

# Least and Greatest Fixpoints

## Transfinite Approximates

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## Fixpoints

We briefly recall few notions on fixpoints.

- Consider the equation:

$$X = f(X)$$

where  $f$  is an operator from  $2^{\mathcal{S}}$  to  $2^{\mathcal{S}}$  ( $2^{\mathcal{S}}$  denotes the set of all subsets of a set  $\mathcal{S}$ ).

- Every solution  $\mathcal{E}$  of this equation is called a **fixpoint** of the operator  $f$
- every set  $\mathcal{E}$  such that  $f(\mathcal{E}) \subseteq \mathcal{E}$  is called **pre-fixpoint**, and
- every set  $\mathcal{E}$  such that  $\mathcal{E} \subseteq f(\mathcal{E})$  is called **post-fixpoint**.
- In general, an equation as the one above may have either no solution, a finite number of solutions, or an infinite number of them. Among the various solutions, the smallest and the greatest solutions (with respect to set-inclusion) have a prominent position, if they exist.
- The the smallest and the greatest solutions are called **least fixpoint** and **greatest fixpoint**, respectively.

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# Tarski-Knaster fixpoint theorem

We say that  $f$  is **monotonic** wrt  $\subseteq$  (set-inclusion) whenever  $\mathcal{E}_1 \subseteq \mathcal{E}_2$  implies  $f(\mathcal{E}_1) \subseteq f(\mathcal{E}_2)$ .

## Theorem (Tarski'55)

Let  $\mathcal{S}$  be a set, and  $f$  an operator from  $2^{\mathcal{S}}$  to  $2^{\mathcal{S}}$  that is monotonic wrt  $\subseteq$ . Then:

- There exists a unique least fixpoint of  $f$ , which is given by  $\bigcap \{\mathcal{E} \subseteq \mathcal{S} \mid f(\mathcal{E}) \subseteq \mathcal{E}\}$ .
- There exists a unique greatest fixpoint of  $f$ , which is given by  $\bigcup \{\mathcal{E} \subseteq \mathcal{S} \mid \mathcal{E} \subseteq f(\mathcal{E})\}$ .

## Proof of Tarski-Knaster theorem: least fixpoint

We start by showing the proof for the **least fixpoint** part. (The proof for the greatest fixpoint is analogous, see later).

Let us define  $\mathcal{L} = \bigcap \{\mathcal{E} \subseteq \mathcal{S} \mid f(\mathcal{E}) \subseteq \mathcal{E}\}$ .

### Lemma

$$f(\mathcal{L}) \subseteq \mathcal{L}$$

### Proof.

- For every  $\mathcal{E}$  such that  $f(\mathcal{E}) \subseteq \mathcal{E}$ , we have  $\mathcal{L} \subseteq \mathcal{E}$ , by definition of  $\mathcal{L}$ .
- By monotonicity of  $f$ , we have  $f(\mathcal{L}) \subseteq f(\mathcal{E})$ .
- Hence  $f(\mathcal{L}) \subseteq \mathcal{E}$  (for every  $\mathcal{E}$  such that  $f(\mathcal{E}) \subseteq \mathcal{E}$ ).
- But then  $f(\mathcal{L})$  is contained in the intersection of all such  $\mathcal{E}$ , so we have  $f(\mathcal{L}) \subseteq \mathcal{L}$ .



## Proof of Tarski-Knaster theorem: least fixpoint

### Lemma

$$\mathcal{L} \subseteq f(\mathcal{L})$$

### Proof.

- By the previous lemma, we have  $f(\mathcal{L}) \subseteq \mathcal{L}$ .
- But then  $f(f(\mathcal{L})) \subseteq f(\mathcal{L})$ , by monotonicity.
- Hence,  $\bar{\mathcal{E}} = f(\mathcal{L})$  is such that  $f(\bar{\mathcal{E}}) \subseteq \bar{\mathcal{E}}$ .
- Thus,  $\mathcal{L} \subseteq f(\mathcal{L})$ , by definition of  $\mathcal{L}$ .



## Proof of Tarski-Knaster theorem: least fixpoint

The previous two lemmas together show that  $\mathcal{L}$  is indeed a fixpoint:  $\mathcal{L} = f(\mathcal{L})$ . We still need to show that is the **least** fixpoint.

### Lemma

$\mathcal{L}$  is the **least** fixpoint: for every  $f(\mathcal{E}) = \mathcal{E}$  we have  $\mathcal{L} \subseteq \mathcal{E}$ .

### Proof.

By contradiction.

