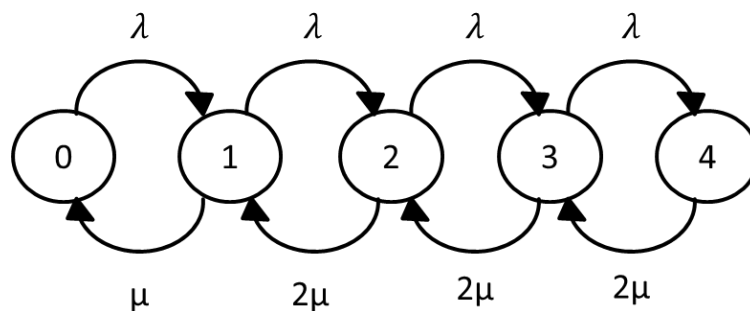


Exam

July 7, 2005

Exercise N. 2

Obtain the markovian process of a queue M/M/2/4 (2 servers, max 4 users in the system). Then calculate the probability, in parametric form, that a user at the request time is rejected from the queue, knowing the arrival rate ($\lambda = 0.5$) and the service rate ($\mu = 0.66$), the throughput and the utilization factor.



Flow-in = Flow-out :

$$\left\{ \begin{array}{l} p_0 \lambda = p_1 \cdot \mu \\ p_1 \lambda = p_2 \cdot 2\mu \\ p_2 \lambda = p_3 \cdot 2\mu \\ p_3 \lambda = p_4 \cdot 2\mu \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} p_1 = p_0 \left(\frac{\lambda}{\mu} \right) \\ p_2 = p_0 \left(\frac{\lambda}{\mu} \right)^2 \cdot \frac{1}{2} \\ p_3 = p_0 \left(\frac{\lambda}{\mu} \right)^3 \cdot \frac{1}{2} \cdot \frac{1}{2} \\ p_4 = p_0 \left(\frac{\lambda}{\mu} \right)^4 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \end{array} \right.$$

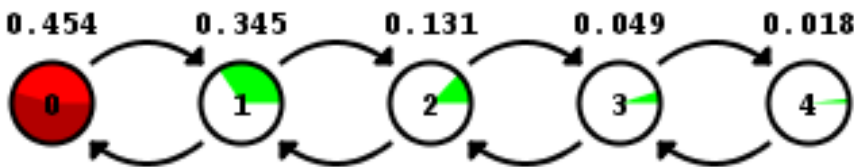
$$p_k = \begin{cases} p_0 \left(\frac{\lambda}{\mu} \right)^k \frac{1}{k!}, & k \leq 2 \\ p_0 \left(\frac{\lambda}{\mu} \right)^k \frac{1}{2!2^{k-2}}, & k > 2 \end{cases}$$

$$\begin{cases} p_1 = p_0 \left(\frac{0.5}{0.66} \right) \\ p_2 = p_0 \left(\frac{0.5}{0.66} \right)^2 \cdot \frac{1}{2} \\ p_3 = p_0 \left(\frac{0.5}{0.66} \right)^3 \cdot \frac{1}{4} \\ p_4 = p_0 \left(\frac{0.5}{0.66} \right)^4 \cdot \frac{1}{8} \end{cases} \Rightarrow \begin{cases} p_1 = p_0 \cdot 0.757 \\ p_2 = p_0 \cdot 0.2869 \\ p_3 = p_0 \cdot 0.108 \\ p_4 = p_0 \cdot 0.0410 \end{cases}$$

$$\sum_{k=0}^4 p_k = 1$$

$$\sum_{k=0}^4 p_k = 1 = \sum_{k=0}^2 p_0 \left(\frac{\lambda}{\mu} \right)^k \frac{1}{k!} + \sum_{k=3}^4 p_0 \left(\frac{\lambda}{\mu} \right)^k \frac{1}{2!2^{k-2}}$$

