteration* introduced by Howard (1960).
- Policy iteration can be shown to be equivalent to *Newton's method* and in finite state, finite action problems it is also globally convergent

$$V_{t+1} = V_t - [I - \Gamma'(V_t)]^{-1}[V_t - \Gamma(V_t)] \quad t = 0, 1, \ldots,$$

- Main cost of policy iteration is inverting the linear operator $[I - \Gamma'(V_t)]$ where $\Gamma'(V_t)$ is the *Fréchet* or *directional derivative* of the Bellman operator Γ

$$\Gamma'(V)(W) = \lim_{\delta \downarrow 0} \frac{\Gamma(V + \delta W) - \Gamma(V)}{\delta}$$

Data

- Harold Zurcher's Maintenance records of 162 busses
- Monthly observations of mileage on each bus (odometer reading)
- Data on maintenance operations
  1. Routine, periodic maintenance (e.g. brake adjustments)
  2. Replacement or repair at time of failure
  3. Major engine overhaul and/or replacement
- Rust focus on 3)

Replacement Data

TABLE IIa

SUMMARY OF REPLACEMENT DATA

(Subsample of buses for which at least 1 replacement occurred)

	Mileage at Replacement				Elapsed Time (Months)				
Bus Group	Max	Min	Mean	Standard Deviation	Max	Min	Mean	Standard Deviation	Number of Observations
1	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0
3	273,400	124,800	199,733	37,459	74	38	59.1	10.9	27
4	387,300	121,300	257,336	65,477	116	28	73.7	23.3	33
5	322,500	118,000	245,291	60,258	127	31	85.4	29.7	11
6	237,200	82,400	150,786	61,007	127	49	74.7	35.2	7
7	331,800	121,000	208,963	48,981	104	41	68.3	16.9	27
8	297,500	132,000	186,700	43,956	104	36	58.4	22.2	19
Full Sample	387,400	83,400	216,354	60,475	127	28	68.1	22.4	124

- No censoring problem (Completed spells)
- Selection Problem? (when focus is only on busses with replacement)

Censored Data

TABLE IIb
CENSORED DATA
(Subsample of buses for which no replacements occurred)

	Mileage at May 1, 1985				Elapsed Time (months)				
Bus Group	Max	Min	Mean	Standard Deviation	Max	Min	Mean	Standard Deviation	Number of Observations
1	120,151	65,643	100,117	12,929	25	25	25	0	15
2	161,748	142,009	151,183	8,530	49	49	49	0	4
3	280,802	199,626	250,766	21,325	75	75	75	0	21
4	352,450	310,910	337,222	17,802	118	117	117.8	0.45	5
5	326,843	326,843	326,843	0	130	130	130	0	1
6	299,040	232,395	265,264	33,332	130	128	129.3	1.15	3
7	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0
Full Sample	352,450	65,643	207,782	85,208	130	25	66.4	34.6	49

- Clearly a selection problem (mean elapsed age and mileage are larger for this sample)

Specification Search for cost function

TABLE VIII

SUMMARY OF SPECIFICATION SEARCH[a]

Cost Function	Bus Group 1, 2, 3	4	1, 2, 3, 4
Cubic	Model 1	Model 9	Model 17
$c(x, \theta_1) = \theta_{11}x + \theta_{12}x^2 + \theta_{13}x^3$	−131.063	−162.885	−296.515
	−131.177	−162.988	−296.411
quadratic	Model 2	Model 10	Model 18
$c(x, \theta_1) = \theta_{11}x + \theta_{12}x^2$	−131.326	−163.402	−297.939
	−131.534	−163.771	−299.328
linear	Model 3	Model 11	Model 19
$c(x, \theta_1) = \theta_{11}x$	−132.389	−163.584	−300.250
	−134.747	−165.458	−306.641
square root	Model 4	Model 12	Model 20
$c(x, \theta_1) = \theta_{11}\sqrt{x}$	−132.104	−163.395	−299.314
	−133.472	−164.143	−302.703
power	Model 5[b]	Model 13[b]	Model 21[b]
$c(x, \theta_1) = \theta_{11}x^{\theta_{12}}$	N.C.	N.C.	N.C.
	N.C.	N.C.	N.C.
hyperbolic	Model 6	Model 14	Model 22
$c(x, \theta_1) = \theta_{11}/(91 - x)$	−133.408	−165.423	−305.605
	−138.894	−174.023	−325.700
mixed	Model 7	Model 15	Model 23
$c(x, \theta_1) = \theta_{11}/(91 - x) + \theta_{12}\sqrt{x}$	−131.418	−163.375	−298.866
	−131.612	−164.048	−301.064
nonparametric	Model 8	Model 16	Model 24
$c(x, \theta_1)$ any function	−110.832	−138.556	−261.641
	−110.832	−138.556	−261.641