- Suppose not. Then there exists an  $\hat{\mathcal{E}}$  such that  $f(\hat{\mathcal{E}}) = \hat{\mathcal{E}}$  and  $\hat{\mathcal{E}} \subset \mathcal{L}$ .
- Being  $\hat{\mathcal{E}}$  a fixpoint (i.e.,  $f(\hat{\mathcal{E}}) = \hat{\mathcal{E}}$ ), we have in particular  $f(\hat{\mathcal{E}}) \subseteq \hat{\mathcal{E}}$ .
- Hence by definition of  $\mathcal{L}$ , we get  $\mathcal{L} \subseteq \hat{\mathcal{E}}$ . Contradiction.



## Proof of Tarski-Knaster theorem: greatest fixpoint

Now we prove the **greatest fixpoint** part.

Let us define  $\mathcal{G} = \bigcup \{ \mathcal{E} \subseteq \mathcal{S} \mid \mathcal{E} \subseteq f(\mathcal{E}) \}$ .

### Lemma

$$\mathcal{G} \subseteq f(\mathcal{G})$$

### Proof.

- For every  $\mathcal{E}$  such that  $\mathcal{E} \subseteq f(\mathcal{E})$ , we have  $\mathcal{E} \subseteq \mathcal{G}$ , by definition of  $\mathcal{G}$ .
- Consider now  $e \in \mathcal{G}$ . Then there exists an  $\hat{\mathcal{E}}$  such that  $\hat{\mathcal{E}} \subseteq f(\hat{\mathcal{E}})$ ,  $e \in \hat{\mathcal{E}}$ , by definition of  $\mathcal{G}$ .
- But  $\hat{\mathcal{E}} \subseteq \mathcal{G}$ , and by monotonicity  $f(\hat{\mathcal{E}}) \subseteq f(\mathcal{G})$ , hence  $e \in f(\mathcal{G})$ .

□

## Proof of Tarski-Knaster theorem: greatest fixpoint

### Lemma

$$f(\mathcal{G}) \subseteq \mathcal{G}$$

### Proof.

- By the previous lemma we have  $\mathcal{G} \subseteq f(\mathcal{G})$
- But then, we have that  $f(\mathcal{G}) \subseteq f(f(\mathcal{G}))$ , by monotonicity.
- Hence,  $\bar{\mathcal{E}} = f(\mathcal{G})$  is such that  $\bar{\mathcal{E}} \subseteq f(\bar{\mathcal{E}})$ .
- Thus,  $f(\mathcal{G}) \subseteq \mathcal{G}$ , by definition of  $\mathcal{G}$ .

□

# Proof of Tarski-Knaster theorem: greatest fixpoint

The previous two lemmas together show that  $\mathcal{L}$  is indeed a fixpoint:  $\mathcal{G} = f(\mathcal{G})$ . We still need to show that is the **greatest** fixpoint.

## Lemma

$\mathcal{G}$  is the **greatest** fixpoint: for every  $\mathcal{E} = f(\mathcal{E})$  we have  $\mathcal{E} \subseteq \mathcal{G}$ .

## Proof.

By contradiction.

- Suppose not. Then there exists an  $\hat{\mathcal{E}}$  such that  $\hat{\mathcal{E}} = f(\hat{\mathcal{E}})$  and  $\mathcal{G} \subset \hat{\mathcal{E}}$ .
- Being  $\hat{\mathcal{E}}$  a fixpoint, we have  $\hat{\mathcal{E}} \subseteq f(\hat{\mathcal{E}})$ .
- Hence by definition of  $\mathcal{G}$ , we get  $\hat{\mathcal{E}} \subseteq \mathcal{G}$ . Contradiction.



# Ordinals

We will need to do more iterations than the natural numbers, so we will use ordinals.

## Ordinals

$$\begin{aligned} 0 &\doteq \emptyset \\ 1 &\doteq \{0\} = \{\emptyset\} \\ 2 &\doteq \{0, 1\} = \{\emptyset, \{\emptyset\}\} \\ 3 &\doteq \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\ &\dots \\ \omega &\doteq \{0, 1, 2, 3, \dots\} \quad \text{-these are the natural numbers, sometimes denoted by } \omega_0 \\ \omega + 1 &\doteq \{\omega, \{\omega\}\} \\ \omega + 2 &\doteq \{\omega, \{\omega\}, \{\omega, \{\omega\}\}\} \\ \omega + 3 &= \{\omega, \{\omega\}, \{\omega, \{\omega\}\}, \{\omega, \{\omega\}, \{\omega, \{\omega\}\}\}\} \\ &\dots \\ \omega_1 & \\ \omega_1 + 1 &\doteq \{\omega_1, \{\omega_1\}\} \\ \omega_1 + 2 &\doteq \{\omega_1, \{\omega_1\}, \{\omega_1, \{\omega_1\}\}\} \\ \omega_1 + 3 &\doteq \{\omega_1, \{\omega_1\}, \{\omega_1, \{\omega_1\}\}, \{\omega_1, \{\omega_1\}, \{\omega_1, \{\omega_1\}\}\}\} \\ &\dots \end{aligned}$$

An ordinal  $\lambda$  is a **limit ordinal** iff  $\lambda = \bigcup_{i < \lambda} i$ , otherwise is a **successor ordinal**.