$$p_0 \left[\sum_{k=0}^2 \left(\frac{\lambda}{\mu} \right)^k \frac{1}{k!} + \sum_{k=3}^4 \left(\frac{\lambda}{\mu} \right)^k \frac{1}{2!2^{k-2}} \right] = 1$$



$$p_0 = \left[(0.7575)^0 \frac{1}{0!} + (0.7575)^1 \frac{1}{1!} + (0.7575)^2 \frac{1}{2!} + (0.7575)^3 \frac{1}{2!2^{3-2}} + (0.7575)^4 \frac{1}{2!2^{4-2}} \right]^{-1} = 0.454$$

$$p_1 = p_0 \cdot 0.7575 = 0.344$$

$$p_2 = p_0 \cdot 0.2869 = 0.131$$

$$p_3 = p_0 \cdot 0.108 = 0.049$$

$$p_4 = p_0 \cdot 0.0410 = 0.018$$



*Probability that a user at request
time is rejected from the queue.*

$$X = p_1 \cdot \mu + \sum_{i=2}^4 p_i \cdot 2\mu = 0,23 + 0,172 + 0,064 + 0,023 = 0,49 \text{ users/sec}$$

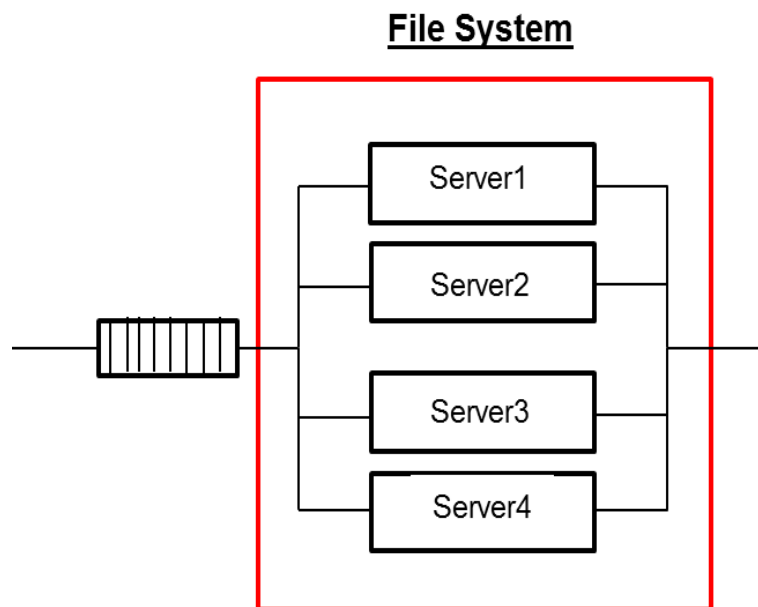
$$U = \frac{\lambda}{\mu} = 0,75$$

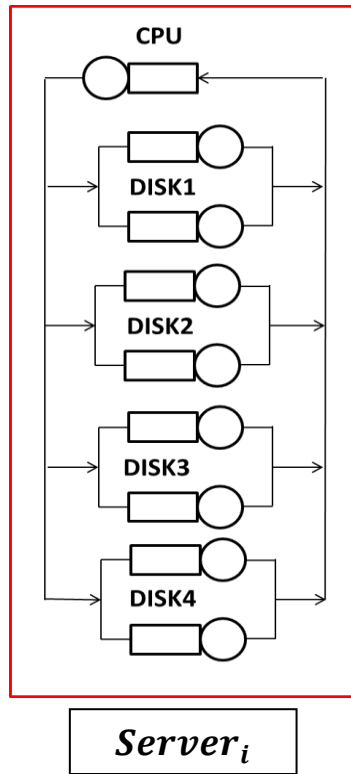
Exercise N. 3

Calculate performability of a read-only file system consists of 4 servers (CPU + memory + RAID1 consists of 8 disks (4+4)). Assume that all servers have the same data. Overall at most 6 users at same time can access to the servers and if all available servers are busy serving a request, then subsequent requests are queued. The average rate of requests, when a user is in the phase of think time, is equal to $1/10 \text{ sec}^{-1}$ and service's rate is equal to $1/5 \text{ sec}^{-1}$, assuming that the server is operating according to the design specifications (i.e. is capable to process and to provide the informations on the system disk).

