Randomized Algorithms for Max-SAT

Max-Cut

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Min-Cut is solvable in polynomial time, Max-Cut is NP-hard.

A Simple Randomized Algorithm

Dumb Rounding

1. For every node v flip a fair coin.
2. If heads, place v in A.
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Corollary

The Max-Cut always contains at least $\frac{|E|}{2}$ edges.

(We'll only prove the theorem.)

Analysis of Dumb Rounding

Theorem

Dumb rounding is a $\frac{1}{2}$-approximation.

Consider an arbitrary edge $e = (u, v)$. We have the following four cases:

- $u \in A, v \in A \Rightarrow e \notin \text{cut}$
- $u \in A, v \in B \Rightarrow e \in \text{cut}$
- $u \in B, v \in A \Rightarrow e \in \text{cut}$
- $u \in B, v \in B \Rightarrow e \notin \text{cut}$

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All cases occur with equal probability. Hence

$$\mathbb{E}[|(A \times B) \cap E|] \quad = \quad \sum_{e \in E} \mathbb{E}[e \in \text{cut}]$$

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$$\begin{aligned}\mathbb{E}[|(A \times B) \cap E|] &= \sum_{e \in E} \mathbb{E}[e \in \text{cut}] = \sum_{e \in E} \mathbb{P}[e \in \text{cut}] \\ &= \sum_{e \in E} \frac{1}{2} = \frac{|E|}{2}\end{aligned}$$

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All cases occur with equal probability. Hence

$$\begin{aligned}
\mathbb{E}[|(A \times B) \cap E|] &= \sum_{e \in E} \mathbb{E}[e \in \text{cut}] = \sum_{e \in E} \mathbb{P}[e \in \text{cut}] \\
&= \sum_{e \in E} \frac{1}{2} = \frac{|E|}{2} \geq \frac{OPT}{2}
\end{aligned}$$

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Remove the randomness from the algorithm.

Derandomization

We will use the method of conditional expectations. Let X be a function of Bernoulli random variables $X_1, \ldots X_n$. Then

$$\mathbb{E}[X] = \mathbb{E}[X|X_1 = 0] \cdot \mathbb{P}[X_1 = 0] + \mathbb{E}[X|X_1 = 1] \cdot \mathbb{P}[X_1 = 1].$$

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This implies $\max(\mathbb{E}[X|X_1 = 0], \mathbb{E}[X|X_1 = 1]) > \mathbb{E}[X]$.

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This implies $\max(\mathbb{E}[X|X_1 = 0], \mathbb{E}[X|X_1 = 1]) > \mathbb{E}[X]$.

Suppose without loss of generality that $\max(\mathbb{E}[X|X_1 = 0], \mathbb{E}[X|X_1 = 1]) = \mathbb{E}[X|X_1 = 0]$. By the same argument

$$\max(\mathbb{E}[X|X_1 = 0, X_2 = 0], \mathbb{E}[X|X_1 = 0, X_2 = 1]) > \mathbb{E}[X].$$

Evaluation of the Expectation

Denote by X_i the random coin toss of node v_i. Let $X_1, \ldots, X_{i-1}$ be the fixed coin tosses (i.e. the nodes that were already placed in A or B. Let $m_1 = |(A \times \{v_i\}) \cap E|$ and $m_2 = |(B \times \{v_i\}) \cap E|$ and $k = |(A \times B) \cap E|$ and $\ell = |((A \times A) \cup (B \times B)) \cap E|$.

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Then

$$\mathbb{E}[X | X_1, \ldots X_{i-1} \text{ fixed}, X_i = 0] = k + m_1 + \frac{|E| - k - m_1 - m_2 - \ell}{2}$$
$$\mathbb{E}[X | X_1, \ldots X_{i-1} \text{ fixed}, X_i = 1] = k + m_2 + \frac{|E| - k - m_1 - m_2 - \ell}{2}$$

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Greedy Max-Cut (via Derandomization)

1. $A \leftarrow \{v_1\}, B \leftarrow \emptyset$
2. For $i = 2$ to n

   place v_i into the set that maximizes the current cut size.