[a] First entry in each box is (partial) log likelihood value ℓ^2 in equation (5.2)) at $\beta = .9999$. Second entry is partial

Structural Estimates

TABLE IX
STRUCTURAL ESTIMATES FOR COST FUNCTION $c(x, \theta_1) = .001\theta_{11}x$
FIXED POINT DIMENSION = 90
(Standard errors in parentheses)

Parameter		Data Sample			Heterogeneity Test	
Discount Factor	Estimates/ Log-Likelihood	Groups 1, 2, 3 3864 Observations	Group 4 4292 Observations	Groups 1, 2, 3, 4 8156 Observations	LR Statistic ($df = 4$)	Marginal Significance Level
$\beta = .9999$	RC	11.7270 (2.602)	10.0750 (1.582)	9.7558 (1.227)	85.46	1.2E − 17
	θ_{11}	4.8259 (1.792)	2.2930 (0.639)	2.6275 (0.618)		
	θ_{30}	.3010 (.0074)	.3919 (.0075)	.3489 (.0052)		
	θ_{31}	.6884 (.0075)	.5953 (.0075)	.6394 (.0053)		
	LL	−2708.366	−3304.155	−6055.250		
$\beta = 0$	RC	8.2985 (1.0417)	7.6358 (0.7197)	7.3055 (0.5067)	89.73	1.5E − 18
	θ_{11}	109.9031 (26.163)	71.5133 (13.778)	70.2769 (10.750)		
	θ_{30}	.3010 (.0074)	.3919 (.0075)	.3488 (.0052)		
	θ_{31}	.6884 (.0075)	.5953 (.0075)	.6394 (.0053)		
	LL	−2710.746	−3306.028	−6061.641		
Myopia test:	LR Statistic ($df = 1$)	4.760	3.746	12.782		
$\beta = 0$ vs. $\beta = .9999$	Marginal Significance Level	0.0292	0.0529	0.0035		

Structural Estimates

TABLE X

STRUCTURAL ESTIMATES FOR COST FUNCTION $c(x, \theta_1) = .001\theta_{11}x$

FIXED POINT DIMENSION = 175

(Standard errors in parentheses)

Parameter		Data Sample			Heterogeneity Test	
Discount Factor	Estimates Log-Likelihood	Groups 1, 2, 3 3864 Observations	Group 4 4292 Observations	Groups 1, 2, 3, 4 8156 Observations	LR Statistic (df = 6)	Marginal Significance Level
$\beta = .9999$	RC	11.7257 (2.597)	10.896 (1.581)	9.7687 (1.226)	237.53	1.89E−48
	θ_{11}	2.4569 (.9122)	1.1732 (0.327)	1.3428 (0.315)		
	θ_{30}	.0937 (.0047)	.1191 (.0050)	.1071 (.0034)		
	θ_{31}	.4475 (.0080)	.5762 (.0075)	.5152 (.0055)		
	θ_{32}	.4459 (.0080)	.2868 (.0069)	.3621 (.0053)		
	θ_{33}	.0127 (.0018)	.0158 (.0019)	.0143 (.0013)		
	LL	−3993.991	−4495.135	−8607.889		
$\beta = 0$	RC	8.2969 (1.0477)	7.6423 (.7204)	7.3113 (0.5073)	241.78	2.34E−49
	θ_{11}	56.1656 (13.4205)	36.6692 (7.0675)	36.0175 (5.5145)		
	θ_{30}	.0937 (.0047)	.1191 (.0050)	.1070 (.0034)		
	θ_{31}	.4475 (.0080)	.5762 (.0075)	.5152 (.0055)		
	θ_{32}	.4459 (.0080)	.2868 (.0069)	.3622 (.0053)		
	θ_{33}	.0127 (.0018)	.0158 (.0019)	.0143 (.0143)		
	LL	−3996.353	−4496.997	−8614.238		
Myopia tests:	LR Statistic ($df = 1$)	4.724	3.724	12.698		
$\beta = 0$ vs. $\beta = .9999$	Marginal Significance Level	0.0297	0.0536	.00037		