Also assume that:

- The disks faults happen with a rate equal to $1/(500 \text{ hours})$ and will be repaired with a rate equal to $1/(50 \text{ hours})$;
- The set CPU + memory faults happen with a rate equal to $1/(1000 \text{ hours})$ and will be repaired with a rate equal to $1/(10 \text{ hours})$.



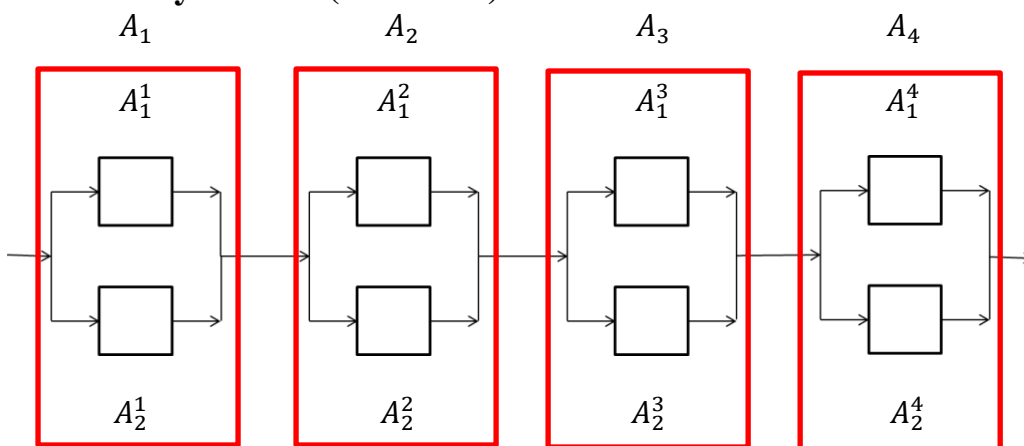


$$A_{SERVER} = A_{CPU} \cdot A_{RAID1}$$

$$A_{CPU} = \frac{MTTF_{CPU}}{MTTF_{CPU} + MTTR_{CPU}} = \frac{1000}{1000 + 10} = 0,99$$

$$A_{DISK} = \frac{MTTF_{DISK}}{MTTF_{DISK} + MTTR_{DISK}} = \frac{500}{500 + 50} = 0,9$$

Availability RAID1 (4+4 disks):



$$A_{RAID1} = A_1 \cdot A_2 \cdot A_3 \cdot A_4$$

$$A_1 = 1 - (1 - A_1^1) \cdot (1 - A_2^1) = 1 - (1 - 0,9) \cdot (1 - 0,9) = 0,99$$

$$A_2 = 1 - (1 - A_1^2) \cdot (1 - A_2^2) = 1 - (1 - 0,9) \cdot (1 - 0,9) = 0,99$$

$$A_3 = 1 - (1 - A_1^3) \cdot (1 - A_2^3) = 1 - (1 - 0,9) \cdot (1 - 0,9) = 0,99$$

$$A_4 = 1 - (1 - A_1^4) \cdot (1 - A_2^4) = 1 - (1 - 0,9) \cdot (1 - 0,9) = 0,99$$

$$A_{RAID1} = 0,99 \cdot 0,99 \cdot 0,99 \cdot 0,99 = 0,96$$

$$A_{SERVER} = A_{CPU} \cdot A_{RAID1} = 0,99 \cdot 0,96 = 0,95$$

To find the availability of the whole system:

$$q_4 = \text{prob}\{0 \text{ server funzionanti}\} = (1 - A_{SERVER})^4 = (1 - 0,95)^4 = 0,00000625$$