Maximum Satisfiability

Given a boolean formula in conjunctive normal form, satisfy as many clauses as possible.

- Formula $(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_2 \vee \overline{x_3} \vee x_4 \vee x_5) \wedge \overline{x_5} \wedge (w_3 \vee x_5) \dots$
- Clause $(x_1 \vee \overline{x_2} \vee \overline{x_3})$
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Max 2-SAT (every clause has exactly two literals) is a special case of the Max-Cut problem.

Integer Linear Program Formulation

- For each clause $(x_1 \vee x_2 \vee \overline{x_3})$ add a variable z
- For each literal x add a binary variable y.
- c is satisfied if and only if $y_1 + y_2 + (1 - y_3) \geq 1$.

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$$\begin{aligned} \text{maximize} \sum_{\text{clause } c} w_c \cdot z_c & \qquad \text{such that} \\ \sum_{x_i \in c} y_i + \sum_{\overline{x_i} \in c} (1 - y_i) \geq z_c & \qquad \text{for all clauses } c \\ y_i \in \{0, 1\} & \\ 0 \leq z_i \leq 1 & \end{aligned}$$

Algorithm

$$\text{maximize} \sum_{\text{clause } c} w_c \cdot z_c \qquad \text{such that}$$

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Randomized Rounding

1. Solve LP relaxation ($0 \leq y_i \leq 1$)
2. Set each x_i to 1 with probability y_i

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$$\text{(simplify notation...)} \sum_{x_i^* \in c} y_i^* \geq z_c \qquad \text{for all clauses } c$$

$$y_i \in \{0, 1\}$$

$$0 \leq z_i \leq 1$$

$$y_i^* = \begin{cases} y_i & \text{if } x_i \in c \\ 1 - y_i \text{if} & \overline{x_i} \in c \end{cases}$$

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A Useful Lemma

AM-GM inequality

Let $y_1, \ldots, y_n$ be a list of n nonnegative numbers. Then

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n}$$

Proof: By induction over n.

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Proof: By induction over n. For $n = 1$ the statement is trivial

Suppose the statement holds for any n non-negative numbers

If all $n + 1$ numbers are equal the two means are equal

If not all numbers are equal, at least one number is greater than the arithmetic mean and one number is smaller than the arithmetic mean. Without loss of generality assume that $x_{n+1} < a := \frac{x_1 + x_2 \ldots + x_{n+1}}{n+1} < x_1$

Proof continued

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Let $y_1, \ldots, y_n$ be a list of n nonnegative numbers. Then

$$a = \frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n}$$

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$\Rightarrow 0 < (a - x_{n+1})(x_1 - a)$

Proof continued

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$(n+1) \cdot a = x_1 + x_2 + \ldots + x_{n+1} \Leftrightarrow n \cdot a = x_2 + \ldots x_1 + x_{n+1} - a$

Proof continued

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$(n+1) \cdot a = x_1 + x_2 + \ldots + x_{n+1} \Leftrightarrow n \cdot a = x_2 + \ldots x_1 + x_{n+1} - a$, i.e. a is also the arithmetic mean of the numbers $\{x_2, \ldots x_n, x_1 + x_{n+1} - a\}$

Proof continued

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$(x_1 + x_{n+1} - a)a - x_1 x_{n+1} = (x_1 - a)(a - x_{n+1}) > 0$

Proof continued

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$(x_1 + x_{n+1} - a)a - x_1 x_{n+1} = (x_1 - a)(a - x_{n+1}) > 0$ which implies $(x_1 + x_{n+1} - a)a > x_1 x_{n+1}$

We apply the inductive hypothesis on $\{x_2, \ldots x_n, x_1 + x_{n+1} - a\}$
$a^{n+1} \geq x_2 \cdot x_3 \ldots \cdot x_n \cdot (x_1 + x_{n+1} - a)a$