Identification of discount factor

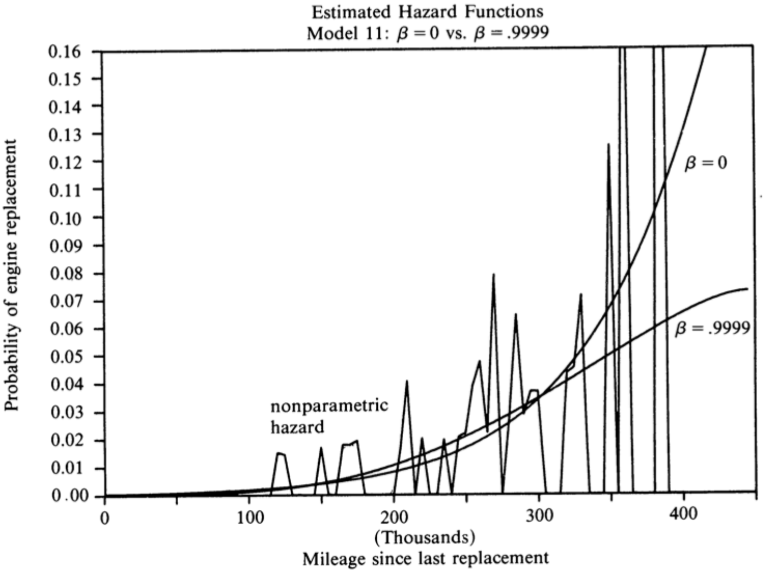
- Given our parametric specification of function cost function (e.g. linear in x), the discount factor is identified (although poorly)
- If we consider a nonparametric specification of $C(x, \theta_1) = \theta_{1,x}$ for $x = 1, .., 90$
    - *not* possible to identify β and the parameters of $C(x, \theta_1)$ separately
    - Identification of β is not robust to the functional form of $C(x, \theta_1)$.
- However, in other dynamic models it can be possible to identify β in a robust way
    - models where some observable variables do not affect current payoffs, but affect the prediction of future payoffs.
    - models with continuous choices (consumption/savings)
- Data on outcomes (costs or profits) can also help to identify β

Why a dynamic model?

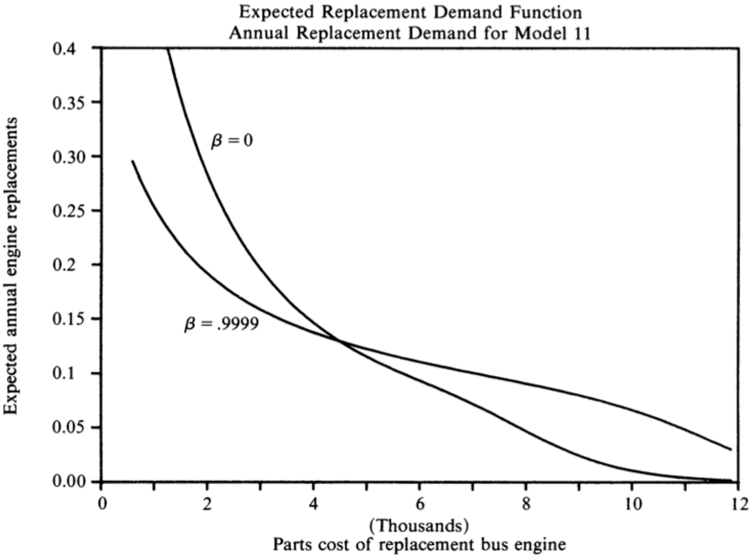
Suppose the "true"β is > 0, but we estimate the model with $\beta = 0$

- Our estimate of the replacement cost function will be biased.
- Parameters *RC* and θ_1 would be biased too
  (*RC* is upward biased and θ_1 is downward biased.)
- Predictions using the estimated model will be biased for two reasons:
  1. parameter estimates are biased
  2. the static model is not correct.
- Though the biases introduced by (1) and (2) might partly compensate each other, it will be a very unlikely coincidence that they compensate each other to make the bias negligible.
- Effect on equilibrium demand and hazard functions are very different!

Estimated Hazard Functions



Demand Function



Why not a reduced form for demand?

Reduced form

- Regress engine replacements on replacement costs

Problem: Lack of variation in replacement costs

- Data would be clustered around the intersection of the demand curves for $\beta = 0$ and $\beta = 0.9999$
  (both models predict that RC is around the actual RC of $4343)
- Demand also depends on how operating costs varies with mileage
- Need exogenous variation in RC
  .... that doesn't vary with operating costs
- Even if we had exogenous variation, this does not help us to understand the underlying economic incentives

Structural Approach

Attractive features

- structural parameters have a transparent interpretation
- evaluation of (new) policy proposals by counterfactual simulations.
- economic theories can be tested directly against each other.
- economic assumptions are more transparent and explicit (compared to statistical assumptions)

Less attractive features

- We impose more structure and make more assumptions
- Truly "structural" (policy invariant) parameters may not exist
- The curse of dimensionality
- The identification problem
- The problem of multiplicity and indeterminacy of equilibria
- Intellectually demanding and a huge amount of work