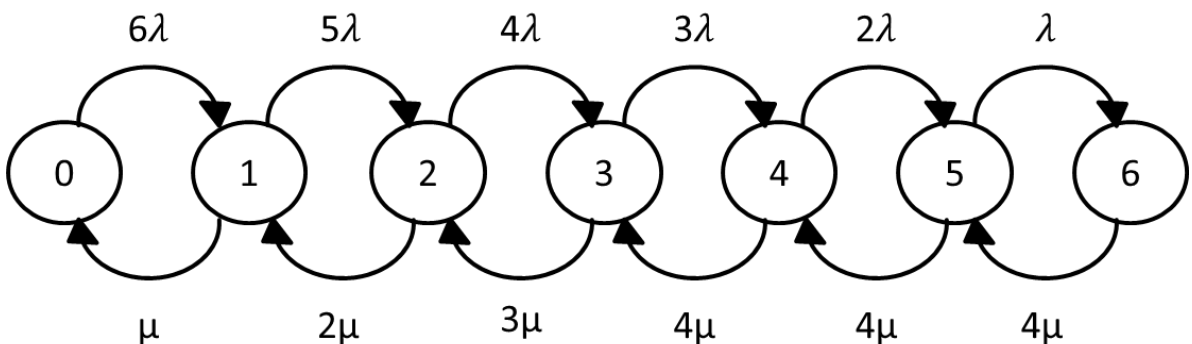
$$q_3 = \text{prob}\{1 \text{ server funzionanti}\} = \binom{4}{3} \cdot A_{SERVER} \cdot (1 - A_{SERVER})^3 = \binom{4}{3} \cdot 0,95 \cdot (1 - 0,95)^3 = 0,000475$$

$$q_2 = \text{prob}\{2 \text{ server funzionanti}\} = \binom{4}{2} \cdot (A_{SERVER})^2 \cdot (1 - A_{SERVER})^2 = \binom{4}{2} \cdot (0,95)^2 \cdot (1 - 0,95)^2 = 6 \cdot 0,90 \cdot 0,0025 = 0,01$$

$$q_1 = \text{prob}\{3 \text{ server funzionanti}\} = 4 \cdot (A_{SERVER})^3 \cdot (1 - A_{SERVER}) = 4 \cdot 0,85 \cdot 0,05 = 0,17$$

$$q_0 = \text{prob}\{4 \text{ server funzionanti}\} = (A_{SERVER})^4 = 0,81$$

Considering that each state represent the number of user in the system, we have:



Flow-in = Flow-out :

$$\left\{ \begin{array}{l} p_0 \cdot 6\lambda = p_1 \cdot \mu \\ p_1 \cdot 5\lambda = p_2 \cdot 2\mu \\ p_2 \cdot 4\lambda = p_3 \cdot 3\mu \\ p_3 \cdot 3\lambda = p_4 \cdot 4\mu \\ p_4 \cdot 2\lambda = p_5 \cdot 4\mu \\ p_5 \cdot \lambda = p_6 \cdot 4\mu \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} p_1 = p_0 \left(\frac{\lambda}{\mu} \right) \cdot 6 \\ p_2 = p_0 \left(\frac{\lambda}{\mu} \right)^2 \cdot \frac{6 \cdot 5}{2} \\ p_3 = p_0 \left(\frac{\lambda}{\mu} \right)^3 \cdot \frac{6 \cdot 5 \cdot 4}{2 \cdot 3} \\ p_4 = p_0 \left(\frac{\lambda}{\mu} \right)^4 \cdot \frac{6 \cdot 5 \cdot 4 \cdot 3}{2 \cdot 3 \cdot 4} \\ p_5 = p_0 \left(\frac{\lambda}{\mu} \right)^5 \cdot \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 3 \cdot 4 \cdot 4} \\ p_6 = p_0 \left(\frac{\lambda}{\mu} \right)^6 \cdot \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 3 \cdot 4 \cdot 4 \cdot 4} \end{array} \right.$$

$$p_j = \left\{ \begin{array}{l} p_0 \left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! j!}, \quad j \leq 4 \\ p_0 \left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! 4! 4^{j-4}}, \quad j > 4 \end{array} \right.$$