Proof continued

AM-GM inequality

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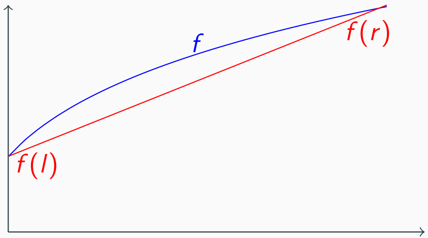
Concave Functions

Definition

A function f is concave on $[l, r]$ if for any $0 \leq \alpha \leq 1$

$$f((1-\alpha) \cdot l + \alpha \cdot r) \geq (1-\alpha) \cdot f(l) + \alpha \cdot f(r)$$



We will use the special case $l = 0$, $r = 1$ and $f(0) = 0$:

$$f(\alpha \cdot r) \geq \alpha \cdot f(1)$$

Analysis of Randomized Rounding

Consider any clause c with k literals. The associated constraint is $\sum_{x_i \in c} y_i + \sum_{\overline{x_i} \in c}(1 - y_i) = \sum_{x_i^* \in c} y_i^* \geq z_c$.

$$\mathbb{P}[c \text{ is satisfied}] \quad = \quad 1 - \prod_{i=1}^{k}(1 - y_i^*)$$

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$$
\begin{aligned}
\mathbb{P}[c \text{ is satisfied}] \quad &= \quad 1 - \prod_{i=1}^{k} (1 - y_i^*) \\
\text{AM-GM-inequality} \quad &\geq \quad 1 - \left(\frac{k - \sum_{i=1}^{k} y_i^*}{k} \right)^k
\end{aligned}
$$

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$$\begin{aligned}
\mathbb{P}[c \text{ is satisfied}] &= 1 - \prod_{i=1}^{k}(1 - y_i^*) \\
\text{AM-GM-inequality} &\geq 1 - \left(\frac{k - \sum_{i=1}^{k} y_i^*}{k}\right)^k \\
\text{LP-constraint} &\geq 1 - \left(1 - \frac{z_c}{k}\right)^k
\end{aligned}$$

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\text{LP-constraint} \quad &\geq \quad 1 - \left(1 - \frac{z_c}{k}\right)^k \\
\text{concave function } (\alpha \equiv z_c) \quad &\geq \quad \left(1 - \left(1 - \frac{1}{k}\right)^k\right) z_c
\end{aligned}
$$

Analysis continued

For any clause, we have $\mathbb{P}[c \text{ is satisfied}] \geq \left(1 - \left(1 - \frac{1}{k}\right)^k\right) z_c$.

$$\mathbb{E}[\sum_{\text{clause } c} w_c \cdot z_c] \quad = \quad \sum_{\text{clause} c} w_c \cdot \mathbb{P}[c \text{ is satisfied}]$$

Analysis continued

For any clause, we have $\mathbb{P}[c \text{ is satisfied}] \geq \left(1 - \left(1 - \frac{1}{k}\right)^k\right) z_c$.

$$\begin{aligned}
\mathbb{E}[\sum_{\text{clause } c} w_c \cdot z_c] &= \sum_{\text{clause} c} w_c \cdot \mathbb{P}[c \text{ is satisfied}] \\
&\geq \min_k \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \sum_{\text{clause } c} w_c z_c
\end{aligned}$$

Analysis continued

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&\geq \min_k \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \sum_{\text{clause } c} w_c z_c \\
&\geq \min_k \left(1 - \left(1 - \frac{1}{k}\right)^k\right) OPT
\end{aligned}
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Analysis continued

For any clause, we have $\mathbb{P}[c \text{ is satisfied}] \geq \left(1 - \left(1 - \frac{1}{k}\right)^k\right) z_c$.

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&\geq \min_k \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \sum_{\text{clause } c} w_c z_c \\
&\geq \min_k \left(1 - \left(1 - \frac{1}{k}\right)^k\right) OPT \\
&\geq \left(1 - \frac{1}{e}\right) OPT
\end{aligned}$$

Further Improvements

Dumb Rounding

1. Set literals to **true** with probability 1/2

For a clause with k literals, the probability of being satisfied is
$\mathbb{P}[c \text{ is satisfied}] \geq \left(1 - \left(\frac{1}{2}\right)^k\right)$.