Eg.:  $1, 2, 3, \dots$  are successor ordinals,  $\omega$  is the first limit ordinal,  $\omega + 1, \omega + 2, \omega + 3, \dots$  are successor ordinals,  $\omega_1$  is the second limit ordinal, etc.

# Approximates of least fixpoints

## Approximates for a least fixpoint

The approximates for a least fixpoint  $\mathcal{L} = \bigcap \{\mathcal{E} \subseteq \mathcal{S} \mid f(\mathcal{E}) \subseteq \mathcal{E}\}$  are as follows:

$$\begin{aligned} Z_0 &\doteq \emptyset \\ Z_1 &\doteq f(Z_0) \\ Z_2 &\doteq f(Z_1) \\ &\dots \\ Z_\omega &\doteq \bigcup_{i < \omega} Z_i \\ Z_{\omega+1} &\doteq f(Z_\omega) \\ Z_{\omega+2} &\doteq f(Z_{\omega+1}) \\ &\dots \\ Z_{\omega_1} &\doteq \bigcup_{i < \omega_1} Z_i \\ Z_{\omega_1+1} &\doteq f(Z_{\omega_1}) \\ Z_{\omega_1+2} &\doteq f(Z_{\omega_1+1}) \\ &\dots \end{aligned}$$

# Approximates of least fixpoints

## Lemma

For all successor ordinals  $i$ ,  $Z_i \subseteq Z_{i+1}$  and for all limit ordinals  $\lambda$ ,  $Z_j \subseteq Z_\lambda$  for all  $j < \lambda$ .

## Proof.

By transfinite induction on  $i$ .

- Zero:  $i = 0$ . By definition  $Z_0 = \emptyset$ , and trivially  $\emptyset \subseteq Z_1$ .
- Successor ordinals:  $i = k + 1$ . By transfinite-inductive hypothesis we assume  $Z_{k-1} \subseteq Z_k$ , and we show that  $Z_k \subseteq Z_{k+1}$ .
  - $f(Z_{k-1}) \subseteq f(Z_k)$ , by monotonicity.
  - But  $f(Z_{k-1}) = Z_k$  and  $f(Z_k) = Z_{k+1}$ , hence we have  $Z_k \subseteq Z_{k+1}$ .
- Limit ordinals:  $i = \lambda$ ,  $Z_\lambda = \bigcup Z_{j < \lambda}$  hence  $Z_j \subseteq Z_\lambda$  for all  $j < \lambda$ .



## Approximates of least fixpoints

### Lemma

For all ordinals  $i$ ,  $Z_i \subseteq \mathcal{L}$ .

### Proof.

By transfinite induction on  $i$ .

- Zero:  $i = 0$ . By definition  $Z_0 = \emptyset$ , and trivially  $\emptyset \subseteq \mathcal{L}$ .
- Successor ordinals:  $i = k + 1$ . By transfinite-inductive hypothesis we assume  $Z_k \subseteq \mathcal{L}$ , and we show that  $Z_{k+1} \subseteq \mathcal{L}$ .
  - ▶  $f(Z_k) \subseteq f(\mathcal{L})$ , by monotonicity.
  - ▶ But then  $f(Z_k) \subseteq \mathcal{L}$ , since  $\mathcal{L} = f(\mathcal{L})$ .
  - ▶ Hence, considering that  $f(Z_k) = Z_{k+1}$ , we have  $Z_{k+1} \subseteq \mathcal{L}$ .
- Limit ordinals:  $i = \lambda$ ,  $Z_\lambda = \bigcup Z_{j < \lambda}$ , since  $Z_j \subseteq \mathcal{L}$  for all  $j < \lambda$ , by transfinite-induction we have that  $Z_\lambda \subseteq \mathcal{L}$ .

□

## Approximates of least fixpoints

### Theorem (Tarski-Knaster on approximates of least fixpoints)

If for an ordinal  $\alpha$ ,  $Z_{\alpha+1} = Z_\alpha$ , then  $Z_\alpha = \mathcal{L}$ . Moreover such ordinal  $\alpha$  always exists!

### Proof.

We show only the first part of the theorem (the second part is highly nontrivial).