Consider k = number of working server, we have:

$$p_j(k) = \left\{ \begin{array}{l} p_0 \left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! j!}, \quad j \leq k \\ p_0 \left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! k! k^{j-k}}, \quad j > k \end{array} \right.$$

$$\sum_{j=0}^6 p_j = 1$$

$$p_0 \left[\sum_{j=0}^k \left(\frac{\lambda}{\mu} \right)^j \binom{6}{j} + \sum_{j=k+1}^6 \left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! k! k^{j-k}} \right] = 1$$

$$p_0 = \left[\sum_{j=0}^k \left(\frac{\lambda}{\mu} \right)^j \binom{6}{j} + \sum_{j=k+1}^6 \left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! k! k^{j-k}} \right]^{-1}$$

If k = 4 (4 working servers):

$$p_0 = \left[\sum_{j=0}^4 \left(\frac{\lambda}{\mu} \right)^j \binom{6}{j} + \sum_{j=4+1}^6 \left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! 4! 4^{j-4}} \right]^{-1} = [1 + 0,5 \cdot 6 + 0,25 \cdot 15 + 0,125 \cdot 20 + 0,0625 \cdot 15 + 0,031 \cdot 7,5 + 0,0156 \cdot 1,875]^{-1} = 0,087$$

If k = 3 (3 working servers):

$$p_0 = \left[\sum_{j=0}^3 \left(\frac{\lambda}{\mu} \right)^j \binom{6}{j} + \sum_{j=3+1}^6 \left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! 3! 3^{j-3}} \right]^{-1} = [1 + 0,5 \cdot 6 + 0,25 \cdot 15 + 0,125 \cdot 20 + 0,0625 \cdot 20 + 0,031 \cdot 13,33 + 0,0156 \cdot 4,44]^{-1} = 0,083$$

If k = 2 (2 working servers):

$$p_0 = \left[\sum_{j=0}^2 \left(\frac{\lambda}{\mu} \right)^j \binom{6}{j} + \sum_{j=2+1}^6 \left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! 2! 2^{j-2}} \right]^{-1} = [1 + 0,5 \cdot 6 + 0,25 \cdot 15 + 0,125 \cdot 30 + 0,0625 \cdot 45 + 0,031 \cdot 45 + 0,0156 \cdot 22,5]^{-1} = 0,062$$

If k = 1 (1 working servers):

$$p_0 = \left[\sum_{j=0}^1 \left(\frac{\lambda}{\mu} \right)^j \binom{6}{j} + \sum_{j=1+1}^6 \left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! 1! 1^{j-1}} \right]^{-1} = [1 + 0,5 \cdot 6 + 0,25 \cdot 30 + 0,125 \cdot 120 + 0,0625 \cdot 360 + 0,031 \cdot 720 + 0,0156 \cdot 720]^{-1} = 0,012$$

Throughput of the web server and its response time when in the file system there are n faulty server:

$$X(n) = \sum_{j=1}^6 p_j (4 - n) X_0(j)$$

$$\begin{aligned} X(0) &= \sum_{j=1}^6 p_j (4) X_0(j) = \sum_{j=1}^4 0,087 \cdot \left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! j!} X_0(j) + \sum_{j=5}^6 0,087 \cdot \\ &\left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! 4! 4^{j-4}} X_0(j) = 0,087 \cdot 0,5 \cdot 6 \cdot 0,2 + 0,087 \cdot 0,25 \cdot 15 \cdot 0,4 + 0,087 \cdot \\ &0,125 \cdot 20 \cdot 0,6 + 0,087 \cdot 0,0625 \cdot 15 \cdot 0,8 + 0,087 \cdot 0,031 \cdot 7,5 \cdot 0,8 + 0,087 \cdot \\ &0,01 \cdot 1,875 \cdot 0,8 = 0,05 + 0,13 + 0,13 + 0,06 + 0,016 + 0,0013 = 0,3873 \end{aligned}$$