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Dumb Rounding

1. Set literals to **true** with probability $1/2$

For a clause with k literals, the probability of being satisfied is $\mathbb{P}[c \text{ is satisfied}] \geq \left(1 - \left(\frac{1}{2}\right)^k\right)$.

Combined Rounding

1. Run Dumb Rounding $\rightarrow W_1$
2. Run Randomized Rounding $\rightarrow W_2$
3. Output the better of the two

Analysis

$$\mathbb{E}[\max(W_1, W_2)] \quad \geq \quad \mathbb{E}\left[\frac{1}{2}W_1 + \frac{1}{2}W_2\right]$$

Analysis

$$
\begin{aligned}
\mathbb{E}[\max(W_1, W_2)] &\geq \mathbb{E}\left[\frac{1}{2}W_1 + \frac{1}{2}W_2\right] \\
\text{Linearity} &\geq \sum_{\text{clause } c} w_c \cdot \left(\frac{1}{2}\mathbb{P}[c \text{ is satisfied by Dumb}]\right. \\
&\qquad \left. + \frac{1}{2}\mathbb{P}[c \text{ is satisfied by Random}]\right)
\end{aligned}
$$

Analysis

$$
\begin{aligned}
\mathbb{E}[\max(W_1, W_2)] &\geq \mathbb{E}\left[\frac{1}{2}W_1 + \frac{1}{2}W_2\right] \\
\text{Linearity} &\geq \sum_{\text{clause } c} w_c \cdot \left(\frac{1}{2}\mathbb{P}[c \text{ is satisfied by Dumb}]\right. \\
&\qquad \left. + \frac{1}{2}\mathbb{P}[c \text{ is satisfied by Random}]\right) \\
&\geq \sum_{\text{clause } c} w_c \cdot \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^k\right) + \frac{1}{2}\left(1 - \left(1 - \frac{1}{k}\right)^k\right) z_c
\end{aligned}
$$

Analysis

$$
\begin{aligned}
\mathbb{E}[\max(W_1, W_2)] &\geq \mathbb{E}\left[\frac{1}{2}W_1 + \frac{1}{2}W_2\right] \\
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\text{boring calculations} &\geq \sum_{\text{clause } c} w_c \cdot \frac{3}{4} z_j
\end{aligned}
$$

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$$
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&\geq \sum_{\text{clause } c} w_c \cdot \frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^k\right) + \frac{1}{2}\left(1 - \left(1 - \frac{1}{k}\right)^k\right) z_c \\
\text{boring calculations} &\geq \sum_{\text{clause } c} w_c \cdot \frac{3}{4} z_j \\
&\geq \frac{3}{4} \sum_{\text{clause } c} w_c z_j \\
&\geq \frac{3}{4} OPT
\end{aligned}
$$

Boring Calculations

Concavity of $1-\left(1-\frac{x}{k}\right)^{k}$ in $[0,1]$:
First Derivative: $\left(1-\frac{x}{k}\right)^{k-1}$
Second Derivative: $-\frac{k-1}{k}\left(1-\frac{x}{k}\right)^{k-2} \leq 0$ (for $k \geq 1$)

For $k=1$: $\frac{1}{2}(1-\frac{1}{2})+\frac{1}{2}z_c=\frac{1}{4}+\frac{1}{2}z_c \geq \frac{3}{4}z_c$

For $k=2$: $\frac{1}{2}(1-\frac{1}{4})+\frac{1}{2}\left(1-\left(1-\frac{1}{2}\right)^{2}\right)z_c \geq \frac{3}{8}+\frac{3}{8}z_c \geq \frac{3}{4}z_c$

For $k \geq 3$: $\frac{1}{2}(1-\frac{1}{8})+\frac{1}{2}\left(1-\frac{1}{e}\right)z_c \geq \frac{7}{16}+\frac{1}{2}(1-1/e)z_c \geq 0.753z_c$