- $Z_\alpha \subseteq \mathcal{L}$  by the above lemma.
- On the other hand, since  $Z_{\alpha+1} = f(Z_\alpha) = Z_\alpha$ , we trivially get  $f(Z_\alpha) \subseteq Z_\alpha$ , and hence  $\mathcal{L} \subseteq Z_\alpha$  by definition of  $\mathcal{L}$ .

□

Observe also that once for some  $\alpha$ ,  $Z_{\alpha+1} = Z_\alpha$ , then for all  $\beta \geq \alpha$  we have  $Z_{\beta+1} = Z_\beta$ , by definition of approximates.

## Approximates of least fixpoints

The above theorem gives us a simple sound procedure to compute the least fixpoint:

### Least fixpoint algorithm

```
 $Z_{old} := \emptyset;$   
 $Z := f(Z_{old});$   
while  $(Z \neq Z_{old})$  {  
     $Z_{old} := Z;$   
     $Z := f(Z);$   
}
```

If in  $\mathcal{L} = \bigcap \{\mathcal{E} \subseteq \mathcal{S} \mid f(\mathcal{E}) \subseteq \mathcal{E}\}$  the set  $\mathcal{S}$  is **finite** then the above procedure **terminates** in  $|\mathcal{S}|$  steps and becomes **sound and complete**.

Notice the above procedure is **polynomial** in the size of  $\mathcal{S}$ .

## Approximates of greatest fixpoints

### Approximates for a greatest fixpoint

The approximates for the greatest fixpoint  $\mathcal{G} = \bigcup \{\mathcal{E} \subseteq \mathcal{S} \mid \mathcal{E} \subseteq f(\mathcal{E})\}$  are:

```
 $Z_0 \doteq \mathcal{S}$   
 $Z_1 \doteq f(Z_0)$   
 $Z_2 \doteq f(Z_1)$   
 $\dots$   
 $Z_\omega \doteq \bigcap_{i < \omega} Z_i$   
 $Z_{\omega+1} \doteq f(Z_\omega)$   
 $Z_{\omega+2} \doteq f(Z_{\omega+1})$   
 $\dots$   
 $Z_{\omega_1} \doteq \bigcap_{i < \omega_1} Z_i$   
 $Z_{\omega_1+1} \doteq f(Z_{\omega_1})$   
 $Z_{\omega_1+2} \doteq f(Z_{\omega_1+1})$   
 $\dots$ 
```





## Approximates of greatest fixpoints

### Theorem (Tarski-Knaster on approximates of greatest fixpoint)

If for some ordinal  $\alpha$ ,  $Z_{\alpha+1} = Z_\alpha$ , then  $Z_\alpha = \mathcal{G}$ . Moreover such ordinal  $\alpha$  always exists!

### Proof.

We show only the first part of the theorem (the second part is highly nontrivial).

- $\mathcal{G} \subseteq Z_\alpha$  by the above lemma.
- On the other hand, since  $Z_{\alpha+1} = f(Z_\alpha) = Z_\alpha$ , we trivially get  $Z_\alpha \subseteq f(Z_\alpha)$ , and hence  $Z_\alpha \subseteq \mathcal{G}$  by definition of  $\mathcal{G}$ .

□

Observe also that once for some  $\alpha$ ,  $Z_{\alpha+1} = Z_\alpha$ , then for all  $\beta \geq \alpha$  we have  $Z_{\beta+1} = Z_\beta$ , by definition of approximates.

## Approximates of greatest fixpoints

The above theorem gives us a simple sound procedure to compute the greatest fixpoint:

### Greatest fixpoint algorithm

```
Zold := S;  
Z := f(Zold);  
while (Z ≠ Zold) {  
    Zold := Z;  
    Z := f(Z);  
}
```

If in  $\mathcal{G} = \bigcup \{ \mathcal{E} \subseteq \mathcal{S} \mid \mathcal{E} \subseteq f(\mathcal{E}) \}$  the set  $\mathcal{S}$  is **finite** then the above procedure **terminates** in  $|\mathcal{S}|$  steps and becomes **sound and complete**.

Notice the above procedure is **polynomial** in the size of  $\mathcal{S}$ .

## Discussion

For simplicity we have considered fixpoint wrt set-inclusion. In fact, the only property of set inclusion that we have used is the **lattice** implicitly defined by it.

We recall that a lattice is a the partial order (defined by set inclusion in our case), with the minimal element ( $\emptyset$  in our case) and maximal element ( $\mathcal{S}$  in our case).

We can immediately extend all the results presented here to arbitrary lattices substituting to the relation  $\subseteq$  the relation  $\leq$  of the lattice.