$$\begin{aligned} X(1) &= \sum_{j=1}^6 p_j (3) X_0(j) = \sum_{j=1}^3 0,083 \cdot \left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! j!} X_0(j) + \sum_{j=4}^6 0,083 \cdot \\ &\left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! 3! 3^{j-3}} X_0(j) = 0,083 \cdot 0,5 \cdot 6 \cdot 0,2 + 0,083 \cdot 0,25 \cdot 15 \cdot 0,4 + 0,083 \cdot \\ &0,125 \cdot 20 \cdot 0,6 + 0,083 \cdot 0,0625 \cdot 20 \cdot 0,6 + 0,083 \cdot 0,031 \cdot 13,33 \cdot 0,6 + 0,083 \cdot \\ &0,01 \cdot 4,44 \cdot 0,6 = 0,05 + 0,12 + 0,12 + 0,062 + 0,020 + 0,002 = 0,3743 \end{aligned}$$

$$\begin{aligned} X(2) &= \sum_{j=1}^6 p_j (2) X_0(j) = \sum_{j=1}^2 0,062 \cdot \left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! j!} X_0(j) + \sum_{j=3}^6 0,062 \cdot \\ &\left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! 2! 2^{j-2}} X_0(j) = 0,062 \cdot 0,5 \cdot 6 \cdot 0,2 + 0,062 \cdot 0,25 \cdot 15 \cdot 0,4 + 0,062 \cdot \\ &0,125 \cdot 30 \cdot 0,4 + 0,062 \cdot 0,0625 \cdot 45 \cdot 0,4 + 0,062 \cdot 0,031 \cdot 45 \cdot 0,4 + 0,062 \cdot \\ &0,01 \cdot 22,5 \cdot 0,4 = 0,037 + 0,093 + 0,093 + 0,069 + 0,034 + 0,0055 = 0,3315 \end{aligned}$$

$$\begin{aligned} X(3) &= \sum_{j=1}^6 p_j (1) X_0(j) = 0,012 \cdot \left(\frac{\lambda}{\mu} \right)^1 \frac{6!}{(6-1)! 1!} X_0(1) + \sum_{j=2}^6 0,012 \cdot \\ &\left(\frac{\lambda}{\mu} \right)^j \frac{6!}{(6-j)! 1! 1^{j-1}} X_0(j) = 0,012 \cdot 0,5 \cdot 6 \cdot 0,2 + 0,012 \cdot 0,25 \cdot 30 \cdot 0,2 + 0,012 \cdot \\ &0,125 \cdot 120 \cdot 0,2 + 0,012 \cdot 0,0625 \cdot 360 \cdot 0,2 + 0,012 \cdot 0,031 \cdot 720 \cdot 0,2 + 0,012 \cdot \\ &0,01 \cdot 720 \cdot 0,2 = 0,0072 + 0,018 + 0,036 + 0,054 + 0,053 + 0,017 = 0,185 \end{aligned}$$

$$N(n) = \sum_{j=1}^6 p_j(4-n) \cdot j$$

$$N(0) = \sum_{j=1}^6 p_j(4) \cdot j = \sum_{j=1}^4 p_0 \cdot \left(\frac{\lambda}{\mu}\right)^j \cdot \frac{6!}{(6-j)! j!} \cdot j + \sum_{j=5}^6 p_0 \cdot \left(\frac{\lambda}{\mu}\right)^j \cdot \frac{6!}{(6-j)! j!} \cdot j$$

$$= 0,087 \cdot 0,5 \cdot 6 \cdot 1 + 0,087 \cdot 0,25 \cdot 15 \cdot 2 + 0,087 \cdot 0,125 \cdot 20 \cdot 3 + 0,087 \cdot 0,0625 \cdot 15 \cdot 4 + 0,087 \cdot 0,031 \cdot 7,5 \cdot 5 + 0,087 \cdot 0,01 \cdot 1,875 \cdot 6 = 0,261 + 0,65 + 0,65 + 0,326 + 0,1 + 0,0097 = 1,996$$

$$N(1) = \sum_{j=1}^6 p_j(3) \cdot j = \sum_{j=1}^3 p_0 \cdot \left(\frac{\lambda}{\mu}\right)^j \cdot \frac{6!}{(6-j)! j!} \cdot j + \sum_{j=4}^6 p_0 \cdot \left(\frac{\lambda}{\mu}\right)^j \cdot \frac{6!}{(6-j)! j!} \cdot j$$

$$= 0,083 \cdot 0,5 \cdot 6 \cdot 1 + 0,083 \cdot 0,25 \cdot 15 \cdot 2 + 0,083 \cdot 0,125 \cdot 20 \cdot 3 + 0,083 \cdot 0,0625 \cdot 20 \cdot 4 + 0,083 \cdot 0,031 \cdot 13,33 \cdot 5 + 0,083 \cdot 0,01 \cdot 4,4 \cdot 6 = 0,249 + 0,622 + 0,622 + 0,415 + 0,171 + 0,021 = 2,1$$

$$N(2) = \sum_{j=1}^6 p_j(2) \cdot j = \sum_{j=1}^2 p_0 \cdot \left(\frac{\lambda}{\mu}\right)^j \cdot \frac{6!}{(6-j)! j!} \cdot j + \sum_{j=3}^6 p_0 \cdot \left(\frac{\lambda}{\mu}\right)^j \cdot \frac{6!}{(6-j)! j!} \cdot j$$

$$= 0,062 \cdot 0,5 \cdot 6 \cdot 1 + 0,062 \cdot 0,25 \cdot 15 \cdot 2 + 0,062 \cdot 0,125 \cdot 30 \cdot 3 + 0,062 \cdot 0,0625 \cdot 45 \cdot 4 + 0,062 \cdot 0,031 \cdot 45 \cdot 5 + 0,062 \cdot 0,01 \cdot 22,5 \cdot 6 = 0,186 + 0,465 + 0,697 + 0,697 + 0,432 + 0,0837 = 2,56$$

$$N(3) = \sum_{j=1}^6 p_j(1) \cdot j = p_0 \cdot \left(\frac{\lambda}{\mu}\right)^1 \cdot \frac{6!}{(6-1)! 1!} \cdot 1 + \sum_{j=2}^6 p_0 \cdot \left(\frac{\lambda}{\mu}\right)^j \cdot \frac{6!}{(6-j)! 1! 1^{j-1}} \cdot j$$

$$= 0,012 \cdot 0,5 \cdot 6 \cdot 1 + 0,012 \cdot 0,25 \cdot 30 \cdot 2 + 0,012 \cdot 0,125 \cdot 120 \cdot 3 + 0,012 \cdot 0,0625 \cdot 360 \cdot 4 + 0,012 \cdot 0,031 \cdot 720 \cdot 5 + 0,012 \cdot 0,01 \cdot 720 \cdot 6 = 0,036 + 0,18 + 0,54 + 1,08 + 1,33 + 0,51 = 3,68$$

$$R(n) = \frac{N(n)}{X(n)}$$

$$R(0) = \frac{N(0)}{X(0)} = \frac{1,996}{0,3873} = 5,15$$

$$R(1) = \frac{N(1)}{X(1)} = \frac{2,1}{0,3743} = 5,61$$

$$R(2) = \frac{N(2)}{X(2)} = \frac{2,56}{0,3315} = 7,72$$

$$R(3) = \frac{N(3)}{X(3)} = \frac{3,68}{0,185} = 19,89$$

The throughput and the response time of the whole file system:

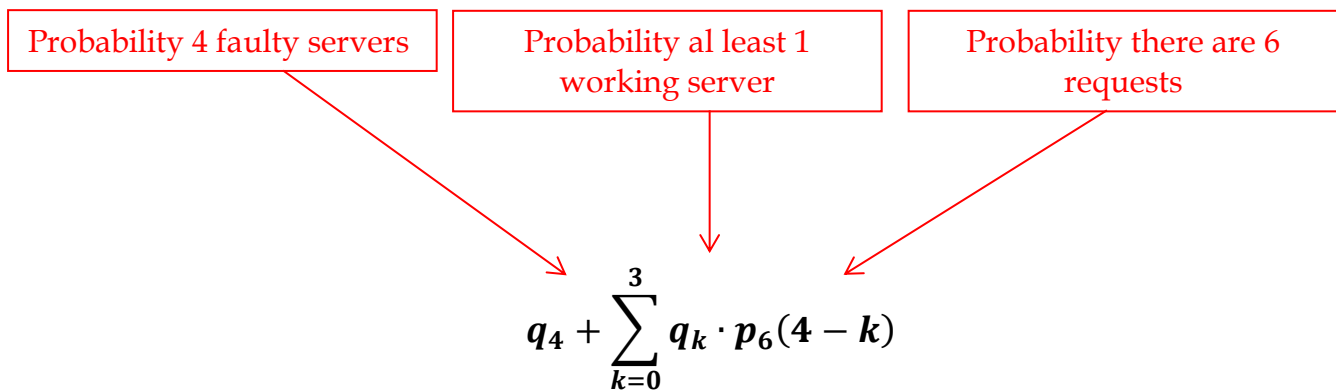
$$X_{tot} = \sum_{n=0}^3 q_n X(n) (4 - n)$$

$$X_{tot} = \sum_{n=0}^3 q_n X(n) (4 - n) = q_0 \cdot X(0) \cdot (4) + q_1 \cdot X(1) \cdot (3) + q_2 \cdot X(2) \cdot (2) + q_3 \cdot X(3) \cdot (1) = 0,81 \cdot 0,3873 \cdot 4 + 0,17 \cdot 0,3743 \cdot 3 + 0,01 \cdot 0,3315 \cdot 2 + 0,000475 \cdot 0,185 \cdot 1 = 1,254 + 0,190 + 0,0066 + 0,00008 = 1,45$$

$$R_{tot} = \frac{1}{1-q_4} \sum_{n=0}^3 q_n R(n)$$

$$R_{tot} = \frac{1}{1-q_4} \sum_{n=0}^3 q_n R(n) = \frac{1}{1-q_4} \cdot q_0 \cdot R(0) + \frac{1}{1-q_4} \cdot q_1 \cdot R(1) + \frac{1}{1-q_4} \cdot q_2 \cdot R(2) + \frac{1}{1-q_4} \cdot q_3 \cdot R(3) = 0,81 \cdot 5,15 + 0,17 \cdot 5,61 + 0,01 \cdot 7,72 + 0,000475 \cdot 19,89 = 4,17 + 0,95 + 0,077 + 0,009 = 5,20$$

Fraction of rejected requests:



$$\begin{aligned}
 q_4 + \sum_{k=0}^3 q_k \cdot p_6(4-k) &= 0,00000625 + 0,81 \cdot 0,0016 + 0,17 \cdot 0,0036 + 0,01 \cdot \\
 0,013 + 0,000475 \cdot 0,086 &= 0,00000625 + 0,0013 + 0,0006 + 0,0001 + \\
 0,00004 &= 0,002
 \end{aligned}$$